THE $\mathcal{A}$-MODULE STRUCTURE FOR THE COHOMOLOGY OF TWO STAGE SPACES.

LEIF KRISTENSEN AND ERIK KJÆR PEDERSEN

0. Introduction

The purpose of the present paper is to give a general method for determining the structure of the cohomology of stable two-stage space (for definitions see Section 1) as a module over the Steenrod algebra $\mathcal{A}$. For convenience we only consider cohomology with coefficients in the integers in $\mathbf{Z}_2$, the integers mod 2. The general part of our paper, however, works equally well with arbitrary coefficients. A two-stage space with stable k-invariant is an H-space and the cohomology consequently a Hopf algebra. We also evaluate the diagonal in the cohomology.

The idea of the paper is to express the $\mathcal{A}$-module structure of the cohomology of a two-stage system in terms of Massey products in $\mathcal{A}$. This is done in Section 3. A Massey product in $\mathcal{A}$ (definition in section 2)

$$< A, B, C > \subset \mathcal{A},$$

is a coset defined for matrices A, B, and C with entries from $\mathcal{A}$ of type $(1, s)$, (s, t) and $(t, 1)$ such that $AB = 0$ and $BC = 0$. The indeterminacy is given by $\{AX + YC\}$ where X and Y run over all matrices of type (s, t) and $(1, t)$. When dealing with Massey products in $\mathcal{A}$ one can usually avoid indeterminacy. This is also the case in the present paper. We shall always be dealing with a particular element from the coset.

The calculation of Massey products in general is not done in the present paper. It is, however, in principle possible to calculate any Massey product in $\mathcal{A}$ as follows. Kristensen and Madsen [6] worked out the reduced diagonal $\psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of a Massey product. It contains Massey products of lower degrees together with some "primary" terms which were left undecided in Kristensen [4]. These terms, which arise from the primary terms in the Cartan formula for a secondary operation, were then computed in Kristensen [5].

Since $\ker \psi$ is spanned by $Q_i = \mathrm{Sq}^{(0,\dots,0,1)}$, this together with evaluation in projective space enables us to determine $< A, B, C >$. This has been carried out in Kristensen [5].

As an example we give in section 5 a complete determination of the $\mathcal{A}$-module structure of $H^*(E_n)$ (for definition of E_n see section 5) This has also been considered by J. Milgram. The theorem in Milgram [9] describing the $\mathcal{A}$-module structure of $H^*(E_n)$, however, is incorrect. (He does not give enough relations.) In Milgram [8] the correct structure of $H^*(E_2)$ and

$H^*(E_3)$ was found. The method used by Milgram is totally different from ours and does not work in the general case.

1. Two stage Postnikov systems

In the following sections we consider a fixed stable two stage Postnikov system

$$\begin{array}{ccc} \Omega B & \longrightarrow & \Omega B \\ \downarrow{\scriptstyle h} & & \downarrow \\ E & \longrightarrow & PB \\ \downarrow{\scriptstyle p} & & \downarrow \\ K(\mathbf{Z}_2, n) & \xrightarrow{\phi} & B \end{array} \tag{1.1}$$

where

$$B = \prod K(\mathbf{Z}_2, n(j)), \qquad \phi^*(z_{n(j)}) = \alpha_j(z_n),$$

$z_{n(j)}$ the generator of $H^{n(j)}K(\mathbf{Z}_2, n(j)); \mathbf{Z}_2), \alpha_j \in \mathcal{A}$.

As an algebra $H^*(E)$ is isomorphic to

$$H^*(K(\mathbf{Z}_2, n))//\operatorname{im}\phi^* \otimes S \tag{1.2}$$

where S is a polynomial algebra, isomorphic under h^* to the subalgebra of $H^*(\Omega B)$ generated by $\sigma(\ker\phi^*)$, σ the suspension (see Kristensen [1],[2] and [3], Smith [10]).

2. Massey products in the Steenrod algebra

Let U and V be finite dimensional vector spaces over $\mathbf{Z}_2$. Further, $\mathcal{A}(U, V)$ is the graded vector space of stable natural transformations $H^*(-, U) \to H^*(-, V)$ and $\mathcal{O}(U, V)$ is the graded vector space of cochain operations, i. e., natural transformations $C^\sharp(-, U) \to C^\sharp(-, V)$, $C^\sharp$ the normalized chain functor.

