IDENTIFYING ASSEMBLY MAPS IN K- AND L-THEORY

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Abstract. In this paper we prove the equivalence of various algebraically or geometrically defined assembly maps used in formulating the main conjectures in K- and L-theory, and C^*-theory.

1. Introduction

Information about the algebraic and geometric topology of a smooth manifold X is contained in the functors $K_n(\mathbb{Z}G)$ and $L_n(\mathbb{Z}G)$ defined by D. Quillen and C. T. C. Wall, where $G = \pi_1(X, x_0)$. Moreover, the topological K-theory of the reduced C^*-algebra of G also contains information about the topology of the manifold, via the indices of elliptic operators on X. These functors are algebraically defined, but are difficult to compute except in special cases, and one of the main problems in this area is to find effective ways to extend the range of these computations.

The most general approach is via *assembly maps*, based on ideas of F. Quinn [30], [31] and F. Waldhausen [39]. In this approach the functors above are reinterpreted as the homotopy groups of spectra $\mathbb{K}(\mathbb{Z}G)$ and $\mathbb{L}(\mathbb{Z}G)$ arising from homotopy invariant functors $F\colon \mathit{Spaces} \longrightarrow \mathit{Spectra}$. An assembly map is a natural transformation $\alpha\colon E \longrightarrow F$ where E is homotopy invariant and strongly excisive functor, with $E(\bullet) \simeq F(\bullet)$. These conditions imply that $\pi_*(E(X))$ satisfies the Eilenberg-Steenrod axioms, except for the dimension axiom, and thus is a generalized homology theory. In this sense the functor E is "known" and computable through spectral sequences. One can then hope to obtain information about F by studying the fibre of the (essentially unique) assembly map associated to the functor F.

This approach has been very successful: a lot of work in geometric topology has been motivated by the Novikov conjecture, which asserts that the assembly map for L-theory applied to the space $X = BG$, for any group G, induces a monomorphism on rational homotopy groups. This has been proved in many cases, suggesting that assembly maps might entirely determine the integral K- and L-theory of torsion-free discrete groups, as conjectured by W.-C. Hsiang [17] and A. Borel. The case of topological K-theory is somewhat special in the context of assembly, because the association of a group G to its reduced C^*-algebra

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is not a functor. However, P. Baum and A. Connes defined a map from the K-homology of BG to the topological K-theory of $C^*_r(G)$, and conjectured that their map induces an isomorphism for G any torsion-free discrete group (see [4] and G. Kasparov's work [19]). See [11] for a detailed survey of results in this area. One of the purposes of this paper is to give a homotopy-theoretic description of the Baum-Connes map and show that it can be interpreted as an assembly map.

The method of "assembly" is very flexible because the K- and L-theory spectra have many different models, which can be chosen to suit the technique of computation. On the other hand, it usually isn't clear from the definition of the different models for the spectra that the associated assembly maps are equivalent, or induce the ones originally defined by explicit algebraic or geometric methods on the homotopy groups. This problem becomes even more critical in the setting of the Farrell-Jones Isomorphism Conjectures, and in the Baum-Connes conjecture, where G is allowed to have torsion. The approach of Davis-Lück [8] provides a nice framework for constructing assembly maps in this generality, but does not identify these new maps with the classically defined assembly maps.

In this paper we present a unified way to show that the assembly maps arising from these different spectrum models are equivalent. From our point of view, the primary assembly map (see § 7) is constructed from functors on the continuously controlled categories introduced in [2]. In § 2 we give some basic material on spectra and introduce the main examples of geometric interest. Our main result is:

Theorem A. *Let G be a discrete group. There are functors F^λ, $\lambda = \begin{cases} \mathbb{K}^{-\infty} \\ \mathbb{L}^{-\infty} \\ \mathbb{K}^{Top} \end{cases}$, from G-CW-Complexes to Spectra, which are G-homotopy invariant and G-excisive, such that $F^\lambda(\mathcal{E}G) \to F^\lambda(\bullet)$ can be identified on homotopy groups with the $\begin{cases} \text{Loday assembly map} \\ \text{Farrell-Jones map} \\ \text{Baum-Connes map} \end{cases}$, where $\mathcal{E}G$ is the universal G-space for $\begin{cases} \text{free } G\text{-actions} \\ G\text{-actions with virtually infinite cyclic isotropy} \\ G\text{-actions with finite isotropy} \end{cases}$. These functors F^λ are compositions of functors from G-CW-Complexes to*

$$\begin{cases} \text{additive categories} \\ \text{additive categories with involution} \\ C^*\text{-categories} \end{cases}$$

followed by $\lambda = \mathbb{K}^{-\infty}$, $\mathbb{L}^{-\infty}$, and $\mathbb{K}^{Top}$ respectively.

See [6], [25] for applications to the Novikov conjectures and further developments of the continuously controlled theory. The proof of the main theorem uses the equivalence of the continuously controlled assembly to the assembly maps of Davis-Lück, established in Theorem 8.3. See Corollary 4.4 for the corresponding equivalence to Loday assembly, and Corollary 9.2, together with Corollary 10.2, for the equivalence to Farrell-Jones assembly.

Our main result on the Baum-Connes assembly map for $\mathbb{K}^{Top}(C^*_r(G))$ is proved in Theorem 7.6. As a consequence, we establish in Corollary 8.4 that the assembly map considered by Davis and Lück for topological K-theory induces the same map on homotopy groups as the Baum-Connes map.

Finally in § 11, we propose a modification of the Farrell-Jones Isomorphism Conjecture [9] applicable to "intermediate" L-theories, including $\mathbb{L}^s$ and $\mathbb{L}^h$.

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Contents

2. Spectra

The purpose of this technical section is to introduce the main objects of study: the spectra $\mathbb{K}^{-\infty}$, $\mathbb{L}^{-\infty}$, and $\mathbb{K}^{Top}$. A spectrum X is a sequence of pointed spaces X_n and maps $\Sigma X_n \to X_{n+1}$. In the literature this is sometimes called a prespectrum, and the term *spectrum* is reserved for the situation where the dual map $X_n \to \Omega X_{n+1}$ is a homeomorphism. The homotopy groups of a spectrum are defined as $\pi_j(X) = \lim_{i\to\infty} \pi_{i+j}(X_i)$. We shall only consider as maps of spectra such maps that preserve all structure, so a map $X \to Y$ is a sequence of continuous pointed maps $X_n \to Y_n$ such that the obvious diagram commutes (precisely, not only up to homotopy). A map of spectra is called a *weak equivalence* if it induces an isomorphism on homotopy groups.

Definition 2.1. Let $\Phi\colon \mathcal{C} \to \mathit{Spectra}$ and $\Psi\colon \mathcal{C} \to \mathit{Spectra}$ be functors. A *homotopy natural transformation* from Φ to Ψ, is a sequence of functors Φ_i and Ψ_i, for $i = 0, 1, \dots, k$, together with a family of natural transformations $\Phi_i \to \Psi_i$ and a family of natural weak equivalences $\Phi_i \to \Psi_{i-1}$ for $i > 0$, where $\Phi_0 = \Phi$ and $\Psi_k = \Psi$. A homotopy natural transformation is called a *weak homotopy equivalence* if it induces an isomorphism on homotopy groups.

A homotopy natural transformation induces a homotopy class of maps on homotopy colimits and homotopy limits if we stay in the category of CW-spectra, so let us briefly recall the concept of CW-spectra.

Ordinary pointed spaces define spectra in an obvious way. In particular we have spheres of *all* integral dimensions in the category of spectra. A spectrum is a CW-spectrum if it is built of spheres using cofibrations in the same way a CW-space is built by ordinary spheres. The standard reference for this is unfortunately written in the equivariant case [20]. The usual theorems hold: Every spectrum is weakly equivalent to a CW-spectrum, and a weak equivalence of CW-spectra is a homotopy equivalence of CW-spectra. We may thus, as mentioned above, replace a homotopy natural transformation by an honest map of spectra. However, we lose naturality that way. In the many cases where the main interest is the induced map on homotopy groups, the homotopy natural transformation gives the best of all worlds.

Definition 2.2. Two homotopy natural transformations from $\Phi\colon \mathcal{C} \to \mathit{Spectra}$, to $\Psi\colon \mathcal{C} \to \mathit{Spectra}$ are *homotopic* if for every object $C \in \mathcal{C}$ the maps of CW-spectra corresponding to $\Phi(C)$ and $\Psi(C)$ are homotopic.

This definition leads to a notion of homotopy equivalence for spectrum-valued functors.

We shall now discuss the various spectra we will be working with.

2.1. Spectra in algebraic K-Theory.

In Quillen's original description of higher Algebraic K-Theory, the main emphasis was on defining the groups $K_i(R)$ for $i \geq 3$, extending the definition for $0 \leq i \leq 2$ in a natural way. Quillen's definition was through the homotopy groups of a space, which happened to be an infinite loop space, hence defines a spectrum, but this assignment was not functorial. Later it became important to realize K-theory as a functor to spectra, notably in the work of R. Thomason [37]. Presently, the easiest way is to use Waldhausens $S_\bullet$ construction [39] on an additive category, and in the case of a ring, replace the ring with the category of finitely generated free (or projective) modules. This last issue of whether to use free or projective modules only influences π_0 of the spectrum and the literature is a bit uneven on this point. In this paper we shall take $\mathbb{K}^{alg}$ to mean the (-1)-connected spectrum with π_0 as K_0 of the additive category in question, and for a ring we shall use the additive category of finitely generated free R-modules, as this seems to fit the literature best.

When studying assembly maps there is a different spectrum that comes up naturally, e. g. in the work of [6], namely $\mathbb{K}^{-\infty}$. Let $\mathcal{A}$ be an additive category and consider the inclusion

$$\mathcal{C}_{\mathbf{R}^i}(\mathcal{A}) \to \mathcal{C}_{\mathbf{R}^{i+1}}(\mathcal{A})$$

induced by the standard inclusion of $\mathbf{R}^i \subset \mathbf{R}^{i+1}$, where the categories are the categories of objects parameterized by $\mathbf{R}^i$ and $\mathbf{R}^{i+1}$ respectively and the morphisms are the bounded morphisms, see e. g. [26]. In K-theory this inclusion is canonically null homotopic in two ways by shifting in a positive or a negative direction in the last coordinate, and this produces a map of spectra

$$\Sigma\, \mathbb{K}^{alg}(\mathcal{C}_{\mathbf{R}^i}(\mathcal{A})) \to \mathbb{K}^{alg}(\mathcal{C}_{\mathbf{R}^{i+1}}(\mathcal{A}))$$

or dually

$$\mathbb{K}^{alg}(\mathcal{C}_{\mathbf{R}^i}(\mathcal{A})) \to \Omega\, \mathbb{K}^{alg}(\mathcal{C}_{\mathbf{R}^{i+1}}(\mathcal{A}))$$

which induces an isomorphism on π_i except possibly on π_0. Each time we increase i we introduce a new negative homotopy group $K_{-i}(\mathcal{A})$, and we define

$$\mathbb{K}^{-\infty}(\mathcal{A}) := \operatorname*{hocolim}_{i\to\infty}(\Omega^i\, \mathbb{K}^{alg}(\mathcal{C}_{\mathbf{R}^i}(\mathcal{A}))\ .$$

This is a spectrum whose (-1)-connective cover is precisely $\mathbb{K}^{alg}(\mathcal{A}^\wedge)$, where $\mathcal{A}^\wedge$ is the idempotent completion of $\mathcal{A}$. But $\mathbb{K}^{-\infty}(\mathcal{A})$ also has negative homotopy groups, for example in case $\mathcal{A}$ is the category of finitely generated free (or projective, it makes no difference here) R-modules, we get Bass's $K_{-i}(R)$.

