ON THE HOMOTOPY INVARIANCE OF THE BOUNDEDLY CONTROLLED ANALYTIC SIGNATURE OF A MANIFOLD OVER AN OPEN CONE

ERIK KJÆR PEDERSEN, JOHN ROE, AND SHMUEL WEINBERGER

1. Introduction

The theorem of Novikov [21], that the rational Pontrjagin classes of a smooth manifold are invariant under homeomorphisms, was a landmark in the development of the topology of manifolds. The geometric techniques introduced by Novikov were built upon by Kirby and Siebenmann [19] in their study of topological manifolds. At the same time the problem was posed by Singer [30] of developing an analytical proof of Novikov's original theorem.

The first such analytic proof was given by Sullivan and Teleman [33, 32, 34], building on deep geometric results of Sullivan [31] which showed the existence and uniqueness of Lipschitz structures on high-dimensional manifolds. (It is now known that this result is false in dimension 4 — see [10].) However, the geometric techniques needed to prove Sullivan's theorem are at least as powerful as those in Novikov's original proof[1]. For this reason, the Sullivan-Teleman argument (and the variants of it that have recently appeared) do not achieve the objective of *replacing* the geometry in Novikov's proof by analysis.

In an unpublished but widely circulated preprint [35], one of us (S.W.) suggested that this objective might be achieved by the employment of techniques from *coarse geometry*. A key part in the proposed proof is played by a certain homotopy invariance property of the 'coarse analytic signature' of a complete Riemannian manifold. We will explain in section 2 below what the coarse analytic signature is, in what sense it is conjectured to be homotopy invariant, and how Novikov's theorem should follow from the conjectured homotopy invariance. In section 3 we will prove the homotopy invariance modulo 2-torsion in the case that the control space is a cone on a finite polyhedron. This suffices for the proof of the Novikov theorem. In section 4 we will show how the methods of this paper can be improved to obtain homotopy invariance 'on the nose'.

Although the coarse signature is an index in a C^*-algebra, our proof is not a direct generalization of the standard proof of the homotopy invariance of signatures over C^*-algebras, as presented for example in [17]. (The assertion to the contrary in [35] is, unfortunately, not

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[1]See the discussion on page 666 of [8].

correct as it stands.) The problem is this: in the absence of any underlying uniformity such as might be provided by a group action, it becomes impossible to prove that the homotopies connecting two different signatures are represented by *bounded* operators on some Hilbert space. We circumvent this problem by comparing two theories, a 'bounded operator' theory and an 'unbounded' theory, by means of a Mayer-Vietoris argument. Homotopy invariance can be proved in the 'unbounded' theory, but since the two theories are isomorphic, it must hold in the 'bounded' theory as well. A somewhat similar argument was used by the first author in a different context [23].

Our 'unbounded' theory is just boundedly[2] controlled L-theory as defined in [26, 25], and to keep this paper to a reasonable length we will freely appeal to the results of this theory. We do not claim, therefore, that this paper gives a 'purely analytic' proof of Novikov's theorem; indeed, if one is prepared, as we are, to appeal to the homological properties of controlled L-theory, then one can prove Novikov's theorem quite directly and independently of any analysis (see [25], for example). Our point is rather the following. Conjecture 2.2 is a natural analogue of theorems about the homotopy invariance of appropriate kinds of symmetric signatures in other contexts. But those theorems have simple general proofs, whereas in our case the proof is indirect and depends strongly on the hypothesis that the control space possesses appropriate geometric properties, of the kind which can also be used to show the injectivity of the assembly map (compare [5]). Moreover, although 2.2 is a conjecture about C^*-algebras, it appears to be necessary to leave the world of C^*-algebras in order to prove it. It may be that conjecture 2.2 is in fact *false* for more general control spaces X, and, if this were so, then it would suggest the existence of some new kind of obstruction to making geometrically bounded problems analytically bounded also.

It is possible that the special case of conjecture 2.2 that is proved in this paper might be approachable by other, more direct, analytic methods, such as a modification of the almost flat bundle theory of [7, 16]; but it seems that similar questions about gaining appropriate analytic control would have to be addressed.

This paper provides a partial answer to a problem that was raised, in one form or another, by several of the participants at the Oberwolfach conference; see (in the problem session) Ferry-Weinberger, problem 1, Rosenberg, problem 2, and Roe, problems 1 and 2.