In $\mathcal{O}(U, V)$ there is a differential

$$\nabla\theta = \delta\theta + \theta\delta,$$

δ the coboundary in $C^\sharp(-, U)$, $C^\sharp(-, V)$, respectively. If we let $\ker\nabla = \mathbf{Z}\mathcal{O}(U, V)$, there is an exact sequence

$$\mathcal{O}(U, V) \to \mathbf{Z}\mathcal{O}(U, V) \to \mathcal{A}(U, V) \to 0 \tag{2.1}$$

(see Kristensen [2]), i. e.

$$\mathcal{A}(U, V) = H(\mathcal{O}(U, V), \nabla).$$

Let $\mathcal{O} = \mathcal{O}(\mathbf{Z}_2, \mathbf{Z}_2)$. For $\alpha \in \mathbf{Z}\mathcal{O}$ the exact sequence (2.1) gives the existence of a cochain operation

$$d(\alpha; -, \dots, -) \in \mathcal{O}(\mathbf{Z}_2, \oplus \dots \oplus \mathbf{Z}_2; \mathbf{Z}_2)$$

such that

$$\nabla d(\alpha; x_1, \dots, x_n) = \alpha(\sum x_i) + \sum \alpha(x_i).$$

We may assume $d(\alpha; -)$ normalized, i. e.

$$d(\alpha; 0, \dots, 0, x, 0, \dots, 0) = 0. \tag{2.2}$$

In the sequel we need the following lemma about cochain operations. Consider a relation in $\mathcal{A}$ of the form

$$\beta \operatorname{Sq}^{n+1+\deg\alpha} \alpha + c,$$

where c is a sum of monomials in Sq^i, either admissible or of excess $> n+1$. Let sq^i, $\overline{\beta}$, $\overline{\alpha}$ and $\overline{c}$ be elements in $\mathbf{Z}\mathcal{O}$ representing Sq^i, β, α and c respectively. (sq^i is further required to vanish on cocycles of dimension $\leq i-1$ and on cochains of dimension $\leq i-2$.) Then (of course)

$$\overline{\beta} \operatorname{sq}^{n+1+\deg\alpha} \overline{\alpha} + \overline{c}$$

in $\mathbf{Z}\mathcal{O}$ is actually a boundary i. e. there exists $\theta \in \mathcal{O}$ with

$$\nabla\theta = \overline{\beta} \operatorname{sq}^{n+1+\deg\alpha} \overline{\alpha} + \overline{c}.$$

Lemma 2.1. *θ may be chosen such that*

$$\theta(x) = 0 \quad \textit{for} \quad \dim x < n,$$
$$\theta(x) = \sum \overline{\beta}'\overline{\alpha}(x)\overline{\beta}''\overline{\alpha}(x), \quad \dim x = n, \delta x = 0,$$

where

$$\psi(\beta) = \sum \beta' \otimes \beta'' + \hat{\beta} \otimes \hat{\beta} + \sum \beta'' \otimes \beta',$$

ψ the diagonal in $\mathcal{A}$

Proof. See Kristensen [2] □

This is called a $\frac{1}{2}$ Cartan formula.
There are pairings

$$\mathcal{A}(V, W) \otimes \mathcal{A}(U, V) \to \mathcal{A}(U, W)$$
$$\mathcal{O}(V, W) \otimes \mathcal{O}(U, V) \to \mathcal{O}(U, W)$$

given by the composition. The latter is left distributive, but because of the lack of additivity of cochain operations it is not right distributive.

Now we proceed to define Massey products. Assume given elements

$$A \in \mathcal{A}(V; \mathbf{Z}_2), \qquad B \in \mathcal{A}(U, V), \qquad C \in \mathcal{A}(\mathbf{Z}_2, U)$$

such that $AB = 0$, $BC = 0$. Choose cochain operations $\overline{A}$, $\overline{B}$, $\overline{C}$ to represent A, B, C. Then there exist $\vartheta \in \mathcal{O}(U; \mathbf{Z}_2)$, $\eta \in \mathcal{O}(\mathbf{Z}_2, V)$ such that

$$\nabla\vartheta = \overline{AB}, \qquad \nabla\eta = \overline{BC}$$

and

$$M = \vartheta\overline{C} + \overline{A}\eta + d(\overline{A}; \eta\delta, \delta\eta)$$

is (as one can easily check) in $\mathbf{Z}\mathcal{O}$, thus determines an element in $\mathcal{A}$. (Note that by (2.2) $d(\overline{A}; \eta\delta, \delta\eta)$ vanish on cocycles.) Varying ϑ and η we get a coset in $\mathcal{A}$, the Massey product $< A, B, C >$. In what follows, however, ϑ and η will be chosen once and for all, so there will be no indeterminacy in the computations.

Since $\mathcal{A}(U, V)$ may be considered $\dim U \times \dim V$ matrices with entries from $\mathcal{A}$, we may reformulate the concept of Massey product in terms of second order relations.