This is a better spectrum to use for assembly purposes, but not the one that was used classically. Note for instance that since $K_{-i}(\mathbf{Z}) = 0$ for $i > 0$, we have $\mathbb{K}^{alg}(\mathbf{Z}) = \mathbb{K}^{-\infty}(\mathbf{Z})$ and the conjecture that

$$B\Gamma_+ \wedge \mathbb{K}^{-\infty}(\mathbf{Z}) \to \mathbb{K}^{-\infty}(\mathbf{Z}\Gamma)$$

is a homotopy equivalence for torsion free groups will then imply $K_{-i}(\mathbf{Z}\Gamma) = 0$ for $i > 0$.

2.2. **Topological K-theory.** A C^*-category is a Banach-category with involution and morphisms satisfying the C^*-identity, $\|TT^*\| = \|T\|^2$. We shall only deal with the special cases that arise in the controlled setting, where the objects are Hilbert spaces and the morphisms are bounded operators, or equivalence classes of bounded operators satisfying the C^*-identity. We are thus only considering C^*-categories with a direct sum functor. Of course we require that the Hom-sets are complete, meaning that the Hom-sets are closed subsets of the bounded operators from one object to another. Typically the Hom-sets are described by some given conditions (depending on the context) followed by completion.

The standard May-Segal machinery [1] takes as input a symmetric monoidal category. Such categories do allow a topology on the Hom-set, the typical, classical example being the category of finitely generated complex vector spaces which produces a connective version of the BU-spectrum. This does produce a spectrum from the kind of C^*-categories considered here. Alternatively one may use Waldhausens $S_.$-construction [39] and apply the simplicial functor to the Hom-sets. This simply gives an extra simplicial direction in something that is already an n-simplicial set.

In [16] the K-theory of some controlled C^*-categories were identified with the K-theory of a corresponding controlled C^*-algebra, defined using the methods of J. Roe. The discussion in [16] is somewhat brief, so we shall elaborate on it here. This is needed to establish Bott periodicity, and various fibrations of spectra.

Let X be a separable, paracompact topological space. Recall from [16] that a bornology on X is a certain system of neighbourhoods of the diagonal in $X \times X$. A bornology permits the definition of controlled categories and controlled C^*-algebras, where the control is with respect to the bornology on X. Typical examples are the bounded bornology and the continuously controlled bornology. The category $\mathcal{C}_{\mathcal{B}}(X; \mathbf{C})$ is defined to have objects A a collection $\{A_x\}_{x\in X}$ of finitely generated Hilbert spaces such that $\{x | A_x \neq 0\}$ is locally finite in X. To get the Hom-sets we start out with the bounded operators from the Hilbert sums $\oplus A_x \to \oplus B_y$ satisfying a control condition on the components $A_x \to B_y$ with respect to the given bornology. We then complete these Hom-sets inside the space of all bounded operators (thus losing strict control with respect to the bornology: a morphism can be approximated arbitrarily well by morphisms satisfying the bornology condition, but it does not itself have to satisfy that condition).

We want to associate spectra to such categories. We proceed as in the last section, with some very small variations. Let $\mathcal{A} = \mathcal{C}_{\mathcal{B}}(X; \mathbf{C})$. $\overline{\mathcal{C}}_{\mathbf{R}^i}(\mathcal{A})$ has the same objects as $\mathcal{C}_{\mathbf{R}^i}(\mathcal{A})$, but the morphisms have to be bounded operators in the analytical sense. We now define

$$\mathbb{K}^{Top}(\mathcal{A}) = \operatorname{hocolim}(\Omega^i \, \mathbf{K}^{\text{May-Segal}}(\overline{\mathcal{C}}_{\mathbf{R}^i}(\mathcal{A})))$$

as in the case of Algebraic K-theory.

Now let $\mathbf{H}$ be the Hilbert space with orthonormal basis $S \times \mathbf{N}$, where S is a countable dense subset of X. This is a $C_0(X)$-module in an obvious way, and it is *sufficiently large*, meaning that no $\phi \in C_0(X)$ acts as a compact operator. We now follow [16, §5] in defining $C^*_{\mathcal{B}}(X)$ as the completion of the subspace of locally compact bounded operators on $\mathbf{H}$ which are controlled with respect to the bornology $\mathcal{B}$. This is a C^*-algebra without unit.

To compare the category with the C^*-algebra we do the following: consider the full subcategory $\mathcal{C}$ of $\mathcal{C}_{\mathcal{B}}(X; \mathbf{C})$ on objects A with $A_x = 0$ when $x \notin S$ and A_x is the Hilbert space with basis $x \times \{1, \cdots, n_x\}$ for some n_x when $x \in S$. (Here we obviously need to include the possibility that $n = 0$ giving $A_x = 0$). Let U_A be the unitary operators on the object A. Stabilizing by the identity, we may define $U = \cup U_A$. This space is, by the definition of the May-Segal construction applied to $\mathcal{C}$, the 0-component of the 0'th space, and by cofinality the same thing holds true for $\mathcal{C}_{\mathcal{B}}(X; \mathbf{C})$. Adding an identity to $C^*_{\mathcal{B}}(X)$ we get $C^*_{\mathcal{B}}(X) \oplus \mathbf{C}$. Denote the units in this C^*-algebra with $\mathbf{C}$-component 1 by $\overline{U}$. There is a map $U \to \overline{U}$ sending $\alpha \in U_A$ to $(\alpha - 1_A, 1)$. The crucial observation is that

Lemma 2.3. *The map $U \to \overline{U}$ is a homotopy equivalence.*

Proof. The elements in $\overline{U}$ are locally compact operators meaning that locally they can be approximated arbitrarily well by finite matrices. The argument may now be completed by a partition of unity argument. □

This lemma shows that we may argue with controlled C^*-algebras or with the corresponding controlled C^*-categories according to taste. Under this correspondence, a filtered subcategory gives rise to an ideal in the C^*-algebra and a fibration of spectra after applying $\mathbb{K}^{Top}$ (see [16, 4.4] and [15]). The categorical approach has the advantage though, of producing a functor to the category of spectra, rather than just a space whose double loops are homotopy equivalent to itself. The C^*-algebra approach, however allows us to conclude Bott periodicity.

The situation is thus completely analogous to that of algebraic K-theory. Using Bott periodicity for a C^*-algebra, we may compute the low-dimensional homotopy groups of U, showing that $\mathbb{K}^{Top}$ is a Bott periodic spectrum for a C^*-category of the type considered here.

2.3. **Spectra in L-theory.** Spectra were introduced into L-theory by F. Quinn [29] in his thesis, based on the foundational Chapter 9 of C. T. C. Wall's book [40]. A tradition of indexing spectra in L-theory in the opposite direction to the usual way was established. The theory with the usual indexing convention goes as follows.

We first define the Δ-set $\mathbf{L}_n(K)$: the i-simplices consist of i-ads of $(i-n)$-dimensional surgery problems with a reference map to K. This Δ-set has base point at the empty surgery problem, and since there are no negative dimensional manifolds, we need to use the convention that the empty set is a manifold of any dimension. We need to be a bit careful about the set-theory, but this can be solved e. g. by assuming that all manifolds and Poincaré spaces come with a given imbedding in $\mathbf{R}^\infty$. It is easy to see that this Δ-set satisfies the Kan-condition (essentially by crossing with I). Therefore an element in the i'th homotopy group of its geometrical realization $|\mathbf{L}_n(K)|$ is given by a single simplex, with boundary at the basepoint. In other words, an element in the i'th homotopy group is represented by an $(i-n)$-dimensional i-ad surgery problem with homotopy equivalences on the boundary components, so that when $i-n \geq 5$ we have $\pi_i(|\mathbf{L}_n(K)|) = L_{i-n}(\mathbf{Z}\pi_1(K))$.

In the category of Δ-sets, the circle is represented by the Δ-set with one simplex (the basepoint) in every dimension, and one extra simplex in dimension 1. In the Δ-set category, the i-simplices of the loop space of a Kan Δ-set will be the degree i maps from the standard circle. For $\mathbf{L}_n(K)$ the i-simplices of the loop space $\Omega\mathbf{L}_n(K)$ are precisely i-ads of $((i+1)-n)$-dimensional surgery problems, that is $(i-(n-1))$ dimensional surgery problems, so in the category of Δ-sets we have

$$\Omega\mathbf{L}_n(K) = \mathbf{L}_{n-1}(K)$$

When Δ-sets are realized, there is a canonical map

$$|\Omega\mathbf{L}_n(K)| \to \Omega|\mathbf{L}_n(K)|$$

which is a homotopy equivalence. This means we get a homotopy equivalence

$$|\mathbf{L}_{n-1}(K)| \to \Omega|\mathbf{L}_n(K)|$$

and the dual $\Sigma|\mathbf{L}_{n-1}(K)| \to |\mathbf{L}_n(K)|$ can be taken to be the structure maps of a spectrum $\mathbb{L}^{geom}(K)$, the Quinn geometric L-spectrum of K. The homotopy groups of this spectrum, $\pi_{n+i}(\mathbf{L}_n(K))$ are surgery problems of dimension $(n+i)-n=i$, so this is 0 when $i \leq 0$, quite mysterious for $1 \leq i \leq 4$, and give the surgery obstruction groups $L_i(\mathbf{Z}\pi_1(K))$ for $i \geq 5$. The low-dimensional problems could be avoided by only allowing the empty manifold in dimensions < 5 in the Δ-set definition above, but this produces a different spectrum.

Crossing with $\mathbf{C}P^2$ sends an i-ad of $(i-(n+4))$-dimensional surgery problems to an i-ad of $(i-n)$-dimensional surgery problems, so it defines a map $\mathbf{L}_{n+4}(K) \to \mathbf{L}_n(K)$. Recall that at the spectrum level, suspension is just a dimension shift, so this gives a map $\Sigma^4\,\mathbb{L}^{geom}(K) \to \mathbb{L}^{geom}(K)$. This map induces an isomorphism on π_j when $j \geq 9$, corresponding to the manifolds involved being of dimension 5 or bigger.

