ON COMPLEMENTS OF CODIMENSION 3 EMBEDDINGS IN S^n

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Abstract. Kervaire proved that a finitely presented group π is the fundamental group of a smooth n-dimensional homology sphere, $n \geq 5$, if and only if π is perfect and $H_2(\pi) = 0$. We give an analogous characterization of fundamental groups of complements of wildly embedded polyhedra in S^n. Our characterization is the same except that in our case the group π need not be finitely presented

Introduction

In [6], Kervaire proved that a finitely presented group π is the fundamental group of a smooth n-dimensional homology sphere, $n \geq 5$, if and only if π is perfect and $H_2(\pi) = 0$. In this note, we give an analogous characterization of fundamental groups of complements of wildly embedded polyhedra in S^n. This characterization is the same except that in our case the group π need not be finitely presented.

Theorem. *Let K^k be a finite polyhedron and let $i : K \to S^n$ be a* PL *embedding with $1 \leq k \leq n-3$, and $n \geq 7$. If π is a countably generated group, then there is a (non-*PL*) reembedding $j : K \to S^n$ with $\pi_1(S^n - j(K)) \cong \pi$ if and only if π is perfect and $H_2(\pi) = 0$.*

In case π is finitely presented, one can prove the theorem for $K = S^1$ as a corollary to Kervaire's theorem and double suspension. Let H^n be a homology n-sphere with $\pi_1(H) = \pi$. By [3], $\Sigma^2 H$ is homeomorphic to S^{n+2}. The suspension circle gives a wild embedding $i : S^1 \to S^{n+2}$ with $\pi_1(S^{n+2} - i(S^1)) \cong \pi$. We prove our theorem by generalizing this technique.

Here is an outline of the proof. For notation and information concerning PL topology, we refer to [10]. We construct a properly embedded acyclic polyhedron $L \subset \mathbb{R}^n$ with $\pi_1(L) \cong \pi$ and let U be an open regular neighborhoods of L in $\mathbb{R}^n$. Setting $X = S^n - U$, we show that X is a cell-like subset of S^n. It follows that S^n/X is an ANR homology sphere and that $\Sigma(S^n/X)$ is homeomorphic to S^{n+1}. The suspension of $[X]$ is an arc α in S^{n+1} such that $\pi_1(S^n - \alpha) \cong \pi$. This establishes our result in case $K = I$, the unit interval. We then prove our main theorem by showing that under the given hypothesis some 1-simplex of K can always be replaces by a copy of α in such a way that the fundamental group of the resulting complement is isomorphic to π.

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The construction of L

Let $\{g_i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$ be generators and relations for π Let $S_1 \vee S_2 \vee S_3 \vee \cdots \subset \mathbb{R}^{n-1}$ be an infinite wedge of PL S^1's and let $L_1 \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ be $S_1 \times [1,\infty) \cup S_2 \times [2,\infty) \cup S_3 \times [3,\infty) \cup \cdots$. L_1 is a locally finite polyhedron in $\mathbb{R}^n$. We think of each S_i as representing a generator corresponding to g_i in $\pi_1(L_1)$. Now, for each $j \geq 1$. let $p(j)$ be the smallest integer such that $r_{p(j)}$ involves only the generators $g_1, \ldots, g_j$ and such that $p(j) \neq p(i)$ for $i < j$. It may be that no integer $p(j)$ satisfies these conditions, in which case we set $p(j) = 0$.

For each j with $p(j) \neq 0$, let $\phi_j : S^1 \to L_1 \times \{j\} \subset \mathbb{R}^{n-1} \times \mathbb{R}^1$ be a PL map representing $r_{p(j)}$ in $\pi_1(L_1)$. ϕ_j extends to a PL map $\overline{\phi}_j : D^2 \to \mathbb{R}^{n-1} \times \{j\}$ which is an embedding on $\mathrm{int}(D^2)$. By general position, we may assume that $\overline{\phi}_i(\mathrm{int}\, D^2) \cap \overline{\phi}_j(\mathrm{int}(D^2) = \emptyset$ for $i \neq j$ and that $\overline{\phi}_i(D^2) \cap L_1 = \overline{\phi}^i(\partial D^2)$ for all i. Now, let $L_2 = L_1 \cup \bigcup_{j=1}^{\infty} \overline{\phi}_j(D^2) \times [j,\infty)$. Following [6]. we invoke a classical theorem of Hopf:

Theorem. (Hopf [2, p.1]). *If K is a connected* CW *complex with $\pi_1(K) = \pi$, then there is an exact sequence:*

$$\pi_2(K) \xrightarrow{\rho} H_2(K) \to H_2(\pi) \to 0$$

where ρ is the Hurewicz homomorphism.

