ON THE BOUNDED AND THIN h-COBORDISM THEOREM PARAMETERIZED BY $\mathbb{R}^k$

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0. Introduction

In this paper we consider bounded and thin h-cobordisms parameterized by $\mathbb{R}^k$. We obtain results similar to those obtained by Quinn [5, 6] and Chapman [2], but in a much more restricted situation. The point of the exercise is to give a self contained proof, based on the algebra developed in [4, 3] in the important special case, where the parameter space is euclidean space. We also get a nice explanation as to why the thin and bounded h-cobordisms theorems have the same obstruction groups. Unlike the general version being developed by D. R. Anderson and H. J. Munkholm [1], we only consider h-cobordisms with constant (uniformly bounded) fundamental group.

In case of the bounded h-cobordism theorem, it is however clear, that the discussion we carry through will generalize to more general metric spaces than $\mathbb{R}^k$, namely to proper metric spaces (every ball compact). We mention this because in this case, we have computed K_1 of some of the relevant categories i. e. the obstruction groups, in joint work with C. Weibel.

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1. Definitions. Statement of results

Definition 1.1. A manifold W parameterized by $\mathbb{R}^k$ consists of a manifold W together with a proper map $W \xrightarrow{p} \mathbb{R}^k$, which is onto.

We use the map p to give a pseudo metric on W by which we measure size. This is distilled in the following definition,

Definition 1.2. Given $K \subseteq W$, W parameterized by $\mathbb{R}^k$ by $p : W \to \mathbb{R}^k$, we define the size of K, $S(K)$ to be

$$S(K) = \inf\{r | \exists y \in \mathbb{R}^k : p(K) \subseteq B(y, r/2)\}$$

where $B(y, r/2)$ is the closed ball in $\mathbb{R}^k$ with radius $r/2$.

$S(K)$ is thus the diameter of the smallest ball containing $p(K)$.

We shall now introduce uniformly bounded and locally constant fundamental groups. Given $t \in \mathbb{R}_+$, we shall define t-bounded fundamental group as follows:

Definition 1.3. The fundamental group of W is t-bounded if the following two conditions hold:

1) For every $\{x, y\} \in W$ and for every homotopy class of paths from x to y, there is a representative $\alpha : (I, 0, 1) \to (W, x, y)$ so that $S(\alpha(I)) < t + S(\{x, y\})$.
2) For every null homotopic map $\alpha \to W$, there is a null homotopy $A : D^2 \to W$ so that $S(A(D^2)) < S(\alpha(S^1)) + t$.

In other words, generators and relations of $\pi_1(W)$ are everywhere representable by something universally bounded. We say the *fundamental group is bounded*, if for some t it is t-bounded, and we say it is *locally constant* if it is t-bounded for all t.

We shall now consider h-cobordisms in the category of manifolds parameterized by $\mathbb{R}^k$.

Definition 1.4. The triple $(W, \partial_0 W, \partial_1 W)$ parameterized by $\mathbb{R}^k$, is a bounded h-cobordism (bounded by t) if the boundary W, ∂W, is the disjoint union of $\partial_0 W$ and $\partial_1 W$, and there are deformations $D_i : W \times I \to W$ of W in $\partial_i W$, so that $S(D_i(w \times I)) < t$ for all $w \in W$.

Given an h-cobordism of this kind, it is natural to ask for a product structure:

Definition 1.5. A bounded product structure (bounded by t) on $(W, \partial_0 W, \partial_1 W)$ is a homeomorphism

$$h : (\partial_0 W \times I, \partial_0 W \times 0, \partial_0 W \times 1) \to (W, \partial_0 W, \partial_1 W)$$

which is the identity on $\partial_0 W$ and satisfies that $S(h(w \times I)) < t$ for all $w \in \partial_0 W$.

We are now able to formulate the thin and bounded h-cobordism theorems.

Bounded h-cobordism theorem. *Let $(W, \partial_0 W, \partial_1 W)$ be a bounded h-cobordism of dimension at least 6, parameterized by $\mathbb{R}^k$ with bounded fundamental group π. Then there is an obstruction in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$, which vanishes if and only if W admits a bounded product structure. All such invariants are realized by bounded h-cobordisms.*

This bounded h-cobordism theorem is a formal consequence of the thin h-cobordism theorem, which we proceed to formulate. However it is much easier to prove the bounded h-cobordism theorem. In the above statement, one could replace $\mathbb{R}^k$ by any other metric space X, which is proper in the sense that every ball is compact, at the price of replacing the obstruction group by $\widetilde{K}_1(\mathcal{C}_X(\mathbb{Z}\pi))$. (see section 4 for definition and discussion of this).