This method produces connective spectra, although we have just seen that the homotopy groups are 4-periodic above dimension 5. In the surgery exact sequence for an n-dimensional manifold M, the maps from normal invariants to the surgery obstruction groups can be identified with the assembly maps

$$H_{n+i}(M;\mathbf{L}(\mathbf{Z})) \to L_{n+i}(\mathbf{Z}\pi_1(M))$$

for $i \geq 0$ [24]. In these dimensions both sides are almost 4-periodic, the difference being a factor of $\mathbf{Z}$ at $i=0$ on the left-hand side. It is however clear, at least in retrospect, that if we want the spectrum-level assembly map

$$M_+ \wedge \mathbf{L}(\mathbf{Z}) \to \mathbf{L}(M)$$

to be a homotopy equivalence when M is a $K(\pi,1)$ (thus proving the Borel conjecture), we have to use a periodic theory. This is because the right-hand side will remain periodic in dimensions $n+i \geq 0$ when using the connective theory, but those on the left-hand side will only be periodic in dimensions $n+i > n$.

We could now proceed as above and use bounded L-theory parameterized by $\mathbf{R}^k$ to construct a periodic L-spectrum out of $\mathbb{L}^{geom}(K)$. This does indeed work, and it automatically removes the low-dimensional problems encountered above, but we encounter K-theory complications because of varying L-theory decorations. This method however gives a good definition of $\mathbb{L}^{-\infty}(K)$, and this approach is explained in detail in [35, §17] in an algebraic setting, but there is no essential difference in the geometric setting. Note that $\mathbb{L}^{-\infty}(K)$ is the same spectrum as the one constructed by Farrell and Jones in [9, pp. 253-254]. It simplifies the technical details encountered in the definition of [9] to cross with $\mathbf{R}^k$ and use the bounded theory, instead of crossing with T^k and splitting.

Another method was designed by Ranicki [32, 34]. Given a ring R with involution, or more generally an additive category $\mathcal{A}$ with involution [33], spectra were defined as above, replacing i-ads of $(i-n)$-dimensional geometric surgery problems with i-ads of $(i-n)$-dimensional quadratic Poincaré complexes over $\mathcal{A}$. Now there are no low-dimensional problems. Originally, Ranicki imposed the condition that a q-dimensional quadratic Poincaré complex has underlying chain complex in dimensions 0 to q. This leads to a (-1)-connected spectrum with the i'th homotopy group isomorphic to $L_i^X(\mathcal{A})$, $i \geq 0$, with the usual K-theory decorations. Later, by relaxing this condition to finite length quadratic chain complexes (thus allowing negative dimensional objects which obviously have no geometric analogue), a strictly 4-periodic spectrum $\mathbb{L}^{alg}$ is constructed. This spectrum has variants according to the K-theory decorations, producing the important examples $\mathbb{L}^p$, $\mathbb{L}^h$ and $\mathbb{L}^s$. An $L^{-\infty}$-version of $\mathbb{L}^{alg}$ is constructed as above by using the bounded categories parametrized by $\mathbf{R}^k$.

Notice that the bounded construction giving the geometrically defined spectrum $\mathbb{L}^{-\infty}(K)$ has a similar flavour: a manifold W^n parameterized by $\mathbf{R}^k$ could have $n < k$, and that does correspond to a "negative dimensional" manifold. In Section 10 we relate the $L^{-\infty}$-versions of $\mathbb{L}^{geom}$ and $\mathbb{L}^{alg}$.

3. Weiss-Williams assembly

We recall some of the definitions and results presented in Weiss-Williams [42]. We will assume that all spaces are homotopy equivalent to CW-complexes. A functor F from spaces to spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant functor F is *excisive* if $F(\emptyset)$ is contractible and F preserves homotopy pushout squares. F is *strongly excisive* if, in addition, F preserves arbitrary coproducts, up to homotopy equivalence.

If F is strongly excisive, then the functor $\pi_*(F)$ from spaces to graded abelian groups is a generalized homology theory satisfying the strong wedge axiom. Moreover, $F \simeq E$ where $E(X) = X_+ \wedge F(\bullet)$ for all CW-complexes X.

Theorem 3.1. ([42, 1.1]) *For any homotopy invariant functor F from spaces to spectra, there exists a strongly excisive and homotopy invariant functor $F^{\%}$ from spaces to spectra and a natural transformation*

$$\alpha_F \colon F^{\%} \longrightarrow F \ ,$$

depending functorially on F, such that $\alpha_F \colon F^{\%}(\bullet) \to F(\bullet)$ is a homotopy equivalence.

Corollary 3.2. ([42, p. 336]) *Let F be a homotopy invariant functor from spaces to spectra. Suppose that $\alpha \colon E \to F$ is a homotopy natural transformation inducing a weak equivalence $E(\bullet) \simeq F(\bullet)$. If E is homotopy invariant and strongly excisive, then α is homotopic to α_F.*

Proof. For any space X, we have a homotopy commutative diagram

$$\begin{array}{ccc} X_+ \wedge E(\bullet) & \xrightarrow{\simeq} & E(X) \\ {\scriptstyle 1\wedge\alpha_\bullet}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\alpha_X} \\ X_+ \wedge F(\bullet) & \longrightarrow & F(X) \end{array}$$

of homotopy natural transformations. Since $F^{\%}(X) \simeq X_+ \wedge F(\bullet)$ the result follows from Theorem 3.1. □

Example 3.3. The theory above applies directly to Waldhausen's functor $A(X)$, but not directly to K-theory (or to Wall's algebraic L-theory) because these functors were originally defined on the category of *Rings* (or rings with involution). As described in [42, § 2], it is possible to extend them to functors on the category of *Ringoids* (or ringoids with involution). This uses the Waldhausen or Quillen construction of K-theory [28], [39] as a functor on exact categories to spectra, or Ranicki's L-theory of additive categories with involution [33]. One can now construct homotopy invariant functors from *Spaces* to *Spectra* by composing with the functor $X \mapsto R\pi(X)$, where $\pi(X)$ denotes the fundamental groupoid of a space equipped with the standard involution and R is a ring with unit. For L-theory we will see in § 10 that this functor agrees on homotopy groups in dimensions ≥ 5 with Wall's geometric definition [40] of the surgery obstruction groups $L_n(\mathbb{Z}\pi(X))$, and with Quinn's construction of the geometric surgery spectra $\mathbb{L}^{geom}(X)$ (a full exposition of this construction has been given by Nicas [24]), and also with the algebraic surgery spectra $\mathbb{L}^{alg}(\mathbb{Z}\pi(X))$ of Ranicki [34]. We will also see in § 4 that the Loday assembly map for K-theory can be recovered by this process.

Non-Example 3.4. The "universal" assembly construction given above does not apply to $\mathbb{K}^{Top}(C^*_r(G))$ since this spectrum is not a functor of G.

4. Loday assembly

In §2, we have already mentioned that the higher algebraic K-theory of a ring A was originally defined by Quillen as the homotopy groups $K_n(A) := \pi_n(BGL(A)^+)$, for $n \geq 1$, of the (+)-construction on $BGL(A) = \lim_{k\to\infty} BGL_k(A)$. Later Quillen [28] introduced the Q-construction and an algebraic K-theory functor $\mathbf{K}\colon Exact \to Spaces$, where $Exact$ denotes the category of exact categories. In [13, p. 228], Quillen showed that when the category $\mathcal{P}(A)$ of projective A-modules is regarded as an exact category in a natural way, then $K_n(A) = \pi_n(\mathbf{K}(\mathcal{P}(A)))$, for $n \geq 1$. The extension of $\mathbf{K}$ to a spectrum-valued functor was done by J. P. May [22], and this gives the (connective) algebraic K-theory functor. An alternate (but homotopy equivalent) description of algebraic K-theory was provided by Waldhausen's $S_\bullet$-construction [39, p. 375], by a method that automatically produces (connective) spectra. Finally, Pedersen and Weibel [26], generalizing work of Gersten [12], Karoubi [18] and Wagoner [38], produced a non-connective functor $\mathbb{K}^{-\infty}\colon AddCat \to Spectra$, which agrees with $\mathbf{K}$ in positive dimensions when restricted to $AddCat \subset Exact$.

The original K-theory assembly map $\alpha_G\colon h_n(BG_+;\mathcal{K}(R)) \to K_n(RG)$, $\mathcal{K}$ defined below, was constructed by Loday [21, 4.1.1], starting with Quillen's (+)-construction definition of K-theory. The basic ingredient [21, 2.4.3] is a pairing on K-groups

$$K_p(A) \times K_q(B) \to K_{p+q}(A \otimes B)$$

for two rings A and B, and any $p,\, q \in \mathbf{Z}$, induced by a sequence of "tensor-product" pairings

$$\widehat{\gamma}\colon BGL(S^n A)^+ \wedge BGL(S^m B)^+ \to BGL(S^{n+m}(A \otimes B))^+$$

for $n, m \geq 0$. Here $S^n A$ denotes the n-th suspension of the ring A (see [21, 1.4.4]). Loday's model $\mathcal{K} = (\mathcal{K}_n)$ for the non-connective K-theory spectrum has spaces $\mathcal{K}_n(A) = K_0(S^n A) \times BGL(S^n A)^+$, for $n \geq 0$. Note that $\mathcal{K}(A)$ does not depend functorially on A, since the spaces and structure maps are well-defined only up to homotopy. The pairings $\widehat{\gamma}$ however induce a weak pairing of spectra $\widehat{\gamma}\colon \mathcal{K}(A) \wedge \mathcal{K}(B) \to \mathcal{K}(A \otimes B)$ (see [21, 2.4.2]).

Now if G is a group and $A = RG$, there is a natural map

$$j\colon BG \to BGL(RG)^+ \to K_0(RG) \times BGL(RG)^+ \equiv \mathcal{K}_0(RG)$$

defined by sending $g \in G$ to a 1×1 matrix and stabilizing. Loday shows that the weak pairings $\widehat{\gamma}$ give maps:

$$BG_+ \wedge \mathcal{K}_n(R) \xrightarrow{\ j\ } \mathcal{K}_0(RG) \wedge \mathcal{K}_n(R) \xrightarrow{\ \widehat{\gamma}\ } \mathcal{K}_n(RG)$$

which are compatible up to homotopy with the suspension maps $\Sigma\mathcal{K}_n \to \mathcal{K}_{n+1}$. This is enough to define a weak pairing of spectra $BG_+ \wedge \mathcal{K}(R) \to \mathcal{K}(RG)$, and the Loday assembly map α_G is the induced map on homotopy groups.

We would like to relate the Loday assembly map to the assembly map of spectra

$$\alpha_X\colon X_+ \wedge \mathbb{K}^{-\infty}(R) \to \mathbb{K}^{-\infty}(R(\pi(X))_\oplus)$$

given by the general theory of § 3. In order to do this, we will show that the Loday pairing is homotopy equivalent to one constructed via the Pedersen-Weibel deloopings. An appropriate technical tool for this purpose is supplied by the work of P. May [23]. We will say that $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a *tensor product pairing* of symmetric monoidal categories if the axioms of [23, § 2] hold. A tensor product pairing of symmetric monoidal categories induces a functor $\mathrm{Iso}(\mathcal{A}) \times \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{C})$ on the subcategories of isomorphisms which is also a tensor product pairing. Any additive category can be regarded as a symmetric monoidal category, and an important example of tensor product pairings arises from the ordinary tensor product of modules over rings.