L_2 is homotopy equivalent to a 2-dimensional CW complex, so $H_2(L_2)$ is free abelian. Since $H_2(\pi) = 0$ by assumption, we can choose $\{[s_i]\}_{i=1}^{\infty} \in \pi_2(L_2)$ so that $\{\rho([s_i])\}_{i=1}^{\infty}$ is a basis for $H_2(L_2)$. For each $j \geq 1$, let $q(j)$ be the smallest integer such that $\rho([s_i])$ is represented by a PL map $\psi_j : S^2 \to L_2 \cap \mathbb{R}^{n-1} \times \{j\}$ and such that $q(j) \neq q(i)$ for $i < j$. If no such integer exists we set $q(j) = 0$.

If $n \geq 7$, general position allows us to extend each ψ_j to a PL map $\overline{\psi}_j : D^3 \to \mathbb{R}^{n-1} \times (j - \frac{1}{4}, j + \frac{1}{4})$ such that $\overline{\psi}_j$ is an embedding on $\mathrm{int}\, D^3$ and such that $\overline{\psi}_j(D^3) \cap L_2 = \overline{\psi}_j(\partial D^3)$. Let $L = L_2 \cup \bigcup_{j=1}^{\infty} \overline{\psi}_j(D^3)$. L is a 3-dimensional locally finite acyclic polyhedron with fundamental group π which is properly embedded in $\mathbb{R}^n$.

If $n = 6$, we form the complex L abstractly and attempt to extend the embedding already given on L_2 using a techniques due to Stallings [11, 5]. General position yields a map with double points in the interiors of 3-simpleces. We join each pair of double points by an arc in L. The image of each such arc is a PL circle in $\mathbb{R}^{n-1} \times (j - \frac{1}{4}, j + \frac{1}{4})$ for some j. By general position, each such circle bounds a disk in $\mathbb{R}^{n-1} \times (j - \frac{1}{4}, j + \frac{1}{4})$ in the complement of the image of L. By simultaneously squeezing the arc out of the domain and the disk out of the range, we can eliminate a pair of double points. Doing this for all such pairs produces a proper embedding of a 3-dimensional polyhedron infinite simple homotopy equivalent to L into $\mathbb{R}^6$.

The proof that X is cell-like

A compact subset X of a manifold M^n is said to be *cell-like* (or CE) if for each neighborhood U of X in M the inclusion $X \to U$ is nullhomotopic. It is shown in [8] that this is a topological property of X and is independent of the choice of manifold M^n or embedding $X \to M^n$. A map $f : X \to Y$ between compact metric spaces is cell-like if it is surjective and if $f^{-1}(y)$ is CE for each $y \in Y$.

Let $L \subset \mathbb{R}^n$ be the polyhedron constructed above. We may assume that L is simplicially embedded in $\mathbb{R}^n$ and that $\mathbb{R}^n$ is triangulated in such a way that L is a full subcomplex. Let $U \subset \mathbb{R}^n$ be an open simplicial neighborhood of L in $\mathbb{R}^n$ and let $\mathbb{R}^n$ be contained in S^n by one-point compactification. U is open in S^n and we denote $S^n - U$ by X. By Alexander duality, $\check{H}^*(X) = \check{H}^*(pt)$. To say that X is CE is the same as saying that the inclusion $* \to X$ of a point into X is a shape equivalence. To prove this, we invoke the following Whitehead Theorem in shape theory

Cohomological Whitehead Theorem. ([9, p.155]). *Let $F : (X, *) \to (Y, *)$ be a pointed shape morphism and let $(X, *)$ and $(Y, *)$ be connected finite-dimensional metric spaces which are shape 1-connected. If F induces an isomorphism of Čech cohomology groups with integer coefficients then F is a pointed shape equivalence.*

While the terminology used in the theorem above may not be entirely familiar, it suffices for our purposes to know that a map $* \to X$ is a pointed shape morphism and that a compact subset of $\mathbb{R}^n$ which is an intersection of simply connected open subsets of $\mathbb{R}^n$ is shape 1-connected. To prove that X is cell-like, it therefore suffices to show that X has a basis of neighborhoods $V_1 \supset V_2 \supset V_3 \supset \cdots$ with $\pi_1(V_i) = 0$.

Let $Q_i \subset L$ be $L \cap (\mathbb{R}^{n-1} \times [0, i + \frac{1}{2}])$. We may assume that each Q_i is a full subcomplex in the given triangulation of $\mathbb{R}^n$. Let $\lambda_i : \mathbb{R}^n \to [0, 1]$ be the linear function which is 1 on vertices of Q_i and 0 on vertices of $\mathbb{R}^n - Q_i$. Let $V_i = \lambda_i^{-1}[0, 1/(i + 2)]$. Each V_i is an open neighborhood of X, $V_i \supset V_{i+1}$, $\bigcap_{i=1}^{\infty} V_i = X$, and $N_i = \mathbb{R}^n - Q_i$ is a regular neighborhood of Q_i in $\mathbb{R}^n$. It is now clear that $\pi_1(V_i) = 0$, because each N_i had Q_i as a codimension 3 spine - a loop in V_i bounds a disk in S^n which can be moved off of Q_i, and, therefore, out of $\text{int}(N_i)$ rel boundary.