Let $r \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ such that $(\mu \otimes 1)(r) = (1 \otimes \mu)(r) = 0$. Then r can be written

$$r = \sum_{i,j} \alpha_i \otimes \beta_{i,j} \otimes \gamma_j, \tag{2.3}$$

where $\sum_i \alpha_i\beta_{i,j}$, $\sum_j \beta_{i,j}\gamma_j$ are relations in $\mathcal{A}$.

Choose cochain operations $\overline{\alpha}_i$, $\overline{\beta}_{i,j}$, $\overline{\gamma}_j$ to represent α_i, $\beta_{i,j}$, γ_j and also R_i, S_j in $\mathcal{O}$ such that

$$\nabla R_j = \sum_i \overline{\alpha}_i\overline{\beta}_{i,j}, \qquad \nabla S_i = \sum_j \overline{\beta}_{i,j}\overline{\gamma}_j.$$

The Massey product corresponding to (2.3) will then be represented by

$$\sum_j R_j\overline{\gamma}_j + \sum_i \overline{\alpha}_i S_i + \sum_i d(\overline{\alpha}_i; \delta S_i, S_i\delta) + \sum_i d(\overline{\alpha}_i; \overline{\beta}_{i,1}\overline{\gamma}_1, \overline{\beta}_{i,2}\cdots).$$

Putting $r_j = \sum \alpha_i\beta_{i,j}$, $s_i = \sum_j \beta_{i,j}\gamma_j$ we normally will write the second order relation as

$$\sum r_j\gamma_j + \sum \alpha_i s_i.$$

In the definition of the Massey product there is indeterminacy arising from the choice of R_j, S_i. However, in this paper the Massey product is evaluated on a class z annihilated by γ_j. Hence the indeterminacy arising from the choice of R_j disappears.

We shall also consider secondary cohomology operations

$$Qu(s_i) : H^* \to H^{*+\deg(s_i)-1}$$

associated with the relations s_i. Let $\overline{z}$ be a cocycle representing z in H^*. Then $Qu(s_i)$ is defined on z if $\gamma_j(z) = 0$. For a given choice of S_i, $Qu(s_i)(z)$ is the class (actually coset) represented by

$$S_i(\overline{z}) + \sum \overline{\beta}_{i,j}(W_j),$$

where $\delta w_j = \overline{\gamma}_j(\overline{z})$. If we alter S_i by adding an element $\overline{\epsilon} \in \mathbf{Z}\mathcal{O}$, $S_i' = S_i + \overline{\epsilon}$ then we get a new secondary operation $Qu(s_i)'$, and

$$Qu(s_i) - Qu(s_i)' = \epsilon, \qquad \epsilon = \operatorname{cls}\overline{\epsilon} \in \mathcal{A}.$$

From this it follows that once we decide a choice of secondary cohomology operation $Qu(s_i)$ we have a fixed S_i up to a boundary ($\epsilon = 0$ iff $\bar{\epsilon} = \nabla\bar{\epsilon}'$). Hence there is no indeterminacy on our Massey products at all when we decide a choice of $Qu(s_i)$.

3. $H^*(E)$ as a module over $\mathcal{A}$

Let $S' = h^*(S) = h^*H^*(E)$ (see (1.1). Then S' is a left $\mathcal{A}$-module, and we have the following

Theorem 3.1. *Let S' be generated as left $\mathcal{A}$-module by*

$$\{\sum_i \beta_i^j(z_{n(i)-1})\}_j$$

and let a generating set of relations be given by

$$\{\sum_j \lambda_j^s(\sum_i \beta_i^j(z_{n(i)-1}))\}_s.$$

Then as a module over the Steenrod algebra $H^*(E)$ is generated by z, $B_j(z)$, where $z = p^*z_n$, and B_j is a secondary operation associated with the relation (3.1) below. A generating set of relations is given by

$$\sum_j \lambda_j^s B_j(z) = M^s(z) + \operatorname{cls}(Q(z)),$$

where M^s is the Massey product associated with the second order relation (3.4) below. The element $\operatorname{cls}(Q(z))$ is of the form $\sum \alpha(z)\beta(z)$, $\alpha, \beta \in \mathcal{A}$. It is associated with the relation q in (3.5) below and is specifically computable using Lemma 2.1.

