BOUNDED AND CONTINUOUS CONTROL

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The purpose of this note is to clarify the relation between bounded and continuous control as it occurs in various proofs of the Novikov conjecture [2, 3]. In [2] the bounded categories $\mathcal{C}$ of [8] are used to prove the algebraic K-theory Novikov conjecture for certain groups. In [3] the continuously controlled categories $\mathcal{B}$ of [1] are used to prove the algebraic K- and L-theory Novikov conjectures for a larger class of groups.

Since continuous control is finer than bounded control, there should be a forgetful functor from continuously controlled algebra to bounded algebra. This is only true, however, after delooping the categories. A functor from bounded algebra to continuously controlled algebra does not always exist, but if the metric space can be given an ideal boundary satisfying conditions of the type that "bounded is small at infinity", one gets a functor from bounded control to continuous control at infinity. To make this precise let us recall some definitions. We shall use the language of algebraic K- or L-theory, but it is relatively straightforward to modify these considerations to topological K-theory or A-theory, using [5, 4].

Let $\mathcal{A}$ be a small additive category, M a proper metric space.

1. **Definition.** The *bounded category* $\mathcal{C}(M;\mathcal{A})$ has objects $A = \{A_x\}_{x\in M}$, a collection of objects from $\mathcal{A}$ indexed by points of M, satisfying $\{x|A_x \neq 0\}$ is locally finite in M. A morphism $\phi : A \to B$ is a collection of morphisms $\phi^x_y : A_x \to B_y$ so that there exists $k = k(\phi)$ so $\phi^x_y = 0$ if $d_M(x, y) > k$.

Composition is defined as matrix multiplication. Given a subspace $N \subset M$, we denote the full subcategory with objects A so that $\{x|A_x \neq 0\}$ is contained in a bounded neighborhood of N by $\mathcal{C}(M;\mathcal{A})_N$. It is easy to see that $\mathcal{C}(M;\mathcal{A})$ is $\mathcal{C}(M,\mathcal{A})_N$-filtered in the sense of Karoubi [6]. We denote the quotient category by $\mathcal{C}(M,\mathcal{A})^{>N}$, the category of germs away from N. $\mathcal{C}(M;\mathcal{A})^{>N}$ thus has the same objects as $\mathcal{C}(M,\mathcal{A})$, but morphisms are identified if they agree except in a bounded neighborhood of N.

Next we recall the definition of continuous control. Let (X, Y) be a pair of Hausdorff spaces, Y closed in X, and contained in $\overline{X - Y}$, $\mathcal{A}$ a small additive category.

2. **Definition.** The *continuously controlled category* $\mathcal{B}(X, Y;\mathcal{A})$ has objects $A = \{A_x\}_{x\in X-Y}$ satisfying $\{x|A_x \neq 0\}$ is locally finite in $X - Y$. A morphism $\phi : A \to B$ is a collection of morphisms $\phi^x_y : A_x \to B_y$ satisfying that for every $z \in Y$ it is continuously

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controlled i. e. for every neighborhood U of z in X, there is a neighborhood V of z in X so that $x \in V$, $y \notin U$ implies $\phi_y^x = 0$ and $\phi_x^y = 0$.

Again composition is defined by matrix multiplication. If Z is a closed subset of Y, we denote the full subcategory of $\mathcal{B}(X, Y; \mathcal{A})$ with objects A so that $\partial(\{x | A_x \neq 0\} \subseteq Z$, by $\mathcal{B}(X, Y; \mathcal{A})_Z$. Here ∂ denotes the *topological* boundary. It is easy to see that $\mathcal{B}(X, Y; \mathcal{A})$ is $\mathcal{B}(X, Y\mathcal{A})_Z$-filtered in the sense of Karoubi. We denote the quotient category by $\mathcal{B}(X, Y; \mathcal{A})^{Y-Z}$ or $\mathcal{B}(X, Y; \mathcal{A})^{>Z}$. The objects are the same as in $\mathcal{B}(X, Y; \mathcal{A})$ but morphisms are identified if they agree in a neighborhood of $Y - Z$.