We now formulate the thin h-cobordism theorem:

Thin h-cobordism theorem. *There is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ so that if $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism of dimension n bigger than 6, parameterized by $\mathbb{R}^k$, bounded by t, with fundamental group bounded by t, then there is a product structure on W bounded by $f(n, k) \cdot t$, if and only if the obstruction to a bounded structure, in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$ vanishes.*

Remark 1.6. The difference between the thin and bounded h-cobordism theorems parameterized by $\mathbb{R}^k$ thus lies in the predictability of the bound of the product structure. This of course implies that one may let t go to 0, whereas in the bounded h-cobordism theorem, that has no effect.

It is natural to relate bounded h-cobordism theorems to classical compact h-cobordism theorem. This is done in the following:

Theorem 1.7. *Let $(M, \partial M_0, \partial M_1)$ be a compact h-cobordism with fundamental group $\pi \times \mathbb{Z}^k$, and let $M \to T^k$ induce the projection $\pi \times \mathbb{Z}^k \to \mathbb{Z}^k$ on fundamental groups. Then the pullback over $\mathbb{R}^k \to T^k$ defines a bounded h-cobordism $(M, \partial_0 W, \partial_1 W)$ (the $\mathbb{Z}^k$-covering) and the torsion invariants are related by the Bass-Heller-Swan epimorphism*

$$\mathrm{Wh}(\pi \times \mathbb{Z}^k) \to \widetilde{K}_{-k+1}(\mathbb{Z}\pi).$$

Remark 1.8. $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$ means $\mathrm{Wh}(\pi)$ for $k = 0$, $\widetilde{K}_0(\mathbb{Z}\pi)$ for $k = 1$ and $K_{-k+1}(\mathbb{Z}\pi)$ for $k > 1$.

2. Reviewing the algebra

In this section, we review some of the algebra from [4, 3]. We also develop the algebra needed to make it possible to treat not only the bounded h-cobordism theorem, but also the thin h-cobordism theorem. This amounts to a discussion of the "size" of the "reason" for the vanishing of an invariant, which is known to vanish. A reader familiar with [4, 3] and only interested in the bounded h-cobordism theorem may thus skip this section.

Given a ring R we define the category $\mathcal{C}_k(R)$ to be $\mathbb{Z}^k$-graded, free, finitely generated , based R-modules and bounded homomorphisms. That means an object A is a collection of finitely generated, free, based R-modules $A(J)$, $J \in \mathbb{Z}^k$, and a morphism $\phi : A \to B$ is a collection $\phi^I_J : A(I) \to B(J)$ of R-module morphisms with the property that there is a $r = r(\phi)$ so that $\phi^I_J = 0$ when $||I - J|| > r$. Here it is convenient to use the max norm on $\mathbb{Z}^k$. A morphism ϕ will be called *degree preserving* or *homogeneous* if $\phi^I_J = 0$ for I different from J.

Another way of thinking of $\mathcal{C}_k(R)$ is to think of A as $\oplus A(J)$. Then the condition on ϕ is that $\phi : A \to B$ is a usual R-module morphism satisfying that $\phi(A(J) \subseteq \bigoplus_{||I-J|| \le r} B(I)$.

The description given here differs from the one given in [4] in that we take based R-modules. This however does not change anything and makes applications to geometry easier. In [4] we proved that $K_1(\mathcal{C}_k(R)) \cong K_{-k+1}(R)$. The definition of $K_1(\mathcal{C}_k(R))$ is, that as generators we take $[A, \alpha]$ where A is an object and α an automorphism and as relations $[A, \alpha\beta] - [A, \alpha] - [A, \beta]$ and $A \oplus B \xrightarrow{\left(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right)} A \oplus B$. The reason it does not make a difference whether we consider based or unbased R-modules, is that $[A, \alpha\beta\alpha^{-1}] = [A, \beta]$. Thus a basis change will have no effect on the invariant.

