GEOMETRICALLY DEFINED TRANSFERS, COMPARISONS

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0. Introduction

There are a number of geometrically defined transfer homomorphisms relating finiteness obstructions, Whitehead torsions, concordances, surgery obstructions etc. in a bundle. It is the purpose of this paper to relate these various homomorphisms. As an example of the kind of results we obtain consider a bundle $F \to Y \to X$ of finite complexes. If two of these satisfy Poincaré duality then the third does as announced by Quinn in [16]. Since then a proof has appeared by Gottlieb [7]. However in both accounts Poincaré torsion is disregarded. The Poincaré torsion of a finite Poincaré Duality pair $(X, \partial X)$ denoted $\tau(X)$ is defined to be the torsion of the chain homotopy equivalence $[X] \cap - : C^\sharp(X) \to C_\sharp(X, \partial X)$ (see §3 for sign convention) We prove

Theorem A. *Let $F \to (Y, \partial Y) \to (X, \partial X)$ be a bundle of Poincaré complexes. Let f^* : $\mathrm{Wh}(\pi_1 X) \to \mathrm{Wh}(\pi_1 Y)$ be the Whitehead transfer homomorphism of Anderson* [1] *(see §1.II for definition) then*

$$\tau(Y) = (-1)^{|F|} \cdot f^*(\tau(X)) + \chi(X) \cdot \tau(F).$$

Here $|F|$ denotes the dimension of F and $\chi(X)$ the Euler characteristic of X.

Remark. The assumption that $(Y, \partial Y)$ is a Poincaré pair is actually redundant. In course of computing $\tau(Y)$ we actually prove $(Y, \partial Y)$ is a Poincaré pair thus generalizing the results of Quinn and Gottlieb to the relative case.

We start by introducing the homomorphisms we want to consider.

1. Geometric Transfer Homomorphisms

I. The L-group Transfer. Let $F \to E \to B$ be a topological bundle with fibre F a manifold and w_B a double cover of B. We obtain a double covering w_E of E as follows: Let $\pi_1 F \to \mathbf{Z}/2\mathbf{Z}$ be the orientation homomorphism and $H(F, *)$ basepoint preserving homotopy equivalences of F. Then $\pi_1 F \to \mathbf{Z}/2\mathbf{Z}$ factors $\pi_1 F \to \pi_0 H(F, *) \to \mathbf{Z}/2\mathbf{Z}$ and $\pi_1 F \to \pi_0 H(F, *)$ factors $\pi_1 F \to \pi_1 E \to \pi_0 H(F, *)$ so there is a canonical extension of $\pi_1 F \to \mathbf{Z}/2\mathbf{Z}$ to $\pi_1 E \to \mathbf{Z}/2\mathbf{Z}$. We add this homomorphism to the composite $\pi_1 E \to \pi_1 B \xrightarrow{w_B} \mathbf{Z}/2\mathbf{Z}$ to obtain a homomorphism $w_E : \pi_1 E \to \mathbf{Z}/2\mathbf{Z}$. This homomorphism defines an isomorphism class of $\mathbf{Z}/2\mathbf{Z}$-bundles over E. We choose one such bundle and denote this by w_E Consider an element of $L^a(\pi_1 B; w_B)$ where a stands for s, h or p, simple, finite or projective surgery problems (we may also consider surgery problems with certain restrictions on torsions or finiteness obstructions). This element is represented [21, Ch. IX] by a degree 1 normal map with a reference map to B

$$\begin{array}{ccccc} \nu_M & \longrightarrow & \nu_X & & \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & X & \xrightarrow{\phi} & B \end{array}$$

where f is a degree 1 map with respect to fundamental classes $[X]$ and $[M]$ in homology with respect to an integral coefficient system given by w_B. Pulling back everything to E defines a transfer homomorphism

$$\tau_L : L_n^a(\pi_1 B, w_B) \to L_{n+|F|}^{a'}(\pi_1 E, w_E)$$

where a may or may not be equal to a'. This is the Wall group transfer homomorphism as defined by Quinn and Wall [15, 21] with the modifications that were observed to be necessary by Farrell and Hsiang [6]. It is essentially only computable in case of a product bundle or more generally in case the fundamental groups behave as in a product bundle [12]. The case of S^1-bundles will be described in a forthcoming joint work with Munkholm [11].