Recall the following from [1] and [2, 3]. Given a small additive category $\mathcal{A}$, we associate a spectrum $K^{-\infty}(\mathcal{A})$ as follows: restricting the morphism to isomorphisms we get a symmetric monoidal category to which there is a functorially associated connective spectrum $K(\mathcal{A})$. The inclusion

$$K(\mathcal{C}(\mathbb{R}^i; \mathcal{A})) \to K(\mathcal{C}(\mathbb{R}^{i+1}; \mathcal{A}))$$

is canonically null homotopic in two ways by Eilenberg swindle towards $+\infty$ and $-\infty$ respectively. Hence we get a map

$$\Sigma K(\mathcal{C}(\mathbb{R}^i; \mathcal{A})) \to K(\mathcal{C}(\mathbb{R}^{i+1}; \mathcal{A}))$$

or adjointly

$$K(\mathcal{C}(\mathbb{R}^i; \mathcal{A})) \to \Omega K(\mathcal{C}(\mathbb{R}^{i+1}; \mathcal{A}))$$

and we define

$$K^{-\infty}(\mathcal{A}) = \operatorname{hocolim}(\Omega^i K(\mathcal{C}(\mathbb{R}^i; \mathcal{A}))) \ .$$

Similarly if $\mathcal{A}$ is a small additive category with involution, we have the functor $\mathbb{L}^{-\infty}$ to spectra [10]. In this note we shall use spt to denote $K^{-\infty}$ or $\mathbb{L}^{-\infty}$. Using [5] the considerations in here generalize to topological K-theory, replacing additive categories by controlled C^*-algebras, and using [4] they generalize to A-theory, replacing additive categories by appropriate categories with cofibrations and weak equivalences.

3. **Definition.** Let spt be any one of $K^{-\infty}$, $\mathbb{L}^{-\infty}$, K^{top} or $A^{-\infty}$.

We shall use the language appropriate for $K^{-\infty}$ throughout.

4. **Theorem.** ([3, 2, 12, 11, 10]) *For any proper metric space M and any subspace $N \subseteq M$ there is a fibration of spectra*

$$\operatorname{spt}(\mathcal{C}(M; \mathcal{A})_N) \to \operatorname{spt}(\mathcal{C}(M; \mathcal{A})) \to \operatorname{spt}(\mathcal{C}(M, \mathcal{A})^{>N})$$

5. **Theorem.** ([1, 3]) *For any compact Hausdorff pair (X, Y) such that $Y \subset \overline{X - Y}$ and a closed subset $Z \subseteq Y$ there is a fibration of spectra*

$$\operatorname{spt}(\mathcal{B}(X, Y; \mathcal{A})_Z) \to \operatorname{spt}(\mathcal{B}(X, Y; \mathcal{A})) \to \operatorname{spt}(\mathcal{B}(X, Y; \mathcal{A})^{Y-Z}) \ .$$

6. **Definition.** A *Steenrod functor* is a homotopy invariant functor

$$\mathrm{St}\ :\ \{\text{compact metrizable spaces}\} \ \to\ \{\text{spectra}\}\ ;\ Y\ \to\ \mathrm{St}(Y)$$

with the following two properties :

(i) St is strongly excisive, i. e. if $Z \subseteq Y$ there is a fibration

$$\mathrm{St}(Z)\ \to\ \mathrm{St}(Y)\ \to\ \mathrm{St}(Y/Z)\ ,$$

(ii) St satisfies the wedge axiom, i. e. for any countable collection $\{X_i\}$ of compact metrizable spaces with wedge $\bigvee\limits_i X_i \subset \prod_i X_i$ the natural map is a homotopy equivalence

$$\mathrm{St}(\bigvee_i X_i)\ \simeq\ \prod_i \mathrm{St}(X_i)\ .$$

Milnor [7] showed that a natural transformation $\mathrm{St}_1 \to \mathrm{St}_2$ of Steenrod functors which induces a homotopy equivalence on S^0 induces a homotopy equivalence $\mathrm{St}_1(Y) \to \mathrm{St}_2(Y)$ for every compact metric space Y.