Lemma 4.1. *Let $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a tensor product pairing of additive categories. Then the natural tensor product pairings $\mathcal{C}_{\mathbf{R}^n}(\mathcal{A}) \times \mathcal{C}_{\mathbf{R}^m}(\mathcal{B}) \to \mathcal{C}_{\mathbf{R}^{n+m}}(\mathcal{C})$, for any $n, m \geq 0$, induce a pairing of spectra $\widehat{\otimes}\colon \mathbb{K}^{-\infty}(\mathcal{A}) \wedge \mathbb{K}^{-\infty}(\mathcal{B}) \to \mathbb{K}^{-\infty}(\mathcal{C})$.*

Proof. The tensor product of objects $(M, f) \otimes (N, g)$ is the object $M \otimes N$ with reference maps $f \times g\colon M \otimes N \to \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{n+m}$. Checking the tensor product axioms for this pairing will also be left to the reader.

A construction for pairings on K-theory has been given in much greater generality by J. P. May [23, p. 332], as explained in [41, § 3]. If $\mathcal{A}$ is a symmetric monoidal category, let $E_0 B\,\mathrm{Iso}(\mathcal{A})$ denote the group completion [41, p. 502] of $\mathrm{Iso}(\mathcal{A})$. For example, if $\mathcal{A}$ is the category of finitely generated free modules over a ring R, then $\mathrm{Iso}(\mathcal{A}) = \coprod_{n=0}^{\infty} GL_n(R)$ and $E_0 B\,\mathrm{Iso}(\mathcal{A}) = \mathbb{Z} \times BGL(R)^+$.

The induced tensor product pairing $\otimes\colon \mathrm{Iso}(\mathcal{A}) \times \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{C})$ functorially determines a pairing $\otimes\colon E_0 B\,\mathrm{Iso}(\mathcal{A}) \times E_0 B\,\mathrm{Iso}(\mathcal{B}) \to E_0 B\,\mathrm{Iso}(\mathcal{C})$ of the group completions as infinite loop spaces. But $E_0 B\,\mathrm{Iso}(\mathcal{A}) = \mathbf{K}(\mathcal{A})$, so we get maps

$$\mathbf{K}(\mathcal{C}_{\mathbf{R}^n}(\mathcal{A})) \wedge \mathbf{K}(\mathcal{C}_{\mathbf{R}^m}(\mathcal{B})) \to \mathbf{K}(\mathcal{C}_{\mathbf{R}^{n+m}}(\mathcal{C}))$$

for all $n, m \geq 0$ (here we just mean the 0'th spaces of the K-theory spectrum). To pass to a pairing of spectra we must check (strict) compatibility with the suspension maps $\Sigma\,\mathbf{K}(\mathcal{C}_{\mathbf{R}^n}(?)) \to \mathbf{K}(\mathcal{C}_{\mathbf{R}^{n+1}}(?))$ used in defining the Pedersen-Weibel spectrum (i.e. we must have strictly commutative versions of the diagrams considered by Loday [21, p. 346]). The details are straightforward from the definitions in [26], and will again be left to the reader. □

Now we specialize to the additive categories of modules over rings.

Theorem 4.2. *For any rings A and B, there is a homotopy equivalence between the Loday (weak) pairing $\widehat{\gamma}\colon \mathcal{K}(A) \wedge \mathcal{K}(B) \to \mathcal{K}(A \otimes B)$ and the Pedersen-Weibel pairing $\widehat{\otimes}\colon \mathbb{K}^{-\infty}(A) \wedge \mathbb{K}^{-\infty}(B) \to \mathbb{K}^{-\infty}(A \otimes B)$.*

Proof. In [27, 6.3], Pedersen and Weibel prove that there is homotopy equivalence of spectra $\theta\colon \mathbb{K}^{-\infty}(A) \simeq \mathcal{K}(A)$ by relating the operations of cone and suspension for $\mathbf{K}(\mathcal{C}_{\mathbf{R}^n}(A))$ to those for $\mathbf{K}(S^n A)$. There is a homotopy commutative diagram

$$\begin{array}{ccc} \mathbf{K}(\mathcal{C}_{\mathbf{R}^n}(A) \wedge \mathbf{K}(\mathcal{C}_{\mathbf{R}^m}(B)) & \xrightarrow{\widehat{\otimes}} & \mathbf{K}(\mathcal{C}_{\mathbf{R}^{n+m}}(A \otimes B) \\ {\scriptstyle \theta_n \wedge \theta_m} \downarrow \simeq & & {\scriptstyle \theta_{n+m}} \downarrow \simeq \\ \mathbf{K}(S^n A) \wedge \mathbf{K}(S^m B) & \xrightarrow{\widehat{\gamma}} & \mathbf{K}(S^{n+m}(A \otimes B) \end{array}$$

and the result follows by checking compatibility with the suspension structure maps. □

We will use the notation $\mathcal{A}_{\oplus}$ to denote the free additive category generated by $\mathcal{A}$ (see section 5). In this notation, $R_{\oplus}$ is the additive category of finitely generated free R-modules.

Theorem 4.3. *For any groupoid $\mathcal{G}$ there is a natural transformation*

$$A_{\mathcal{G}}\colon B\mathcal{G}_+ \wedge \mathbb{K}^{-\infty}(R) \to \mathbb{K}^{-\infty}(R(\mathcal{G})_{\oplus}),$$

such that $A_{(\bullet)}$ is a homotopy equivalence. If the groupoid $\mathcal{G}$ is a group G, the Loday assembly map α_G is isomorphic to the map induced by A_G on homotopy groups.

Proof. The natural transformation $A_{\mathcal{G}}$ is constructed by generalizing Loday's method from groups to groupoids, using the Pedersen-Weibel pairing $\widehat{\otimes}$ from Lemma 4.1. We may use the tensor product pairing

$$\otimes\colon \mathbb{Z}(\mathcal{G})_{\oplus} \times R_{\oplus} \to R(\mathcal{G})_{\oplus}$$

of additive categories. The inclusion of categories $\mathcal{G} \to \mathbb{Z}(\mathcal{G})_\oplus$ gives the natural map $j\colon B\mathcal{G} \to \mathbf{K}(\mathbb{Z}(\mathcal{G})_\oplus)$, and Theorem 4.2 identifies the resulting assembly maps on the homotopy groups $h_n(BG_+; \mathbb{K}^{-\infty}(R)) \to K_n(RG)$. □

Corollary 4.4. *Let X be a connected space and $G = \pi_1(X, x_0)$. Then the Loday assembly map α_G is isomorphic to the map induced on homotopy groups by the Weiss–Villiams assembly maps $\alpha_{B\pi(X)}$.*

Proof. Let $c\colon X \to B\pi(X)$ be the classifying homotopy natural transformation of the universal covering. Note that $G = \pi_1(X, x_0) \cong \pi(X)_{x_0}$, where $\pi(X)_{x_0} \subset \pi(X)$ denotes the subgroupoid with one object $\{x_0\}$. This inclusion induces an isomorphism on K-theory and a homotopy equivalence $BG \simeq B\pi(X)$. By Theorem 4.3 we have a homotopy natural transformation

$$\widehat{\alpha}_X\colon\ X_+ \wedge \mathbb{K}^{-\infty}(R) \xrightarrow{c\wedge 1} B\pi(X)_+ \wedge \mathbb{K}^{-\infty}(R) \xrightarrow{A_{\pi(X)}} \mathbb{K}^{-\infty}(R(\pi(X))_\oplus)$$

such that $\widehat{\alpha}_\bullet$ is a homotopy equivalence. By Corollary 3.2, $\widehat{\alpha}_X$ is homotopy equivalent to the generalized assembly map α_X. For $X = BG$, the map c induces a homotopy equivalence, so A_G is homotopy equivalent to $\alpha_{B\pi(X)}$. □

5. Davis-Lück assembly

In [8], Davis and Lück introduced a new viewpoint on assembly, generalizing the construction of Section 3 to an equivariant setting. Let G be any discrete group, and consider functors

$$H\colon G\text{-}CW\text{-Complexes} \longrightarrow \mathit{Spectra} \tag{5.1}$$

which are G-homotopy invariant and G-excisive [8, p. 243]. The main features of this theory are:

(i) For any G-CW complex X, and any such functor H, the Davis-Lück generalized assembly map is the map

$$\alpha_X\colon H(X) \to H(\bullet)$$

induced by the projection $X \to \bullet$. Important examples arise from taking $X = \mathcal{E}_\mathcal{F}(G)$, the classifying space of a family $\mathcal{F}$ of subgroups of G, closed under taking subgroups and conjugation.

(ii) Any functor $E\colon \mathbf{Or}(G) \to \mathit{Spectra}$ out of the orbit category of G may be extended uniquely (up to homotopy) to a functor $E_\%\colon G\text{-}CW\text{-Complexes} \to \mathit{Spectra}$ which is G-homotopy invariant and G-excisive [8, 5.1]. The *orbit category* $\mathbf{Or}(G)$ is the category with objects G/K, for K any subgroup of G, and the morphisms are G-maps.

(iii) Any functor $F\colon \mathit{Spaces} \to \mathit{Spectra}$ gives a functor $\mathbf{Or}(G) \to \mathit{Spectra}$ by composition with the functor $E\colon \mathbf{Or}(G) \to \mathit{Spaces}$ defined on objects by $G/K \mapsto G/K \times_G EG$. By part (ii) there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{Or}(G) & \xrightarrow{\ E\ } & \mathit{Spaces} & \xrightarrow{\ F\ } & \mathit{Spectra} \\ \big\downarrow & & & \nearrow_{(F\circ E)_{\%}} & \\ G\text{-}CW\text{-Complexes} & & & & \end{array}$$

(iv) If $F\colon \mathit{Spaces} \to \mathit{Spectra}$ is a homotopy invariant functor and $\bar{F} := (F \circ E)_{\%}$ denotes the corresponding functor on G-CW-Complexes, then the induced map $\bar{F}(EG) \to \bar{F}(\bullet)$ is homotopy equivalent to the Weiss-Williams assembly map $F^{\%}(BG) \to F(BG)$. This follows immediately from the formula

$$\bar{F}(EG) \simeq \operatorname*{hocolim}_{\Delta \subset BG} \bar{F}(G \times \Delta) \simeq \operatorname*{hocolim}_{\Delta \subset BG} \bar{F}(G) \simeq BG_+ \wedge F(\bullet) \simeq F^{\%}(BG)$$

since $\bar{F}(G) = F(G \times_G EG)$ and $G \times_G EG \simeq \bullet$. Moreover, $\bar{F}(\bullet) = F(\bullet \times_G EG) = F(BG)$, and if we replace the spectra by CW-spectra it is easy to establish a homotopy commutative diagram relating the two maps.

