ALGEBRAIC K-THEORY WITH CONTINUOUS CONTROL AT INFINITY

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Abstract. Let $(\overline{E}, \Sigma)$ be a pair of spaces consisting of a compact Hausdorff space $\overline{E}$ and a closed subspace Σ. Let $\mathfrak{A}$ be an additive category. This paper introduces the category $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ of geometric modules over E with coefficients in $\mathfrak{A}$ and with continuous control at infinity. One of the main results is to show that the functor that sends a CW comples X to the algebraic K-theory of $\mathcal{B}(cX, X; \mathfrak{A})$ is a homology theory. Here cX is the closed cone on X and X is its base.

The categories $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ are generalizations of the categories $\mathcal{C}(Z; \mathfrak{A})$ of geometric modules and bounded morphisms introduced by Pedersen and Weibel [8]. Here (Z, ρ) is a complete metric space. If X is a finite CW complex and $\mathcal{O}(X)$ is the metric space open cone on X considered in [9], then there is an inclusion of categories $\mathcal{C}(\mathcal{O}(X); \mathfrak{A}) \to \mathcal{B}(cX, X; \mathfrak{A})$. A second main result is that this inclusion induces an isomorphism on K-theory.

One advantage of the present approach is that $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ depends only on the topology of $(\overline{E}, \Sigma)$ and not on any metric properties. This should make application of these ideas to problems involving stratified spaces, for example, more direct and natural.

0. Introduction

Let Z be a metric space with complete metric ρ, $\mathfrak{A}$ be an additive category and $\mathcal{C}(Z; \mathfrak{A})$ be the category of geometric modules on Z with coefficients in $\mathfrak{A}$ and bounded morphisms introduced and studied in [8] and [9]. We recall that an object in $\mathcal{C}(Z, \mathfrak{A})$ is a collection $A = \{A_x \in \mathfrak{A} | x \in Z\}$ with the property that for every bounded set $K \subset Z$, $\{x \in K | A_x \neq 0\}$ is finite. A morphism $f : A \to B$ is a collection $f = \{f^x_y | f^x_y : A_x \to B_y$ is a morphism in $\mathfrak{A}\}$ for which there is a number d such that $f^x_y = 0$ if $\rho(x, y) > d$.

The categories $\mathcal{C}(Z; \mathfrak{A})$ have been of interest and use both in algebra and topology. In [8], the categories $\mathcal{C}(\mathbb{R}^{n+1}; \mathfrak{A})$ are used to construct a nonconnective spectrum $\mathbf{K}(\mathfrak{A})$ delooping the usual K-theory spectrym for $\mathfrak{A}$. In [9], the categories $\mathcal{C}(\mathcal{O}(X); \mathfrak{A})$, where $\mathcal{O}(X)$ is the open cone on X, are used to construct a homology theory on the category of finite CW complexes. Finally in [4], an equivariant version of these categories is used in the study of group actions.

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In this paper we describe a new approach to these ideas that depends only on the topology of the space Z and not on its metric properties. The advantage of this approach is that it can be applied more naturally and more directly to some topological problems than the "bounded" approach. This is especially true of problems that arise in stratified settings. An application that exploits the greater flexibility this gives will be given in [1].

Let $\mathcal{E}$ be the category whose objects are pairs $(\overline{E};\Sigma)$ where $\overline{E}$ is a compact Hausdorff space and Σ is a closed subspace. Let $E = \overline{E} - \Sigma$. (Althought we do not require it, we often think of E as being dense in $\overline{E}$. We also think of Σ as the "space at infinity" of E.) A morphism in $\mathcal{E}$ is a (not necessarily continuous) function $f : \overline{E}_1 \to \overline{E}_2$ having $F^{-1}(\Sigma_2) = \Sigma_1$, for which $f|E_1 : E_1 \to E_2$ is *proper* in the sense that if $K \subset E_2$ is compact, then $f^{-1}(K)$ has compact closure in E_1, and for which f is continuous at every point $z \in \Sigma_1$.

Let $(\overline{E}, \Sigma) \in \mathcal{E}$ and $\mathfrak{A}$ be an additive category. We define a new additive category $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ as follows: An object of $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ is a collection $A = \{A_x \in \mathfrak{A} | x \in E\}$ of objects of $\mathfrak{A}$ with the property that for each compact set $K \subset E$, $\{x \in K | A_x \neq 0\}$ is finite. A morphism $f : A \to A'$ is a collection $f = \{f^x_y | f^x_y : A_x \to A'_y$ is a morphism in $\mathfrak{A}$ and $x, y \in E\}$ that has the properties that for all x, $\{y | f^x_y \neq 0\}$ is finite and that for every point $z \in \Sigma$ and every neighborhood U of z in $\overline{E}$, there is a neighborhood V of z such that $f^x_y = 0$ whenever $x \in V$ and $y \notin U$. (In particular, if $y \in E$ and $U = \overline{E} - \{y\}$, the compactness of Σ implies that $\{x | f^x_y \neq 0\}$ is finite.) We describe this condition by saying that f is *continuously controlled at infinity.* The additive structure of $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ is given by setting $A \oplus A' = \{A_x \oplus A'_x\}$.

We may regard $\mathcal{B}(-, -; \mathfrak{A})$ as a functor from $\mathcal{E}$ to additive categories as follows: Let $h : (\overline{E}_1, \Sigma_1) \to (\overline{E}_2 \to \Sigma_2)$ be a morphism in $\mathcal{E}$. We define a functor $h_* : \mathcal{B}(\overline{E}_1, \Sigma_1; \mathfrak{A}) \to \mathcal{B}(\overline{E}_2, \Sigma_2; \mathfrak{A})$ as follows: If $A = \{A_x | x \in E_1\}$ is an object of $\mathcal{B}(\overline{E}_1, \Sigma_1; \mathfrak{A})$, let $h_*(A)$ be the object $\{B_z\}$ of drawboxed() $\mathcal{B}(\overline{E}_2, \Sigma_2; \mathfrak{A})$ with $B_z = \Sigma A_x$ where the sum runs over $x \in h^{-1}(z)$. Since the set $h^{-1}(z)$ has compact closure in E_1, there are only finitely many x in $h^{-1}(z)$ with $A_x \neq 0$. Hence the sum is finite. Similarly for any compact set $K \subset E_2$, $\{z \in K | B_z \neq 0\}$ is finite. Hence $\{B_z\}$ is an object of $\mathcal{B}(\overline{E}_2, \Sigma_2; \mathfrak{A})$. Let $f : A \to A'$ be the morphism $f = \{f^x_y | x.y \in E_1\}$. Then $h_*(f) = \{g^z_w\} : h_*(A) \to h_*(A")$ is the morphism with $g^z_w = \sum f^x_y$ where the sum runs over $(x, y) \in h^{-1}(z) \times h^{-1}(w)$. It is easily checked that $h_*(f)$ is a morphism in $\mathcal{B}(\overline{E}_2, \Sigma_2; \mathfrak{A})$ and that h_* is an additive functor.