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2. The coarse signature

Let X be a proper metric space. We refer to [28, 15, 14] for the construction of the C^*-algebra $C^*(X)$ of locally compact finite propagation operators and of the assembly map $\mu\colon K_*(X) \to K_*(C^*(X))$. We recall that the groups $K_*(C^*(X))$ are functorial under *coarse*

[2]Notice that there are two senses in which the word 'bounded' is used in this paper; we may distinguish them as *geometrically bounded* and *analytically bounded*.

maps, that is, proper maps f such that the distance between $f(x)$ and $f(x')$ is bounded by a function of the distance between x and x'. Such maps need not be continuous; but on the subcategory of continuous coarse maps the groups $K_*(X)$ are functorial also, and assembly becomes a natural transformation.

If X is a proper metric space, a *(smooth) manifold over* X is simply a manifold[3] M equipped with a control map $c\colon M \to X$; c must be proper but it need not be continuous. It is elementary that any such manifold M can be equipped with a complete Riemannian metric such that the control map c becomes a coarse map, and that any two such Riemannian metrics can be connected by a path of such metrics.

2.1. Definition. Let (M, c) be a manifold over X. The *coarse analytic signature* of M over X is defined as follows: equip M with a Riemannian metric such that c becomes a coarse map, let D_M denote the signature operator on M. According to [28] this operator has a 'coarse index' $\mathrm{Ind}(D_M) \in K_*(C^*(M))$, which is in fact the image of the K-homology[4] class of D under the assembly map μ. We define

$$\mathrm{Sign}_X(M) = c_*(\mathrm{Ind}\, D_M) \in K_*(C^*(X)).$$

Remark. For clarity we should make explicit what is meant by the 'signature operator', especially on an odd-dimensional manifold. We use the language of Dirac operators on Clifford bundles (see [27], for example). Let C be the bundle of Clifford algebras associated to the tangent bundle TM, and let $\omega \in C$ be the volume form. Then $\omega^2 = \pm 1$, the sign depending on the dimension of M modulo 4, and so there is a decomposition of C into two eigenspaces $C^+ \oplus C^-$ of ω. If the dimension of M is even, ω anticommutes with the Clifford action of TM on C, and so C becomes a graded Clifford bundle, and we define the *signature operator* to be the associated graded Dirac operator. If the dimension of M is odd, then ω commutes with the action of TM, and so C^+ and C^- individually are Clifford bundles; we define the *signature operator* in this case to be the (ungraded) Dirac operator of C^+.

It is implicit in the definition of the coarse analytic signature that $c_*(\mathrm{Ind}\ D_M)$ is independent of the choice of Riemannian metric on M. This may be proved by the following development of the theory of [14]. Recall that in that paper the assembly map μ was defined to be the connecting map in the six-term K-theory exact sequence arising from an extension of C^*-algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0,$$

where $D^*(X)$ is the C^*-algebra of *pseudolocal* finite propagation operators. Using Pashcke's duality theory [22], it was shown that the K-theory of the quotient algebra $D^*(X)/C^*(X)$ was isomorphic to the (locally finite) K-homology of X. Now let us generalize the whole set-up

[3] All manifolds will be assumed to be oriented.

[4] In this paper, we define the *K-homology* of a locally compact metrizable space M to be the Kasparov group [18] $KK(C_0(M), \mathbb{C})$; this is the same as the *locally finite Steenrod K-homology* of X as defined in algebraic topology.

to the case of a manifold M over X, which we write $\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right)$. We define algebras $C^*\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right)$ and $D^*\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right)$ to be the (completions of) the algebras of locally compact and pseudolocal operators, respectively, on M, that have finite propagation when measured in X. Then it is not hard to see on the one hand that the K-theory of $C^*\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right)$ maps canonically to the K-theory of $C^*(X)$ (in fact it is equal to it if the range of the control map is coarsely dense), and on the other hand that the K-theory of $D^*\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right) \Big/ C^*\left(\begin{smallmatrix} M \\ \downarrow \\ X \end{smallmatrix}\right)$ is canonically isomorphic to the (locally finite) K-homology of M. Thus we obtain an assembly map $\mu\colon K_*(M) \to K_*(C^*(X))$ which is independent of any choice of Riemannian metric on M; and naturality of the construction shows that $\mu(D_M)$ coincides with the coarse signature as defined above for any choice of metric.

The usual notions of algebraic topology may be formulated in the category of manifolds over X. In particular we have the concepts of *boundedly controlled map*, *boundedly controlled homotopy*, and *boundedly controlled homotopy equivalence*. A map

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \varphi\ } & M_2 \\ & {\scriptstyle c_1}\searrow \quad \swarrow{\scriptstyle c_2} & \\ & X & \end{array}$$

is thus boundedly controlled if φ is continuous, and c_1 is at most a uniformly bounded distance from $c_2 \cdot \varphi$. Similarly a boundedly controlled homotopy is a boundedly controlled map

$$\begin{array}{ccc} M_1 \times I & \xrightarrow{\ H\ } & M_2 \\ & {\scriptstyle c_1 \cdot p}\searrow \quad \swarrow{\scriptstyle c_2} & \\ & X & \end{array}$$

where p is projection on the first factor. Notice this means that $c_2(H(m \times I))$ has uniformly bounded diameter. The notion of a boundedly controlled homotopy equivalence now follows in an obvious manner.