II. The Whitehead Torsion Transfer. Consider a bundle $F \to E \to B$ of finite complexes. Anderson [1] defines a transfer homomorphism

$$\tau_1 : \mathrm{Wh}(\pi_1 B) \to \mathrm{Wh}(\pi_1 E)$$

such that if

$$\begin{array}{ccc} F & \xrightarrow{f} & F_1 \\ {\scriptstyle i}\downarrow & & \downarrow \\ E & \xrightarrow{g} & E_1 \\ \downarrow & & \downarrow \\ B & \xrightarrow{h} & B_1 \end{array}$$

is a commutative diagram of bundles with f, g and h homotopy equivalences, then the Whitehead torsions are related by

$$t(g) = \tau_1(t(h)) + i_*(t(f)) \cdot \chi(B)$$

where $\chi(B)$ is the Euler characteristic of B. When no confusion is possible we will some times denote τ_1 by p^* where $p : E \to B$.

III. The $\widetilde{K}_0$-finiteness Obstruction Transfer. Let $F \to E \to B$ be a fibration of finitely dominated spaces. Ehrlich [5] defines a transfer homomorphism

$$\tau_0 : \widetilde{K}_0(\mathbf{Z}\pi_1 B) \to \widetilde{K}_0(\mathbf{Z}\pi_1 E)$$

relating the finiteness obstructions of F, E and B according to the formula

$$\sigma(E) = \tau_0(\sigma(B)) + i_*(\sigma(F))\chi(B).$$

As in the Wh-case we will denote τ_0 by p^* when there is no chance of confusion, $p : E \to B$.

IV. A K_{-i} Transfer. Let $F \to E \to B$ be a bundle of finite complexes. It is proved by Munkholm and Pedersen [10] that the diagram

$$\begin{array}{ccc} \mathrm{Wh}(\pi_1 E \times \mathbf{Z}) & \longrightarrow & \widetilde{K}_0(\mathbf{Z}\pi_1 E) \\ \uparrow \tau_1 & & \uparrow \tau_0 \\ \mathrm{Wh}(\pi_1 B \times \mathbf{Z}) & \longrightarrow & \widetilde{K}_0(\mathbf{Z}\pi_1 B) \end{array}$$

is commutative.

Here the horizontal maps are the Bass-Heller-Swan homomorphisms [3] and the vertical maps are the Wh- and $\widetilde{K}_0$-transfers.

Since by definition $K_{-n}(\mathbf{Z}(\pi\times\mathbf{Z}^n))$ is the intersection of various inclusions of $\widetilde{K}_0(\mathbf{Z}(\pi\times\mathbf{Z}^n))$ in $\mathrm{Wh}(\pi \times \mathbf{Z}^{n+1})$ this implies there is a well defined homomorphism

$$\tau_{-n} : K_{-n}(\mathbf{Z}\pi_1 B) \to K_{-n}(\mathbf{Z}\pi_1 E)$$

associated with this bundle such that the diagram

$$\begin{array}{ccc} \mathrm{Wh}(\pi_1 E \times \mathbf{Z}^{n+1}) & \longrightarrow & K_{-n}(\mathbf{Z}\pi_1 E) \\ \uparrow & & \uparrow \tau_{-n} \\ \mathrm{Wh}(\pi_1 B \times \mathbf{Z}^{n+1} & \longrightarrow & K_{-n}(\mathbf{Z}\pi_1 B) \end{array}$$

