CONTINUOUSLY CONTROLLED ALGEBRAIC K-THEORY OF SPACES AND THE NOVIKOV CONJECTURE.

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1. Introduction

Let M be a smooth orientable manifold. Hirzebruch's index theorem states that the index of M is equal to the L-polynomial $\mathcal{L}(M)$ evaluated on the top class of M i. e.

$$I(M) = \mathcal{L}(M)[M].$$

Here the L-polynomial is a certain rational polynomial in the Pontrjagin classes. This theorem is surprising from many points of view: the left hand side is obviously a homotopy invariant and an integer, whereas the right hand side a priory is only a smooth invariant and a rational number. This led Novikov to the following conjecture: If M has fundamental group π and x is an element in $H^*(B\pi)$, f the classifying map, could it be that the higher signatures, $x \cup \mathcal{L}(M)[M]$ are homotopy invariants of the manifold M? From this point of view Hirzebruch's theorem is a verification of the Novikov conjecture for simply connected manifolds. Later Wall [17, section 17H] realized that the Novikov conjecture can be expressed using the assembly map $h_*(B\pi; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\pi)$. The Novikov conjecture is equivalent to the assembly map being a rational monomorphism.

Over the years it has turned out that there are lots of assembly maps. In algebraic K-theory, in C^*-theory and in A-theory among others. It has become common practice to call the statement that the assembly map is a rational monomorphism, the Novikov conjecture in that theory. In the case of C^*-theory, monicity of the assembly map implies, but is not equivalent to the classical Novikov conjecture.

In this paper we treat the A-theory case. We wish to extend the results in [6] and [7] to A-theory, using a variation of the continuously controlled A-theory in [13, 15] to replace the continuously controlled K-theory in [1]. One of the main problems here is, that as a computational device, slightly discontinuous maps were allowed in [1] and [6], and A-theory does not respond nicely to that. The answer is to work with spaces that are so locally contractible, that the slightly discontinuous maps (eventually continuous maps) can be replaced by continuous maps. Otherwise the strategy is to follow [7, 6] and [13, 15], and we shall assume the reader has some familiarity with these papers. We prove the following theorem

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Theorem A. *Let Γ be a group with finite $B\Gamma$, and assume the universal cover $E\Gamma$ admits a compactification D satisfying*

(i) *The Γ-action extends to D.*
(ii) *D is contractible.*
(iii) *Compact subsets of E become small near $F = D - E$ i. e. for every point $y \in D$, for every compact subset $K \subset E$ and for every neighborhood U of y in D, there exists a neighborhood V of y in D so that if $g \in \Gamma$ and $gK \cap V \neq \emptyset$ then $gK \subset U$.*

Let X be a space satisfying that $K_{-i}(\mathbb{Z}\pi_1(X)) = 0$ for i sufficiently large. Then the assembly map

$$B\Gamma_+ \wedge A_{-\infty}(X) \to A_{-\infty}(B\Gamma \times X)$$

is a split monomorphism of spectra.

By [2] the above conditions are satisfied for word hyperbolic groups.

In the above theorem $A_{-\infty}$ denotes the non-connective delooping of the usual A-theory spectrum, see [14]. Since the strategy of this paper follows the K- and L-theory proofs so closely, we shall mainly be emphasizing the points where the A-theory arguments are different.

The assembly map in A-theory has been extensively studied under various assumptions on the group, and with various conclusions such as rational splitting, integral splitting, integral isomorphism, see e. g. [9], [3], and [10].

2. Continuously controlled A-theory

In this section we recall and expand results from [15].

Definition 2.1. Let X be a topological space. $\mathcal{R}(X)$ denotes the category of *retractive spaces over* X. An *object* is a triple (Y, r, s) where $r : Y \to X$ is a map of topological spaces, and s is a section of the map r. A *morphism* $(Y, r, s) \to (Y', r', s')$ in $\mathcal{R}(X)$ is by definition a map $f : Y \to Y'$ satisfying that $r'f = r$, and $s' = fs$.

We will also insist that an object of this category is embedded as a subspace of $X \times \mathbb{R}^\infty$, such that the retraction is given by the projection map to X. This is important not only for set-theoretic reasons but also because we want to be able to talk about fixed points of a group action on this category in a meaningful way.

Next we want to impose control conditions on this category.

Definition 2.2. A *control pair* is a pair of compact Hausdorff spaces (D, F) such that $E = D - F$ is an open dense subset of D.

(In the applications later in this paper, E usually corresponds to $E\Gamma$ for some group Γ, and D is a compactification of E.)