The following is the homotopy invariance property that we wish to use:

2.2. Conjecture. If two smooth manifolds M and M' over X are homotopy equivalent by a boundedly controlled orientation-preserving homotopy equivalence, then their coarse analytic signatures agree:

$$\mathrm{Sign}_X(M) = \mathrm{Sign}_X(M') \in K_*(C^*(X)).$$

We make a few comments on the difficulty in proving this along the lines of the analytic proof in [17]. One wants to construct chain homotopies which intertwine the L^2-de Rham complexes of M and M' (or some simplicial L^2-complexes constructed from an approximation procedure). Because we are working in the world of C^*-algebras, everything has to be a bounded operator on appropriate L^2-spaces. This means that one needs suitable estimates on the derivatives of the maps and homotopies involved, and such estimates do not seem automatically to be available unless one works in a 'bounded geometry' context. This would be appropriate for a proof of the *bi-Lipschitz* homeomorphism invariance of the Pontrjagin classes, but not, it seems, of the topological invariance.

We will now show that conjecture 2.2 implies Novikov's theorem. In fact, we will show a little more, namely that the conjecture implies that the K-homology class of the signature operator of a smooth manifold is invariant under homeomorphism. This is also the conclusion of Sullivan and Teleman's proof [32, 33] which uses Lipschitz approximation. To simplify the later proofs a little, we work away from the prime 2.

2.3. PROPOSITION. *Suppose that Conjecture 2.2 is true modulo 2-torsion for control spaces X which are open cones on finite polyhedra. Then, if N and N' are homeomorphic compact smooth manifolds, the K-homology signatures of N and N' are equal in $K_*(N) \otimes \mathbb{Z}[\frac{1}{2}]$.*

2.4. COROLLARY. *In the situation above, the rational Pontrjagin classes of M and M' agree.*

Proof. By the Atiyah-Singer index theorem [2], the homology Chern character of the signature class is the Poincaré dual of the $\mathcal{L}$-class (which is the same as the Hirzebruch L-class except for some powers of 2); the rational Pontrjagin classes can be recovered from this. □

Proof. (OF THE PROPOSITION): We begin by considering manifolds M and M' which are smoothly $N \times \mathbb{R}$ and $N' \times \mathbb{R}$ respectively. Let g_{ij} be a Riemannian metric on N. We equip M with a warped product metric of the form

$$dt^2 + \varphi(t)^2 g_{ij} dx^i dx^j,$$

where $\varphi(t)$ is a smooth function with $\varphi(t) = 1$ for $t < -1$ and $\varphi(t) = t$ for $t > 1$. The exact form of the metric is not especially important, provided that it has one cylinder-like and one cone-like end, so that N^+ (that is, N with a disjoint point added) is a natural Higson corona of M. Let $X = M$ considered as a metric space. Then M is obviously boundedly controlled over X (via the identity map!). We use the homeomorphism between N and N' to regard M' as boundedly controlled over X as well. A simple smoothing argument shows that M and M' are boundedly controlledly (smoothly) homotopy equivalent over X; thus by the conjecture their coarse analytic signatures agree.

We now identify the coarse analytic signature of M with the ordinary K-homology signature of N. To do this recall from [28, 14] that there is a natural map defined by Paschke duality

$$b\colon K_*(C^*(M)) \to \widetilde{K}_{*-1}(N^+) = K_{*-1}(N),$$

with the property that $b(\operatorname{Ind} D)$, for any Dirac-type operator D on M, is equal to $\partial[D]$, where $\partial\colon K_*(M) \to K_{*-1}(N)$ is the boundary map in K-homology. (In fact, b is an isomorphism for cone-like spaces such as M, but this fact will not be needed here.) On the other hand, it is a standard result in K-homology that 'the boundary of Dirac is Dirac' [13, 36]. It is not true that 'the boundary of signature is signature', but this *is* true up to powers of 2. In fact $\partial[D_M] = k[D_N]$, where $[D_N]$ is the class of the signature operator of N and k is 2 if M is even-dimensional, 1 if M is odd-dimensional. We will discuss the factor k in section 4.