For any additive category $\mathfrak{A}$, let $\mathbb{K}\mathfrak{A}$ denote the classifying space $B\mathcal{A}^{-1}\mathcal{A}$ where $\mathcal{A} = \operatorname{Iso}\mathfrak{A}$ is the category of isomorphisms in $\mathfrak{A}$. The algebraic K-theory of $\mathfrak{A}$ is given by $K_m(\mathfrak{A}) = \pi_m(\mathbb{K}\mathfrak{A})$ for $m \geq 0$. The algebraic K-theory of the additive category $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ is denoted by $K_*\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ and is called the *algebraic K-theory of* $\overline{E}$ *with continuous control at infinity.* If $h : (\overline{E}_1, \Sigma_1 \to (\overline{E}_2, \Sigma_2)$ is a morphism in $\mathcal{E}$, h induces a homomorphism

$$h_* : K_*\mathcal{B}(\overline{E}_1, \Sigma_1; \mathfrak{A}) \to K_*(\overline{E}_2, \Sigma_2; \mathfrak{A})$$

and we may may regard $K_*\mathcal{B}(-;\mathfrak{A})$ as a functor from $\mathcal{E}$ to the category of graded abelian groups.

The closed-cone construction can be used to obtain many examples of pairs $(\overline{E},\Sigma)$ in $\mathcal{E}$. In particular if X is a non-empty compact Hausdorff space, we let cX be the closed cone on X (i. e. the quotient space obtained from $X \times [0,1]$ by identifying $X \times \{0\}$ to a point c) and (cX, X) be the pair in which X is the image of $X \times \{1\}$ in cX. In $X = \emptyset$, then we let $(cX, X) = (c, \emptyset)$. In either case, (cX, X) is in $\mathcal{E}$. In fact, the closed cone construction defines a functor $c : \mathcal{F} \to \mathcal{E}$ where $\mathcal{F}$ is the category of compact Hausdorff spaces. The image of $X \times [0,1)$ in cX is denoted oX and is called the *small open cone* on X.

If $\mathfrak{A}$ is an additive category, we let $\hat{\mathfrak{A}}$ be the *idempotent completion* [3] of $\mathfrak{A}$; that is, $\hat{\mathfrak{A}}$ is the category with objects (A, p) where $A \in \mathfrak{A}$ and $p : A \to A$ has $p^2 = p$ anf with morphisms $f : (A, p) \to (B, q)$ those morphisms $f : A \to B$ in $\mathfrak{A}$ for which $f = qfp$.

Let X be a compact metrizable space and S^nX be its nth suspension. We will prove that the collection of spaces $\widetilde{\mathbf{K}}(X;\mathfrak{A}) = \{\mathbb{K}\hat{\mathcal{B}}(cS^nX, S^nX;\mathfrak{A}) | n = 0, 1, 2, \ldots\}$ is a spectrum. This spectrum is nonconnecteive and its stable homotopy is denoted by $\widetilde{K}_*(X;\mathfrak{A})$. It is studied in Section 4.

Theorem I. *For any additive category $\mathfrak{A}$, the functor $X \mapsto \widetilde{K}_*(X;\mathfrak{A})$ is a reduced homology theory on the category of compact metrizable spaces.*

The reader should see Theorem 5.2 for a more complete statement of this theorem.

The main steps in the proof of Theorem I are showing that if $X = vY$ is the closed cone on Y, the K-theory classifying space of X is contractible and establishing a Mayer-Vietoris property. Thes steps are done in Corollary 2.3 and Theorem 4.4 respectivley. Here we have written $X = vY$ instead of $X = cY$ since we must form cX to get the homology of X and it is convenient to dictinguish the two cone points c and v.

For any pair of compact metrizable spaces (X, A), let $K_*(X, A;\mathfrak{A}) = \widetilde{K}_*(X \cup_A vA;\mathfrak{A})$ where vA is the cone on A with vertex v and we take vA to be a point if $A = \emptyset$. To simplify notation, throughout this paper we shall write $X \cup_v A)$ instead of the more precise $X \cup_A vA$.

Theorem II. *The functor $K_*(-;\mathfrak{A})$ is a homology theory on the category of pairs of compact metrizable spaces. If we restrict this homology theory to the category of pairs of finite* CW *complexes, its representing spectrum $\mathbf{K}(pt;\mathfrak{A}) = \widetilde{\mathbf{K}}(S^0;\mathfrak{A})$ has $\Omega\mathbf{K}(pt;\mathfrak{A}) = \mathbf{K}(\mathfrak{A})$.*

The reader should see Theorem 5.3 for a more complete statement of this theorem. In this theorem $\mathbf{K}(\mathfrak{A})$ is the nonconnective spectrum for the K-theory of $\mathfrak{A}$ constructed in [8]. It follows from this theorem and the main result of [9] that $K_*(X,\mathfrak{A}) = H_{*-1}(X, \mathbf{K}(\mathfrak{A}))$ for every compact polyhedron X where the right-hand side is the homology of X with coefficients in the spectrum $\mathbf{K}(\mathfrak{A})$.

In proving the last part of Theorem II, the idea is to show that $\widetilde{\mathbf{K}}(S^0;\mathfrak{A})$ and the spectrum $\widetilde{\mathbf{K}}'(pt;\mathfrak{A})$ of [9] have isomorphic homotopy groups. In positive dimensions, this is accomplished in Lemma 5.5 and in nonpositive dimensions, in Lemma 5.6. The proof of

Lemma 5.6 is based on the results in section 2 and the construction of a "Bass-Heller-Swan" homomorphism motivated by the discussion in [7, Section 2].

Let X be a finite CW complex and $\mathcal{O}(X)$ be the *large open cone* on X (i. e. the space obtained from $X \times [0, \infty)$ by identifying $X \times \{0\}$ to a point c). If we identify $\mathcal{O}(X)$ with oX under the homeomorphism that sends $[\xi, t]$ to $[\xi, 1 - t/(1 + t)]$, there is an inclusion of categories $i : \mathcal{C}(\mathcal{O}(X); \mathfrak{A}) \to \mathcal{B}(cX, X; \mathfrak{A})$ obtained by noticing that any morphism in $\mathcal{C}(\mathcal{O}(X); \mathfrak{A})$ is continuously controlled at infinity under the above identification. Let

$$i_* : K_*^b(X; \mathfrak{A}) \to K_*(X; \mathfrak{A})$$

be the induced homomorphism on homology where $K_*^b(X; \mathfrak{A})$ is the homology of X using bounded morphisms as described in [9]. The next result follows easily from Theorem II.

Corollary III. *For any finite* CW *complex X, $i_* : K_*^b(X; \mathfrak{A}) \to K_*(X; \mathfrak{A})$ is an isomorphism.*

In a recent paper [12], Vogell describes another approach to proving Theorems I and II for finite CW complexes. Although his approach in that case is simpler than the one taken here, it does not work in the full generality of this paper.

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1. Some elementary properties of the categories $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$

We collect here some elementary properties of the categories $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ that will be useful later in this paper and illustrate the topological flavor of this category.