commutes. The left hand vertical map is the Wh-transfer associated with $F \to E \times T^{n+1} \to B \times T^{n+1}$. The diagram associated with the finiteness obstruction transfer of the bundle $F \to E \times T^n \to B \times T^n$

$$\begin{array}{ccc} \widetilde{K}_0(\mathbf{Z}(\pi_1 E \times \mathbf{Z}^n)) & \longrightarrow & K_{-n}(\mathbf{Z}\pi_1 E) \\ \uparrow & & \uparrow \\ \widetilde{K}_0(\mathbf{Z}(\pi_1 B \times \mathbf{Z}^n)) & \longrightarrow & K_{-n}(\mathbf{Z}\pi_1 B) \end{array}$$

will also be commutative. Thus τ_{-n} will inherit all properties known of the $\widetilde{K}_0$- transfer e. g. the composite $f_* \cdot \tau_{-n}$ is multiplication by a generalized Euler characteristic, see [14], and τ_{-n} only depends on the homotopy fundamental group data of the bundle [12].

V. The Concordance Transfer. Consider a bundle of closed manifolds $F \to E \to B$. Transfers of concordance spaces have been defined by Burghelea and Lashof, Hatcher and others [4, 8]. These transfers are maps of concordance spaces, which are topological groups, so we get induced homomorphisms on π_0 of these spaces, the isotopy classes of concordances. For a special class of concordances these isotopy classes are completely described by Anderson and Hsiang [2] as follows: Let $h : M \times \mathbf{R}^{n+2} \times I \to M \times \mathbf{R}^{n+2} \times I$ be a concordance bounded in the $\mathbf{R}^{n+2}$ factor (with usual metric). Then the isotopy class of h is completely determined by an invariant in $K_{-n}(\mathbf{Z}\pi_1 M)$ for $n \geq 0$ and in $\mathrm{Wh}(\pi_1 M)$ for $n = -1$. We have the following definition: Let

$$\begin{array}{ccc} E \times \mathbf{R}^{n+2} \times I & \xrightarrow{k} & E \times \mathbf{R}^{n+2} \times I \\ \downarrow & & \downarrow \\ B \times \mathbf{R}^{n+2} \times I & \xrightarrow{h} & B \times \mathbf{R}^{n+2} \times I \end{array}$$

be a commutative diagram of bounded concordances. There is a homomorphism

$$\tau_c : K_{-n}(\mathbf{Z}\pi_1 B) \to K_{-n}(\mathbf{Z}\pi_1 E)$$

relating the concordance invariants of h and k, i. e. $\tau_c(\sigma(h)) = \sigma(k)$. We need to show that τ_c is everywhere defined and well defined. That it is everywhere defined follows from e. g. [8] and well definedness will be shown in the next section.

2. The Concordance Transfer

Let $F \to E \to B$ b a bundle of manifolds. We wish to show that the concordance transfer is well defined

Theorem 2.1. *Assume we have a commutative diagram of bounded concordances*

$$\begin{array}{ccc} E \times \mathbf{R}^{n+2} \times I & \xrightarrow{k} & E \times \mathbf{R}^{n+2} \times I \\ \downarrow & & \downarrow \\ B \times \mathbf{R}^{n+2} \times I & \xrightarrow{h} & B \times \mathbf{R}^{n+2} \times I \end{array}$$

Then $\sigma(k) = \tau_{-n}(\sigma(h))$ *where* $\tau_{-n} : K_{-n}(\mathbf{Z}\pi_1 B) \to K_{-n}(\mathbf{Z}\pi_1 E)$ *is the homomorphism IV.*

This shows that the concordance transfer is well defined since $\tau_c = \tau_{-n}$.