Proof. The proof will consist of a sequence of lemmas. Let

$$\sum \beta_i z_{n(i)-1} \in S'.$$

Then since S' is generated on $\sigma \ker \phi^*$), we have

$$\phi^*(\sum_i \beta_i z_{n(i)}) = \sum \beta_i \alpha_i z_n = 0.$$

Hence $\sum \beta_i \alpha_i$ is an unstable relation in $\mathcal{A}$, that is, there exists a unique sum c of admissible monomials in $\mathcal{A}$ of excess larger than n such that

$$\sum \beta_i \alpha_i + c \tag{3.1}$$

is a relation. Let B be the corresponding unstable secondary operation. Since $\alpha_j(z) = 0$, the operation B is defined on $z \in H^*(E)$, and we have

Lemma 3.2. $H^*(B(z)) = \sum_i \beta_i(z_{n(i)-1})$.

Proof. Let $\overline{z}_n$ be the basic cocycle of $K(\mathbf{Z}_2, n)$, $\overline{z} = p^*(\overline{z}_n)$, and let $\overline{\beta}_i$, $\overline{\alpha}_i$, $\overline{c}$ be cochain operations representing β_i, α_i, c. Choose cochains u_i such that $\delta u_i = \overline{\alpha}_i(\overline{z})$ and a cochain operation R such that $\nabla R = \sum_i \overline{\beta}_i \overline{\alpha}_i + \overline{c}$. Then

$$R(\overline{z}) + \sum_i \overline{\beta}_i(u_i)$$

is a cocycle representing $B(z)$. We get

$$h^\sharp(R(\overline{z}) + \sum \overline{\beta}_i(u_i)) = R((ph)^\sharp(\overline{z}_n)) + \sum_i \overline{\beta}_i(h^\sharp u_i).$$

In the Serre spectral sequence of the fibration

$$\Omega B \to E \to K(\mathbf{Z}_2, n)$$

we get from diagram (1.1) by naturality the differentials

$$d_{n(j)}(z_{n(j)-1}) = \alpha_j(z_n).$$

Since $\delta u_i = \overline{\alpha}_i(\overline{z})$ and $d_{n(j)}$ is induced by the coboundary in $C^\sharp(E)$, we get

$$\mathrm{cls}(h^\sharp u_i) = z_{n(i)-1}.$$

With notation as in Theorem 3.1, Lemma 3.2 gives

$$h^*(\sum_j \lambda_j B_j(z)) = \sum_j \lambda_j (\sum_i \beta_i^j(z_{n(i)-1})) = 0,$$

and therefore

$$\sum_j \lambda_j \beta_i^j(z_{n(i)-1}) = 0$$

for all i. Hence $e_i \in \mathcal{A}$ such that e_i is a sum of admissible monomials, $\mathrm{exc}\, e_i > n(i) - 1$ and

$$\sum_j \lambda_j \beta_i^j + e_i \tag{3.2}$$

is a relation.

We write e_i as a sum of two terms, $\mathrm{Sq}^p \gamma_i$ (for dimensional reasons p does not depend on i) of excess $n(i)$ and f_i of excess $> n(i)$:

$$e_i = \mathrm{Sq}^p \gamma_i + f_i, \qquad p - \deg(\gamma_i) = n(i). \tag{3.3}$$

From (3.2) and (3.3) we get

$$(\sum \gamma_i \alpha_i z_n)^2 = (\sum \gamma_i \phi^* z_{n(i)})^2 = \phi^*(\sum \mathrm{Sq}^p \gamma_i z_{n(i)}) = \phi^*(\sum_i e_i z_{n(i)})$$
$$= \phi^*(\sum_j \lambda_j (\sum_i \beta_i^j z_{n(i)}) = \sum_l \lambda_j \phi^*(\sum_i \beta_i^j z_{n(i)}) = 0.$$

Consequently

$$\sum \gamma_i \alpha_i(z_n) = 0$$

and there is a unique sum g of admissible monomials of excess larger than n such that

$$\sum \gamma_i \alpha_i + g = 0$$

Consider the second order relation

$$\sum_j \lambda_j r_j + \sum_i s_i \alpha_i + \operatorname{Sq}^p t + q, \tag{3.4}$$

$$\begin{aligned} r_j &= \sum_i \beta_i^j \alpha_i + c_j, \\ s_i &= \sum_j \lambda_j \beta_i^j + \operatorname{Sq}^p \gamma_i + f_i, \\ t &= \sum_i \gamma_i \alpha_i + g, \\ q &= \operatorname{Sq}^p g + \sum_j \lambda_j c_j + \sum_i f_i \alpha_i, \end{aligned} \tag{3.5}$$

and choose cochain operations R_i, S_i, T, Q such that

$$\nabla R_i = r_i, \quad \nabla S_i = s_i, \quad \nabla T = t, \quad \nabla Q = q.$$