Theorem 5 is one of the key ingredients in proving:

7. **Theorem.** ([3]) *For any compact metrizable pair* (X, Y) *there is a homotopy equivalence*

$$\mathrm{spt}(\mathcal{B}(X, Y; \mathcal{A}))\ \simeq\ \mathrm{spt}(\mathcal{B}(CY, Y; \mathcal{A}))$$

where CY *is the cone on* Y*. Moreover,*

$$\begin{aligned} \mathrm{st}\ &: \{\textit{compact metrizable spaces}\}\ \to\ \{\textit{spectra}\}\ ;\\ Y\ &\to\ \mathrm{st}(Y) = \mathrm{spt}(\mathcal{B}(CY, Y; \mathcal{A})) \end{aligned}$$

is a Steenrod functor.

8. **Definition.** The functor

$$M \to h^{l.f.}(M; \Sigma\,\mathrm{spt}(\mathcal{A})) = \mathrm{st}(M_+)$$

is the *homology with locally finite coefficients in the spectrum* $\Sigma\,\mathrm{spt}(\mathcal{A})$ on the category of spaces M with metrizable one-point compactification M_+ and proper maps.

Recall that a metric space M is proper if closed balls are compact. The one-point compactification M_+ of a locally compact space M is metrizable if and only if M is second countable.

9. **Definition.** Let M be a proper metric space with a compactification $\overline{M}$, such that M is dense in $\overline{M}$. Thus $\partial M = \overline{M} - M$ is the topological boundary of M in $\overline{M}$. The compactification is *small at infinity* if for every $z \in \partial M$, $k \in \mathbb{N}$, and neighborhood U of z in $\overline{M}$, there is a neighborhood V of z in $\overline{M}$ so that $B(x, k) \subset U$ for all $x \in V \cap M$.

The one-point compactification M_+ of a proper metric space M is an obvious example of such a compactification. A more interesting class of examples is provided by radial compactifications of nonpositively curved manifolds.

10. **Proposition.** *If M is a proper metric space and $\overline{M}$ is a compactification which is small at infinity there is a "forget some control" map*

$$\mathcal{C}(M;\mathcal{A}) \to \mathcal{B}(\overline{M},\partial M;\mathcal{A})$$

which is the identity on objects.

Proof. One simply observes that bounded morphisms are automatically continuously controlled at points of ∂M. □

We thus get an induced map of spectra

$$\mathrm{spt}(\mathcal{C}(M;\mathcal{A})) \to \mathrm{spt}(\mathcal{B}(\overline{M},\partial M);\mathcal{A}) \simeq \mathrm{st}(\partial M)$$

11. **Definition.** The *continuously controlled assembly map* is the connecting map in the Steenrod theory

$$\partial : \mathrm{st}(M_+) = \mathrm{st}(\overline{M}/\partial M) \to \Sigma\,\mathrm{st}(\partial M)$$

The relation between bounded and continuous control is derived from the following :

12. **Theorem.** *If M is a proper metric space with a metrizable one-point compactification there is a* bounded assembly map

$$h^{l.f.}(M;\Sigma\,\mathrm{spt}(\mathcal{A})) \to \Sigma\,\mathrm{spt}(\mathcal{C}(M;\mathcal{A}))\ .$$

If $\overline{M}$ is a compactification of M which is small at infinity the diagram

$$\begin{array}{ccc} h^{l.f.}(M;\Sigma\,\mathrm{spt}(\mathcal{A})) & \longrightarrow & \Sigma\,\mathrm{spt}(\mathcal{C}(M;\mathcal{A})) \\ \| & & \downarrow \\ \mathrm{st}(\overline{M}/\partial M) & \xrightarrow{\ \partial\ } & \Sigma\,\mathrm{st}(\partial M) \end{array}$$

is commutative.