Proof of Theorem 2.1. The invariant $\sigma(h)$ of Hsiang and Anderson is computed as follows: Wrapping around a torus [17] produces a concordance

$$\overline{h} : B \times T^{n+1} \times \mathbf{R} \times I \to B \times T^{n+1} \times \mathbf{R} \times I$$

such that the cover of $\overline{h}$, $\tilde{h}$ agrees with h on an open subset of $\mathbf{R}^{n+1}$ crossed with $B \times \mathbf{R} \times I$. Choosing c sufficiently big, the space between $B \times T^{n+1} \times 0 \times I$ and $\overline{h}(B \times T^{n+1} \times c \times I)$ is an h-cobordism with a torsion on $\mathrm{Wh}(\pi_1 B \times \mathbf{Z}^{n+1})$. Now Hsiang and Anderson show this invariant actually lies in the summand $K_{-n}(\mathbf{Z}\pi_1 B)$ and completely determines the isotopy class of the concordance. However, wrapping around a torus is functorial, so we may obtain a commutative diagram

$$\begin{array}{ccc} \overline{k} : E \times T^{n+1} \times \mathbf{R} \times I & \longrightarrow & E \times T^{n+1} \times \mathbf{R} \times I \\ \downarrow & & \downarrow \\ \overline{h} : B \times T^{n+1} \times \mathbf{R} \times I & \longrightarrow & B \times T^{n+1} \times \mathbf{R} \times I \end{array}$$

and it is now evident that the invariants become related by the Wh-transfer which restricts to the K_{-n}-transfer. □

We obtain the following

Corollary 2.2. *Let $M \to E \to B$ be a bundle with M a closed manifold. Then the* Wh-*transfer of $M \to E \times T^{n+2} \to B \times T^{n+2}$ preserves the K_{-i} summands of the* Wh-*groups of the fundamental groups.*

We also obtain the result that all known information on Wh- and K_0-transfers hold for concordance transfers as well. One of these results is that the concordance transfer only depends on the fundamental group data. This result has also been proved by Burghelea and Lashof [4] by a geometrical argument. We however further obtain that the various computational results for K_0- and Wh-transfers [10, 12, 14] extend to the concordance transfer.

3. Poincaré Torsion

Let $(X; \partial_1 X, \partial_2 X)$ be a finite Poincaré triple in the sense of Wall [20]. The homotopy equivalence

$$[X] \cap - : C^\sharp(X, \partial_1 X) \to C_\sharp(X, \partial_2 X)$$

of based chain complexes over $\mathbf{Z}\pi_1 X$ has a well defined torsion $\tau(X, \partial_1 X)$, the Poincaré torsion. We need a sign convention in computing this torsion, and we choose the degrees of the cochain complex i. e. we think of $C_i(X, \partial_2 X)$ as degree $|X| - i$. Clearly $\tau(X, \partial_1 X)$ only depends on the simple homotopy type of $(X; \partial_1 X, \partial_2 X)$. The Poincaré torsion in a bundle is determined by the following

Theorem 3.1. *Let $F \to (Y, \partial Y) \to (X, \partial X)$ be a bundle of finite complexes. Then*

$$\tau(Y) = (-1)^{|F|} \cdot f^*(\tau(X)) + \chi(X) \cdot \tau(F)$$

where f^* is the Wh-*transfer of* §*2.II.*

To prove this theorem we need a number of lemmas and various results about different ways to compute torsion. When we have an exact sequence $0 \to A \to B \to C \to 0$ of based chain complexes we shall allow ourselves to think of $0 \to B \to C \to sA \to 0$ as a short exact sequence of based chain complexes even though C should really be replaced by the simply homotopy equivalent mapping cylinder of $B \to C$. We use * to denote duality in the Whitehead group as well as in modules.