Definition 2.3. Let (D, F) be a control pair, and let X denote any topological space. Let $\mathcal{R}_f(D, F; X)$ be the full subcategory of $\mathcal{R}(X \times E)$ consisting of those objects (Y, r, s) satisfying that

(i) Y is a finite-dimensional, locally finite CW-complex rel. $X \times E$
(ii) Cells of Y become small near F, i. e. for every $z \in F$ and every neighborhood U of z there exist a neighborhood V of z such that $e \cap V \neq \emptyset$ implies $e \subset U$ for every cell e in Y .
(iii) The retraction r is proper.

Eventually we want to relax the finiteness condition in this definition to include spaces which are locally finite up to homotopy, or even finitely dominated in a suitable sense.

Definition 2.4. Let Y and Y' be objects of $\mathcal{R}_f(D, F; X)$. A map (a continuous map, but not necessarily a *morphism* in the category) $f : Y \to Y'$ is called *F-controlled* if for every $z \in F$, and for every neighborhood U of z in D there exists a neighborhood $V \subset U$ such that

$$f(Y_V) \subset Y'_U$$

(where $Y_V = r^{-1}(X \times (V \cap E)), Y'_U = r'^{-1}(X \times (U \cap E)))$.

Observe that a *morphism* in $\mathcal{R}_f(D, F; X)$ is automatically F-controlled. (Just choose $V = U$.)

Definition 2.5. A morphism $f : Y \to Y'$ in $\mathcal{R}_f(D, F; X)$ is an *F-controlled homotopy equivalence* if there exists a map $g : Y' \to Y$ (not necessarily a morphism !), together with F-controlled homotopies $fg \simeq 1_{Y'}$, and $gf \simeq 1_Y$.

Observe that it is implied that the map g is F-controlled.

Definition 2.6. Let G be an open subset of F. A morphism $f : Y \to Y'$ in $\mathcal{R}_f(D, F; X)$ is called *germ of an F-controlled homotopy equivalence at G* if there exist open sets $V \subset U$ and W in D, all intersecting F in G, and a map $g : Y'_U \to Y$, such that $fg|Y'_V$ is F-controlled homotopic to the identity and such that $f(Y_W) \subset Y'_U$ and $gf|Y_W$ is F-controlled homotopic to the identity

Notice the control conditions imply the existence of W so $f(Y_W) \subset Y'_U$

We use the notations $c_F\mathcal{R}_f(D, F; X)$ resp. $c_F^G\mathcal{R}_f(D, F; X)$ to denote the subcategories of $\mathcal{R}_f(D, F; X)$ defined by restricting the morphisms to be F-controlled (resp. germs at G of F-controlled) homotopy equivalences.

With these concepts we can give the definition of the category we really are interested in, namely $\mathcal{R}_{fd}(D, F; X)$. For the objects of this category we want to allow spaces which are *retracts up to c_F-equivalence* of objects of $\mathcal{R}_f(D, F; X)$. In the following we shall denote

$\mathcal{R}_{fd}(D, F; X)$ simply by $\mathcal{R}(D, F; X)$ but the reader is cautioned to remember that it is allowing finite domination rather than just finiteness that is responsible for the non-connective nature of the spectra.

The category $\mathcal{R}(D, F; X)$ is a category with cofibrations in the sense of [16]. The class of cofibrations is given by the morphisms satisfying the obvious F-controlled version of the homotopy extension property. The c_F-equivalences (resp. c_F^G-equivalences) qualify as classes of weak equivalences in the sense of [16]. Applying Waldhausen's $S.$-construction to the categories just described we obtain the continuously controlled versions of A-theory.

Definition 2.7.

$$A^c(D, F; X) = \Omega \,|\, c_F S.\mathcal{R}(D, F; X) \,|$$

$$A^c(D, F; X)^{germ\ G} = \Omega \,|\, c_F^G S.\mathcal{R}(D, F; X) \,|$$

We want to study the functorial behavior of this construction.

Lemma 2.8. *Let $f : (D, F) \to (D', F')$ be a continuous map of pairs satisfying that $f(D - F) \subset (D' - F')$. Then there is an induced map*

$$f_\sharp : A^c(D, F; X) \to A^c(D', F'; X) \ .$$

If f is an F'-controlled homotopy equivalence, then $f_\sharp$ is a weak homotopy equivalence.