By a similar argument we may identify the coarse analytic signature of M' with k times the ordinary K-homology signature of N', pulled back to $K_*(N)$ via the homeomorphism $N' \to N$. The desired result therefore follows from the equality of these two signatures. □

Remark. With a little more effort, this argument might be made to work with the hypothesis that N and N' are ε-controlled homotopy equivalent for all ε, rather than homeomorphic. Of course one knows from the α-approximation theorem [6] that N and N' are in fact homeomorphic under this hypothesis, but the point is that one can avoid appealing to this geometric result.

In the next section we will need to know that the coarse analytic signature is bordism invariant. In other words, we will require

2.5. Proposition. *Suppose that N is an X-bounded manifold which is the boundary of an X-bounded manifold-with-boundary M. Then* $\mathrm{Sign}_X(N)$ *is a 2-torsion element in* $K_*(C^*(X))$.

Proof. Let M° denote the interior of M. A portion of the exact sequence of K-homology is

$$K_{*+1}(M^\circ) \to K_*(N) \to K_*(M).$$

As remarked above, there is an integer k equal to 1 or 2 such that $\partial[D_{M^\circ}] = k[D_N]$, and it follows from exactness that the image of $[D_N]$ in $K_*(M)$ is a k-torsion element. But from the naturality of the assembly map there is a commutative diagram

$$\begin{array}{ccc} K_*(N) & \longrightarrow & K_*(M) \\ & \searrow \quad \swarrow & \\ & K_*(C^*(X)) & \end{array}$$

in which both vertical arrows are assembly maps. Therefore the image of $[D_N]$ under the assembly map, that is the coarse analytic signature, is k-torsion. □

3. Proof of homotopy invariance modulo 2-torsion

We begin by recalling the definition of the L^h-groups of an additive category with involution[26]. Given an additive category $\mathfrak{U}$ an involution on $\mathfrak{U}$ is a contravariant functor $*\colon \mathfrak{U} \to \mathfrak{U}$, sending U to U^*, and a natural equivalence $** \cong 1$. One of the defining properties of an additive category is that the Hom-sets are abelian groups, that is $\mathbb{Z}$-modules. All the categories that we will consider will have the property that the Hom-sets are in fact modules over the ring $\mathbb{Z}[i, \frac{1}{2}]$, and we will make this assumption from now on. This yields two simplifications in L-theory: the existence of $i = \sqrt{-1}$ makes L-theory 2-periodic, since dimensions n and $n+2$ get identified through scaling by i, and the existence of $\frac{1}{2}$ removes the difference between quadratic and symmetric L-theory. We therefore get the following description of L-theory. In degree 0 an element is given as an isomorphism $\varphi\colon A \to A^*$ satisfying $\varphi = \varphi^*$. Elements of the form $B \oplus B^* \cong B^* \oplus B$, with the obvious isomorphism, are considered trivial, and L_0^h is the Grothendieck construction determining whether a selfadjoint isomorphism is stably conjugate to a trivial isomorphism. In the definition of L_2^h the condition $\varphi = \varphi^*$ is replaced by $\varphi = -\varphi^*$ but in the presence of i these groups become scale equivalent. In odd degrees the groups are given as automorphisms of trivial forms.

Remark. Suppose the additive category $\mathfrak{U}$ is the category of finitely generated projective modules over a C^*-algebra A and the involution is given by the identity on objects and the $*$-operation on morphisms. One defines the projective L-groups $L_*^p(A)$ to equal $L_*^h(\mathfrak{U})$ for this category $\mathfrak{U}$. In this situation, the availability of the Spectral Theorem for C^*-algebras allows one to separate out the positive and negative eigenspaces of a non-degenerate quadratic form and thus to assign a signature in $K_*(A)$ (a formal difference of projections) to any element of $L_*^p(A)$. This construction goes back to Gelfand and Mischenko [11], and a very careful account may be found in Miller [20]; the exposition in Rosenberg [29] is couched in language similar to ours, and also includes a proof that one obtains in this way a homomorphism $L_*^p(A) \to K_*(A)$, which becomes an isomorphism after inverting 2.

Remark. Notice that we are using *projective* modules in the above statement, so one calls the corresponding L-group $L^p(A)$. In general L^p of an additive category with involution is just L^h of the idempotent completion of the category. To simplify these issues we will work modulo 2-torsion, so from now on when we write $L(A)$ without upper index we shall mean $L^h(A) \otimes \mathbb{Z}[\frac{1}{2}]$, noting that by the Ranicki-Rothenberg exact sequences tensoring with $\mathbb{Z}[\frac{1}{2}]$ removes the dependency on the upper decoration. To retain the above mentioned isomorphism we obviously have to tensor K-theory with $\mathbb{Z}[\frac{1}{2}]$ as well.