Given an object A of $\mathcal{C}_{k+1}(R)$ there is an obvious object $A[t,t^{-1}]$ of $\mathcal{C}_{k+1}(R[t,t^{-1}])$. This object has a homogeneous automorphism β_t which is the identity on homogeneous elements whose last coordinate is negative, and multiplication by t when the last coordinate is positive. If α is an automorphism of A bounded by r, then the commutator $[\alpha,\beta_t]$ is the identity on any element whose last coordinate is numerically bigger than r, since α both commutes with multiplication by t and with the identity. This means that $[\alpha,\beta_t]$ only does something interesting in a certain band. If we then restrict to that band, and forget the last coordinate in the grading (by taking direct sum), then we get a $\mathbb{Z}^k$ graded automorphism in $\mathcal{C}_k(R[t,t^{-1}])$. This is the Bass-Heller-Swan monomorphism

$$K_{-k}(R) = K_1(\mathcal{C}_{k+1}(R)) \to K_1(\mathcal{C}_k(R[t,t^{-1}])) = K_{-k+1}(R[t,t^{-1}]).$$

The details are given in [4]. Here we want to use this for some simple observations:

Let K be a fixed integral k-tuple. We may then regrade $\mathbb{Z}^k$ by vector addition of K. This will clearly induce a functor of $\mathcal{C}_k(R)$.

Lemma 2.1. *The map on $K_{-k+1}(R)$ induced by the regrading given by vector addition of K is the identity.*

Proof. The map $A \to$ (regraded) A induced by the identity is bounded, and the map on $K_{-k+1}(R)$ is thus given by conjugation by this map. □

This lemma is used to prove the more interesting

Lemma 2.2. *Let A be an object of $\mathcal{C}_k(R)$ and α and β two automorphisms of A bounded by r. Suppose there is a $K \in \mathbb{Z}^k$ so that α and β agree on all $A(J)$ with $||J-K|| \le r$, i. e. on some box with sides $2r$, α and β agree. Then $[A,\alpha]=[A,\beta]$ in $K_{-k+1}(R)$.*

Proof. Using lemma 2.1 we may assume $K=0$. Now consider $\gamma = \alpha\beta^{-1}$. We have $\gamma = \mathrm{id}$ on a box with side length $2r$, and after application of the Bass-Heller-Swan monomorphism this is still the case. After k applications of the B-H-S monomorphism, we thus have the identity. □

The above lemma is used to show that parameterized torsion is well defined under subdivision.

Now consider the map $r : \mathbb{Z}^k \to \mathbb{Z}^k$ multiplying by $r > 0$. This induces a functor $r_* : \mathcal{C}_k(R) \to \mathcal{C}_k(R)$ sending A to r_*A with $r_*A(J) = A(rJ)$ and 0 otherwise, morphism induced by the identity.

Lemma 2.3. *The map induced by multiplication by $r > 0$ is the identity on $K_{-k+1}(R)$.*

Proof. After k applications of the Bass-Heller-Swan monomorphism, we clearly have the identity. □

Finally we have to do the algebra needed to get the thin h-cobordism theorem, rather than just the bounded h-cobordism theorem. At this point we need to remind the reader as to what we mean by an elementary automorphism α of A. By this we mean there is a direct sum decomposition $A = A_1 \oplus A_2$ of based submodules, so that α may be given the matrix presentation $\left(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right)$. We also need to remind the reader that there is another description of $K_{-k+1}(R)$ as the Grothendieck construction of $\mathbb{Z}^{k-1}$-graded projections. We call a projection *geometric* when it sends any basis element either to itself or 0.

Lemma 2.4. *There is a function $f : \mathbb{N} \to \mathbb{N}$, so that the following is true:*

1) *If $A \in \mathcal{C}_{k-1}(R)$ and $p : A \to A$ is a projection bounded by r and so that $[A, p] = 0$ in $K_{-k+1}(R)$. Then after stabilization there is an automorphism β bounded by $f(k)\cdot r\cdot 24$ so that $\beta p \beta^{-1}$ is geometric.*
2) *If $A \in \mathcal{C}_k(R)$ and $\alpha : A \to A$ is an automorphism bounded by r, so that $[A, \alpha] = 0 \in K_{-k+1}(R)$. Then stably α may be written as a product of 24 elementary automorphisms, each of which is bounded by $f(k) \cdot r$.*