Proof. The map $f_\sharp$ is induced by the functor defined by $Y \mapsto Y \cup_{X \times E} X \times E'$. Observe that (by the compactness of D and D') f is automatically proper. Therefore $Y \cup_E E'$ is again locally finite. The continuity of $f|D$ guarantees that c_F-equivalences are mapped to c'_F-equivalences. To show that $f_\sharp$ is a homotopy equivalence if f is an F'-controlled homotopy equivalence one uses the same argument as in [13, Lemma 1.3] □

It is easy to verify that we obtain a functor $(D, F) \mapsto A^c(D, E; X)$ on a suitable category of pairs, which is homotopy invariant in a way specified in the lemma. This functor has been studied in some detail in [15]. The main result there was the following:

Theorem 2.9. *Let F be a finite complex, and let $D = cF$ be the* closed cone *on F. Then the functor $F \mapsto A^c(D, F; X)$ is a generalized reduced homology theory. The coefficients of this homology theory are given by $\Sigma A_{-\infty}(X)$.*

(Here by a generalized reduced homology theory we mean a homotopy invariant functor which maps cofibre sequences into fibrations up to homotopy, and which maps the one-point space to a contractible space.)

We will need a more general version of this theorem here. Namely for the applications we have in mind, we have to admit more general spaces than just finite complexes, e.g. compact metrizable, or even more general spaces. Accordingly we shall have to work with generalized Steenrod homology theories, whose definition we briefly recall [8, 11].

Definition 2.10. A *Steenrod homology theory* is a functor

$$f : (\text{compact metrizable spaces}) \to (\text{spectra})$$

satisfying the following axioms

(i) f is homotopy invariant

(ii) If $A \subset X$ is a closed subset then there is a fibration up to homotopy

$$f(A) \to f(X) \to f(X/A) \ .$$

(iii) If $X = \bigvee_{i=0}^{\infty} X_i$ is a (strong) countable wedge, then there is a weak homotopy equivalence

$$f(X) \to \prod_{i=0}^{\infty} f(X_i)$$

induced by the projection maps.

We will now describe the main technical tool in analyzing the excision properties of continuously controlled A-theory.

Let us fix the following data: (D, F) is a control pair as considered before, X is any topological space, C is a closed subset of F.

Definition 2.11. For any object (Y, r, s) of the category $\mathcal{R}(D, F; X)$ define its *support* by

$$\operatorname{supp}(Y) = \{e \in E \,|\, r^{-1}(X \times \{e\}) \neq s(X \times \{e\})\} \ .$$

Let $\mathcal{R}(D, F; X)_C$ be the subcategory of $\mathcal{R}(D, F; X)$ consisting of those objects Y which have *support near* C, i. e. such that $\operatorname{supp}(Y)$ is outside some neighborhood of $F - C$ in D. Moreover, let

$$A^c(D, F; X)_{\text{supp } C} = \Omega \,|\, c_F S.\mathcal{R}(D, F; X)_C \,| \ .$$

Proposition 2.12. *Let (D, F) be a control pair, and let $C \subset F$ be a closed subset. Then there is a fibration up to homotopy*

$$A^c(D, F; X)_{\textit{supp } C} \to A^c(D, F; X) \to A^c(D, F; X)^{\textit{germ } F-C} \ .$$

Proof. Let $\widetilde{\mathcal{R}}(D, F; X)$ denote the subcategory of $\mathcal{R}(D, F; X)$ of those objects (Y, r, s) which are c_F^{F-C}*-acyclic*, i. e. such that $s : X \times E \to Y$ is the germ near $F - C$ of an F-controlled homotopy equivalence. By the generic fibration theorem [16, Theorem 1.6.4] we obtain a fibration up to homotopy

$$c_F S.\widetilde{\mathcal{R}}(D, F; X) \to c_F S.\mathcal{R}(D, F; X) \to c_F^{F-C} S.\mathcal{R}(D, F; X) \ .$$

(The verification of the extension axiom for the c_F^{F-C}-equivalences which is required for the application of the generic fibration theorem involves a slight modification of the categories involved, cf. [16, proof of 3.3.1])

Observe that an object which has support near C is automatically c_F^{F-C}-acyclic. To conclude the proof of the proposition we therefore have to verify that the inclusion of categories

$c_F\mathcal{R}(D,F;X)_C \subset c_F\widetilde{\mathcal{R}}(D,F;X)$ induces a weak homotopy equivalence on K–theory. This in turn follows from an application of the *approximation theorem*, [16, thm 1.6.7]. We have to verify the following condition:

Given a morphism $f : Y \to Z$ in $c_F\widetilde{\mathcal{R}}(D,F;X)$ where Y is in $c_F\mathcal{R}(D,F;X)_C$ there exists a cofibration $i : Y \rightarrowtail Y'$ in $c_F\mathcal{R}(D,F;X)_C$, and a c_F=equivalence $f' : Y' \to Z$ such that $f = f'i$.

By a mapping cylinder argument we may without loss of generality assume that f is a cofibration. If Z is c_F^{F-C}-acyclic, there exists a controlled deformation $H : Z \times I \to Z$ (rel. $X \times E$) from the identity map to a map which factors through a certain subspace Z' which has support outside some neighborhood of $F - C$. By choosing this neighborhood small enough we can assume that Z' is a subcomplex of Z and contains Y as a subcomplex, since by assumption Y has support near C. Because the deformation is F-controlled, there exists Z'' satisfying that

(i) Z'' contains Z' as a subcomplex
(ii) $H(Z' \times I) \subset Z''$
(iii) Z'' has support near C.