Let $(\overline{E}, \Sigma) \in \mathcal{E}$, $\mathfrak{A}$ be an additive category and $A = \{A_x | x \in E\} \in \mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. The set $\text{supp}_\infty(A) = cl\{x \in E | A_x \neq 0\} \cap \Sigma$ is called the *support of A at infinity*. Notice that $z \in \text{supp}_\infty(A)$ if and only if every neighborhood U of z contains a point $x \in E$ with $A_x \neq 0$.

Lemma 1.1. *Let $A, B \in \mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. If A is isomorphic to B, the* $\text{supp}_\infty(A) = \text{supp}_\infty(B)$.

Proof. Let $f = \{f_y^x : A_x \to B_y\}$ be an isomorphism from A to B. Let $z \in \text{supp}_\infty(A)$ and U be a neighborhood of z in $\overline{E}$. Then there is a smaller neighborhood V of z so that if $x \in V$ and $f_y^x \neq 0$, then $y \in U$. Since $z \in \text{supp}_\infty(A)$, there is an $x \in V$ with $A_x \neq 0$. Since f is an isomorphism, there is a y with $f_y^x \neq 0$. Hence $B_y \neq 0$ for some y in U, $z \in \text{supp}_\infty(B)$, and $\text{supp}_\infty(A) \subset \text{supp}_\infty(B)$. Lemma 1.1 follows easily. □

Let $(\overline{E}, \Sigma) \times I = (\overline{E} \times I, \Sigma \times I)$. Then lemma 1.1 shows that there are objects in $\mathcal{B}(\overline{E}, \Sigma) \times I, \mathfrak{A})$ that are not isomorphic to objects in the subcategory $\mathcal{B}((\overline{E}, \Sigma) \times \{0\}, \mathfrak{A})$. Hence the inclusion

$$i : \mathcal{B}((\overline{E}, \Sigma) \times \{0\}, \mathfrak{A}) \to \mathcal{B}((\overline{E}, \Sigma) \times I, \mathfrak{A})$$

is not an equivalence of categories.

Lemma 1.2. *Let $(\overline{E}, \Sigma) \in \mathcal{E}$, $K \subset \overline{E}$ be closedand $W \subset \Sigma$ be disjoint from K. Let $A, B \in \mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ and $f : A \to B$. Then there exists a neighborhood U of W in $\overline{E} - K$ so that if either (x, y) or (y, x) is in $U \times K$, then $f^x_y = 0$.*

Proof. Suppose first that W is a single point w. Then there are disjoint open sets U' and V' with $w \in U'$ and $K \subset V'$. For each $z \in K \cap \Sigma$, there is a neighborhood V_z so that if $f^x_y \neq 0$ and $x \in V_z$, then $y \in V'$. Let $V = \bigcup\{V_z | z \in K \cap \Sigma\}$. For all $x \in V$, if $f^x_y \neq 0$, then $y \in V'$. Since $C = K \cap (\overline{E} - V)$ is compact and contained in E, the sets $\{x \in C | A_x \neq 0\}$ and $\{y \in E | f^x_y \neq 0, x \in C\}$ are finite. Let $U = U' - \{y \in E | f^x_y \neq 0, x \in K\}$. Then U is an open neighborhood of w and for all $(x, y) \in K \times U$, $f^x_y = 0$. Since f is continuous at w, we may replace U by a smaller neughborhood of w, if necessary to insure that if $(x, y) \in U \times K$, then $f^x_y = o$. The lemma follows in case W is a single point.

In the general case choose a neighborhood U_w for each $w \in W$ as in the preceeding paragraph. Then $U = \bigcup\{U_w | w \in W\}$ works. □

Let $f : (\overline{E}_1, \Sigma_1) \to (\overline{E}_2, \Sigma_2)$ be a map in $\mathcal{E}$ for which $f|_{\Sigma_1} : \Sigma_1 \to \Sigma_2$ is a homeomorphism. We say that$f$ is an *equivalence at infinity* if for every $p \in \Sigma_1$ and every neighborhood U of p in $\overline{E}_1$, there exists a neighborhood V of $f(p)$ in $\overline{E}_2$ with $f^{-1}(V) \subset U$. This term is explained in the following lemma:

Lemma 1.3. *Let $f : (\overline{E}, \Sigma) \to (\overline{E}, \Sigma$ be a map in $\mathcal{E}$ for which $f|_\Sigma = 1$. Then there is a natural transformation $\eta : I \to f_*$ where I is the identity functor of $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. If f is an equivalence at infinity, then η is a natural equivalence. Hence f induces the identity on K-theory.*

Proof. Let $A = \{A_x\}$ be an object of $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ and set $B = f_*(A)$. Let $\eta(A) = \{\eta^x_y\} : A \to B$ be the morphism for which η^x_y is the inclusion of the A_x summand into B_y if $y = f(x)$ and is 0 otherwise. Since f is a m map in $\mathcal{E}$, for every $p \in \Sigma$ and every neighborhood U of p there is a neighborhood V so that $f(V) \subset U$. hence $\eta(A)$ is a morphism in $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. It is easily checked that $\eta = \{\eta(A) : I \to f_*$ is a natural transformation.

Suppose that f is an equivalence at infinity. To see that η is a natural equivalence, it suffices to show that for each A, $\eta(A)$ is an isomorphism. Let $\rho(A) : \{\rho^x_y\} : B \to A$ be the morphism for which ρ^x_y is the projection of B_x on the A_y summand if $y \in f^{-1}(x)$ and is 0 otherwise. Since the set $\{y | \rho^x_y \neq 0\} = \{x | \eta^y_x \neq 0\}$, it is finite. Let $p \in \Sigma$ and U be a neighborhood of p. By hypothesis there is a neighborhood V of p with $f^{-1}(V) \subset U$. hence $\rho(A)$ is a morphism in $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. It is easily checked that $\rho(A)$ is an inverse of $\eta(A)$.

The following general observations show that f induces the identity on K-theory: Let $F, G : \mathfrak{A} \to \mathfrak{B}$ be additive functors between additive categories and $\eta : F \to G$ be a natural equivalence. Then F, G induce functors $\overline{F}, \overline{G} : \mathfrak{A}^{-1}\mathfrak{A} \to \mathfrak{B}^{-1}\mathfrak{B}$ and η induces a natural transformation $\overline{\eta} : \overline{F} \to \overline{G}$. Hence F and G induce the same homomorphism on K-theory. □

Corollary 1.4. *Let $h_1, h_2 : (\overline{E}, \Sigma) \to (\overline{E}, \Sigma)$ be homeomorphisms with $h_1|_\Sigma = h_2|_\Sigma$. Then there is a natural equivalence $\eta : h_{1*} \to h_{2*}$. Hence h_1 and h_2 induce the same map on K-theory.*

Proof. Since $f = (h_1^{-1})h_2$ satisfies the hypothesis of Lemma 1.3, there is a natural equivalence $\eta' : I \to f_*$ where I is the identity functor. Then $\eta = h_{1*}(\eta')$ is the desired natural equivalence. □

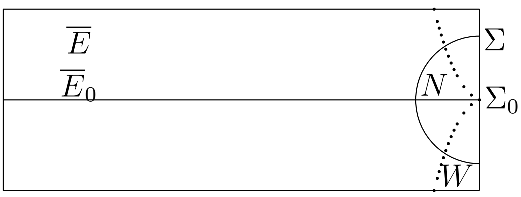
A morphism $i : (\overline{E}_1, \Sigma_1) \to (\overline{E}_2, \Sigma_2)$ is an inclusion in $\mathcal{E}$ if $i : \overline{E}_1 \to \overline{E}_2$ is an inclusion and $\Sigma_1 = \overline{E}_1 \cap \Sigma_2$.