Lemma 3.2. *Let $f : A \to B$ be a homotopy equivalence of based chain complexes with torsion τ. The torsion of $f^* : B^* \to A^*$ is then $-\tau^*$.*

Proof. trivial □

Lemma 3.3. *Let $(X, \partial_1 X, \partial_2 X)$ be a Poincaré triple. Then*

$$\tau(X, \partial_1 X) = -(-1)^{|X|} \cdot \tau(X, \partial_2 X)^*$$

Proof. The dual of

$$C^\sharp(X, \partial_1 X) \to C_\sharp(X, \partial_2 X)$$

is

$$C^\sharp(X, \partial_2 X) \to C_\sharp(X, \partial_1 X)$$

We get the factor $(-1)^{|X|}$ since we change the way we count degrees by $|X|$. □

Lemma 3.4. *Let $(X; \partial_1 X, \partial_2 X)$ be a Poincaré triple. Then*

$$\tau(X, \partial_2 X) = \tau(X) - i_* \tau(\partial_2 X).$$

Proof. Noting that $\partial\partial_1 X = \partial_1 X \cap \partial_2 X = \partial\partial_2 X$ we have a diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C^\sharp(\partial_2 X, \partial\partial_2 X) & \longleftarrow & C^\sharp(X, \partial_1 X) & \longleftarrow & C^\sharp(X, \partial X) & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle [\partial_2 X]\cap -} & & \downarrow{\scriptstyle [X]\cap -} & & \downarrow{\scriptstyle [X]\cap -} & & \\ 0 & \longleftarrow & s(C_\sharp(\partial_2 X)) & \longleftarrow & C_\sharp(X, \partial_2 X) & \longleftarrow & C_\sharp(X) & \longleftarrow & 0 \end{array}$$

giving $\tau(X, \partial_1 X) = \tau(X, \partial X) + i_* \tau(\partial_2 X, \partial\partial_2 X)$. Using Lemma 3.3 and dualizing we get the result. □

We note that in case $\partial_2 X = \partial X$, $\partial_1 X = \emptyset$ we obtain the usual

$$\tau(\partial X) = \tau(X) - \tau(X, \partial X) = \tau(X) + (-1)^{|X|} \tau(X)^*.$$

We also need the following Mayer-Vietoris type result.

Lemma 3.5. *Let $(A; \partial_1 A, \partial_2 A)$ and $(B; \partial_1 B, \partial_2 B)$ be Poincaré triples with $\partial_1 A = \partial_1 B = A \cap B$. Then $(A \cup B, \partial_2 A \cup \partial_2 B)$ is a Poincaré pair and $\tau(A \cup B) = \tau(A) + \tau(B) - \tau(A \cap B)$.*

Proof. We have Mayer-Vietoris short exact sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C^\sharp(A \cap B, \partial(A \cap B)) & \longleftarrow & C^\sharp(A \cup B, \partial(A \cup B)) & \longleftarrow & C^\sharp(A, \partial A) \oplus C^\sharp(B, \partial B) & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & sC_\sharp(A \cap B) & \longleftarrow & C_\sharp(A \cup B) & \longleftarrow & C_\sharp(A) \oplus C_\sharp(B) & \longleftarrow & 0 \end{array}$$

giving

$$\tau(A \cup B, \partial(A \cup B)) = \tau(A \cap B, \partial(A \cap B)) + \tau(A, \partial A) + \tau(B, \partial B).$$

Now use Lemma 3.3 and dualize. □

Lemma 3.6. *Let F be a Poincaré duality space. Then $\tau(F \times D^m) = \tau(F)$ and $\tau(F \times S^m) = (1 + (-1)^m) \cdot \tau(F)$.*

Proof. It is easy to see that $\tau(F \times D^m) = \tau(F)$. It now follows from Lemma 3.5 that $\tau(F \times S^m) = \tau(F) + \tau(F) - \tau(F \times S^{m-1})$ which completes the proof using induction. □

Lemma 3.7. *Let $(A, \partial A)$ be a Poincaré duality pair which is the union of $(B, \partial B)$ and $D^m \times F$, where $(B, \partial B)$ is a Poincaré duality pair and F is a Poincaré space and*

$$B \cap D^m \times F = \partial B \cap S^{m-1} \times F = S^{i-1} \times D^m \times F.$$