Now consider the composite map $p : Z'' \rightarrowtail Z \xrightarrow{H|Z\times 1} Z' \rightarrowtail Z''$. This map is *idempotent up to F-controlled homotopy*, and one verifies that the mapping telescope of p maps to Z by a c_F-equivalence. By construction the mapping telescope has support near C, and it is dominated up to c_F-controlled homotopy by the locally finite complex Z''. This proves the proposition. □

To obtain more concrete results we will have to restrict our control pairs as suggested by the main result of [15]. It turns out though that the role of the cone in that result has to be replaced by a more sophisticated concept in order to be applicable to more general spaces. The idea of this construction is due to Milnor, [12]. For any compact metrizable space C one constructs a pair of spaces $(T(C), M(C)$ satisfying the following conditions

(i) $T(C)$ is a contractible, compact metrizable space
(ii) $M(C)$ is a contractible, countable, locally finite CW complex
(iii) $M(C)$ is an open dense subspace of $T(C)$
(iv) $T(C) - M(C) = C$
(v) $T(C)/C = M(C)^\infty$ (the one-point compactification)

The space $T(C)$ is constructed as follows: Any compact metrizable space C can be written as a countable inverse limit of *finite simplicial complexes* $C = \varprojlim C_i$. This follows by choosing a sequence of finite open covers $\mathcal{U}_i$ of C whose mesh tends to zero. C_i is then defined as the Čech nerve of $\mathcal{U}_i$. The maps $C_i \to C_{i-1}$ are *not* the induced maps, but have to chosen more carefully, see [12]. We will always assume that the covering $\mathcal{U}_0$ is trivial, so that $C_0 = *$. Define

$$M_i(C) = M(C_0 \leftarrow \cdots \leftarrow C_i)$$

the iterated mapping cylinder, and let furthermore

$$M(C) = \varinjlim M_i(C) \ .$$

There are canonical retractions $M_{i+1}(C) \to M_i(C)$, so that we obtain an inverse system, and we can therefore define

$$T(C) = \varprojlim M_i(C) \ .$$

The verification of the stated properties is easy. The construction of this "Milnor cone" can be made almost functorial. In fact, one chooses the sequence of coverings of C in a special way (a "convergent sequence of coverings") and obtains the following lemma, due to Milnor, [12].

Lemma 2.13. *Let C, D be compact metrizable spaces, and let $f : C \to D$ be a continuous map. Then there exists a map of triples*

$$\overline{f} : (T(C), M(C), C) \to (T(D), M(D), D)$$

such that $\overline{f}|C = f$, $\overline{f}|M(C)$ is proper, and cellular. Moreover, choosing another (convergent) sequence of coverings gives a pair $(T'(C), C)$ which is C–controlled homotopy equivalent to $(T(C), C)$. □

In the following we will study the controlled A-theory associated to the control pair $(T(F), F)$ for a compact metrizable space F. Though $A^c(T(F), F; X)$ is not a functor on the nose, given a map $F \to F'$, we still have an induced map $A^c(T(F), F; X) \to A^c(T(F'), F'; X)$ well defined up to homotopy, by the preceding lemma, and by lemma 2.8.

Remark 2.14. With some effort one could indeed produce an honest functor. Namely observe that the category $\mathcal{I}$ of convergent sequences of coverings is filtering. So we obtain an inverse system $\{\alpha \mapsto A^c(T_\alpha(F), F; X)\}_{\alpha \in \mathcal{I}}$ which defines a functor

$$A^c(\alpha) : (\text{compact metric spaces}) \to (\text{pro-objects of spectra})$$

and we can define $A^c(T(F), F; X) = \operatorname{holim} A^c(\alpha)$.

Proposition 2.15. *Let C be a closed subset of F. The canonical map*

$$A^c(T(C), C; X) \to A^c(T(F), F; X)_{supp\ C}$$

is a weak homotopy equivalence.

Proof. There is a filtration of $T(F)$ given by $T_i(F) = M_i(F) \cup T(C)$. The canonical map $(T(C), C) \to (T_i(F), C)$ is clearly a C-controlled homotopy equivalence. Hence by lemma 2.8 we have a homotopy equivalence $A^c(T(C), C; X) \xrightarrow{\simeq} A^c(T_i(F), C; X)$. Since $\mathcal{R}(T(F), F; X)_C = \bigcup_i \mathcal{R}(T_i(F), C; X)$, and A–theory commutes with filtering colimits, the proposition follows. □

Before proceeding further we need a (very) special case.