We now recall the (geometrically) bounded additive categories defined in [24]. Let X be a metric space, and R a ring with anti-involution. This turns the category of left R-modules into an additive category with involution, since the usual dual of a left R-module is a right R-module, but by means of the anti-involution this may be turned into a left R-module.

The reader should keep in mind the model case in which X is the infinite open cone $\mathcal{O}(K)$ on a complex $K \subseteq S^n \subset \mathbb{R}^{n+1}$ and $R = \mathbb{C}$. The category $\mathfrak{C}_X(R)$ is defined as follows:

3.1. Definition. An *object* A of $\mathfrak{C}_X(R)$ is a collection of finitely generated based free right R-modules A_x, one for each $x \in X$, such that for each ball $C \subset X$ of finite radius, only finitely many A_x, $x \in C$, are nonzero. A *morphism* $\varphi\colon A \to B$ is a collection of morphisms $\varphi_y^x\colon A_x \to B_y$ such that there exists $k = k(\varphi)$ such that $\varphi_y^x = 0$ for $d(x, y) > k$.

The composition of $\varphi\colon A \to B$ and $\psi\colon B \to C$ is given by $(\psi \circ \varphi)^x_y = \sum_{z\in X} \psi^z_y \varphi^x_z$. Note that $(\psi \circ \varphi)$ satisfies the local finiteness and boundedness conditions whenever ψ and φ do.

3.2. Definition. The *dual* of an object A of $\mathfrak{C}_X(R)$ is the object A^* with $(A^*)_x = A^*_x = \mathrm{Hom}_R(A_x, R)$ for each $x \in X$. A^*_x is naturally a left R-module, which we convert to a right R-module by means of the anti-involution. If $\varphi\colon A \to B$ is a morphism, then $\varphi^*\colon B^* \to A^*$ and $(\varphi^*)^x_y = h \circ \varphi^y_x$, where $h\colon B_x \to R$ and $\varphi^y_x\colon A_y \to B_x$. φ^* is bounded whenever φ is. Again, φ^* is naturally a left module homomorphism which induces a homomorphism of right modules $B^* \to A^*$ via the anti-involution.

If we choose a countable set $E \subset X$ such that for some k the union of k-balls centered at points of E covers X, then it is easy to see that the categories $\mathfrak{C}_E(R)$ and $\mathfrak{C}_X(R)$ are equivalent.

It is convenient to assume that such a choice has been made once and for all. Then we may think of the objects of $\mathfrak{C}_X(\mathbb{C})$ as based complex vector spaces with basis a subset of $E \times \mathbb{N}$ satisfying certain finiteness conditions. Any based complex vector space has a natural inner product, and therefore a norm, and we define a morphism in $\mathfrak{C}_X(\mathbb{C})$ to be *analytically bounded* if it becomes a bounded operator when its domain and range are equipped with these natural ℓ^2 norms.

3.3. Definition. The category $\mathfrak{C}^{b.o.}_X(\mathbb{C})$ has the same objects as $\mathfrak{C}_X(\mathbb{C})$, but the morphisms have to satisfy the further restriction that they define analytically bounded operators on $\ell^2(E \times \mathbb{N})$

It is apparent that there is a close connection between the category $\mathfrak{C}^{b.o.}_X(\mathbb{C})$ and the C^*-algebra $C^*(X)$. In fact, the way we have arranged things any object A in the category $\mathfrak{C}^{b.o.}_X(\mathbb{C})$ can be thought of as a projection in $C^*(X)$ defined by the generating set for A and hence as a projective $C^*(X)$-module, and an endomorphism of A respects the $C^*(X)$-module structure. Since the involution on $\mathfrak{C}^{b.o.}_X(\mathbb{C})$ is given by duality, it corresponds to the $*$-operation on $C^*(X)$. Hence we get a map

$$L_*(\mathfrak{C}^{b.o}_X(\mathbb{C})) \to L_*(C^*(X)) = K_*(C^*(X)).$$

Similarly the forgetful functor $\mathfrak{C}^{b.o.}_X(\mathbb{C}) \to \mathfrak{C}_X(\mathbb{C})$ induces a map

$$L_*(\mathfrak{C}^{b.o}_X(\mathbb{C})) \to L_*(\mathfrak{C}_X(\mathbb{C})).$$

Notice that whenever we have a manifold $\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix}$ bounded over a metric space X, we may triangulate M in a bounded fashion so the cellular chain complex of M can be thought of as a chain complex in $\mathfrak{C}_X(\mathbb{Z})$ and, more relevantly, the chain complex with complex coefficients can be thought of as a chain complex in $\mathfrak{C}_X(\mathbb{C})$. Poincaré duality thus gives rise to a self-dual map and hence an element $\sigma_X[M] \in L_0(\mathfrak{C}_X(\mathbb{C}))$, the bounded symmetric signature of the manifold. The bounded symmetric signature is an invariant under bounded homotopy equivalence, since a bounded homotopy equivalence gives rise to a chain homotopy equivalence in the category $\mathfrak{C}_X(\mathbb{C})$ and the L-groups by their definition are chain homotopy invariant [26].