Conclusion

For any given space Y, we will let ΣY denote $Y \times [-1, 1]/\sim$, where $(y, 1) \sim (y', 1)$ and $(y, -1) \sim (y', -1)$ for all $y, y' \in Y$. To see that $\Sigma(S^n/X)$ is homeomorphic to S^{n+1}, we invoke:

Theorem. ([3, 10.2]). *Let $n \geq 4$ and let $f : M^n \to X^{\mathrm{ANR}}$ be a CE map such that the nonmanifold part of X is contained in a topological polyhedron of codimension ≥ 3. Then $X \times \mathbb{R}^1$ is a manifold.*

S^n/X is an ANR by [4],[6], or [8]. The reader interested in general theorems of this sort should consult pp. 259-261 of [9]. The general principle is that whether S^n/X is an ANR depends on the shape of X. Since X has the shape of a point the result follows. That S^n/X is a homology manifold follows from the Vietoris-Begle Theorem.

Cannon's Theorem implies that $S^n/X \times \mathbb{R}^1$ is a manifold. Since $H^4(S^n/X \times \mathbb{R}^1, \mathbf{Z}_2) = 0$, $S^n/X \times \mathbb{R}^1$ has a PL structure. By Browder's $M \times \mathbb{R}$ Theorem or by the boundary theorem of Browder, Levine, and Livesay [1], $S^n/X \times \mathbb{R}^1 \cong S^n \times \mathbb{R}^1$. It follows that $\Sigma(S^n/X) \cong S^{n+1}$. Let $[X] \in S^n/X$ be the singular point and let $\alpha = \Sigma[X] \subset \Sigma(S^n/X)$ be its suspension. Then $S^{n+1} - \alpha \cong U \times \mathbb{R}^1$ and $\pi_1(S^{n+1} - \alpha) = \pi_1(U) = \pi$. This proves our main theorem for $\alpha = I$, the unit interval.

To extend the theorem to the case of an arbitrary finite polyhedron $K^k \subset \mathbb{R}^n$ with $n-k \geq 3$ and $k \geq 1$, we will make use of the Bing shrinking criterion.

Bing Shrinking Criterion. *If $f : X \to Y$ is a surjection of compact metric spaces, then f is a uniform limit of homeomorphisms if and only if for each $\epsilon > 0$ there is an $h : X \to X$ such that $d(h \circ f, f) \leq \epsilon$ and such that for all $y \in Y$ $\operatorname{diam}(h(f^{-1}(y)) < \epsilon$. The homeomorphisms h are called shrinking homeomorphisms for f.*

Let $q \in S^n/X - [X]$ and let $\beta : [0,1] \to S^n/X$ be an embedding so that $\beta(0) = [X]$, $\beta(1) = q$, and so that $\beta|(0,1]$ is PL. This last makes sense because $S^n/X - [X] \cong U$, which is an open subset of S^n. Let $B = \beta(I) \subset S^n/X$. If $\epsilon > 0$ is given, let P be a regular neighborhood of $\beta(\epsilon, 1]$ in $S^n/X - [X]$ and let N be a regular neighborhood of the point $\beta(\epsilon)$ in $\operatorname{int}(P)$. By regular neighborhood theory, given any neighborhood V of P in U there is a PL isotopy h_t, with compact support in V such that $h_0 = \operatorname{id}$ and $h_1(P) = N$. Such h's give shrinking homeomorphisms for the quotient map $S^n/X \to S^n/X/B$ and show that $S^n/X \cong (S^n/X)/B$. (See Fig. 1)

If $i : K^k \to S^{n+1}$ is a PL embedding, we may assume that $i(K) \subset U \times \mathbb{R}^1$. To simplify notation, we will write $K \subset U \times \mathbb{R}^1$. Let $\sigma \in K$ be a 1-simplex. Since $q \times [-\frac{1}{2}, \frac{1}{2}] \subset \Sigma(S^n/X)$ is a standard PL arc in $U \times \mathbb{R}^1$, there is a PL homeomorphism $T : U \to U$ with $T(\sigma) = q \times [-\frac{1}{2}, \frac{1}{2}]$. Thus, we may assume that $q \times [-\frac{1}{2}, \frac{1}{2}] \subset K$ is a 1-simplex of K. By general position, we may assume that $K \cap B \times (-1,1) = q \times [-\frac{1}{2}, \frac{1}{2}]$. (See Fig. 2)