Lemma 2.16. $A^c(T(*), *; X) = A^c([0, 1], 1; X) \simeq *$.

Proof. This follows from the main theorem of [15], or alternatively directly from an Eilenberg swindle argument. □

Corollary 2.17. $A^c(T(F),F;X) \xrightarrow{\simeq} A^c(T(F),F;X)^{germ\ F-*}$ □

Proposition 2.18. *There is a chain of homotopy equivalences between*

$$A^c(T(F),F;X)^{germ\ F-C} \quad \text{and} \quad A^c(T(F/C),F/C;X)$$

Proof. Since we are interested in germs at $F-C$, we may assume (by an easy application of the approximation theorem) that the objects in $c_F^{F-C}\mathcal{R}(T(F),F;X)$ are trivial in a neighborhood of C, and even that they are trivial in a neighborhood of T(C). In this way we obtain a functor

$$c_F^{F-C}\mathcal{R}(T(F),F;X)_{(T(C))} \to c_{F/C}^{F/C-*}\mathcal{R}(T(F)\cup_{T(C)}T(*),F/C-*;X)\ .$$

which also satisfies the approximation hypothesis. The previous corollary, together with the identification $T(F/C)=T(F)\cup_{T(C)}T(*)$ proves the proposition. □

Next we have to verify the strong wedge axiom.

Proposition 2.19. *There is a weak homotopy equivalence*

$$A^c(T(\bigvee F_i),\bigvee F_i;X) \xrightarrow{\simeq} \prod A^c(T(F_i),F_i;X)\ .$$

Proof. Using the previous corollary again, we obtain that the categories

$$c_{\vee F_i}\mathcal{R}(T(\bigvee F_i),\bigvee F_i;X) \text{ and } c_{\vee F_i}^{\vee F_i-*}\mathcal{R}(T(\bigvee F_i),\bigvee F_i;X)$$

have the same K–theory up to homotopy. In the latter category we may assume as before that objects are trivial near $T(*)-*$. Hence this category decomposes as a product of categories $\prod_i c_{F_i}^{F_i-*}\mathcal{R}(T(F_i),F_i;X)$. By [5] K-theory of a category with cofibrations and weak equivalences commutes with infinite products up to homotopy, provided there is a cylinder functor, and the weak equivalences satisfy the cylinder axiom, which is certainly true here. This ends the proof of the proposition. □

Finally we need to show that $A^c(T(F),F;X)$ is homotopy invariant as a functor of F when F is compact metrizable. This follows from the following

Lemma 2.20. *Let F be compact metrizable. Then $A^c(T(CF),CF;X)$ is weakly contractible.*

Proof. Choose K_i so that $F=\varprojlim K_i$, so $T(F)$ may be chosen to be the inverse limit of iterated mapping cylinders. Clearly $CF=\varprojlim CK_i$, and we may choose $T(CF)$ to be the iterated mapping cylinder of CK_i compactified by CF. But this model of $T(CF)$ allows selfmaps f_i sending the i-th mapping cylinder to the cone point in CK_i, and for $j>i$ sending the cone $CK_j=K_j\times[0,1]/K_j\times 0$ linearly to $K_j\times[0,j-i/n]/K_j\times 0$, and in the mapping cylinders by linear interpolation. These maps may be used to produce an Eilenberg swindle which shows $A^c(T(CF),CF;X)$ is weakly contractible. □

Notice that since the $S.$ construction provides a functorial delooping we may consider $A^c(D, F; X)$ as taking values in the category of spectra, and we get a map of spectra

$$A^c(D, F; X) \to A^c(\Sigma D, \Sigma F; X)$$

which is functorially nullhomotopic in two ways by Eilenberg swindle thus producing maps

$$A^c(D, F; X) \to \Omega A^c(\Sigma D, \Sigma F; X) \to \dots .$$

We denote the homotopy colimit by $A^c_{-\infty}$. The preceding propositions now lead to

Theorem 2.21. *Let F be a compact metrizable space and X a space such that $K_{-i}(\mathbb{Z}\pi_1(X)) = 0$ for i sufficiently large. Then $A^c_{-\infty}(T(F), F; X)$ is a reduced Steenrod functor with value $\Sigma A_{-\infty}(X)$ on S^0. In particular if F is a finite CW complex $T(F)$ can be taken to be the cone on F and $\Omega A^c_{-\infty}(CF, F; X)$ is weakly homotopy equivalent to $F \wedge A_{-\infty}(X)$.*

Proof. The condition on X ensures that the hocolim of spectra above is finite, and thus commutes with infinite products. □

Remark 2.22. The spectrum $A_{-\infty}(X)$ is the spectrum given by the nonconnective delooping in [14]. It has connective covering the usual A-theory spectrum $A(X)$ and the negative homotopy groups are given by $K_{-i}(\mathbb{Z}[\pi_1(X)])$.