Lemma 3.3. *Let notation be as above and let M denote the Massey product corresponding to (3.4). Then*

$$\sum \lambda_j B_j(z) = M(z) + \operatorname{cls}(Q(\overline{z})).$$

Furthermore since $\operatorname{exc} q > n$, *$Q(\overline{z})$ is given by a $\frac{1}{2}$ Cartan formula.*

Proof. $\sum_j \lambda_j B_j(z)$ is represented by

$$\begin{aligned} &\sum_j (\overline{\lambda}_j (R_j(\overline{z}) + \sum_i \overline{\beta}_i^j u_i)) \\ &\qquad = \sum_j \overline{\lambda}_j R_j(\overline{z}) + \sum_i \sum_j \overline{\lambda}_j \overline{\beta}_i^j u_i + \sum_j \nabla d(\overline{\lambda}_j, R_j(\overline{z}) + \sum_i \overline{\beta}_i^j u_i, \overline{\beta}_1^j, \ldots). \end{aligned}$$

This expression is related to the Massey product. From (3.5) we have

$$\sum_i \sum_j \overline{\lambda}_j \overline{\beta}_i^j u_i = \sum_i \nabla S_i u_i + \sum_i \operatorname{sq}^p \overline{\gamma}_i u_i + \sum_i \overline{f}_i u_i$$

and

$$\overline{f} u_i = 0 \quad \text{since} \quad \dim u_i < \operatorname{exc} f_i - 1.$$

$$\sum_i \operatorname{sq}^p \overline{\gamma}_i u_i$$
$$= \operatorname{sq}^p(T(\overline{z}) + \sum_i \overline{\gamma}_i u_i) + \operatorname{sq}^p(T\overline{z}) + \nabla d(\operatorname{sq}^p, T(\overline{z}) + \sum_i \overline{\gamma}_i u_i, \overline{\gamma}_1 u_1, \ldots)$$

and $T(\overline{z}) + \sum \overline{\gamma}_i u_i$ is a cocycle of dimension $p-1$ such that

$$\sum_i \operatorname{sq}^p \overline{\gamma}_i u_i = \operatorname{sq}^p(T\overline{z}) + \nabla d(\operatorname{sq}^p; T(\overline{z}) + \sum_i \overline{\gamma}_i u_i, \overline{\gamma}_1 u_1, \ldots).$$

Furthermore

$$\nabla d(\overline{\lambda}_j; R_j(\overline{z}) + \sum_i \overline{\beta}^j_i u_i, \beta^j_1 u_1, \ldots) \sim d(\overline{\lambda}_j; \overline{c}_j(\overline{z}), \overline{\beta}^j_1 \overline{\alpha}_1 \overline{z}, \ldots),$$
$$\nabla d(\operatorname{sq}^p; T(\overline{z}) + \sum \overline{\gamma}_i u_i, \overline{\gamma}_1 u_1, \ldots) \sim d(\operatorname{sq}^p; \overline{g}(\overline{z}), \overline{\gamma}_1 \overline{\alpha}_1 \overline{z}_1, \ldots),$$
$$\sum_i \nabla S_i u_i \sim \sum S_i \overline{\alpha}_i(\overline{z}).$$

Hence

$$\sum_j \overline{\lambda}_j (R_j(\overline{z}) + \sum_i \overline{\beta}^j_i u_i) \sim \overline{M}(\overline{z}) + Q(\overline{z}).$$

□

This ends the proof of Theorem 3.1.

4. Diagonal in $H^*(E)$

Since E is an H-space, there is a Hopf-algebra structure on $H^*(E)$ with diagonal ψ^*, where ψ is the multiplication. By Theorem 1 a generator of $H^*(E)$ is of the form $B(z)$, where B is a secondary operation corresponding to a relation of the form

$$\sum_i \beta_i \alpha_i + c$$

where $\operatorname{exc} c > n$. To compute $\psi^*(B(z))$ write $c = \operatorname{Sq}^{j_i} \operatorname{Sq}^{J_i} + d$, where $j_i - \deg J_i = n+1$, $\operatorname{exc} d > n+1$. We have

$$\psi^*(z) = z \times 1 + 1 \times z,$$

and by naturality

$$\begin{aligned} \psi^*(B(z)) &= B(z \times 1 + 1 \times z) \\ &= B(z \times 1) + B(1 \times z) + d(c; z \times 1, 1 \times z) \\ &= 1 \times B(z) + B(z) \times 1 + \sum_i \operatorname{Sq}^{J_i}(z \times 1) \operatorname{Sq}^{J_i}(1 \times z) \\ &= 1 \times B(z) + B(z) \times 1 + \sum_i \operatorname{Sq}^{J_i}(z) \times \operatorname{Sq}^{J_i}(z). \end{aligned}$$

In this computation we have used Kristensen [2].