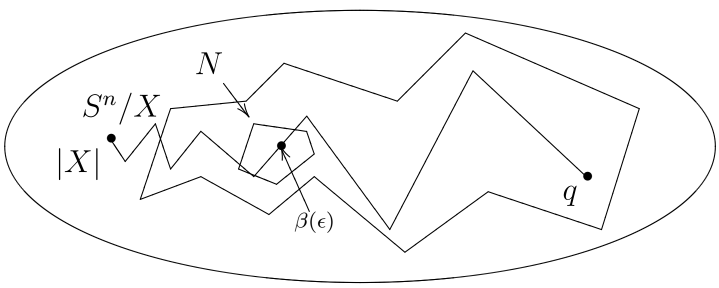

Fig. 1

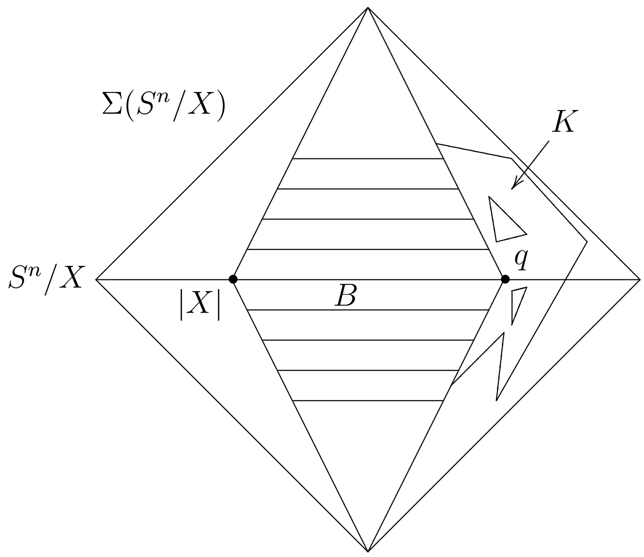

Fig. 2

Consider the partition $\mathcal{G}$ of $\Sigma(S^n/X)$ whose nondegenerate elements are $B \times \{t\}$ for $t \in (-\frac{1}{2}, \frac{1}{2})$, $B \times \{\frac{1}{2}\} \cup [X] \times [\frac{1}{2}, 1]$, and $B \times \{-\frac{1}{2}\} \cup [X] \times [-1, -\frac{1}{2}]$. To complete the proof of our main theorem it suffices to show that the projection $c : \Sigma(S^n/X) \to \Sigma(S^n/X)/\sim$ is shrinkable, where $x \sim x'$ if $x, x' \in G \in \mathcal{G}$, since $c(K) \subset \Sigma(S^n/X) \sim$ is a copy of K in S^{n+1} such that $S^{n+1} - K$ is homeomorphic to $\Sigma(S^n/X) - (\alpha \cup B \times [-\frac{1}{2}, \frac{1}{2}] \cup K) \cong U \times \mathbb{R}^1 - ((B - [X]) \times [-\frac{1}{2}, \frac{1}{2}] \cup K)$. Since $(B - [X]) \times [-\frac{1}{2}, \frac{1}{2}] \cup K$ is a polyhedron with codimension ≥ 3 in $U \times \mathbb{R}^1$, its absence does not affect π_1, so $\pi_1(S^{n+1} - K) \cong \pi_1(U) \cong \pi$, as desired.

A shrinking homeomorphism for c is constructed as the composition of homeomorphisms, H and K. H is a homeomorphism of the form $H(x,t) = (h_{\rho(t)}(x), t)$ where $\rho(t) = 1$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$ and where ρ decreases rapidly to 0 outside of this interval. K is a homeomorphism which shrinks $[X] \times [\frac{1}{2}, 1]$ and $[X] \times [-1, -\frac{1}{2}]$ to small neighborhoods of the suspension points. K is easily constructed using the product structure on $\Sigma(S^n/X)$. One constructs K so that there is a small neighborhood $\mathcal{O}$ of $[X]$ in S^n/X such that $\mathcal{O} \times [\frac{1}{2}, 1]$ and $\mathcal{O} \times [-1, -\frac{1}{2}]$ are shrunk by $\mathrm{id} \times k$, where $k : [-1, 1] \to [-1, 1]$ is a suitable PL homeomorphism. Then one constructs H so that the arcs $B \times \{t\}$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, are shrunk into $\mathcal{O} \times [-\frac{1}{2}, \frac{1}{2}]$. Since H is level preserving, the images of the $B \times \{t\}$'s are not stretched by such a K and the composition shrinks c.

Remark. As pointed out by the referee, our theorem remains true when the polyhedron K is replaces by any compact metric space Z which contains an arc. If Z is embedded in S^n codimension three, work of Štanko [12, 13] guarantees that the embedding can be approximated by a 1-LCC embedding and that the 1-LCC embedding has the general position properties used in the proof.

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