3. Splitting the A-theory assembly map

At this point it is natural to proceed along the lines of [6]. This is indeed possible to a certain extent, with some restrictions

(i) We need to add the condition that D be metrizable in theorem A.
(ii) We need to assume that the pair (D, F), where $F = D - E\Gamma$ is controlled homotopy equivalent to $(T(F), F)$

Condition (i) does exclude certain groups from being treated. Considering (ii) it seems natural to conjecture that (D, F) is automatically controlled homotopy equivalent to $(T(F), F)$ given the other conditions on the compactification, but we have not been able to prove that. This problem does not arise in the K- and L-theory situation because allowing eventually continuous maps, we do not need $(T(F), F)$, we can make do with (CF, F). Thus it is not obvious that $A^c_{-\infty}(D, F; X)$ is given as the Steenrod homology theory of F with coefficients in $A_{-\infty}(X)$. To avoid both these problems we instead utilize the Čech homology techniques developed in [7]. The point here is that we can always construct a homotopy natural transformation

$$A^c_{-\infty}(D, F; X) \to \check{h}(F; A_{-\infty}(X))$$

which will be a homotopy equivalence in case (D, F) is a metrizable pair of compact Hausdorff spaces, which is controlled homotopy equivalent to $(T(F), F)$.

Lemma 3.1. *Let E be a countable finite dimensional locally finite simplicial complex, and $\tilde{C}(E_+) = (E \times (0,1])_+$ denotes the reduced cone of the one point compactification. Then there is a strict map of pairs $f : (\tilde{C}(E_+), E_+) \to (T(E_+), E_+)$ which is a homeomorphism in a neighborhood of E, inducing homotopy equivalence*

$$A^c_{-\infty}(\tilde{C}(E_+), E_+; X) \simeq A^c_{-\infty}(T(E_+), E_+; X)$$

Proof. Filtering E by subcomplexes K_i with frontier F_i we may display E_+ as the inverse limit $E = \quad K_i/F_i$ under the projection maps. We get a map of inverse systems

$$\begin{array}{ccccccccc}
* & \longleftarrow & E_+ & \longleftarrow & E_+ & \longleftarrow & E_+ & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
* & \longleftarrow & K_1/F_1 & \longleftarrow & K_2/F_2 & \longleftarrow & K_3/F_3 & \longleftarrow & \cdots
\end{array}$$

From this, we get an induced map of the iterated mapping cylinders, and hence of the inverse limits of the iterated mapping cylinders i. e. $C(E_+) \to T(E_+)$. It is easy to construct a selfmap of $T(E_+)$ which is the identity on E_+ and on a neighborhood of E which sends the line going through the $[F_i]$-points to the point $+$ in E_+ so this way we get the map $C(E_+) \to T(E_+)$ to factor through $\tilde{C}(E_+)$. Consider the diagram of fibrations

$$\begin{array}{ccccc}
A^c_{-\infty}(\tilde{C}(E_+), E_+; X)_* & \longrightarrow & A^c_{-\infty}(\tilde{C}(E_+), E_+; X) & \longrightarrow & A^c_{-\infty}(\tilde{C}(E_+), E_+; X)^E \\
\downarrow & & \downarrow {\scriptstyle f_*} & & \downarrow \\
A^c_{-\infty}(T(E_+), E_+; X)_* & \longrightarrow & A^c_{-\infty}(T(E_+), E_+; X) & \longrightarrow & A^c_{-\infty}(T(E_+), E_+; X)^E
\end{array}$$

We have c is a homotopy equivalence since f is a homeomorphism in a neighborhood of E. To show a is a homotopy equivalence we need to show that $A^c_{-\infty}(\tilde{C}(E_+), E_+; X)$ is contractible, but this is easy since with trivial support in a neighborhood of E we may use the cone structure to produce an Eilenberg swindle towards the reduced cone point □

We briefly recall the construction of Čech homology with coefficients in a spectrum $\mathbb{T}$, and the natural transformation from controlled theory [7].

Let F be a compact Hausdorff space, and consider a finite open covering of F,

$$\alpha : \{U_1, U_2, \ldots U_n\}.$$

Define a functor G_α from subsets of $\{1, 2, \ldots, n\}$ to spectra by

$$G_\alpha(i_1, i_2, \ldots, i_s\} = \begin{cases} 1, & \text{if } U_{i_1} \cap \ldots U_{i_s} \neq \emptyset \\ \mathbb{T} & \text{if } U_{i_1} \cap \ldots U_{i_s} = \emptyset \end{cases}$$

then

$$\check{h}(F; \mathbb{T}) = \Omega \operatorname{holim}_{\alpha} \operatorname{hocolim} G_\alpha.$$

It was proved in [7] that $\check{h}(F;\mathbb{T})$ when restricted to compact metrizable spaces is a Steenrod functor.