In the Davis-Lück theory, the interesting examples of functors H as in (5.1) are all constructed by extension from functors out of $\mathbf{Or}(G)$ factoring through *Groupoids* (or a variant of this category called *Groupoids*^{inj}). Recall that a groupoid is a small category with all morphisms invertible. Then *Groupoids* is the category with groupoids as objects and functors of groupoids as morphisms. The subcategory *Groupoids*^{inj} (see [8, p. 210]) has the same objects but only allows *faithful* functors $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ as morphisms. A functor F is faithful if for any two objects x, $y \in \mathcal{G}_1$, the induced map $\mathrm{mor}_{\mathcal{G}_1}(x, y) \to \mathrm{mor}_{\mathcal{G}_2}(F(x), F(y))$ is injective. This subcategory is needed for applications of the theory to C^*-algebras.

The following construction (see [8, § 2]) gives a functor

$$Gr\colon \mathbf{Or}(G) \to \mathit{Groupoids}^{inj} .$$

A left G-set G/K defines a groupoid $\overline{G/K}$ whose objects are the cosets xK, for $x \in G$, and morphisms $\mathrm{mor}(xK, yK) = \{g \in G \,|\, y^{-1}gx \in K\}$. Composition is given by group multiplication and a map of left G-sets defines a functor on the associated groupoids. Now any spectrum valued functor out of *Groupoids* or *Groupoids*^{inj} can be composed with Gr and the theory above will apply. The most useful applications so far involve those functors which factor through *AddCat*, the category whose objects are additive categories (with involution) or C^*Cat, the category whose objects are C^*-categories. Here are the three main examples of Davis-Lück [8, § 2]:

5.2. *Algebraic K-theory*: This uses a functor from *Groupoids* to *AddCat*, the category of additive categories. If $\mathcal{C}$ is a category and R is a commutative ring with unit, then $R\mathcal{C}$

denotes the associated R-category to $\mathcal{C}$. The objects of $R\mathcal{C}$ are the same as those of $\mathcal{C}$, and the morphism set $\mathrm{mor}_{R\mathcal{C}}(x, y)$ is the free R-module generated by $\mathrm{mor}_{\mathcal{C}}(x, y)$. For any groupoid $\mathcal{G}$, we let $R(\mathcal{G})_\oplus$ denote the free additive R-category generated by $R\mathcal{G}$. This is the symmetric, monoidal R-category described in [8, p. 214]. We get a K-theory functor

$$\mathbb{K}^{-\infty}\colon Groupoids \to Spectra$$

sending a groupoid $\mathcal{G}$ to the spectrum $\mathbb{K}^{-\infty}(R(\mathcal{G})_\oplus)$, by composing with the (non-connective) algebraic K-theory functor

$$\mathbb{K}^{-\infty}\colon AddCat \to Spectra$$

of Pedersen and Weibel [26].

5.3. *L-theory*: This uses a functor from *Groupoids* to $AddCat^-$, the category of additive categories with involution. Suppose that R has an involution $r \mapsto \bar{r}$, and $\mathcal{G}$ is a groupoid. Then there is an involution on $R\mathcal{G}$ defined by the R-linear extension of the map $f \mapsto f^{-1}$ for each $f \in \mathrm{mor}_{\mathcal{C}}(x, y)$. This gives $R\mathcal{G}$ the structure of an R-category with involution, and $R(\mathcal{G})_\oplus$ becomes an additive category with involution. We get an L-theory functor

$$\mathbb{L}^{-\infty}\colon Groupoids \to Spectra$$

sending a groupoid $\mathcal{G}$ to the spectrum $\mathbb{L}^{-\infty}(R(\mathcal{G})_\oplus)$, by composing with the $L^{-\infty}$ version of the periodic algebraic L-theory functor

$$\mathbb{L}^{alg}\colon AddCat^- \to Spectra$$

constructed by Ranicki [33], [34, p. 139]. There are versions of this construction for other forms of L-theory, including L^s, L^h and L^p. In this paper we don't consider assembly for non-oriented involutions, or more general antistructures.

5.4. *C^*-theory*: This uses a functor from $Groupoids^{inj}$ to C^*-Categories. Here $R = \mathbb{C}$ is the complex numbers, and we must complete the morphism sets of $(\mathbb{C}\mathcal{G})_\oplus$ as described on [8, pp. 212-213] to get a C^*-category $C^*_r\mathcal{G}$. Then we get a functor

$$\mathbb{K}^{Top}\colon Groupoids^{inj} \to Spectra$$

sending a groupoid $\mathcal{G}$ to $\mathbb{K}^{Top}(C^*_r\mathcal{G})$ by composing with the topological K-theory functor

$$\mathbb{K}^{Top}\colon C^*Cat \to Spectra$$

constructed by Davis-Lück [8, pp. 216-217]. They have achieved functoriality of C^*_r by restricting the allowable morphisms. We remark that there is a minor error in the construction in [8, p. 217, line 8], where it is asserted that the exterior tensor product extends functorially to the group completion before passing to the classifying spaces.[1] On the other hand, what is really needed for the construction is the pairing μ further down. Just as in our proof of Theorem 4.3, this pairing can be constructed by the categorical machinery of J. P. May [23, p. 332], as explained in [41, § 3].

[1] We are indebted to W. Lück and M. Joachim for pointing this out.

6. Quinn assembly

In the Appendix of [30], F. Quinn introduced an even more general form of assembly (more details are given also in [31]). Let $p\colon Y \to Z$ be a simplicially stratified fibration. If $F\colon \mathit{Spaces} \to \mathit{Spectra}$ is a homotopy invariant functor, then one can define an assembly map

$$\alpha_Q\colon \operatorname*{hocolim}_{\sigma\subset Z} F(p^{-1}(\sigma)) \to F(Y)$$

induced by naturality and the homotopy equivalence $\operatorname{hocolim}_{\sigma\subset Z}(\sigma) \simeq Z$. The indexing category in the homotopy colimit is the simplices of Z with inclusions. The assembly map α_Q is independent of the choices for sufficiently fine triangulations $\{\sigma\}$ of Z. Notice that for the special case $p\colon X \to X$ is the *identity* map, $\alpha_Q\colon X_+ \wedge F(\bullet) \to F(X)$ is homotopy equivalent to α_X (by Corollary 3.2).

This construction can also be applied to any G-CW complex X and the simplicially stratified fibration

$$\begin{array}{c} X \times_G EG \\ \downarrow{\scriptstyle p} \\ X/G \end{array}$$

to produce a functor $F_Q\colon G\text{-}CW\text{-Complexes} \to \mathit{Spectra}$ which is G-homotopy invariant and G-excisive. By definition,

$$F_Q(X) = \operatorname*{hocolim}_{\Delta\subset X/G} F(p^{-1}(\Delta))$$

so by restricting to $\mathbf{Or}(G)$ we get $F_Q(G/K) = F(G/K \times_G EG)$. On the other hand, the Davis-Lück construction given in (5.1)(iii) provides another G-homotopy invariant and G-excisive functor $\bar{F} := (F\circ E)_\%\colon G\text{-}CW\text{-Complexes} \to \mathit{Spectra}$ associated to the same functor F.

Theorem 6.1. *For X a contractible G-space, and $F\colon \mathit{Spaces} \to \mathit{Spectra}$ a homotopy invariant functor, the generalized Quinn assembly map $\alpha_Q\colon F_Q(X) \to F(X \times_G EG)$ is homotopy equivalent to the map $\bar{F}(X) \to \bar{F}(\bullet)$ induced by the projection $X \mapsto \bullet$.*

Proof. There is a homotopy equivalence $\bar{F}(X) \simeq F_Q(X)$ induced by the inclusion $X \to X \times_G EG$ and the homotopy colimit expression $\bar{F}(X) = \operatorname{hocolim}_{\Delta\subset X/G} F(q^{-1}(\Delta))$, where $q\colon X \to X/G$ is the projection to the orbit space. We therefore have a homotopy commutative diagram

$$\begin{array}{ccccc} F_Q(X) & \xleftarrow{\simeq} & \bar{F}(X) & \xrightarrow{\alpha_X} & \bar{F}(\bullet) \\ \downarrow{\scriptstyle \alpha_Q} & & \downarrow & & \downarrow{\scriptstyle \simeq} \\ F(X \times_G EG) & = & F(X \times_G EG) & \longrightarrow & F(BG) \end{array}$$

for any G-space X. If X is contractible, the homotopy invariance of F implies that $F(X \times_G EG) \simeq F(BG)$ (induced by the second factor projection $X \times_G EG \to BG$). □

An important example of this construction is $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G. The associated assembly maps for $F = \mathbb{K}^{-\infty}$ and for $F = \mathbb{L}^{-\infty}$ are the ones used in the Farrell-Jones Isomorphism Conjectures [9, 1.6]. More precisely, for K-theory F is the composition of the fundamental groupiod functor $X \to \pi(X)$ with $\mathbb{K}^{-\infty}\colon$ *Groupoids* $\to$ *Spaces*. For L-theory, F is the $L^{-\infty}$-version of Quinn's geometric surgery spectrum $\mathbb{L}^{geom}(X)$. In Section 9 and Section 10, these assembly maps will be related to the ones defined by Davis-Lück out of the orbit category $\mathbf{Or}(G)$. Actually, Farrell and Jones state more general conjectures involving simplicially stratified fibrations $p\colon \widetilde{Y} \times_G X \to X/G$, where $\widetilde{Y}$ is the universal covering of a space Y with $\pi_1(Y) = G$. They use the composite

$$\operatorname*{hocolim}_{\Delta \subset X/G} F(p^{-1}(\Delta)) \xrightarrow{\alpha_Q} F(\widetilde{Y} \times_G X) \xrightarrow{f_*} F(Y)$$

of the Quinn assembly map with the induced map from the projection $f\colon \widetilde{Y} \times_G X \to Y$. This version is appropriate for the stable pseudo-isotopy functor, where $F(Y)$ depends on the space Y and not just on its fundamental group. For the functors considered in this paper $Y = EG$ and the last map f_* is a homotopy equivalence since $X = \mathcal{E}_{\mathcal{F}}(G)$ is contractible.

7. Assembly via Controlled Categories

The controlled categories of Pedersen [25], Carlsson-Pedersen [6], [7] are our main tool for identifying various different assembly maps. We will recall the definition of these categories and then describe three important examples of functors $H\colon G$-CW-Complexes $\to$ *Spectra* as in (5.1), leading to the proofs of our main results.

Let G be any discrete group, and let Z be a G-CW complex. If $X \subset Z$ is a closed G-invariant subspace, we will use the notation $Y = Z - X$. A subset $C \subset Z$ is called relatively G-compact if $C \cdot G/G$ is relatively compact in Z/G.

The isotropy subgroup of $y \in Y$ is denoted G_y. The category $\mathcal{B}_G(Z, X; R)$ has objects (A, f), where A is a free RG-module, and $f\colon A \to Fin(Y)$ is a G-equivariant map to the finite subsets of Y, satisfying:

(i) $f(a+b) \subseteq f(a) \cup f(b)$.
(ii) $A_y = \{a \in A \mid f(a) \subseteq \{y\}\}$ is a finitely generated RG_y-module, for each $y \in Y$.
(iii) $A = \bigoplus_{y \in Y} A_y$ as an R-module.
(iv) $\{y \in Y \mid A_y \neq 0\}$ is locally finite in Y, and relatively G-compact in Z.