Then

$$\tau(A) = \tau(B) + (-1)^i \tau(F).$$

Proof. Follows trivially from Lemma 3.5 and 3.6. □

The main application of this lemma is

Proposition 3.8. *Let $F \to (E, \partial E) \to (W, \partial W)$ be a bundle with F a Poincaré space and W a manifold. Then $\tau(E) = \chi(W) \cdot \tau(F)$ (where $\chi(W)$ is the Euler characteristic of W).*

Proof. W has a handlebody decomposition and the bundle restricted to a handle is trivial. Now use Lemma 3.7 □

Let $(A, \partial A) \to X$ be a map of CW-complexes with homotopy fibre of homotopy type (D^m, S^{m-1}). We may use Stasheff's construction [19] to construct a pair of CW-complexes $(E_A, \partial E_A)$ and a homotopy commutative diagram

$$\begin{array}{ccc} (A, \partial A) & & \\ \uparrow{\scriptstyle E(f)} & \overset{f}{\searrow} & X \\ & \nearrow & \\ (E_A, \partial E_A) & & \end{array}$$

such that $E(f)$ is a homotopy equivalence and such that the inverse image of the interior of all D^n in X under $(E_A, \partial E_A) \to X$ is homeomorphic to $D^n \times (D^m, S^{m-1})$. Even though $(E_A, \partial E_A) \to X$ is not a fibration we do have a Thom isomorphism

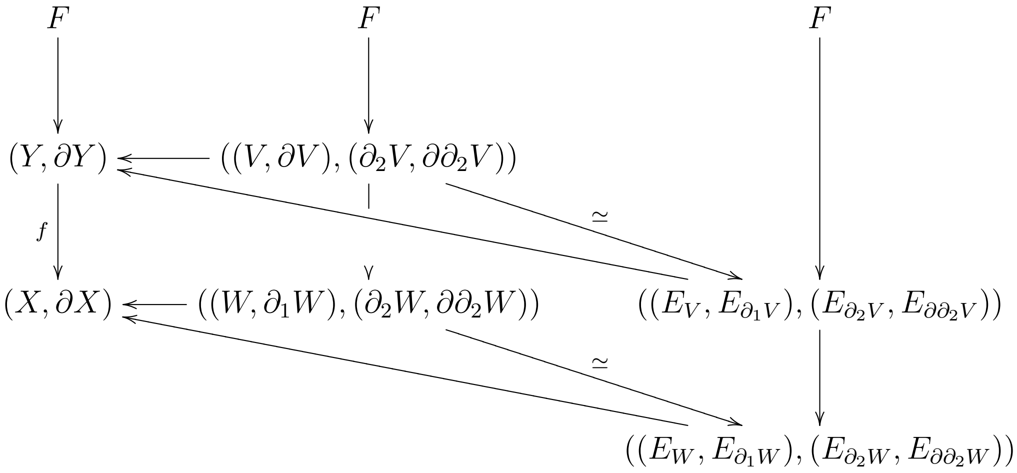
$$C^\sharp(X) \to C^\sharp(E_A, \partial E_A).$$

Assume X, A and ∂A are finite complexes. Then $(E_A, \partial E_A)$ is a finite pair and the Thom isomorphism $C^\sharp(X) \to C^\sharp(E_A, \partial E_A)$ is clearly simple, sending a basis to a basis. The diagram