As mentioned above we get maps

$$K_*C^*(X) \xleftarrow{\alpha} L_*(\mathfrak{C}_X^{b.o.}(\mathbb{C})) \xrightarrow{\beta} L_*(\mathfrak{C}_X(\mathbb{C}))$$

3.4. Theorem. *In case $X = \mathcal{O}(K)$, the open cone on a finite complex, the map β is an isomorphism. Moreover,* $\mathrm{Sign}_{\mathcal{O}(K)} M = \alpha\beta^{-1}\sigma_{\mathcal{O}(K)}[M]$

Proof. Let $\mathcal{F}$ be any of the functors

$$K \mapsto L_*(\mathfrak{C}_{\mathcal{O}(K)}^{b.o.}(\mathbb{C})), \quad K \mapsto L_*(\mathfrak{C}_{\mathcal{O}(K)}(\mathbb{C})).$$

Then $\mathcal{F}$ is a reduced generalized homology theory on the category of finite complexes. In case $\mathcal{F}(K) = L_*(\mathfrak{C}_{\mathcal{O}(K)}(\mathbb{C}))$ this is proved by Ranicki [26]. In the case $\mathcal{F}(K) = L_*(\mathfrak{C}_{\mathcal{O}(K)}^{b.o.}(\mathbb{C}))$ the proof needs the extensions to Ranicki's results provided in [5] but goes along exactly the same lines, noting that restricting the morphisms to the ones defining analytically bounded operators does not prevent Eilenberg swindles[5], and thus the basic Karoubi filtration technique goes through. Moreover, β is plainly a natural transformation of homology theories[6], and it is an isomorphism for $K = \emptyset$, so it is an isomorphism for all finite complexes. This proves the first statement.

To prove the second statement note that if M has a bounded triangulation of bounded geometry (meaning that for each $r > 0$ there is a number N_r such that the number of simplices meeting $c^{-1}(B(x;r))$, for any $x \in X$, is at most N_r), then the natural representative of $\sigma_{\mathcal{O}(K)}[M]$ is in fact an analytically bounded operator (since Poincaré duality is given by sending a cell to its dual cell combined with appropriate subdivision maps). Moreover, by following the line of proof given by Kaminker and Miller [17, 20], and using the de Rham theorem in the bounded geometry category [9], one may show that this bounded operator passes under α to the class of the signature operator in $K_*(C^*(\mathcal{O}(K))$ (compare [17], theorem 5.1). In case M is not of bounded geometry we need to notice that both $\sigma_{\mathcal{O}(K)}[M]$ and $\mathrm{Sign}_{\mathcal{O}(K)}[M]$ are $\mathcal{O}(K)$-bordism invariants modulo 2-torsion, the latter by proposition 2.5, and that any manifold $\begin{pmatrix} M \\ \downarrow \\ \mathcal{O}(K) \end{pmatrix}$ is $\mathcal{O}(K)$-bordant to a bounded geometry manifold. To see this latter statement make $M \to \mathcal{O}(K)$ transverse to a level $t \cdot K \subset \mathcal{O}(K)$, and let V be the inverse image of $(\geq t) \cdot K$, W the inverse image of $(\leq t) \cdot K$. We then get a bordism from M to $W \cup_{\partial W} \partial W \times [0, \infty)$ by $M \times I \cup_V V \times [0, \infty)$ and the map p extends to a proper map from the bordism to $\mathcal{O}(K)$ by sending $(m, t) \in M \times I$ to $p(m)$ and $(v, s) \in V \times [0, \infty)$ to $(s + u) \cdot k$ where $u \cdot k = p(v)$. This is easily seen to be a proper map, and we do get a bordism over $\mathcal{O}(K)$ to a manifold of bounded geometry. □

[5]The key point is that the operator norm of an orthogonal direct sum is the *supremum* of the operator norms of its constituents. See [15] for the details of an Eilenberg swindle in the analytic situation.

[6]We do *not* assert that α is a natural transformation of homology theories.