Corollary 1.5. *Let $(\overline{E}_1 \Sigma_1)$ and $(\overline{E}_2, \Sigma_2)$ be objects in $\mathcal{E}$ with $\Sigma_1 = \Sigma_2$. Let $i : (\overline{E}_1, \Sigma_1) \to (\overline{E}_2, \Sigma_2)$ be an inclusion and $r : (\overline{E}_2, \Sigma_2) \to (\overline{E}_1, \Sigma_1)$ be a retraction in $\mathcal{E}$. Suppose that r is an equivalence at infinity. Then $i_* : \mathcal{B}(\overline{E}_1, \Sigma_1; \mathfrak{A}) \to \mathcal{B}(\overline{E}_2, \Sigma_2; \mathfrak{A})$ is an equivalence of categories.*

Proof. Since r is a retraction, $ri = 1$ and $r_* i_*$ is the identity functor. Since i is clearly an equivalence at infinity and r is an equivalence at infinity by hypothesis, ir is an equivalence at infinity. Hence $i_* r_*$ is naturally equivalent to the idnetity by Lemma 1.3. Corollary 1.5 follows □

We now introduce a technical concept that is used in the proof of theorem 3.1. Let $(\overline{E}_0, \Sigma_0) \to (\overline{E}, \Sigma) \in \mathcal{E}$ be an inclusion in $\mathcal{E}$. We say that $(\overline{E}_0, \Sigma_0)$ is an *eventual neighborhood retract* in $(\overline{E}, \Sigma)$ if there is a closed neighborhood N of Σ_0 in $\overline{E}$ and a function $r : (N - \Sigma) \cup \overline{E}_0 \to \overline{E}_0$ with $r^{-1}(\Sigma_0) = \Sigma_0$ and $r|_{\overline{E}_0} = \mathrm{id}$ with the following property: For every open neighborhood W of $\Sigma - \Sigma_0$ in $\overline{E} - \overline{E}_0$, $r_W = r| : [(N - \Sigma - W) \cup \overline{E}_0] \to (\overline{E}_0, \Sigma_0)$ is an equivalence at infinity. This amounts to requiring that for every triple (p, U, W) consisting of a point $p \in \Sigma_0$, a neighborhood U of p in $\overline{E}$, and an open neighborhood W of $\Sigma - \Sigma_0$ in $\overline{E}$, there is a neighborhood V of p in $\overline{E}_0$ with $r^{-1}(V) \subset U \cup W$. The reader will note that only the restrictions r_W of r, and not r itself need be morphisms in $\mathcal{E}$ and that r need to be continuous only at infinity. The function r is called an *eventual retraction.*

The diagram below illustrates the definition of an eventual neighborhood retraction r. In it, r is defined on $(N - \Sigma) \cup \overline{E}_0$. It is "well behaved" on $(N - \Sigma - W) \cup \overline{E}_0$ for every W.



Proposition 1.6. *Suppose that $\Sigma_0 \subset \Sigma$ is a neighborhood retract. Then $(c\Sigma_0, \Sigma_0)$ is an eventual neighborhood retract in $(c\Sigma, \Sigma)$.*

Proof. By replacing Σ with an appropriate closed neighborhood of Σ_0, we may assume there is a continuous retraction $\rho : \Sigma \to \Sigma_0$. Let $N = c\Sigma$ and $r : N \to c\Sigma_0$ be the cone on ρ. Let W be an open neighborhood of $\Sigma - \Sigma_0$ not meeting $c\Sigma_0$. Then $r_W = r|_{(c\Sigma - W)} : (c\Sigma - W, \Sigma_0) \to (c\Sigma_0, \Sigma_0)$ is a morphism in $\mathcal{E}$ and $ri = 1$ where $i : (c\Sigma_0, \Sigma_0) \to (c\Sigma - W, \Sigma_0)$ is the inclusion.

Let U be any open neighborhood of $p \in \Sigma_0$ in $c\Sigma$. Then there is an open neighborhood V of p in $c\Sigma_0$ with $(r_W)^{-1}(V) \subset U$. If not, then for each neighborhood V of p, there is a point $x_V \in c\Sigma - W - U$ with $r(x_V) \in V$. Then the net $\{r(x_V)|V \in \mathcal{V}\}$ converges to p where $\mathcal{V}$ is the set of neighborhood of p. Let $\mathcal{B}$ be the set of neighborhoods of p that do not contain c. Then $\mathcal{B}$ is a base of the neighborhoods of p. hence $\{r(x_V) = (c\rho)(x_V)|V \in \mathcal{B}\}$ also converges to p. On the other hand, $\{x_V|V \in \mathcal{B}\}$ converges to some point q of $c\Sigma$. Since $x_V \in c\Sigma - W - U$ which is compact, $q \in \overline{E} - W - U \subset c\Sigma - \Sigma$. In particular q and $(c\rho)(q)$ are not in Σ. Hoevere, since $c\rho$ is continuous, $(c\rho)(q) = p \in \Sigma_0$. This is a contradiction. □

If X is a nonempty space, let SX be the unreduced suspension of X (i. e. space obtained from $X \times [-1, 1]$ by identidying $X \times \{i\}$ to $\{i\}$ for $i = \pm 1$). If $X = \emptyset$, let $SX = S\emptyset = S^0$ be the discrete two-point space.

Proposition 1.7. *If $\Sigma_0 \subset \Sigma$ is a neighborhood retract, then so is $S\Sigma_0 \subset \Sigma$.*

Proof. Let $r : N \to \Sigma_0$ be a continuous retraction of a closed neighborhoos N of Σ_0 in Σ onto Σ_0. Let $v\Sigma$ be the cone on Σ with vertex v. We show first that there is a closed neighborhood N_1 of $v\Sigma_0$ and a continuous retraction $r_1 : N_1 \to v\Sigma_0$ with $N \subset N_1$, $N_1 \cap \Sigma_0 = N$, and $r_1|_N = r$. To see this, let $h : I \times I \to I \times I$ be a function with the following properties:

(i) For all $s \in I$, $h(s, 0) = (s, 0)$ and $h(s, 1) = (s, 1)$.
(ii) If $\frac{3}{4} \leq s \leq 1$ and $\frac{1}{2} \leq t \leq 1$, then $h(s, t) = (s, 1)$.