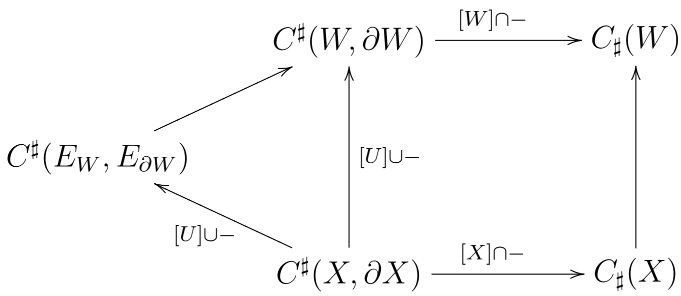
$$\begin{array}{ccc} C^\sharp(X) & \longrightarrow & C^\sharp(E_A, \partial E_A) \\ & \searrow & \uparrow{\scriptstyle E(f)^*} \\ & & C^\sharp(A, \partial A) \end{array}$$

shows that the torsion of the Thom isomorphism $C^\sharp(X) \to C^\sharp(A, \partial A)$ is the tors ion of $E(f)^*$. We are now ready for

Proof of Theorem 3.1. Consider the following diagram

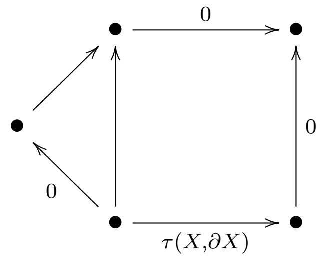


which we proceed to explain. We embed $(X, \partial X)$ in halfspace H^n, some big n such that $X \cap \mathbf{R}^{n-1} = \partial X$. Now W is a regular neighborhood of X in H^n meeting R^{n-1} regularly in $\partial_2 W = W \cap \mathbf{R}^{n-1}$. We define $\partial_1 W = \text{closure}(\partial W - \partial_2 W)$, so $\partial_1 W \cap \partial_2 W = \partial\partial_2 W$. According to Spivak [18] the homotopy fibre of $(W, \partial_1 W) \to X$ and $(\partial_2 W, \partial\partial_2 W) \to \partial X$ is (D^m, S^{m-1}) where $m = n - |X|$. We pull everything back to total space level to get $((V, \partial_1 V), (\partial_2 V, \partial\partial_2 V))$ and note that Stasheff's construction on $((E_W, E_{\partial_1 W}), (E_{\partial_2 W}, E_{\partial\partial_2 W}))$ pulled back to total space level gives Stasheff's construction on $((V, \partial_1 V), \partial_2 V, \partial\partial_2 V))$. We have a Thom isomorphism $H^*(X, \partial X) \to H^*(W, \partial W)$ which at chain level (with $\mathbf{Z}\pi_1 X$-coefficients) is given by cup product with some element $[U]$ representing the Thom class. We get the homotopy commutative diagram



where $E_{\partial W} = E_{\partial_1 W} \cup E_{\partial_2 W}$.

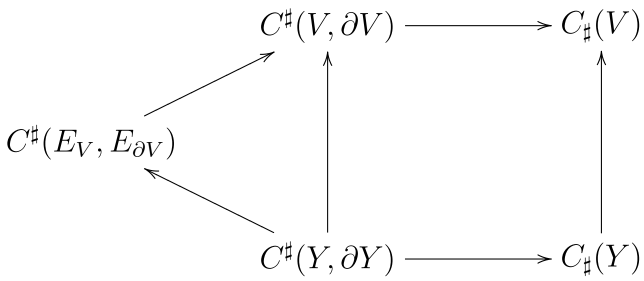
The torsions of the various homotopy equivalences are as indicated in the diagram



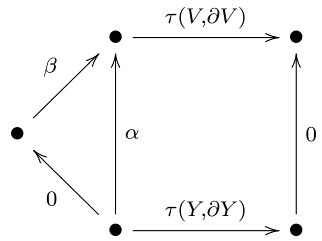
We conclude that the torsion of $C^\sharp(E_W, E_{\partial W}) \to C^\sharp(W, \partial W)$ is $(-1)^m \cdot \tau(X, \partial X)$ (the factor $(-1)^m$ because of the degree shift in the Thom isomorphism). It follows from Lemma 3.2 that the torsion of $C_\sharp(W, \partial W) \to C_\sharp(E_W, E_{\partial W})$ is