Remark. In the above argument we needed to reduce the manifold M to bounded geometry, and to do this we used the fact that it is always possible to split M over an open cone. If one could similarly reduce a homotopy equivalence bounded over an open cone to a bounded geometry homotopy equivalence, the proof of our theorem would be considerably simplified. However, it appears that the proof of such a result would require a lengthy excursion into bounded geometry surgery [3].

3.5. Corollary. *In the situation above* $\mathrm{Sign}_{\mathcal{O}(K)}(M)$ *is an invariant modulo 2-torsion under boundedly controlled homotopy equivalence.*

As has already been explained, this suffices for a proof of Novikov's theorem.

4. Integral homotopy invariance

In the previous section we worked modulo 2-torsion, for simplicity. We will now justify the title of this paper by showing that it is not in fact necessary to invert 2 in corollary 3.5. In this section we will therefore, of course, suspend the convention made previously that all L and K groups are implicitly tensored with $\mathbb{Z}[\frac{1}{2}]$.

There are two points at which 2-torsion issues were neglected: the proof of the bordism invariance of the signature in section 2, and the identification of the various decorations on L-theory in section 3. We will address these in turn.

Bordism invariance. We begin by discussing in somewhat greater detail the reason for the appearance of the factor k in the formula for the boundary of the signature operator. The informal statement that 'the K-homology boundary of Dirac is Dirac' can be expressed more precisely as follows:

4.1. Proposition. *Let M be a manifold with boundary N, C a bundle of Clifford modules on M, D_C the corresponding Dirac operator. If M is even-dimensional, we assume that C is graded by a grading operator ε. Let n be a unit normal vector field to N. Define a bundle of Clifford modules ∂C on N as follows:*

1. *If M is odd-dimensional, $\partial C = C$, graded by Clifford multiplication by in;*
2. *If M is even-dimensional, ∂C is the $+1$ eigenspace of the involution $in\varepsilon$ on C (which commutes with the Clifford action of TN).*

Then the boundary, in K-homology, of the class of the Dirac operator $[D_C]$ is the class of the Dirac operator of ∂C.

While this particular statement does not appear to be in the literature, related results are proved in [12, 13, 36]. Now let us take C to be the bundle of Clifford modules that defines the signature operator on M (see the remark after definition 2.1). Then a simple calculation shows that if M is odd-dimensional, ∂C defines the signature operator on N, but that if M is even-dimensional, ∂C defines the direct sum of two copies of the signature operator on N. Hence the factor k in 2.3 and 2.5.

To get rid of the factor 2 in the even-dimensional case we employ an idea of Atiyah [1]. Suppose that the normal vector field n extends to a unit vector field (also called n) on M, and define an operator R_n on C by *right* Clifford multiplication by in. Then R_n is an involution and its ± 1 eigenspaces are bundles of (left) Clifford modules. Let D_{C^+} be the Dirac operator associated to the $+1$ eigenbundle of R_n. Using the result above, we find that $\partial[D_{C^+}]$ is exactly the signature operator of N. Thus, the same method of proof as that of 2.5 gives us

4.2. Proposition. *Suppose that N is an X-bounded manifold which is the boundary of an X-bounded manifold-with-boundary M. In addition, if M is even-dimensional, suppose that the unit normal vector field to N extends to a unit vector field on M. Then* $\mathrm{Sign}_X(N) = 0$ *in $K_*(C^*(X))$.*

Now we remark that there is no obstruction to extending the field n over any non-compact connected component of M. Moreover, provided that the control space X is non-compact and coarsely geodesic, there is no loss of generality in assuming that every connected component of M is non-compact; for, in any compact component, we may punch out a disc, replace it with an infinite cylinder, and control this cylinder over a ray in X. Thus we conclude that over any such space X, and in particular over an open cone on a finite polyhedron, the coarse signature is bordism invariant on the nose.

Decorations. With the issue of bordism invariance settled, the integral boundedly controlled homotopy invariance of the coarse analytic signature will follow (as in section 3) from:

4.3. Theorem. *The functors $K \mapsto L^h_*(\mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C}))$ and $K \mapsto L^h_*(\mathfrak{C}_{\mathcal{O}(K)}(\mathbb{C}))$ are isomorphic homology theories (under the forgetful map).*

Proof. All we need to do is to prove that both functors are homology theories, since they agree on $\emptyset$. Since $\mathbb{C}$ is a field we have $K_{-i}(\mathbb{C}) = 0$ for $i > 0$ [4, Chap. XII]. Hence

$$L^h_*(\mathfrak{C}_{\mathcal{O}(K)}(\mathbb{C})) = L^{-\infty}_*(\mathfrak{C}_{\mathcal{O}(K)}(\mathbb{C}))$$

is a homology theory. To prove $L^h_*(\mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C}))$ is also a homology theory we use the excision result [5, Theorem 4.1]. Combining this with [5, Lemma 4.17] we only need to see that idempotent completing any of the categories $\mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C})$ does not change the value of L^h i. e. that K_0 of the idempotent completed categories is trivial. This is the object of the next proposition. □

4.4. Proposition. *With terminology as above we have*

$$K_0(\mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C})^\wedge) = 0$$

for K a non-empty finite complex.