Such a function can be constructed using standard elementary techniques. Let $\phi : \Sigma \to I$ be a continuous map for which $\phi(\Sigma_0) = 0$ and $\phi(\Sigma - \operatorname{Int} N) = 1$. Define $H : \Sigma \times I \to \Sigma \times I$ by setting $H(x, t) = (x, h(\phi(x), t))$. By (1), $H(x, 0) = (x, 0)$ and $H(x, I) = (x, 1)$ for all x. Hence H induces a continuous map $H' : v\Sigma \to v\Sigma$ for which $H'[x, 0] = [x, 0]$ for all x and $H'(v) = v$. Let $N_1 = q(N \times I \cup \Sigma \times [\frac{1}{2}, 1])$. Then N_1 is a closed neighborhood of $v\Sigma_0$ in $v\Sigma$. Since $H(x, t) = (x, 1)$ for all $(x, t) \in \phi^{-1}[\frac{3}{4}, 1] \times]\frac{1}{2}, 1]$ by (2), for such (x, t), $H'[x, t] = v$ and $H'(N_1) \subset vN$. We now set $r_1 = (vr)H' : N_1 \to v\Sigma_0$ and observe that r_1 has the properties claimed.

Let $S\Sigma = v\Sigma \cup w\Sigma$ where the union is along Σ. By gluing two copies of the map r_1 together along the common subspace N, we obtain a neighborhood N_2 of $S\Sigma_0$ in $S\Sigma$ and continuous retraction $r_2 : N_2 \to S\Sigma_0$. This completes the proof of Proposition 1.7 □

2. The structure of the category $\mathcal{B}(cX, X; \mathfrak{A})$

Let (X, d) be a compact metric space X together with a metric for which $\operatorname{diam} X \leq 2$. In this section we shal determine the structure of $\mathcal{B}(cX, X; \mathfrak{A})$. We also use this result to show that $K_*\mathcal{B}(cX, X; \mathfrak{A}) = 0$ if $X = vY$ where Y is a compact metrizable space.

Let $\mathcal{O}(X)$ be the *large open cone* on X (i. e. the space obtained from $X \times [0, \infty)$ by identifying $X \times \{0\}$ to a point c). It was observed in [11] (cf. also [2] that setting

$$\rho(x, y) = \min\{s, t\} d(\xi, \eta) + |s - t| \tag{1}$$

for $x = [\xi, t]$ and $y = [\eta, s]$ in polar coordinates, gives a metric on $\mathcal{O}(X)$. We identify $\mathcal{O}(X)$ with $o(X)$ under the homeomorphism that sends $[\xi, t]$ to $[\xi, 1 - t/(1 + t)]$ and write cX as $\mathcal{O}(X) \cup X(\infty)$ where $X(\infty)$ is a copy of X "at infinity".

For any numbers $r < s \leq \infty$, let $< r, s >= \{[\xi, t] \in \mathcal{O}(X) | r \leq t < s\}$. If $\xi \in X$ and $\beta > 0$, let $\operatorname{ang}(\xi, \beta) = \{[\zeta, s] | 0 < s \text{ and } d(\zeta, \xi) < \beta\}$. By an abuse of language if $x = [\xi, t] \neq c$ in $\mathcal{O}(X) \cup X(\infty)$ and $\beta > 0$, we shall write $\operatorname{ang}(x, \beta)$ for $\operatorname{ang}(\xi, \beta)$. Finally for $f \in \mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ and $x \in \mathcal{O}(X)$, let $S_x(f) = \{y \in \mathcal{O}(X) | f^x_y \neq 0\}$.

Let $\underline{r} = (r_1, r_2, r_3, \ldots)$ be an increasing sequence of positive integers. Let $\mathcal{B}_{\underline{r}}$ be the subcategory of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ containing all the objects of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ but only those morphisms $f : A \to B$ for which there are constants α, β and γ so that

For almost all k, if $x \in < r_k, r_{k+1} >$, then

$$S_x(f) \subset < r_{k-\alpha} r_{k+1+\alpha} > \cap \operatorname{ang}(x, \frac{\beta}{k - \gamma}). \tag{2}$$

Of course for (2) to make sense, we must have $k > \gamma$ and $k \geq \alpha$. A simple calculation shows that if f' and f'' are in $\mathcal{B}_{\underline{r}}$, then so is $f'' f'$ and thus that $\mathcal{B}_{\underline{r}}$ really is a subcategory of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$.

The structure of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ is given by the following theorem:

Theorem 2.1. *Let X be a compact metric space with* $\operatorname{diam} X \leq 2$ *and $\mathfrak{A}$ be any additive category. Then the collection $\{\mathcal{B}_{\underline{r}}$ forms a direct system of subcategoies of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ ordered by inclusion. that is*

(i) $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A}) = \bigcup \mathcal{B}_{\underline{r}}$

(ii) *if $\underline{r}'$ and $\underline{r}''$ are two such sequences, then there is a sequence $\underline{r}$ with $\mathcal{B}_{\underline{r}'} \subset \mathcal{B}_{\underline{r}}$ and $\mathcal{B}_{\underline{r}''} \subset \mathcal{B}_{\underline{r}}$.*

Furthermore, for every increasing sequence $\underline{r}$, $\mathcal{B}_{\underline{r}}$ is isomorphic to $\mathcal{C}(\mathcal{O}(X); \mathfrak{A})$.

Theorem 2.1 is a direct consequence of Lemma 2.4-2.8.

Corollary 2.2. *Let X be a compact metric space with* $\operatorname{diam} X \leq 2$ *and $\mathfrak{A}$ be an additive category. Then $K_* \mathcal{B}(cX, X; \mathfrak{A}) = \underrightarrow{\operatorname{colim}} K_* \mathcal{B}_{\underline{r}}$.*

The colimit here is over the family $\{\mathcal{B}_{\underline{r}}\}$ of subcategories of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ and we use the identification of this category with $\mathcal{B}(cX, X; \mathfrak{A})$ given above to simplify notation. The corollary is an immediate consequence of Theorem 2.1.

We recall that for any additive category $\mathfrak{A}$, its *idempotent completion* [3] is the category $\hat{\mathfrak{A}}$ with objects (A, p) where $A \in \mathfrak{A}$ and $p : A \to A$ has $p^2 = p$ and with morphisms $f : (A, p) \to (B, q)$ those morphisms $f : A \to B$ in $\mathfrak{A}$ for which $f = qfp$.