$$-(-1)^m \tau(X, \partial X)^* = (-1)^m \cdot (-1)^{|X|} \cdot \tau(X).$$

By definition of the Wh-transfer homomorphism the torsion of $C_\sharp(V, \partial V) \to C_\sharp(E_V, E_{\partial V})$ where $E_{\partial V} = E_{\partial_1 V} \cup E_{\partial_2 V}$ is $f^*((-1)^m(-1)^{|X|} \cdot \tau(X))$. In the homotopy commutative diagram at totalspace level



the torsions are



where $-\beta^* = f^*((-1)^m \cdot (-1)^{|X|} \cdot \tau(X))$.
We get $\alpha = (-1)^m \cdot \beta$ and

$$\alpha + (-1)^m \tau(V, \partial V) = \tau(Y, \partial Y)$$

using Proposition 3.8 to prove $\tau(V) = \chi(X) \cdot \tau(F)$ and dualizing we get

$$\tau(Y) = (-1)^{|F|} \cdot f^*(\tau(X)) + \chi(X) \cdot \tau(F).$$

□

4. Comparing L-Group and Wh-Transfers

We obtain the following result on L-group transfers: Let $M \to E \to B$ be a bundle, M a closed manifold of dim n, $\pi_1(E) = \pi$, $\pi_1 B = \rho$.

Theorem. *There are commutative diagrams of Rothenberg exact sequences*

$$(*)\qquad \begin{array}{ccccccccc} \longrightarrow & L_i^s(\rho) & \longrightarrow & L_i^h(\rho) & \longrightarrow & \hat{H}(\mathbf{Z}_2; \mathrm{Wh}\,\rho) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & L_{i+n}^s(\pi) & \longrightarrow & L_{i+n}^h(\pi) & \longrightarrow & \hat{H}^{i+n}(\mathbf{Z}_2; \mathrm{Wh}\,\pi) & \longrightarrow \end{array}$$

$$(**)\qquad \begin{array}{ccccccccc} \longrightarrow & L_i^h(\rho) & \longrightarrow & L_i^p(\rho) & \longrightarrow & \hat{H}^i(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}[\pi])) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & L_{i+n}^h(\pi) & \longrightarrow & L_{i+n}^p(\pi) & \longrightarrow & \hat{H}^{i+n}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}[\pi])) & \longrightarrow \end{array}$$

where the vertical maps are (induced by) the Wall-group, K_0- and $(-1)^n$ times the Wh-*transfer.*

Proof. According to [13] $\hat{H}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}\pi))$ is represented by surgery problems on finitely dominated Poincaré complexes with finite boundary. It follows then from the definition of the $\widetilde{K}_0$-transfer that $\hat{H}^i(\mathbf{Z}_2, \widetilde{K}_0(\mathbf{Z}\rho)) \to \hat{H}^{i+n}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}\pi))$ is induced by the $\widetilde{K}_0$-transfer. To see that $\hat{H}^i(\mathbf{Z}_2, \mathrm{Wh}\,\rho) \to \hat{H}^{i+n}(\mathbf{Z}_2, \mathrm{Wh}\,\pi)$ is induced by $(-1)^n$ times the Wh-transfer we now use the similar geometric representation of $\hat{H}^i(\mathbf{Z}_2, \mathrm{Wh}\,\rho)$ as surgery problems on simple Poincaré complexes with finite boundary, and Theorem 3.1. □

Final Remarks. Consider an orientable bundle $S^1 \to E \to B$, $\pi_1 B = \rho$, $\pi_1 E = \pi$ both finite. Then the $\widetilde{K}_0$-transfer is trivial [10] so we get a transfer homomorphism $L^p(\mathbf{Z}\rho) \to L^h(\mathbf{Z}\pi)$. This and similar situations pose some interesting problems concerning what happens in the Rothenberg exact sequence.

References

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