Proof. is by induction on k on the statements 1) and 2) for any ring. We will show that if the ring is of the form $R = S[t, t^{-1}]$ and the given automorphism (projection) only involves finitely many t-powers, then the automorphisms produced have the same property. We shall allow ourselves to refer freely to [4]. To facilitate the reading, we do the first steps rather than the general step. For $k = 1$ statement 1 disappears, so consider statement 2. The map $p_0 : A \to A$ is the identity in positive gradings and the 0-map in negative gradings. The map $\alpha p_0 \alpha^{-1}$ restricted to $\bigoplus_{i=-r}^{r} A(i)$ is conjugate to p_0 at least after stabilization of say $A(0)$, so there is an automorphism β of $\bigoplus_{i=-r}^{r} A(i)$ so that $\beta\alpha p_0 \alpha^{-1}\beta^{-1} = p_0$ or $\beta\alpha p_0 = p_0\beta\alpha$.

Extending β to all of A by the identity, we have an automorphism β bounded by $2r$ so that $\beta\alpha p_0 = p_0\beta\alpha$. We thus get $\alpha = \beta^{-1}(\beta\alpha)$ where β^{-1} and $\beta\alpha$ both are bounded by $2r$. Since β is the identity away from the interval $-r$ to r, β preserves the two halves when we split up A say at r. Denote $\beta\alpha$ or β^{-1} by γ. The trick used in [4] is the equation

$$(\gamma \oplus 1 \oplus 1 \oplus \ldots) = (\gamma \oplus \gamma^{-1} \oplus \gamma \ldots)(1 \oplus \gamma \oplus \gamma^{-1} \ldots)$$

each term on the right hand side may be written as a product of 6 elementary isomorphisms each of which is bounded by $4r$, so $f(1)$ may be taken to be 4. If the ring R is of the form $S[t, t^{-1}]$, and the automorphism α only involves finitely many t-powers, then clearly all the elementary automorphisms produced have that same property.

For $k = 2$ consider a $\mathbb{Z}$-graded projection p of A as in statement 1). Then $pt + (1 - p)$ is a $\mathbb{Z}$-graded automorphism of $R[t, t^{-1}]$ modules involving only finitely many t-powers and bounded by r. By what we just proved $pt + (1 - p)$ may be written as a product of 24 elementary matrices, each only involving finitely many t-powers and each bounded by $4r$, i. e. $pt + (1 - p) = \prod_{i=1}^{24} E_i$. Turning t-powers into a grading, and conjugating the projection p_0 by this automorphism, delivers back the projection at t-degree 0, the id in positive t-degree and the 0-map in negative t-degrees. Considering $(pt + (1 - p))p_0(pt + (1 - p))^{-1}$ in a band around t-degree 0 corresponds to stabilization. Using the trick of lemma 1.10 in [4] which turns an elementary matrix into a product of one with support in a band around t-degree 0 and one far away, we obtain β bounded by $24 \cdot 4 \cdot r$ so that in a broad band (of t-degrees) $\beta p \beta^{-1} = p_0$. The trick being employed is that it does not matter how high t-powers get involved, because the grading introduced by the t-powers will immediately be forgotten.

It is now clear how the induction proceeds, one essentially uses the same words. □

3. Bounded simple homotopy theory parameterized by $\mathbb{R}^k$

In this section we elaborate a little on the results of [3], and carry these results into the manifold category. First we recall

Definition 3.1. A finite, bounded CW-complex parameterized by $\mathbb{R}^k$ consists of the following: A finite dimensional CW-complex X together with a map $X \to \mathbb{R}^k$ which is onto and proper, so that there is a $t \in \mathbb{R}^+$ so that the size, $S(C) < t$ for each cell C.

Definition 3.2. Let K be a space parameterized by $\mathbb{R}^k$. A simple homotopy type on K consists of

1) a bounded, finite CW complex X parameterized by $\mathbb{R}^k$.
2) a bounded homotopy equivalence $K \to X$.

Two such are said to be equivalent if the induced bounded homotopy equivalence of finite bounded CW-complexes has 0 torsion in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$ (see [3] for definitions)

Theorem 3.3. *A manifold W parameterized by $\mathbb{R}^k$ with bounded fundamental group, has a well defined simple homotopy type given by a triangulation with bounded simplices (in the* PL *oe* Diff *categories) or by a bounded handlebody structure in the* TOP *category.*