Corollary 2.3. *Suppose $X = vY$ is the closed cone on the compact metrizable space Y. Then the spaces $\mathbb{K}\mathcal{B}(cX, X; \mathfrak{A})$ and $\mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A})$ are contractible.*

Proof. Any compact, metrizable space has a metric d of diameter ≤ 2. For this metric $\mathcal{B}(cX, X; \mathfrak{A}) = \bigcup \mathcal{B}_{\underline{r}}$ and hence $\hat{\mathcal{B}}(cX, X; \mathfrak{A}) = \bigcup \hat{\mathcal{B}}_{\underline{r}}$. By Theorem 2.1, $\mathcal{B}_{\underline{r}}$ is isomorphic to $\hat{\mathcal{C}}(\mathcal{O}(X); \mathfrak{A})$ which is flasque by [9]. hence $K_*\hat{\mathcal{B}}_{\underline{r}} = K_*\hat{\mathcal{C}}(\mathcal{O}(X); \mathfrak{A}) = 0$ and $K_*\hat{\mathcal{B}}(cX, X; \mathfrak{A}) = \underrightarrow{\text{colim}} K_*\hat{\mathcal{B}}_{\underline{r}} = 0$. It follows that $\mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A})$ is contractible. The proof that $\mathbb{K}\mathcal{B}(cX, X; \mathfrak{A})$ is contractibel is similar. □

Lemma 2.4. *Let $f \in \mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ and suppose that $r > 0$ and $\beta > 0$ are given. then there is an $R > 0$ so that for all $x \in < R, \infty >$, $S_x(f) \subset < r, \infty > \cap \operatorname{ang}(x, \beta)$.*

Proof. Suppose no such R exists. Then there exist sequences $x_n = [\xi_n, t_n]$ and $y_n = [\zeta_n, s_n]$ $(n = 1, 2, 3, \ldots)$ so that for all n, $x_n \in < n, \infty >$, $y_n \in S_{x_n}(f)$ and either $s_n \leq r$ or $d(\zeta_n, \xi_n) \geq \beta$. By choosing a subsequence if necessary, we may assume $x_n \to x \in X(\infty)$. Since f is continuously controlled at infinity, $y_n \to x$. Hence $s_n \to \infty$ and $d(\zeta_n, \xi_n) \to 0$. This contradicts the choices of x_n and y_n. □

Lemma 2.5. *For any $f \in \mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty); \mathfrak{A})$ there is a sequence $\underline{r}$ with $f \in \mathcal{B}_{\underline{r}}$.*

Proof. The proof is inspired by an argument of Hughes given in [5, Section 5]. We construct the sequence $\underline{r}$ by induction. Use Lemma 2.4 to choose r_1, so that if $x \in < r_1, \infty >$, then $S_x(f) \subset \operatorname{ang}(x, 1)$. Suppose $r_1, \ldots, r_n$ constructed so that

(i) if $x \in < r_k, \infty >$, then $S_x(f) \subset < r_{k-1}, \infty > \cap \operatorname{ang}(x, 1/k)$ for $k \leq n$; and
(ii) if $x \in < 0, r_k >$, then $S_x(f) \subset < 0, r_{k+1} >$ for $k < n$.

Now use Lemma 2.4 to choose r_{n+1} so that if $x \in < r_n, \infty >$, then (i) holds for $k = n+1$. Since there are only finitely many non-zero f_y^x with $x \in < 0, r_n >$, we may also choose r_{n+1} large enough that (ii) also holds. this completes the construction of $\underline{r}$.

If $x \in < r_n, r_{n+1} >$, then $S_x(f) \subset < r_{n-1}, r_{n+2} > \cap \operatorname{ang}(x, 1/n)$. By letting $\alpha = 1, \beta = 1$ and $\gamma = 0$ in (2), this shows that $f \in \mathcal{B}_{\underline{r}}$. □

Lemma 2.6. *Let $\underline{r}_0 = (1, 2, 3, \ldots)$. Then $\mathcal{B}_{\underline{r}_0} = \mathcal{C}(\mathcal{O}(X); \mathfrak{A})$.*

Proof. Let $f \in \mathcal{B}_{\underline{r}_0}$, $x = [\xi, t]$, $y = [\eta, s]$, and suppose that $f_y^x \neq 0$. Then if $x \in < k, k+1 >$,

$$\begin{aligned}\rho(x, y) &= \min\{s, t\} d(\xi, \eta) + |s - t| \\ &\leq \min\{s, t\} \frac{\beta}{k - \gamma} + \alpha + 1 \leq (k + \alpha + 1) \frac{\beta}{k - \gamma} + \alpha + 1.\end{aligned}$$

Since $(k + 1 + \alpha)(k - \gamma) \to 1$ as $k \to \infty$, $\rho(x, y) < 2\beta + \alpha + 1$ for all $x = [\xi, t]$ with t sufficiently large. Since there are only finitely many f_y omitted by this process, there is a number B so that $\rho(x, y) < B$ for all x, y with $f_y \neq 0$. Thus $\mathcal{B}_{\underline{r}_0} \subset \mathcal{C}(\mathcal{O}(X); \mathfrak{A})$.

Let $x \in < n, n+1 >$ and suppose $d > 0$ is given. Simple estimates using (1) show that $B(x, d) \subset < n - d, n + d + 1 > \cap \operatorname{ang}(x, d/(n - d))$. Hence if $f \in \mathcal{C}(\mathcal{O}(X); \mathfrak{A})$ has the property that $\rho(x,y) < d$ when $f_y \neq 0$, then $f \in \mathcal{B}_{\underline{r}_0}$ with constants $\alpha = \beta = \gamma = d$ in (2). Thus $\mathcal{C}(\mathcal{O}(X);\mathfrak{A}) \subset \mathcal{B}_{\underline{r}_0}$. This completes the proof of Lemma 2.6. ☐

Lemma 2.7. *For every sequence $\underline{r}$, $\mathcal{B}_{\underline{r}}$ is a subcategory of $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty);\mathfrak{A})$ and there is an isomorphism of categories $h_{\underline{r}^*} : \mathcal{B}_{\underline{r}_0} \to \mathcal{B}_{\underline{r}}$.*

Proof. Let $\rho_{\underline{r}} : [0,\infty) \to [0,\infty)$ be the homeomorphism that maps $[k, k+1]$ linearly onto $[r_k, r_{k+1}$ and set $h_{\underline{r}}[\xi, t] = [\xi, \rho_{\underline{r}}(t)]$. Then $h_{\underline{r}}$ induces an automorphism $h_{\underline{r}^*}$ of the category $\mathcal{B}(\mathcal{O}(X) \cup X(\infty), X(\infty);\mathfrak{A})$ onto itself whose inverse is $(h_{\underline{r}}^{-1})_*$. Straighfoprward calculation shows that $h_{\underline{r}^*}(\mathcal{B}_{\underline{r}_0}) \subset \mathcal{B}_{\underline{r}}$ and that $(h_{\underline{r}}^{-1})_*(\mathcal{B}_{\underline{r}}) \subset \mathcal{B}_{\underline{r}_0}$ from which Lemmma 2.7 follows. ☐

Lemma 2.8. *Let $\underline{r}' = (r'_1, r'_2, r'_3, \ldots)$ and $\underline{r}'' = (r''_1, r''_2, r''_3, \ldots)$ be increasing sequences of integers. Then there is a sequence $\underline{r}$ so that $\mathcal{B}_{\underline{r}'} \subset \mathcal{B}_{\underline{r}} \supset \mathcal{B}_{\underline{r}''}$.*

We say that a squence $\underline{s}$ *splices* the sequences $\underline{t}$ if $t_n < s_n < t_{n+1}$ for all n. Notice that if $\underline{s}$ splices $\underline{t}$, the $\mathcal{B}_{\underline{t}} = \mathcal{B}_{\underline{s}}$.