Proof. We need an appropriate categorical model for $\Sigma\,\mathrm{spt}(\mathcal{C}(M;\mathcal{A}))$ to be able to relate bounded control to continuous control, since continuous control cannot be discussed without the extra parameter in the cone. We shall use ad hoc notation. The following is a diagram of categories as above. The objects are parameterized by $M\times(0,1)$ in all cases, but the control conditions vary. As part of the proof we construct a map from $\mathrm{spt}(\mathcal{B}(C(M_+),M_+;\mathcal{A}))\to \mathrm{spt}(\mathcal{C}(M,\mathcal{A}))$ for any metric space M justifying the claim that there is always a forget control map from continuous to bounded control. Here is a diagram which we proceed to explain

$$\begin{array}{ccccc} \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathcal{A}_3 \\ \downarrow & & & & \uparrow \\ \mathcal{A}_4 & & \longrightarrow & & \mathcal{A}_5 \end{array}$$

Here

$$\mathcal{A}_4 = \mathcal{B}(C\overline{M}, C\partial M \cup \overline{M}; \mathcal{A})$$
$$\mathcal{A}_5 = \mathcal{B}(\Sigma\overline{M}, \Sigma\partial M; \mathcal{A})$$

and the map $\mathcal{A}_4 \to \mathcal{A}_5$ is induced by collapsing $\overline{M}$. This means that

$$\mathrm{spt}(\mathcal{A}_4) \to \mathrm{spt}(\mathcal{A}_5)$$

is the boundary map

$$\partial : \mathrm{st}(\overline{M} \cup \partial M) \to \mathrm{st}(\Sigma\partial M) \simeq \Sigma\, \mathrm{st}(\partial M)$$

The category

$$\mathcal{A}_1 = \mathcal{B}^b(C\overline{M}, C\partial M \cup \overline{M}; \mathcal{A})$$

is the subcategory of $\mathcal{A}_4$, where the morphisms are required also to have bounded control. The map $\mathcal{A}_1 \to \mathcal{A}_4$ forgets this extra control requirement. It induces a homotopy equivalence because in both cases the subcategories with support at $C(\partial M)$ allow Eilenberg swindles and the germ categories at M are isomorphic. The spectra $\mathrm{spt}(\mathcal{A}_1)$ and $\mathrm{spt}(\mathcal{A}_4)$ are thus models for $h^{l.f.}(M; \Sigma\, \mathrm{spt}(\mathcal{A}))$.

$\mathcal{A}_2$ is the categorical model for $\Sigma\, \mathrm{spt}(\mathcal{C}(M; \mathcal{A}))$. We think of $M \times (0, 1)$ as compactified by $\Sigma\partial M$, and require continuous control at the suspension points and bounded control everywhere. The map $\mathcal{A}_1 \to \mathcal{A}_2$ is induced by collapsing $\overline{M}$ and forgetting the continuous control along $\partial M \times (0, 1)$. The subcategories of $\mathcal{A}_2$ with support at 0 and 1 respectively, admit obvious Eilenberg swindles and intersect in a category isomorphic to $\mathcal{C}(M; \mathcal{A})$. Hence $\mathrm{spt}(\mathcal{A}_2)$ is a deloop of $\mathrm{spt}(\mathcal{C}(M; \mathcal{A}))$. Finally

$$\mathcal{A}_3 = \mathcal{B}(\Sigma\overline{M}, \Sigma\partial M, p_{\overline{M}}; \mathcal{A})$$

is the category with continuous control at the suspension points, but along $\partial M \times (0, 1)$ we only require control in the ∂M-direction. Arguing as for $\mathcal{A}_2$ we find $\mathrm{spt}(\mathcal{A}_3)$ is a deloop of $\mathrm{spt}(\mathcal{B}(\overline{M}, \partial M; \mathcal{A}))$. The map $\mathcal{A}_2 \to \mathcal{A}_3$ is a forget control map sending bounded control to control in the ∂M-direction. The map $\mathcal{A}_2 \to \mathcal{A}_3$ is precisely the deloop of the map $\mathcal{C}(M; \mathcal{A}) \to \mathcal{B}(\overline{M}, \partial M; \mathcal{A})$ mentioned earlier, the map forgetting bounded control, but remembering the control at infinity. To see this consider the diagram

$$\begin{CD} \mathcal{B}(I, S^0; \mathcal{C}(M;R)) @>>> \mathcal{B}(I, S^0; \mathcal{B}(\overline{M}, \partial M; R)) \\ @VVV @VVV \\ \mathcal{A}_4 @>>> \mathcal{A}_5 \end{CD}$$

where all the maps are induced by the identity on objects, and the vertical maps induce homotopy equivalences. □

Looping the map

$$h^{l.f.}(M; \Sigma\,\mathrm{spt}(\mathcal{A})) = \mathrm{st}(\overline{M}/\partial M) \to \Sigma\,\mathrm{spt}(\mathcal{C}(M;\mathcal{A}))$$

gives a map from $h^{l.f.}(M; \mathrm{spt}(\mathcal{A}))$ to the bounded K-theory of M. This is the map used in [2], where $h^{l.f.}(M; \mathrm{spt}(\mathcal{A}))$ was given a more homotopy theoretic description.