Proof. We give the argument in the PL category. This extends to the Diff category by smooth triangulations. The TOP category requires the usual modifications in the argument. Given $t \in \mathbb{R}_+$, we choose a triangulation with simplices of size less that t. This is a bounded finite CW complex, hence the identity defines a simple homotopy type on W. We have to compare this to another triangulation with simplices of size less than t'. The two triangulations have a common subdivision, so as in compact topology it suffices to show that the identity is a homotopy equivalence with trivial torsion when thought of as a map from W with some triangulation K to a subdivision $\overline{K}$. We pick out one of the coordinates in $\mathbb{R}^k$ say the last, and call this x. Rather than comparing the triangulation and its subdivision directly, we introduce an intermediate subdivision cell complex K' which is a subdivision of K and has $\overline{K}$ as a subdivision. Furthermore if a simplex of K' has barycenter with x-value bigger than $3t$ the simplex is also a simplex of K, whereas if the x-value is smaller than $-3t$, the simplex is also a simplex of $\overline{K}$. In other words the cell decomposition agrees with K for large positive values of x and with $\overline{K}$ for large negative values of x. It is not possible to have K' be a triangulation, because we have to subdivide a face of a simplex without subdividing the simplex itself. This however is no problem when we only want a cell complex. We now compare K and K'. At the level of chain complexes the identity induces a map sending a generator corresponding to a cell to the sum of the simplices it is being divided into, and the homotopy inverse sends one of these back to the generator and the rest to 0. For large positive x-values there is no subdivision, so the map is the identity. By Lemma 2.2 it suffices to know the map on a big chunk, so we are done. Comparing K' and $\overline{K}$ is treated similarly, but now using the fact that the cell decomposition agree for large negative x-value.

Note that the reason we can not simply refer to the usual compact proofs, that we may not subdivide equally much everywhere, so there may be more than finitely many steps in the subdivision procedure. We are now ready to define the obstruction and prove the theorem. □

4. Proof of thin and bounded h-cobordism theorem parameterized by $\mathbb{R}^k$

Consider an h-cobordism $(W, \partial_0 W, \partial_1 W)$ parameterized by $\mathbb{R}^k$ and bounded by t, with fundamental group π bounded by t. For the purpose of the bounded h-cobordism theorem, these can be taken to be the same number by taking the bigger, while for the thin h-cobordism theorem it is part of the assumption. By assumption the inclusion $\partial_0 W \subseteq W$ is a bounded homotopy equivalence. Since $\partial_0 W$ as well as W have well defined simple homotopy types by theorem 3.3, this homotopy equivalence has a well defined torsion in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$. If $(W, \partial_0 W, \partial_1 W)$ is boundedly equivalent to $(\partial_0 W \times I, \partial_0 W \times 0, \partial_0 W \times 1)$ then W is obtained from $\partial_0 W$ by attaching no handles, and it is clear that this torsion must vanish. Assuming the invariant vanishes, we give W a filtration as $\partial_0 W \times I \cup 0-\text{handles} \cup 1-\text{handles} \cup \ldots \cup n+1-\text{handles} \cup \partial_0 W \times I$ in such a way that the size of each handle is bounded by t, and the size of each $w \times I$ in $\partial_0 W \times I$ or $\partial_1 W \times I$ is bounded by t. The aim is to get rid of all the handles in-between, without changing the size of the product structure lines too badly. The procedure is the usual handlebody theory, with attention paid to size, and the arguments are very similar to those applied by Quinn in [5], but of course with different algebra.

Cancelling 0-handles is done in standard fashion, but one has to worry that one does not get too long sequences of 0 and 1 handles, letting the size get out of control. We have a t-bounded deformation retraction of W to $\partial_0 W$. The restriction to 0-handles defines a map

$$(0-\text{handles}) \times I \to W$$

defining a path from the core of each 0-handle to $\partial_0 W$. Using (very small) general position, one may assume this path runs in the 1-skeleton of W, relative to $\partial_0 W$, so from the core of every 0-handle, there is a path through cores of 1 and 0-handles to $\partial_0 W$, bounded by t when measured in $\mathbb{R}^k$. If this path has any loop, we may simply discard the loop. That does not increase the size.Also if the path from one 0-handle is a part of a longer path from another 0-handle, we may forget the shorter path. In the end we would like to have an embedding

$$(\text{cores of some 0-handles}) \times I \to W$$

which goes through all 0-handles and retaining the control of size. This is done by subdividing every 0-handle with more than 1 path going through into so may 0 and 1-handles, that they have been made disjoint. We now have a disjoint embedding of paths from $\partial_0 W$ going through 0 and 1 handles and with size being bounded by t. Cancelling these 0-handles accordingly will change the boundedness of the collar structure on the boundary by a controlled multiple of t.