Proof. Clearly one can choose subsequences $\underline{s}'$ and $\underline{s}''$ of $\underline{r}'$ and $\underline{r}''$ respectively so that $\underline{s}'$ splices $\underline{s}''$. Then $\mathcal{B}_{\underline{s}'} = \mathcal{B}_{\underline{s}''}$. Since $\underline{s}''$ is a subsequence of $\underline{r}'$, we see that $\mathcal{B}_{\underline{r}'} \subset \mathcal{B}_{\underline{s}'}$. Similarly $\mathcal{B}_{\underline{r}''} \subset \mathcal{B}_{\underline{s}''}$. The proof is completed by letting $\underline{r} = \underline{s}''$. ☐

3. The Mayer-Voetories property

This section establishes a Mayer-Vietories property. Although the result given here is technical, the consequences of it obtained in the next section are far less so and are sufficient for proving the main theorems of this paper. The approach taken here is a modification of that in [2]. The difference is that the categories in [2] were filtered; whereas the categories that arise here are not. The reader who is interested in only the main results of this paper can sjip this section and refer back to it only as needed.

We recall some definitions and notation.

If

fb is a full, additive subcategory of the additive category $\mathfrak{A}$, then the *idempotent semicompletion* (or simply, *semicompletion*) of $\mathfrak{A}$ *with respect to* $\mathfrak{B}$ is the full, additive subcategory of $hat\mathfrak{A}$ containing those objects (A,p) isomorphic to $(B,q) \oplus (C,1)$ with $(B,q) \in \hat{\mathfrak{B}}$ and $C \in \mathfrak{A}$. The semicompletion is denoted by $\widetilde{\mathfrak{A}}$.

Theorem 3.1. *Let $(\overline{E}, \Sigma)$ be the pushout of the diagram $(\overline{E}_1, \Sigma_1) \overset{i_1}{\longleftarrow} (\overline{E}_0, \Sigma_0) \overset{i_2}{\longrightarrow} (\overline{E}_2, \Sigma_2)$ of inclusions in $\mathcal{E}$ in which $\overline{E}_0 - \Sigma_0 \neq \emptyset$. If $(\overline{E}_0, \Sigma_0)$ is an eventual neighborhood retract in $(\overline{E}, \Sigma)$, then the square of spaces*

$$\begin{array}{ccc} \mathbb{K}\hat{\mathcal{B}}(\overline{E}_0, \Sigma_0;\mathfrak{A}) & \xrightarrow{i_1} & \mathbb{K}\widetilde{\mathcal{B}}(\overline{E}_1, \Sigma_1;\mathfrak{A}) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathbb{K}\widetilde{\mathcal{B}}(\overline{E}_2, \Sigma_2;\mathfrak{A}) & \xrightarrow{j_2} & \mathbb{K}\widetilde{\mathcal{B}}(\overline{E}, \Sigma;\mathfrak{A}) \end{array}$$

is a pullback up to homotopy. Here the semicompletions are with respect to $\mathcal{B}(\overline{E}_0, \Sigma_0; \mathfrak{A})$. *Hence there is a long exact sequence*

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathcal{B}) \to \pi(\mathbb{K}\mathcal{B}_0) \to \pi_n(\mathbb{K}\mathcal{B}_1) \oplus \pi_n(\mathbb{K}\mathcal{B}_2) \to \pi_n(\mathbb{K}\mathcal{B}) \to \cdots$$

$(n \geq 0)$ *where* $\mathbb{K}\mathcal{B}_i = \mathbb{K}\widetilde{\mathcal{B}}(\overline{E}_i, \Sigma_i; \mathfrak{A})(i = 1, 2, \emptyset)$ *and* $\mathbb{K}\mathcal{B}_0 = \mathbb{K}\hat{\mathcal{B}}(\overline{E}_0, \Sigma_0; \mathfrak{A})$.

The reader will recall that the definition of Σ_0 being an eventual retract in $\overline{E}$ is given in section 1. Therem 3.1 is an immediate consequence of Lemmas 3.2 and 3.3.

Suppose given a commutative diagram (*) of additive categories and additive functors

$$(\star) \qquad \begin{array}{ccc} \mathfrak{A}_0 & \xrightarrow{i_1} & \mathfrak{A}_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathfrak{A}_2 & \xrightarrow{j_2} & \mathfrak{A} \end{array}$$

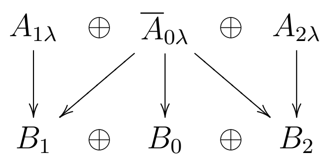
and a directed set Λ, We say that $(\star)$ has the *Mayer-Vietories property* relative to Λ if the following conditions hold:

(1) The functors i_1, i_2, j_1 and j_2 are full and faithful.

(2) For every pair of objects $A, B \in \mathfrak{A}$ there is a family of subgroups $\{F_\lambda \operatorname{Hom}(A, B) | \lambda \in \Lambda\}$ such that if $\mu < \lambda$, then $F_\mu \operatorname{Hom}(A, B) \subset F_\lambda \operatorname{Hom}(A, B)$ and with $\operatorname{Hom}(A, B) = \bigcup F_\lambda \operatorname{Hom}(A, B)$.

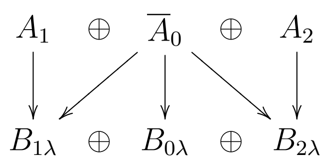
(3) For every $A \in \mathfrak{A}$, there is a $\lambda(A) \in \Lambda$ so that for every $\lambda \geq \lambda(A)$ there is a *preferred decomposition* of A as $A = A_{1\lambda} \oplus \overline{A}_{0\lambda} \oplus A_{2\lambda}$ with $A_{i\lambda} \in \mathfrak{A}_i (i = 1, 2)$ and an isomorphism $\sigma_{0\lambda} : \overline{A}_{0,\lambda} \to A_{0\lambda}$ to an object $A_{0\lambda} \in \mathfrak{A}_)$. It is further required that 1_A and $1_{1\lambda} \oplus \sigma_{0\lambda} \oplus 1_{2\lambda}$ be in $F_\lambda \operatorname{Hom}(A, A')$ and that $1_{1\lambda} \oplus (\sigma_{0\lambda})^{-1} \oplus 1_{2\lambda} \in F_\lambda \operatorname{Hom}(A', A)$ for all $\lambda \geq \lambda(A)$ where $A' = A_{1\lambda} \oplus A_{0\lambda} \oplus A_{2\lambda}$.

(4) If $\lambda \geq \lambda(A)$ and $f \in F_\lambda \operatorname{Hom}(A, B)$, then f decomposes as



relative to the preferred decomposition of A and any decomposition $B = B_1 \oplus B_0 \oplus B_2$ with $B_i \in \mathfrak{A}_i \quad (i = 1, 2)$.