This gives a complete description of the diagonal.

5. EXAMPLES

Let us consider the two-stage Postnikov system

$$\begin{array}{ccc} K\mathbf{Z}_2, 2n-1) & \longrightarrow & K(\mathbf{Z}_2, 2n-1) \\ \downarrow & & \downarrow \\ E_n & \longrightarrow & P(\mathbf{Z}_2, n-1) \\ \downarrow & & \downarrow \\ K(\mathbf{Z}_2, n) & \xrightarrow{\phi} & K(\mathbf{Z}_2, 2n) \end{array}$$

where $\phi^*(z_{2n}) = z_n^2$.

Let ζ be the dual of the squaring homomorphism $\mathcal{A}^* \to \mathcal{A}^*$. Then $\zeta(\mathrm{Sq}^{2i}) = \mathrm{Sq}^i$, $\zeta(\mathrm{Sq}^{2i+1}) = 0$, and we have for $\alpha \in \mathcal{A}$,

$$\phi^*(\alpha z_{2n}) = \alpha\phi^* z_{2n} = \alpha z_n^2 = ((\zeta\alpha)(z_n))^2.$$

So wee see that

$$H^*(E_n) = \bigwedge[\mathrm{Sq}^I(z), I\mathrm{adm}, \mathrm{exc}\, I < n] \otimes S,$$

where $z = p^*(z_n)$ and S is isomorphic to the subalgebra of $H^*(K(\mathbf{Z}_2, 2n-1))$ generated by $\alpha(z_{2n-1})$, $\alpha \in \ker\zeta$.

By Theorem 3.1 we need to find generators and relations for $S'h^*(S)$ as a left $\mathcal{A}$-module.

Lemma 5.1. *As a left $\mathcal{A}$-module S' is generated by $t_1, \ldots, t_{2i+1}, \ldots, t_{2n-1}$, where $t_{2i+1} = \mathrm{Sq}^{2i+1}(z_{2n-1})$. A generating set of relations is*

$$r^{p,q} = \mathrm{Sq}^{2p+1}\, t_{2q+1} + \sum_{j=0}^{[p-\frac{1}{2}]} \binom{q-1-j}{p-1-2j} \mathrm{Sq}^{2p+2q+1-2j}\, t_{2j+1}, \qquad p \leq 2q < n$$

and

$$\begin{aligned} s^{p,q} = \mathrm{Sq}^{4p+2}\, t_{2q+1} &+ \mathrm{Sq}^{2(p+q+1}\, t_{2j+1}, \\ &+ \sum_{j+q+1}^{2p+q+1} \binom{p-1-j}{-q+j} \mathrm{Sq}^{2(2p+q+1-j)}\, t_{2j+1}, \qquad p < q < n. \end{aligned}$$

Remark. The binomial coefficient $\binom{n}{i}$ is the coefficient to t^i in the power series

$$(1+t)^n \in \mathbf{Z}_2[[t]], \qquad n, i \in \mathbf{Z}$$

Proof. In Milgram [8] it is proved that $\mathrm{Sq}^t z_{2n-1}$, $t = 2^i - 1$, generate S' as a left $\mathcal{A}$-module. For convenience we take more generators, so the first assertion is obvious. To prove we have enough relations, remember that S' is a polynomial algebra, so we will prove that $\mathrm{Sq}^i t_{2j+1}$ can be written as a sum of polynomial generators by means of the relations $r^{p,q}$, $s^{p,q}$. Using Adem relations it is easy to show that we can take

$$\{\mathrm{Sq}^i t_{2q+1} | i \geq 4q+2 \text{ or } i \equiv 0 \mod 4\}$$

as some of the polynomial generators, so assume $i < 4q+2$ and $i = 2p+1$ or $i = 4p+2$. If $i = 2p+1$, then $r^{p,q}$ immediately expresses $\mathrm{Sq}^i t_{2q+1}$ as a sum of polynomial generators, and if $i = 4p+2$, then repeated application of $s^{p,q}$ will express $\mathrm{Sq}^i t_{2q+1}$ as a sum of polynomial generators. □

Remark. Note that the relations $r^{p,q}$ and $s^{p,q}$ are derived from the Adem relation

$$\mathrm{Sq}^{2p+1} \mathrm{Sq}^{2q+1} + \sum \binom{2q-j}{2p+1-2j} \mathrm{Sq}^{2p+2q+2-j} \mathrm{Sq}^j$$

and from the relation $r(2(p+q+1), 2p+1; 4p+2q+3)$ in Kristensen [4]

By Theorem 3.1 we have that $H^*(E_n)$ is generated by

$$z, B_1(z), \dots, B_{2i+1}(z), \dots, B_{2n-1}(z),$$

where B_{2i+1} is a secondary operation associated with the relation

$$\mathrm{Sq}^{2i+1} \mathrm{Sq}^n + (\sum \binom{n-1-j}{2i+1-2j} \mathrm{Sq}^{n+2i+1-j} \mathrm{Sq}^j).$$

A generating set of relations is given by Massey products of certain second-order relations. These Massey products will be computed now.