The cancelling of 1-handles is now done in standard fashion by introducing 2 and 3 handles, and using the 2 handles to cancel the 1-handles. Having done this from both ends of the handlebody, we have a handlebody without any 0, 1, n, and $n+1$ handles, and the product structure on the collars of the boundary is bounded by a constant times t. All 2 and $n-1$ handles must be attached to the boundary by homotopically trivial maps (otherwise they would change the fundamental group), so we now have the same fundamental group π at all levels of the decomposition.

The cellular $\mathbb{Z}\pi$ chain complex of $(W, \partial_0 W)$ may be thought of as a chain complex in $\mathcal{C}_k(\mathbb{Z}\pi)$ by associating to each cell an integral lattice point in $\mathbb{R}^k$ near the points in $\mathbb{R}^k$ over which the cell sits. As elaborated in [3]. this cellular chain complex

$$0 \to C_{n-1} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} \dots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \to 0$$

will be contractible in $\mathcal{C}_k(\mathbb{Z}\pi)$, with a contraction s whose bound is directly related to the bound of the deformation of W in $\partial_0 W$. We now proceed to cancel handles following the scheme indicated by the algebra: we introduce cancelling 3 and 4 handles corresponding to all the 2-handles, and sitting over the points in $\mathbb{R}^k$ where the 2-handles sit, to obtain a chain complex in $\mathcal{C}_k(\mathbb{Z}\pi)$ which in low dimensions is

$$C_2 \oplus C_4 \xrightarrow{1 \oplus \partial} C_2 \oplus C_3 \xrightarrow{(0,\partial)} C_2 \to 0$$

At the level of 3-handles we now perform handle additions, so that to each handle x in C_2 we add $s(x)$ in C_3. Since s is bounded, this will increase the cell size by a controllable amount. Since in dim 2 we have $\partial s = 1$, the chain complex, after having performed this handle addition, now has the form

$$C_2 \oplus C_4 \to C_2 \oplus C_3 \xrightarrow{(1,*)} C_2 \to 0.$$

We are now in a situation to cancel the 3-handles we introduced against the 2-handles, since we have obtained algebraic intersection 1, and after some small Whitney isotopies we will have geometric intersection 1 and can cancel handles. After the cancelation the chain complex has the form $C_2 \oplus C_4 \to C_3 \to 0$, and is of course still contractible in $\mathcal{C}_k(\mathbb{Z}\pi)$.

Continuing this procedure, we get into a two-index situation

$$0 \longrightarrow C_{r+1} \underset{s}{\overset{\partial}{\rightleftarrows}} C_r \longrightarrow 0$$

and the collars have bounded product structures, bounded by predictable (even computable as a function of $\dim(W)$) constant times t. The invariant in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi) = K_1 \mathcal{C}_k(\mathbb{Z}\pi)$ is given by the torsion of this chain complex, which is exactly the isomorphism ∂. Of course ∂ is not an automorphism but an isomorphism. The point is that if ∂ is of the type sending a generator to a generator, then we may cancel handles. It is however easy to see that at least stably (see e. g. [4]) C_r and C_{r+1} are isomorphic by an isomorphism sending generators to generators, hence composing ∂ with such an isomorphism, we obtain an automorphism. At this point there is a choice involved, but for $k > 1$ the torsion of an automorphism sending generators to generators is 0. This is Lemma 1.5 of [4]. When $k = 1$ this is not true, and what Quinn calls a flux phenomenon occurs. The invariant thus only becomes well defined after dividing out by automorphisms that send generators to generators, which amounts to saying the invariant lives in reduced K-groups. At this point one might mention that the choices involved in finding representing cells of the $\mathbb{Z}\pi$ modules have no effect since an automorphism multiplying generators by elements of π will have 0 torsion, because it is homogeneous.

Since we have assumed the invariant is 0 in $\widetilde{K}_{-k+1}(\mathbb{Z}\pi)$, the automorphism can be written as a product of elementary automorphisms after stabilization. After stabilizing geometrically by introducing cancelling handles, we may then change ∂ to cancel one of these elementary automorphisms at a time, at the expense of letting the handles grow bigger. At this point, as in all handle addition arguments, we of course use the boundedness of the fundamental group, to be able to judge how much bigger the handles get. In the end ∂ will be equal to the isomorphism from C_{r+1} to C_r chosen, that sends generators to generators. We now cancel handles and are done.