When Y has more than one point it follows from these conditions that $f(a) = \emptyset$ if and only if $a = 0$. When Y is precisely one point, this has to be added as an extra assumption. A morphism $\phi\colon (A, f) \to (B, g)$ consists of an RG-homomorphism $\phi\colon A \to B$, such that ϕ is *continuously controlled* at $X \subset Z$. In terms of the components $\phi_y^z\colon A_y \to B_z$ of the map (for y, $z \in Y$), this means that for every $x \in X$, and for every G_x-invariant neighbourhood U of x in Z, there is a G_x-invariant neighbourhood V of x in Z so that $\phi_y^z = 0$ and $\phi_z^y = 0$ whenever $y \in Y - U$ and $z \in V \cap U \cap Y$. [2]

Remark 7.1. It is advantageous to have this kind of equivariant category be equivalent to the fixed category of some category with group action, see e. g. [6]. For the case in question a suggestion for such a category could be the category $\mathcal{D}_G(Z, X; R)$ defined in [25, 3.1]. This will be the case if the definition of continuous control in [25] is changed to use G_x-invariant neighborhoods. This gives us a continuously controlled version of the bounded equivariant categories introduced in [14]. On the other hand, the category $\mathcal{B}_G(Z, X; R)$ is not itself a fixed-point category in any obvious way, as required for the applications to the Novikov conjecture in [6], [7].

The following lemma is proved in [3]

Lemma 7.2. *The category $\mathcal{B}_G(Z, X; R)$ depends functorially on the pair (Z, X).*

In our applications to assembly maps, we will take X to be a G-CW complex and let $Z = X \times [0, 1]$ with the subspace $X = X \times 1 \subset Z$. To simplify the notation, let

$$\mathcal{B}_G(X \times [0, 1); R) := \mathcal{B}_G(X \times [0, 1], X \times 1; R) \ .$$

Let $\mathcal{A} = \mathcal{B}_G(X \times [0, 1); R)_\emptyset$ denote the full subcategory of $\mathcal{U} = \mathcal{B}_G(X \times [0, 1); R)$ with objects (A, f) such that the intersection with the closure

$$\mathrm{supp}(A, f) = \overline{\{(x, t) \in X \times [0, 1) \mid A_{(x,t)} \neq 0\}} \cap (X \times 1)$$

is the empty set. This category is equivalent to the category of finitely generated free RG-modules.

The quotient category will be denoted $\mathcal{B}_G(X \times [0, 1); R)^{>0}$, and we remark that this is a germ category since $\mathcal{U}$ is $\mathcal{A}$-filtered [27], [6]. The objects are the same as in $\mathcal{B}_G(X \times [0, 1); R)$ but morphisms are identified if they agree close to $X = X \times 1$ (i.e. on the complement of a neighbourhood of $X \times 0$).

The category $\mathcal{B}_G(X \times [0, 1); R)^{>0}$ is an additive category with involution, and we obtain a functor G-CW-Complexes $\to$ $AddCat^-$. The results of [5, 1.28, 4.2] now show that the functors F^λ: G-CW-Complexes $\to$ *Spectra* defined by

$$F^\lambda(X) := \begin{cases} \mathbb{K}^{-\infty}(\mathcal{B}_G(X \times [0, 1); R)^{>0}) \\ \mathbb{L}^{-\infty}(\mathcal{B}_G(X \times [0, 1); R)^{>0}) \end{cases} , \tag{7.3}$$

where $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$ respectively, are G-homotopy invariant and G-excisive. This defines two out of the three functors needed for the statement of our main theorem in the Introduction.

[2]We would like to thank Holger Reich for pointing out a technical problem in our first version of this continuous control condition.

Applying $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$ to the sequence of additive categories (with involution):

$$\mathcal{B}_G(X\times[0,1);R)_\emptyset \to \mathcal{B}_G(X\times[0,1);R) \to \mathcal{B}_G(X\times[0,1);R)^{>0}$$

gives a fibration of spectra [6]. This leads to another description for the assembly map.

Theorem 7.4. ([25, 4.1]) *The continuously controlled assembly map $F^\lambda(X) \to F^\lambda(\bullet)$ is homotopy equivalent to the connecting map*

$$\lambda(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to \Omega^{-1}\lambda(\mathcal{B}_G(X\times[0,1);R)_\emptyset$$

for $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$.

Proof. We first compare the functors under the maps induced by the projection $X \to \bullet$ to a basepoint. Consider the diagram:

$$\begin{array}{ccccc}
\mathcal{B}_G(X\times[0,1);R)_\emptyset & \longrightarrow & \mathcal{B}_G(X\times[0,1);R) & \longrightarrow & \mathcal{B}_G(X\times[0,1);R)^{>0} \\
\downarrow a & & \downarrow b & & \downarrow c \\
\mathcal{B}_G(\bullet\times[0,1);R)_\emptyset & \longrightarrow & \mathcal{B}_G(\bullet\times[0,1);R) & \longrightarrow & \mathcal{B}_G(\bullet\times[0,1);R)^{>0}
\end{array}$$

The functor a is an equivalence of categories, and the K-theory or L-theory of $\mathcal{B}_G(\bullet\times[0,1);R)$ is trivial. Comparing the fibration sequences we obtain an identification of $c\colon F^\lambda(X) \to F^\lambda(\bullet)$, which is just

$$c\colon \lambda(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to \lambda(\mathcal{B}_G(\bullet\times[0,1);R)^{>0}),$$

to the connecting map $\lambda(\mathcal{B}_G(X\times[0,1);R)^{>0}) \to \Omega^{-1}\lambda(\mathcal{B}_G(X\times[0,1);R)_\emptyset)$, up to homotopy equivalence. □

For applications to topological K-theory and C^*-algebra, the definition of $\mathcal{B}_G(X\times[0,1);R)^{>0}$ must be modified. We produce a functor

$$C_r^*\mathcal{B}_G\colon G\text{-}CW\text{-Complexes} \to C^*Cat$$

such that by composing with $\mathbb{K}^{Top}$ we again get a functor with the properties of (5.1). This will define the third functor F^λ for $\lambda = \mathbb{K}^{Top}$ needed in the statement of our main theorem in the Introduction.

To define the C^*-category $C_r^*\mathcal{B}_G(X\times[0,1);\mathbb{C})$ we start with the category $\mathcal{B}_G(X\times[0,1);\mathbb{C})$. We further require that each object A comes with a G-invariant inner product. The objects are thus pre-Hilbert spaces with the group G acting by isometries. A pre-Hilbert space A uniquely determines a Hilbert space $\bar{A}$ by completion. The morphisms in $C_r^*\mathcal{B}_G(X\times[0,1);\mathbb{C})$ are obtained by first considering the subgroup of the morphisms in $\mathcal{B}_G(X\times[0,1);\mathbb{C})$ which are bounded linear operators $\phi\colon A\to B$ on the pre-Hilbert spaces. We then extend these morphisms to bounded linear operators $\bar{\phi}\colon \bar{A}\to\bar{B}$ on the Hilbert space completions, and finally take the closure of this collection of maps inside the space of all bounded operators from $\bar{A}$ to $\bar{B}$. This means that the morphisms in $C_r^*\mathcal{B}_G(X\times[0,1);\mathbb{C})$ can be approximated arbitrarily closely by continuously controlled morphisms, but may not themselves be continuously controlled. This gives $C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})$ the structure of a C^*-category.

The subcategory $C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})_\emptyset$ is again the full subcategory with objects (A,f) such that $\mathrm{supp}(A,f) = \emptyset$. The quotient category $C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0}$ has the same objects as $\mathcal{B}_G(X \times [0,1);\mathbb{C})$ and two morphisms are identified in the quotient category if their difference can be approximated arbitrarily closely by morphisms factoring through objects of the subcategory. We define

$$F^\lambda(X) := \mathbb{K}^{Top}(C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0})$$

for $\lambda = \mathbb{K}^{Top}$.

Theorem 7.5. *The functor $F^\lambda\colon G$-CW-Complexes $\to$ Spectra, $\lambda = \mathbb{K}^{Top}$, is G-homotopy invariant and G-excisive and has the following properties:*

(i) *If X is a cocompact G-space with finite isotropy then*

$$\pi_*(F^\lambda(X)) = \mathrm{KK}^G_{*-1}(C_0(X),\mathbb{C}))$$

where KK *denotes Kasparov's KK-groups, and $C_0(X)$ denotes the complex valued functions of X vanishing at infinity.*

(ii) *For any subgroup H of G, we have $\Omega F^\lambda(G/H) \simeq \mathbb{K}(C^*_r(H))$, the K-theory spectrum of the reduced C^*-algebra of H. Here $\simeq$ is weak equivalence of spectra.*

(iii) *For any G-space X, $F^\lambda(X)$ is the homotopy colimit of F^λ applied to the G-compact subspaces.*

Proof. Part (iii) follows immediately from the definition and the fact that the K-theory of a C^*-category is the direct limit of the K-theory of the endomorphisms of the objects of the category. The G-compact condition in axiom (iv) of the definition of $C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})$ means we might as well take G-compact subspaces and then the direct limit. This has as a consequence that we only need to consider cocompact spaces in what follows.

To see that the functor is excisive consider a Mayer-Vietoris situation in the category of G-CW-Complexes, with X and Y intersecting $X \cap Y$ with union $X \cup Y$. Identifying the category $C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0}_{X\cap Y}$ with $C^*_r\mathcal{B}_G(X \cap Y \times [0,1);\mathbb{C})^{>0}$ and similarly identifying $C^*_r\mathcal{B}_G(X \cup Y \times [0,1);\mathbb{C})^{>0}_Y$ with $C^*_r\mathcal{B}_G(Y \times [0,1);\mathbb{C})^{>0}$, we obtain a diagram of categories

$$\begin{array}{ccccc}
C^*_r\mathcal{B}_G(X\cap Y \times [0,1);\mathbb{C})^{>0} & \longrightarrow & C^*_r\mathcal{B}_G(Y \times [0,1);\mathbb{C})^{>0} & \longrightarrow & C^*_r\mathcal{B}_G(Y \times [0,1);\mathbb{C})^{>X\cap Y} \\
\downarrow & & \downarrow & & \downarrow \\
C^*_r\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0} & \longrightarrow & C^*_r\mathcal{B}_G(X\cup Y \times [0,1);\mathbb{C})^{>0} & \rightarrow & C^*_r\mathcal{B}_G(X\cup Y \times [0,1);\mathbb{C})^{>X}
\end{array}$$

where each row leads to a fibration of spectra when applying $\mathbb{K}^{Top}$ by the results of section 2. The right hand vertical arrow induces a homotopy equivalence since there is an obvious inverse given by projection. Using Lemma 2.3, this argument is a slight (equivariant) variant of the arguments given in [16] . This proves excision.