First consider the second order relation

$$\mathrm{sq}^{2p+1} \tau_{2q+1} + \sum_j \binom{q-1-j}{p-1-2j} \mathrm{sq}^{2p+2q+1-2j} \tau_{2j+1} + r^{p,q} \mathrm{sq}^n + q'$$

where

$$\tau_{2j+1} = \mathrm{sq}^{2j+1} \mathrm{sq}^n + \sum_m \binom{n-1-m}{2j+1-2m} \mathrm{sq}^{2j+1+n-m} \mathrm{sq}^m,$$

$$r^{p,q} = \mathrm{sq}^{2p+1} \mathrm{sq}^{2q+1} + \sum_j \binom{q-1-j}{p-1-2j} \mathrm{sq}^{2p+2q+1-2j} \mathrm{sq}^{2j+1}$$

and

$$q = \mathrm{sq}^{2p+1}(\sum_m \binom{n-1-m}{2q+1-2m} \mathrm{sq}^{2q+1+n-m} \mathrm{sq}^m)+ \\ + \sum_j \binom{q-1-j}{p-1-2j} \mathrm{sq}^{2p+2q-2j+1}(\sum_m \binom{n-1-m}{2j+1-2m} \mathrm{sq}^m),$$

and denote the corresponding Massey product by $M^{p,q}$.

Write $M^{p,q} = a + b$, where a is a sum of admissible monomials each of excess less than or equal to $n-1$ and $\operatorname{exc} b \geq n$, then since $z^2 = 0$ in $H^*(E_n)$,

$$M^{p,q}(z) = a(z)$$

We can compute a by evaluating $M^{p,q}$ in $H^*(K(\mathbf{Z}_2, n-1))$. Let T_{2j+1}, $R^{p,q}$ and Q be cochain operations such that $\nabla T_{2j+1} = \tau_{2j+1}$, $\nabla R^{p,q} = r^{p,q}$ and $\nabla Q = q$, chosen as in Lemma 2.1. Let $\overline{z}_{n-1} \in C^{n-1}(K(\mathbf{Z}_2, n-1))$ be the basic cocycle. Then

$$\begin{aligned} \overline{M}^{p,q}(\overline{z}_{n-1}) &= \mathrm{sq}^{2p+1} T_{2q+1}(\overline{z}_{n-1}) + \sum_j \binom{q-1-j}{p-1-2j} \mathrm{sq}^{2p+2q+1-2j} T_{2j+1}(\overline{z}_{n-1})+ \\ &\quad + R^{p,q} \mathrm{sq}^n(\overline{z}_{n-1}) + Q(\overline{z}_{n-1}) \\ &= \mathrm{sq}^{2p+1}(\sum_{i\leq q} \mathrm{sq}^{2j+1-i} \overline{z}_{n-1} \mathrm{sq}^i \overline{z}_{n-1})+ \\ &\quad + \sum_j \binom{q-1-j}{p-1-2j} \mathrm{sq}^{2p+2q+1-2j}(\sum_{i\leq j} \mathrm{sq}^{2j+1-i} \overline{z}_{n-1} \mathrm{sq}^i \overline{z}_{n-1}). \end{aligned} \tag{5.1}$$

Since $M^{p,q}$ is stable, this must be equal to

$$(\alpha(\overline{z}_{n-1}))^2 = \mathrm{sq}^{p+q+n} \alpha(\overline{z}_{n-1}),$$

for some α in $\mathcal{A}$. (An alternate argument for this is that suspension and reduced diagonal both vanish).