To prove the thin h-cobordism theorem, we have to worry about how many handle additions we perform, but by lemma 2.4 this is controlled. To sum up the difference between the thin and the bounded h-cobordism theorem, to do the thin version one needs to do the following: First multiply the reference map in $\mathbb{R}^k$ by $1/\epsilon$ so the h-cobordism will be bounded by 1. Here we use lemma 2.3 to show this does not change the obstruction. To get into the 2 index situation, there is no difference between the two proofs. In the 2-index situation, we need lemma 2.4 to see that we can control how many handle additions we need to perform, and how far away the handles that have to be added can sit.

Proof of Theorem 1.7. Consider a compact h-cobordism $(M, \partial_0 M, \partial_1 M)$ with fundamental group $\pi \times \mathbb{Z}^k$. The torsion of this h-cobordism will be represented by the torsion of the based chain complex of the universal cover of $(M, \partial_0 M)$ as $\mathbb{Z}[\pi \times \mathbb{Z}^k]$ modules. This is exactly the same chain complex as that of the $\mathbb{Z}^k$-covering, but now $\mathbb{Z}^k$ has been turned into a $\mathbb{Z}^k$-grading. On the other hand, the description of the Bass-Heller-Swan epimorphism given in [4] is exactly that. □

Realizability of obstructions. Given a manifold $\partial_0 W \to \mathbb{R}^k$ with uniformly bounded fundamental group π and an element $\sigma \in \widetilde{K}_{-k+1}(\mathbb{Z}\pi)$, we wish to construct an h-cobordism $(W, \partial_0 W, \partial_1 W)$ with obstruction σ. However σ is represented by a $\mathbb{Z}^k$-graded bounded automorphism $\alpha : C \to C$, where C is some object of $\mathcal{C}_k(\mathbb{Z}\pi)$. We start out with $\partial_0 W \times I$. Then we attach infinitely many trivial handles of the same dimension r corresponding to the generators of C, and each placed at a point which in $\mathbb{R}^k$ is near by the integral lattice point of the generator in C. As in the standard realizability theorem we now attach $r+1$-handles by maps given by α above. It is easy to extend the reference map to $\mathbb{R}^k$ and we get a manifold $(W, \partial_0 W, \partial_1 W)$ with the chain complex $0 \to C \xrightarrow{\alpha} C \to 0$ and will thus have a torsion given by the class of α which is σ. To prove it is a bounded h-cobordism, we do however need to invoke the Whitehead theorem type results of Anderson and Munkholm [1].

5. Parameterizing by other metric spaces

In the proof of the bounded h-cobordism theorem (not the thin h-cobordism theorem) we have nowhere used that the metric space we parameterized by is $\mathbb{R}^k$. Any other metric space X will do, as long as X satisfies that every ball in X is compact (A proper metric space in the sense of [1]). The groups in which the obstructions will then take values will then be $\widetilde{K}_1(\mathcal{C}_X(\mathbb{Z}\pi))$ where $\mathcal{C}_X(R)$ is an additive category described as based, finitely generated, free R-modules parameterized by X and bounded homomorphisms. That means an object A is a set of based, finitely generated , free R-modules $A(x)$, one for each $x \in X$ with the property, that for any ball $B \subset X$, $A(x) = 0$ for all but finitely many $x \in B$. A morphism $\phi : A \to B$ is a set of R-module morphisms $\phi^x_y : A(x) \to B(y)$, so that there exists $k = k(\phi)$ with the property that $\phi^x_y = 0$ for $d(x, y) > k$. The study of this sort of category is the subject of forthcoming joint work with C. Weibel, in which we obtain results about the K-theory of such categories. In the case of $X = \mathbb{R}^k$ we have preferred to have the modules sitting at the integral lattice points, but this is not an important difference. In general when the fundamental group is uniformally bounded with respect to the metric space X, the proof of the bounded h-cobordism theorem will go through word for word. The obstructions will be elements of $\widetilde{K}_1(\mathcal{C}_X(\mathbb{Z}\pi))$, where $\widetilde{\ }$ stands for the reduction by automorphisms sending generators to generators. The case where the fundamental group is not necessarily being assumed to be uniformally bounded is presently being studied by D. R. Anderson and H. J. Munkholm.

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