Now consider $\mathbb{T} = A_{-\infty}(X)$ and let (D, F) be a pair of compact Hausdorff spaces such that $D - F$ is contractible. Given a covering $\alpha : \{U_1, \dots, U_n\}$, the excision results in section 1 generalize to

$$\operatorname{hocolim}(A^c_{-\infty}(D, F; X)_{U_{i_1} \cap \dots U_{i_s}}) \simeq A^c_{-\infty}(D, F; X)$$

and hence taking the homotopy limit over all finite coverings we get

$$\operatorname*{holim}_{\alpha} \operatorname{hocolim}(A^c_{-\infty}(D, F; X)_{U_{i_1} \cap \dots U_{i_s}}) \simeq A^c_{-\infty}(D, F; X)$$

Notice the natural map

$$A^c_{-\infty}(D, F; X)_{\emptyset} \to A_{-\infty}(X)$$

is a weak homotopy equivalence. This follows from the usual homotopy invariance of A-theory and the fact that $D - F$ is assumed to be contractible. Sending $A^c_{-\infty}(D, F; X)_U$ to a point when U is nonempty, and to $A_{-\infty}(X)$ when U is empty produces a map

$$\operatorname*{holim}_{\alpha} \operatorname*{hocolim}_{\alpha=(U_1,\dots,U_n)} (A^c_{-\infty}(D, F; X)_{U_{i_1} \cap \dots \cap U_{i_s}}) \to \Sigma\check{h}(F; A_{-\infty}(X)),$$

and hence we get

Recall from [4] that a *homotopy natural transformation* from a functor Φ to Ψ is a sequence of functors Φ_i and Ψ_i, for $i = 0, 1, \dots, k$, together with a family of natural transformations $\Phi_i \to \Psi_i$ and a family of natural equivalences $\Phi_i \to \Psi_{i-1}$ for $i > 0$, where $\Phi_0 = \Phi$ and $\Psi_k = \Psi$.

Theorem 3.2. *There is a homotopy natural transformation*

$$A^c_{-\infty}(D, F; X) \to \Sigma\check{h}(F; A_{-\infty}(X))$$

which is a weak equivalence when F is metrizable and (D, F) is controlled homotopy equivalent to $(T(F), F)$.

Proof. For $D = *$, $F = \emptyset$ this is clearly an equivalence, and since both sides are homology theories we immediately get that this is an equivalence for F a finite CW complex and D the cone on F. The extension to the case when F is metrizable and (D, F) is controlled equivalent to $(T(F), F)$ is proved in [12], but for the readers convenience we briefly give the argument in the following Lemma. □

Lemma 3.3. *Let h and k be reduced Steenrod homology theories and ν a natural transformation $h \to k$ inducing isomorphism on $\emptyset$. Then ν is an isomorphism on all compact metric spaces. Moreover for one of the homology theories it suffices that the wedge axiom holds for a wedge of finite CW complexes.*

Proof. It follows by standard techniques that ν is an isomorphism on finite CW complexes. Let F be a compact metric space. Milnor proves that F may be displayed as the inverse limit of finite complexes $F = \varprojlim(K_i)$,

$$\ldots \to K_i \to K_{i-1} \to \ldots K_1 \to *.$$

Denoting the mapping cylinder of $K_i \to K_{i-1}$ by M_i we have $T(F) = \cup M_i \cup F$. Since $T(F)$ is contractible ν is an isomorphism on F if and only if it is an isomorphism on $T(F)/F$. Let

$$M_{\text{even}} = \cup_{i \text{ even}} M_i \cup F.$$

We have $\cup K_i \cup F$, a subset of M_{even}, and when we collapse F we get

$$\cup K_i \cup * = \bigvee K_{i+}$$

with the strong topology. $M_{\text{even}}/\cup K_i \cup F$ also becomes a strong wedge and $T(F)/M_{\text{even}}$ is a strong wedge, so we find that ν is an isomorphism on $T(F)/F$ and hence on F. All the wedges we considered were wedges of finite complexes. $\square$