Now we consider the second order relation

$$\mathrm{sq}^{4p+2} \tau_{2q+1} + \mathrm{sq}^{2(p+q+1)} \tau_{2p+1} + \sum_{j>q} \binom{p-1-j}{-q+j} \mathrm{sq}^{2(2p+q+1-j)} \tau_{2j+1} + s^{p,q} \mathrm{sq}^n + q,$$

where

$$\tau_{2j+1} = \mathrm{sq}^{2j+1} \mathrm{sq}^n + \sum_k \binom{n-1-k}{2j+1-2k} \mathrm{sq}^{n+2j+1-k} \mathrm{sq}^k,$$

$$s^{p,q} = \mathrm{sq}^{4p+2} \mathrm{sq}^{2p+1} + \mathrm{sq}^{2(p+q+1)} \mathrm{sq}^{2p+1} + \sum_{j>q} \binom{p-1-j}{-q+j} \mathrm{sq}^{2(2p+q+1-j)} \mathrm{sq}^{2j+1},$$

and

$$\begin{aligned}q = &\operatorname{sq}^{4p+2}(\sum_k \binom{n-1-k}{2p+1-2k} \operatorname{sq}^{n+2q+1-k} \operatorname{sq}^k) \\ &+ \operatorname{sq}^{2(p+q+1)}(\sum_k \binom{n-1-k}{2p+1-2k} \operatorname{sq}^{n+2p+1-k} \operatorname{sq}^k) \\ &+ \sum_{j>q} \binom{p-1-j}{-q+j} \operatorname{sq}^{2(2p+q+1-j)}(\sum_k \binom{n-1-k}{2j+1-2k} \operatorname{sq}^{n+2j+1-k} \operatorname{sq}^k),\end{aligned}$$

and denote the corresponding Massey product by $N^{p,q}$. Again we get $N^{p,q}(z_{n-1})$ written as a sum of products of primary operations on z_{n-1}, but now $\dim N^{p,q}(z_{n-1}$ is odd so $N^{p,q}(z_{n-1}) = 0$ Hence $N^{p,q}(z) = 0$. The only thing missing is a determination of $\operatorname{cls} Q(z)$. This, however, is given by a $\frac{1}{2}$ Cartan formula (Lemma 2.1).

The results concerning the cohomology of E_n are summarized in

Theorem 5.2. *As a module over the Steenrod algebra, $H^*(E_n)$ is generated by*

$$z, B_1(z), \ldots, B_{2j+1}(z), \ldots, B_{2n+1}(z),$$

where B_{2j+1} is a secondary operation associated with the relation

$$\operatorname{Sq}^{2j+1} \operatorname{Sq}^n + \sum_k \binom{n-1-k}{2j+1-2k} \operatorname{Sq}^{n+2j+1-k} \operatorname{Sq}^k .$$

The diagonal is given by

$$\psi^*(B_{2j+1}(z)) = B_{2j+1}(z) \times 1 + (n-1-j) \operatorname{Sq}^j(z) \times \operatorname{Sq}^j(z) + 1 \times B_{2j+1}(z).$$

A generating set of relations is given by

$$\begin{aligned}&\operatorname{Sq}^{2p+1} B_{2q+1}(z) + \sum_j \binom{q-1-j}{p-1-2j} \operatorname{Sq}^{2p+2q+1-2j} B_{2j+1}(z) \\ &\quad = \operatorname{Sq}^{p+q+n} \alpha(z) + (n-1-q)(\sum_{k\le p} \operatorname{Sq}^{2p+1-k} \operatorname{Sq}^q(z) \operatorname{Sq}^k \operatorname{Sq}^q(z)) + \\ &\qquad + \sum_j \binom{q-1-j}{p-1-2j} (n-1-j)(\sum_{k\le p+q-j} \operatorname{Sq}^{2p+2q+1-2j-k} \operatorname{Sq}^j(z) \operatorname{Sq}^k \operatorname{Sq}^j(z)),\end{aligned}$$

$p \leq 2q \leq n$; α as in (5.1), and

$$\begin{aligned}
\mathrm{Sq}^{4p+2} B_{2q+1}(z) + \mathrm{Sq}^{2(p+q+1)} B_{2p+1}(z) + \sum_{j>q} \binom{p-1-j}{-q+j} \mathrm{Sq}^{2(2p+q+1-j)} B_{2j+1}(z) \\
= (n-1-q)\Big(\sum_{k\leq 2p} \mathrm{Sq}^{4p+2-k} \mathrm{Sq}^q(z) \mathrm{Sq}^k \mathrm{Sq}^q(z)\Big) + \\
+ (n+1-p)\Big(\sum_{k\leq p+q} \mathrm{Sq}^{2(p+q+1)-k} \mathrm{Sq}^p(z) \mathrm{Sq}^k \mathrm{Sq}^p(z)\Big) + \\
+ \sum_{j>q} \binom{p-1-j}{-q+j} (n-1-j)\Big(\sum_{k<2p+q-j} \mathrm{Sq}^{2(2p+q+1-j)-k} \mathrm{Sq}^j(z) \mathrm{Sq}^k \mathrm{Sq}^j(z)\Big),
\end{aligned}$$

$p < q < n$.

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Aarhus Universitet, Aarhus Denmark

University of Chicago, U.S.A.