Proof. The proof follows the methods in [23] and [24] quite closely, and the reader is supposed to be familiar with these papers. Let L be a finite complex, $K = L \cup_\alpha D^n$. Consider the category $\mathfrak{U} = \mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C})$ and the full subcategory $\mathfrak{A} = \mathfrak{C}^{b.o.}_{\mathcal{O}(K)}(\mathbb{C})_{\mathcal{O}(L)}$ with objects having support in a bounded neighborhood of $\mathcal{O}(L)$. $\mathfrak{A}$ is isomorphic to $\mathfrak{C}^{b.o.}_{\mathcal{O}(L)}(\mathbb{C})$, and $\mathfrak{U}$ is $\mathfrak{A}$-filtered in the sense of Karoubi, so following [24], we get an exact sequence

$$\dots K_1(\mathfrak{U}) \to K_1(\mathfrak{U}/\mathfrak{A}) \to K_0(\mathfrak{A}^\wedge) \to K_0(\mathfrak{U}^\wedge) \to$$

but $\mathfrak{U}/\mathfrak{A}$ is isomorphic to

$$\mathfrak{C}^{b.o.}_{\mathcal{O}(D^n)}(\mathbb{C})/\mathfrak{C}^{b.o.}_{\mathcal{O}(S^{n-1})}(\mathbb{C})$$

which has the same K-theory as $\mathfrak{C}^{b.o.}_{\mathbb{R}^n}(\mathbb{C})$. So by induction over the cells in K, it suffices to prove that

$$K_1(\mathfrak{C}^{b.o.}_{\mathbb{R}^n}(\mathbb{C})) = 0 \qquad n > 1$$

and

$$K_0(\mathfrak{C}^{b.o.}_{\mathbb{R}^{n-1}}(\mathbb{C})^\wedge) = 0 \qquad n > 1$$

but following the arguments in [23] it is easy to see these groups are equal. Now consider the ring $\mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$. The category

$$\mathfrak{C}^{b.o.}_{\mathbb{R}^n}(\mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}])$$

with geometrically bounded morphisms, inducing analytically bounded operators on the Hilbert space where the t_i powers are also used as basis has a subcategory

$$\mathfrak{C}^{b.o., t_1, \dots, t_k}_{\mathbb{R}^n}(\mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}])$$

where the morphisms are required to use uniformly bounded powers of the t_i's. Turning t_i-powers into a grading produces a functor

$$\mathfrak{C}^{b.o., t_1, \dots, t_k}_{\mathbb{R}^n}(\mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]) \to \mathfrak{C}^{b.o., t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k}_{\mathbb{R}^{n+1}}(\mathbb{C}[t_1, t_1^{-1}, \dots, t_{i-1}, t_{i+1}, \dots, t_k, t_k^{-1}]).$$

We claim this is a split epimorphism on K_1. Consider the automorphism β_{t_i} which is multiplication by t_i on the upper half of $\mathbb{R}^{n+1}$ and the identity on the lower half. Here upper and lower refers to the coordinate introduced when the t_i-powers were turned into a grading. The splitting is given by sending an automorphism α to the commutator $[\alpha, \beta_{t_i}]$ and restricting to a band. Since both the bounded operator and the bounded t-power conditions are responsive to the Eilenberg swindle arguments used in [23] the argument carries over to this present situation. From this it follows there is a monomorphism

$$K_1(\mathfrak{C}^{b.o.}_{\mathbb{R}^n}(\mathbb{C})) \to K_1(\mathfrak{C}^{b.o., t_1, \dots, t_n}_{*}(\mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])).$$

But the bounded t-power condition is vacuous, when the metric space is a point, and the uniformity given by the $\mathbb{Z}^n$-action renders the bounded operator condition vacuous too. Since the inclusion maps given by the commutator with β_{t_i} commute up to sign we find that the image of $K_1(\mathfrak{C}^{b.o.}_{\mathbb{R}^n}(\mathbb{C}))$ is contained in

$$K_{-i}(\mathbb{C}) \subset K_1(\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$$

which is 0 since $\mathbb{C}$ is a field and we are done. □

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Department of Mathematics, State University of New York, Binghamton, Binghamton, NY 13901

Jesus College, Oxford OX1 3DW, England

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104