Proof of Theorem A. Consider a group Γ admitting a compactification D as in theorem A, and consider the projection $p : D \times (0,1) \to D$ where $D \times (0,1)$ is identified with an open subset of the cone CD. We define $A^c_{-\infty}(CD, CF, p; X)$ as we defined $A^c_{-\infty}(CD, CF; X)$, except along points of $F \times (0,1)$ we do not require control in the cone direction, so in Definition 2.4 the condition only has to satisfied for open sets U of the form $p^{-1}(U')$. Consider the following diagram

$$\begin{array}{ccccc}
\Omega A^c_{-\infty}(\tilde{C}E\Gamma_+, E\Gamma_+; X) & \xleftarrow{a} & \Omega A^c_{-\infty}(CD, CF \cup D, p; X) & \xrightarrow{b} & \Omega A^c_{-\infty}(\Sigma D, \Sigma F, p; X) \\
\downarrow c & & \downarrow d & & \downarrow l \\
\breve{h}(E\Gamma_+; A_{-\infty}(X)) & \xleftarrow{f} & \breve{H} & \xrightarrow{g} & \breve{T} \\
& \nwarrow h & \uparrow i & & \uparrow \\
& & \breve{h}(CF \cup D; A_{-\infty}(X)) & \xrightarrow{k} & \breve{h}(\Sigma F; A_{-\infty}(X)) l_j
\end{array}$$

Which we shall proceed to explain using the notation

$$\begin{array}{ccccc}
S & \xleftarrow{a} & H & \xrightarrow{b} & T \\
\downarrow c & & \downarrow d & & \downarrow l \\
\breve{S} & \xleftarrow{f} & \breve{H} & \xrightarrow{g} & \breve{T} \\
& \nwarrow h & \uparrow i & & \uparrow j \\
& & Y & \xrightarrow{k} & Z
\end{array}$$

First consider $\check{H}$ and $\check{T}$. We shall call a finite covering $\alpha = \{U_1, U_2, \ldots, U_n\}$ of $CF \cup D$ or of ΣF,p-saturated if $p^{-1}p(U_i) \subset U_i$ or in other words if $(x,t) \in U_i$ for some t, $0 < t < 1$ implies $(x,t) \in U_i$ for all t, $0 < t < 1$. We now define

$$\check{H} = \operatorname*{holim}_{\alpha\ p\text{-saturated}} \operatorname{hocolim}(G_\alpha),$$

and $\check{T}$ is defined similarly. The map c is the homotopy natural transformation constructed above, the maps d and l are constructed similarly to c but using only p-saturated open sets. The map a is induced by collapsing CF. The map b is induced by collapsing D. The maps i and j are induced by restriction to a subcategory of coverings. The maps h and k are induced by collapsing CF and D respectively. Notice the whole diagram is equivariant with respect to a natural action by Γ. The proof of the splitting of the A-theory assembly map is now completed by the following statements

(i) $S^\Gamma \simeq B\Gamma_+ \wedge A_{-\infty}(X)$ as spectra
(ii) a^Γ is a homeomorphism.
(iii) $S^\Gamma \to S^h\Gamma$ is a weak homotopy equivalence.
(iv) $T^\Gamma \simeq A_{-\infty}(B\Gamma \times X)$
(v) The maps c,h,i,k, and j are homotopy equivalences.

Since equivariant maps that are weak homotopy equivalences unequivariantly induce weak homotopy equivalence on homotopy fixed sets. This produces a splitting of the assembly map $S^\Gamma \to T^\Gamma$.

To see (i) notice that we defined A-theory on the basis of subsets of $E\Gamma \times X \times \mathbb{R}^\infty$. The fixed sets will be based on the setwise invariant subsets under the Γ action in other words identifiable with $A^c_{-\infty}(CB\Gamma, B\Gamma_+; X)$, but we have seen this is equivalent as spectra to $B\Gamma_+ \wedge A_{-\infty}(X)$. To see (ii) notice that the control condition along CF in the F-direction is automatic by equivariance and in the p-direction it is not required. The proof of (iii) proceeds by a cell by cell argument exactly as in [6]. To see (iv) once again notice that the control conditions along ΣF are automatically fulfilled except at the suspension points, hence we may identify

$$A^c_{-\infty}(\Sigma D, \Sigma F, p; X)^\Gamma = A^c_{-\infty}(\Sigma B\Gamma, S^0; X),$$

but a simple interpretation of the two sides reveals that

$$A^c_{-\infty}(\Sigma B\Gamma, S^0; X) = A^c_{-\infty}(I, S^0; X \times B\Gamma)$$

which is $\Sigma A_{-\infty}(X \times B\Gamma)$ by theorem 2.9 above. Finally to see (v), c is a weak homotopy equivalence by Theorem 3.2 because $E\Gamma_+$ is metrizable. h and k are homotopy equivalences since we are collapsing a contractible set in a Čech homology theory. i and j are homotopy equivalences since we are only restricting the coverings on a cone part of the space, and the restricted coverings have nerves that are cones of that subspace. □

Remark 3.4. Finally we should notice that the proof that the map considered here is indeed the assembly map proceeds exactly as in the K-theory case [6, 18]

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