13. **Remark.** The bounded L-theory assembly map of [9, p. 327]

$$A^{l.f.} : H_*^{l.f.}(X; \mathbb{L}.(R)) \to L_*(\mathcal{C}(X;R))$$

is the special case of the bounded assembly map of Theorem 12 when spt is the quadratic L-spectrum $\mathbb{L}.(R)$ of a ring with involution R.

In the applications to the Novikov conjecture in [3, 2] $M = E\Gamma$ is a contractible space with a free action of a discrete group Γ.

14. **Definition.** An *equivariant split injection* of spectra with Γ-actions $S \to T$ is an equivariant map of spectra for which there exists a spectrum R with Γ-action and an equivariant map of spectra $T \to R$ such that the composite $S \to T \to R$ is a homotopy equivalence (non-equivariantly).

15. **Theorem.** *Let Γ be a discrete group such that $B\Gamma$ is finite and the free Γ-action on $M = E\Gamma$ has a metrizable Γ-equivariant compactification $\overline{M}$ which is small at infinity.*

(i) *The continuously controlled assembly map factors through the bounded assembly map*

$$h^{l.f.}(E\Gamma; \Sigma\,\mathrm{spt}(\mathcal{A})) \to \Sigma\,\mathrm{spt}(\mathcal{C}(E\Gamma;\mathcal{A})) \to \mathrm{st}(\Sigma\partial M)\ .$$

(ii) *If the bounded assembly map*

$$\Sigma\,\mathrm{spt}(\mathcal{C}(E\Gamma;\mathcal{A})) \to \mathrm{st}(\Sigma\partial M)$$

is an equivariant split injection then the Novikov conjecture holds for Γ, meaning that the assembly map

$$h(B\Gamma; \mathrm{spt}(R)) \to \mathrm{spt}(R[\Gamma])$$

is a split injection of spectra for any commutative ring R with $K_{-i}(R) = 0$ for sufficiently large i.

(iii) *If the continuously controlled assembly map*

$$h^{l.f.}(E\Gamma; \Sigma\,\mathrm{spt}(\mathcal{A})) \to \mathrm{st}(\Sigma\partial M)$$

is an equivariant split injection the Novikov conjecture holds for Γ.

(iv) *If* $\overline{M}$ *is contractible then the Novikov conjecture holds for* Γ.

Proof. (i) Immediate from Theorem 12.
(ii) Immediate from [2].
(iii) Immediate from [3]. Alternatively, combine (i) and (ii).
(iv) Proved in [3]. Alternatively, note that if $\overline{M}$ is contractible then the continuously controlled assembly map is an equivariant map which is a homotopy equivalence. □

16. **Conjecture.** *If* M *is a proper metric space with a contractible compactification* $\overline{M}$ *which is small at infinity then the map* $\mathcal{C}(M;R) \to \mathcal{B}(\overline{M}, \partial M; R)$ *induces a homotopy equivalence of spectra* $\mathrm{spt}(\mathcal{C}(M;R)) \to \mathrm{spt}(\mathcal{B}(\overline{M}, \partial M; R))$ *for any ring* R *with* $K_{-i}(R) = 0$ *for sufficiently large* i.

17. **Example.** Conjecture 16 is true if $M = O(K)$ is the open cone on a finite CW complex K, since in this case $\mathrm{spt}(\mathcal{C}(M;R)) \to \mathrm{spt}(\mathcal{B}(\overline{M}, \partial M; R))$ is a natural transformation of generalized homology theories which is a homotopy equivalence on the coefficients ([8, 1, 11, 4, 12]).

18. **Example.** Conjecture 16 is true if M is a complete nonpositively curved manifold ([5]).

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