Denoting the one point compactification of X by $X_+ = X \cup \{+\}$, we note that $C_r^*\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0}$ and $C_r^*\mathcal{B}_G(X_+ \times [0,1);\mathbb{C})^{>\{+\}}$ are the same categories, so they have the same K-theory. This in turn has the same K-theory as $C_r^*\mathcal{B}_G(X_+ \times [0,1);\mathbb{C})$, so we are reduced to the case where the space is compact. Given excision, homotopy invariance is equivalent to showing that $C_r^*\mathcal{B}_G(CX_+ \times [0,1);\mathbb{C})$ has trivial K-theory. The proof of this follows the arguments of [2], see also [16].

The proof of (i) is given in [16], but we briefly recall the argument. An object A in the category $C_r^*\mathcal{B}_G(X \times [0,1);\mathbb{C})$ is a $C_0(X)$-module in an obvious way. A morphism $T : A \to B$ is a bounded operator of Hilbert spaces, and it was proved in [16] that the closure of the continuously controlled morphisms is precisely the pseudolocal operators. The condition of taking germs away from zero corresponds precisely to disregarding locally compact operators, and the isomorphism is obtained essentially by saying that the K-theory elements are the same in the two cases. An alternative proof would be to note that both functors are excisive, and they have the same values on orbits of type G/H when H is finite as stated in (ii). Note that the KK-groups are only defined on the category of G-CW-Complexes and proper G-maps, whereas $C_r^*\mathcal{B}_G(X \times [0,1);\mathbb{C})$ is defined for all G-spaces X and all G-maps. Part (i) says that these functors agree on the category of proper G-CW-Complexes and proper G-maps.

To verify (ii) notice that on $G/H \times [0,1)$, the continuous control condition means that near $G/H \times 1$ no component of an operator can reach from one branch to another. Since we are disregarding operators factoring through an object B with $\operatorname{supp}(B) = \emptyset$, this means we may as well assume an operator preserves $e/H \times [0,1)$ and is given by equivariance on the other branches. By definition of the homsets, the endomorphisms of an object with non-empty support will be just $C_r^*(H)$ tensored with the bounded operators, divided out by $C_r^*(H)$ tensored with the compact operators, because we disregard operators factoring through an object B with $\operatorname{supp}(B) = \emptyset$. We thus obtain the suspension of $C_r^*(H)$ and the result follows. □

We now relate the continuously controlled assembly map

$$\alpha_X \colon \mathbb{K}^{Top}(C_r^*\mathcal{B}_G(X \times [0,1);\mathbb{C})^{>0}) \to \mathbb{K}^{Top}(C_r^*\mathcal{B}_G(\bullet \times [0,1);\mathbb{C})^{>0})$$

to the Baum-Connes map, thus proving our main theorem for topological K-theory.

Theorem 7.6. *Let $X = \mathcal{E}_{\mathcal{F}}G$ denote the classifying space for G-actions with finite isotropy. Then the continuously controlled assembly map α_X induces the Baum-Connes assembly map*

$$KK_i^G(C_0(X);\mathbb{C}) \to K_i(C_r^*G)$$

on homotopy groups.

Proof. Every element T in $KK_i^G(C_0(X);\mathbb{C})$ can be represented by a bounded operator of Hilbert spaces with G-action. Moreover, this operator can be chosen to satisfy a Fredholm condition, so the equivariant index is well-defined, giving an element in $K_i(C^*_r G)$. This is the Baum-Connes map (see [4] for more details).

The domain $K_{i+1}(C^*_r\mathcal{B}_G(X\times[0,1);\mathbb{C})^{>0})$ of the continuously controlled assembly map is isomorphic to $KK^G_i(C_0(X);\mathbb{C})$ by Part (i) of Theorem 7.5 above. Now consider an element in $K_{i+1}(C^*_r\mathcal{B}_G(X\times[0,1);\mathbb{C})^{>0})$. The effect of sending X to a point is to forget all control conditions getting an element of $K_{i+1}(C^*_r\mathcal{B}_G(\bullet\times[0,1);\mathbb{C})^{>0})$. The group $K_{i+1}(C^*_r\mathcal{B}_G(\bullet\times[0,1);\mathbb{C})^{>0})$ is isomorphic to $K_i(C^*_r\mathcal{B}_G(\bullet\times[0,1);\mathbb{C})_\emptyset)\cong K_i(C^*_r G)$, and this last isomorphism is again given by the equivariant index. By the C^*-version of Theorem 7.4, this composite is homotopy equivalent to the continuously controlled assembly map. □

8. Groupoids to Controlled Categories

In this section we will relate the Davis-Lück assembly maps constructed from functors $F\colon \mathbf{Or}(G)\to \mathit{AddCat}$ via groupoids to those constructed on G-CW-Complexes using the continuously controlled categories. Since the extension of functors $\mathbf{Or}(G)\to \mathit{Spectra}$ to G-CW-Complexes is unique, given the properties (5.1), it is enough to compare these functors for $X=G/H$.

We define a subcategory $\mathcal{B}^{(c)}_G(X\times[0,1);R)$ of $\mathcal{B}_G(X\times[0,1);R)$, with the same objects but we allow only morphisms $\phi\colon A\to B$ such that the components $\phi^{(y,t)}_{(x,s)}=0$ whenever $s\neq t$, for $0\le s,t<1$. We will also need the corresponding subcategory of $C^*_r\mathcal{B}_G(X\times[0,1);\mathbb{C})$, based on the definitions of Section 7.

Proposition 8.1. *There is a functor*

$$I\colon R(\overline{G/H})_\oplus \to \mathcal{B}^{(c)}_G(G/H\times[0,1);R)_\emptyset$$

inducing isomorphisms in $\mathbb{K}^{-\infty}$-theory, $\mathbb{L}^{-\infty}$-theory, and similarly for $\mathbb{K}^{Top}$ applied to $C^*_r(\overline{G/H})$.

Proof. We will discuss the functor on $R(\overline{G/H})_\oplus$. A similar discussion holds for the C^*-category version. If $S=G/H$, then objects in $\overline{G/H}$ are just cosets $s\in S$. Let $H_s=\{g\in G\,|\,gs=s\}$. This is a conjugate subgroup of H in G. We define I on objects by sending $s\mapsto RH_s$ over $(s,0)$, and then extend by equivariance to $RG\otimes_{H_s}RH_s$. A morphism $g\in \mathrm{mor}(s,t)$ has the property that $g_2^{-1}gg_1^{-1}=h\in H$, where $s=g_1H$ and $t=g_2H$. Then $I(g)(a):=ag^{-1}$ for all $a\in RH_s$, and

$$I(g)\in \mathrm{Hom}_{RH_s}(R[g_1Hg_1^{-1}],R[g_1Hg_2^{-1}])$$

where both $R[g_1Hg_1^{-1}]=RH_s$ and $R[g_1Hg_2^{-1}]$ are left RH_s-modules, since $H_s=g_1Hg_1^{-1}$. The module $R[g_1Hg_2^{-1}]$ is isomorphic (as a left RH_s-module) to $I(H_t)_s=(g_1^{-1}g_2)\otimes_{H_t}RH_t$, so we have defined a component

$$I(g)^s_s\colon I(s)_s\to I(t)_s$$

of the morphism. The components $I(g)^t_s=0$ when $s\neq t$.

This is again extended by equivariance to get an element in $\mathrm{mor}(I(s), I(t))$. Notice that these morphisms preserve the levels $\{s\} \times [0,1)$, and have empty support at $G/H \times 1$, so we get a functor into $\mathcal{B}_G^{(c)}(X \times [0,1); R)_\emptyset$.

To complete the proof we need to show that I induces isomorphisms on homotopy groups. For any coset $s \in G/H$, let $FMod(RH_s)$ denote the category of finitely generated free left RH_s-modules. Consider the commutative diagram:

$$\begin{array}{ccc} R(\overline{G/H})_\oplus & \xrightarrow[I]{\approx} & \mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset \\ \uparrow & & \uparrow \\ FMod(RH_s) & \longrightarrow & \mathcal{B}^{(c)}([0,1); RH_s)_\emptyset \end{array}$$

where the horizontal arrows give equivalences of categories (by our stronger control condition on morphisms). The left vertical arrow is induced by an equivalence of groupoids, namely the inclusion of the sub-groupoid generated by the object $\{s\} \in \overline{G/H}$ (see [8, 2.4(ii)]). We get isomorphisms on $\mathbb{K}^{-\infty}$, $\mathbb{L}^{-\infty}$, and on $\mathbb{K}^{Top}$ using the C^*-category versions of $R(\overline{G/H})_\oplus$ (see [8, p. 212-213]) and $\mathcal{B}_G^{(c)}(G/H \times [0,1); R)$.

The right vertical arrow is given by induction of modules. Since each object in $\mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset$ is isomorphic to an induced object $RG \otimes_{RH_s} RH_s$, we obtain an equivalence of categories. The same holds for the C^*-version. □

Theorem 8.2. *Let F denote one of the functors $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$. The functors $F_i \colon \mathbf{Or}(G) \to Spectra$, for $i = 1, 2$, defined by $F_1(G/H) = F(R(\overline{G/H})_\oplus)$, and $F_2(G/H) = \Omega F(\mathcal{B}_G(G/H \times [0,1); R)^{>0})$ are homotopy equivalent. The same conclusion holds for $F = \mathbb{K}^{Top}$ applied to the C^*-category versions.*

Proof. In this argument, no special modifications are needed for the $\mathbb{K}^{Top}$-version beyond taking the C^*-categories in the definition of F_1 and F_2. We will construct a natural transformation between the functors F_1 and F_2 which induces isomorphisms on the homotopy groups.

Consider the commutative diagram of fibrations:

$$\begin{array}{ccccc} \mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset & \longrightarrow & \mathcal{B}_G^{(c)}(G/H \times [0,1); R) & \longrightarrow & \mathcal{B}_G^{(c)}(G/H \times [0,1); R)^{>0} \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ \mathcal{B}_G(G/H \times [0,1); R)_\emptyset & \longrightarrow & \mathcal{B}_G(G/H \times [0,1); R) & \longrightarrow & \mathcal{B}_G(G/H \times [0,1); R)^{>0} \end{array}$$

where the spectrum $F(\mathcal{B}_G^{(c)}(G/H \times [0,1); R))$ has trivial homotopy groups, and the map c induces an equivalence of categories. It follows that the connecting map from the upper fibration, together with c, induce a homotopy equivalence

$$\Omega F(\mathcal{B}_G(G/H \times [0,1); R)^{>0}) \xrightarrow{\simeq} F(\mathcal{B}_G^{(c)}(G/H \times [0,1); R)_\emptyset)$$

of spectra. Now the composite of the natural isomorphism I from Lemma 8.1, and the inverse of this homotopy equivalence gives the required natural isomorphism between F_1 and F_2. □

We have now reached one of our main goals.

Theorem 8.3. *The Davis-Lück assembly maps arising from* $\mathbb{K}^{-\infty}$, $\mathbb{L}^{-\infty}$, *and* $\mathbb{K}^{Top}$ *are homotopy equivalent to the continuously controlled assembly maps.*

Proof. The description in (5.2)-(5.4) of the Davis-Lück functors shows that they all factor through the functor Gr (or its C^* variant) out of the orbit categrory $\mathbf{Or}(G)$. On the other hand, the continuously controlled functors H_C from (7.3) are all determined by K-theory or L-theory applied to $\mathcal{B}_G(G/H \times [0,1); R)^{>0}$. The result now follows from Theorem 8.2. □

Corollary 8.4. *The Davis-Lück assembly map arising from* $\mathbb{K}^{Top}$ *applied to* $\mathcal{E}_{\mathcal{F}}(G)$*, for* $\mathcal{F}$ *the family of finite subgroups of* G*, isomorphic (on homotopy groups) to the Baum-Connes assembly map.*

Proof. The result of Theorem 7.6 identifies the Baum-Connes assembly map with the one induced by $\mathbb{K}^{Top}$ and the continuously controlled categories (applied to the G-space $\mathcal{E}_{\mathcal{F}}(G)$). The final step is provided by Theorem 8.3. □

9. $\mathbf{Or}(G)$ to fundamental groupoids

In this section we factor the functor $Gr\colon \mathbf{Or}(G) \to Groupoids$ of Davis-Lück through the functor $E\colon \mathbf{Or}(G) \to Spaces$ defined on objects by $G/H \mapsto G/H \times_G EG$, where EG denotes the universal contractible G-space on which G acts freely. The passage $\pi\colon Spaces \to Groupoids$ is just $X \mapsto \pi(X)$, where $\pi(X)$ denotes the fundamental groupoid of X. As a consequence, we can identify the Davis-Lück assembly maps with those of Farrell-Jones for algebraic K-theory. By Theorem 8.3, this also completes our identification of the Farrell-Jones assembly maps in K-theory with the continuously controlled assembly maps.

Proposition 9.1. *There is a functor* $\Phi\colon \overline{G/H} \to \pi(G/H \times_G EG)$ *inducing isomorphisms in algebraic* K*-theory and* L*-theory.*

Proof. We fix a basepoint $\bullet \in EG$, and let $(\bullet) \subset \pi(G/H \times_G EG)$ denote the full subcategory with one object $[eH, \bullet]$. Similarly, let $(\star) \subset \overline{G/H}$ denote the full subcategory with one object eH. For any object $s \in G/H$ we set $\Phi(s) = [s, \bullet] \in G/H \times_G EG$. For a morphism $g \in \mathrm{mor}(s,t)$, we set $\Phi(g) = [s, \gamma(u)]$, using a path $\gamma(u)$, $0 \le u \le 1$, in EG joining $\bullet$ and $g^{-1} \cdot \bullet$. The path is unique up to homotopy, and $[s, g^{-1} \cdot \bullet] = [t, \bullet]$. Since $\Phi(eH) = [eH, \bullet]$, we get an equivalence of the subcategories $(\star) \to (\bullet)$.

Consider the commutative diagram:

$$\begin{array}{ccc} R(\star)_\oplus & \longrightarrow & R(\bullet)_\oplus \\ \downarrow & & \downarrow \\ R(\overline{G/H})_\oplus & \xrightarrow{\Phi} & R(\pi(G/H \times_G EG))_\oplus \end{array}$$

where the inclusion $R(\star)_\oplus \subset R(\overline{G/H})_\oplus$ induces isomorphisms on $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$, since $\overline{G/H}$ is a connected groupoid, as in the proof of Proposition 8.1 (see [36, 2.1], [8, 2.4(ii)]). Similarly, the inclusion $R(\bullet)_\oplus \subset \pi(G/H \times_G EG)$ induces isomorphisms on $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$, and the result for Φ follows. □

Corollary 9.2. *The Davis-Lück assembly map for algebraic K-theory is homotopy equivalent to the Farrell-Jones assembly map.*

Proof. The Farrell-Jones assembly map in K-theory is just the Quinn construction of Section 6 applied to the functor $F = \mathbb{K}^{-\infty} \circ \pi\colon \mathit{Spaces} \to \mathit{Spectra}$ and the space $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ is the family of virtually infinite cyclic subgroups of G. To identify this assembly map with the Davis-Lück assembly map, we just combine Proposition 9.1 above with Theorem 6.1. □

10. Assembly in L-theory

In this section we show that the Davis-Lück assembly maps in L-theory are compatible with the original geometric L-theory assembly maps

$$X_+ \wedge \mathbf{L}(\mathbb{Z}) \to \mathbf{L}(\mathbb{Z}\pi_1(X, x_0))$$

of Quinn [29] (or more precisely with the $L^{-\infty}$-version of Yamasaki [43]), or with the subsequent algebraic L-spectrum construction of Ranicki [34], and those used in the Farrell-Jones Isomorphism Conjectures. By Theorem 8.3, this completes our identification of these maps with the continuously controlled assembly map.

The L-theory assembly maps of Farrell-Jones uses Quinn's construction (described in Section 6) applied to the functor $\mathbb{L}^{-\infty}\colon \mathit{Spaces} \to \mathit{Spectra}$. As explained in § 2, this is constructed from the connective geometric surgery spectrum $\mathbb{L}^{geom}$ defined by Quinn [29] whose homotopy groups $\pi_n(\mathbb{L}^{geom}(X)) \cong L_n(\mathbb{Z}\pi_1(X, x_0))$ are the geometric surgery obstruction groups of C. T. C. Wall [40]. In particular, $\pi_n(\mathbb{L}^{geom}(\bullet)) = L_n(\mathbb{Z})$ are the surgery obstruction groups for the trivial group.

We also have the algebraic L-spectrum $\mathbb{L}^{alg}$ and its corresponding $L^{-\infty}$-version. In this section, we want to compare the $L^{-\infty}$-versions of $\mathbb{L}^{geom}$ and $\mathbb{L}^{alg}$, and it is convenient to use these two symbol to emphasize the contrast between algebraically and geometrically defined functors. We mean the $L^{-\infty}$ versions of both functors throughout.

We have already shown in Theorem 6.1 that the generalized Quinn assembly for the functor $F = \mathbb{L}^{alg}$ or $F = \mathbb{L}^{geom}$ is homotopy equivalent to the corresponding Davis-Lück assembly map.

We compare the $L^{-\infty}$-versions of $(\alpha_X, \mathbb{L}^{alg})$ to $(\alpha_X, \mathbb{L}^{geom})$ by checking what happens over the orbit category $\mathbf{Or}(G)$.

The first step is Proposition 9.1, where a functor $\Phi\colon \overline{G/H} \to \pi(G/H \times_G EG)$ is defined inducing equivalences $\mathbb{L}^{alg}(\mathbb{Z}(\overline{G/H})_\oplus) \simeq \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)$. The remaining step is the following result.

Proposition 10.1. *There is a homotopy natural transformation*

$$\Theta\colon \mathbb{L}^{geom}(G/H \times_G EG) \simeq \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)$$

which is a homtopy equivalence of spectrum-valued functors on $\mathbf{Or}(G)$*.*

Proof. There is a homotopy equivalence $\mathbb{L}^{geom}(G/H \times_G EG) \simeq \mathbb{L}^{alg}(\mathbb{Z}H)$ by one of the main results of geometric surgery theory [40, Ch. 9]. The equivalence depends on a choice of base-point in $G/H \times_G EG \simeq BH$, so this isn't natural.

Similarly, algebraic surgery theory show that there is a homotopy equivalence

$$\mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus)) \simeq \mathbb{L}^{alg}(\mathbb{Z}H),$$

since the category $\mathbb{Z}(\overline{G/H})_\oplus$ is equivalent to the category of finitely generated free $\mathbb{Z}H$-modules (again by picking a base point as in § 8).

However, the algebraic chain complex theory of Ranicki [33], [34] over the additive category $\mathbb{Z}(\pi(G/H \times_G EG)_\oplus$, provides a natural setting for the passage from geometry to algebra without choosing base points. We triangulate EG and apply the construction of the algebraic quadratic Poincaré complexes over the semi-simplicial Δ-set models for $\mathbb{L}^{geom}$ and $\mathbb{L}^{alg}$. This gives a homotopy natural transformation Θ and a homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{L}^{geom}(G/H \times_G EG) & \xrightarrow{\simeq} & \mathbb{L}^{alg}(\mathbb{Z}H) \\ \Theta \big\downarrow & \nearrow_{\simeq} & \\ \mathbb{L}^{alg}(\mathbb{Z}(\pi(G/H \times_G EG)_\oplus) & & \end{array}$$

showing that Θ is a homotopy equivalence. □

Corollary 10.2. *The Farrell-Jones assembly map for geometric $L^{-\infty}$-theory is homotopy equivalent to the Davis-Lück assembly map $(\alpha_X, \mathbb{L}^{-\infty})$ for $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G.*

11. Isomorphism conjectures in L-theory

The Farrell-Jones Isomorphism Conjecture in L-theory is stated only for $\mathbb{L}^{-\infty}$. Recently it was noticed that the corresponding assertion for the functor $\mathbb{L}^h$ is *not* valid for $G = \mathbf{Z}^2 \times \mathbf{Z}/5$ [10]. From the conceptual point of view, this is not surprising since the computation of the domain $F^{\%}(\mathcal{E}_{\mathcal{F}}(G))$ for $F = \mathbb{L}^h$ will involve a stratified system of splitting obstructions in various K-groups. However, neither the Davis-Lück theory nor the Quinn generalized assembly framework seems to have a suitable "intermediate" functor with which to formulate an $\mathbb{L}^h$ conjecture. We would like to propose the following:

Conjecture 11.1. *For any discrete group G and any commutative ring R, the assembly map*

$$\mathbb{L}^s(\mathcal{B}_G(X \times [0,1); R)^{>0}) \to \mathbb{L}^s(\mathcal{B}_G(\bullet \times [0,1); R)^{>0} \cong \Omega^{-1}\mathbb{L}^h(RG)$$

is a homotopy equivalence for $X = \mathcal{E}_{\mathcal{F}}(G)$, where $\mathcal{F}$ denotes the family of virtually infinite cyclic subgroups of G.

Since $\Omega\mathbb{L}^s(\mathcal{B}_G(\bullet \times [0,1); R)^{>0} \simeq \mathbb{L}^h(RG)$, the conjecture provides a possible description of the L-groups of RG with K-theory decorations in $Wh(RG)$, $\tilde{K}_0(RG)$, or $K_i(RG)$ for $i < 0$.

We remark that this conjecture holds for any group G over a ring R such that the Farrell-Jones Isomorphism Conjectures are valid for $\mathbb{L}^{-\infty}(RG)$, and for $\mathbb{K}^{-\infty}(RG)$ in dimensions ≤ 1, by a 5-lemma argument on the Ranicki-Rothenberg sequences relating L-groups with varying K-theory decorations.

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