(5) If $\lambda \geq \lambda(B)$ and $f \in F_\lambda \operatorname{Hom}(A, B)$, then f decomposes as



re;ative to any decomposition $A = A_1 \oplus A_0 \oplus A_2$ with $A_i \in \mathfrak{A} \quad (i = 0, 1, 2)$ and the preferred decomposition of B.

Lemma 3.2. *If (*) has the Mayer-Vietories property with respect to Λ and $\mathfrak{A}_0$ is idempotent complete then*

$$\begin{array}{ccc} \mathbb{K}\mathfrak{A}_0 & \xrightarrow{i_1} & \mathbb{K}\mathfrak{A}_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \mathbb{K}\mathfrak{A}_2 & \xrightarrow{j_2} & \mathbb{K}\mathfrak{A} \end{array}$$

is a pullback up to homotopy. Hence for $n \geq 0$, there is a long exact sequence

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathfrak{A}) \to \pi(\mathbb{K}\mathfrak{A}_0) \to \pi_n(\mathbb{K}\mathfrak{A}_1) \oplus \pi_n(\mathbb{K}\mathfrak{A}_2) \to \pi_n(\mathbb{K}\mathfrak{A}) \to \cdots$$

Proof. The proof is similar to the proofs of the corresponding rsults in [8],[9] and [2]. Let $\mathfrak{B}$ be Thomasson's simplified double mapping cylinder ogf the diagram $\mathfrak{A}_1 \overset{\mathfrak{A}}{\longleftarrow}_0 \overset{\mathfrak{A}}{\longrightarrow}_2$ and $\Sigma : \mathfrak{B} \to \mathfrak{A}$ be the obvious functor. We show Σ is a homotopy equivalence on K-theory classifying spaces using Quillen's Theorem A by showing that for each object $A \in \mathfrak{A}$, the category $A \downarrow \Sigma$ is contractible.

For any $\lambda \in \Lambda$, let $F_\lambda(A \downarrow \Sigma)$ be the full subcategory of $A \downarrow \Sigma)$ with objects those morphisms $\alpha : A \to B_1 \oplus B_0 \oplus B_2 = B$ for which $\alpha \in F_\lambda \operatorname{Hom}(A, B)$ and $\alpha^{-1} \in F_\lambda \operatorname{Hom}(B, A)$. Let σ be the composite

$$\sigma = 1 \oplus \sigma_{0\lambda} \oplus 1 : A = A_{1\lambda} \oplus \overline{A}_{0\lambda} \oplus A_{2\lambda} \to A_{1\lambda} \oplus A_{0\lambda} \oplus A_{2\lambda}.$$

The argument given in [8],[9] or [2] shows that if $\lambda \geq \max\{\lambda(A), \lambda(B)\}$, then σ is an initial object in $F_\lambda(A \downarrow \Sigma)$. Hence this category is contractible. Since Λ is a directed set, it follows that $A \downarrow \Sigma$ is the directed union of contractible subcategories. Hence $A \downarrow \Sigma$ is contractible and Lemma 3.2 follows. □

Let $(\overline{E}, \Sigma) \in \mathcal{E}$. Let A be an object of $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$. The *support* of A is the set $\operatorname{supp} A = \{x \in E | A_x \neq 0\}$. If $X \subset \overline{E}$, we write $A|_X$ for the object $\{B_y | y \in E\}$ where $B_y = A_x$ if $y = x \in X$ and B_y is zero if $y \notin X$. If $f = \{f^x_y : A_x \to B_y | x, y \in E\} : A \to B$ is a morphism in $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$, we let $f|_X : A|_X \to B|_X$ be the morphism $\{f^x_y | x, y \in X\}$.

Lemma 3.3. *Let $(\overline{E}, \Sigma)$ be the pushout of a diagram $(\overline{E}_1, \Sigma_1) \overset{i_1}{\longleftarrow} (\overline{E}_0, \Sigma_0) \overset{i_2}{\longrightarrow} (\overline{E}_2, \Sigma_2)$ of inclusions in $\mathcal{E}$ in which $\overline{E}_0 - \Sigma_0 \neq \emptyset$. Let $\Lambda = \{U | U \subset \overline{E} - \overline{E}_0$ is a neighborhoos of $\Sigma - \Sigma_0$ with $U \cap \Sigma_0 \neq \emptyset$ and set $U \geq V$ if $U \subset V$. If $(\overline{E}_0, \Sigma_0)$ is an eventual neighborhood retract in $(\overline{E}, \Sigma)$, then the following diagram of additive categories and functors has the*

Mayer-Vietories property relative to Λ

$$(\star\star)\qquad \begin{array}{ccc} \hat{\mathcal{B}}(\overline{E}_0,\Sigma_0;\mathfrak{A}) & \xrightarrow{i_1} & \widetilde{\mathcal{B}}(\overline{E}_1,\Sigma_1;\mathfrak{A}) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \widetilde{\mathcal{B}}(\overline{E}_2,\Sigma_2;\mathfrak{A}) & \xrightarrow{j_2} & \widetilde{\mathcal{B}}(\overline{E},\Sigma;\mathfrak{A}) \end{array}$$

where the semicompletions are taken with respect to $\hat{\mathcal{B}}(\overline{E}_0,\Sigma_0;\mathfrak{A})$

Proof. The semicompletion $\hat{\mathcal{B}}(\overline{E}_k,\Sigma_k;\mathfrak{A})(k=1,2,\emptyset$ can be described more concretely as the full subcategory of $\hat{\mathcal{B}}(\overline{E},\Sigma;\mathfrak{A})$ with objects those (A,p) for which there is a neighborhood U_k of $\Sigma_k-\Sigma_0$ in $\overline{E}_k-\overline{E}_0$ so that

(i) $\operatorname{supp} A\subset\overline{E}_k$;
(ii) $p^x_y=0$ for $(x,y)\in U_k\times(E_k-U_k)\cup(E_k-U_k)\times U_k\cup(U_k\times U_k-\Delta)$; and
(iii) $p^x_x=1$ if $x\in U_k$.

Here $p=\{p^x_y : A_x\to A_y|x,y\in E\}$ and Δ is the diagonal in $U_k\times U_k$. Notice that (i)-(iii) implyu that $p|_{U_k}=1$, that $p|_{E_k-U_k}$ is a projection, and that $(A,p)=(A|_{U_k},1)\oplus(A|_{E_k-U_k},p|_{E_k-U_k})$.

We now verify that $(\star\star)$ has the Mayer-Vietories property relative to Λ by verifying conditions (1)-(5) above.

It is clear from the description of $\widetilde{\mathcal{B}}(\overline{E}_k,\Sigma_k;\mathfrak{A})(k=1,2,\emptyset$ that the functors i_1,i_2,j_1 and j_2 are full and faithfull, so (1) holds.

Let $(A,p)\in\hat{\mathcal{B}}(\overline{E}_k,\Sigma_k;\mathfrak{A})$, choose $U\subset\overline{E}-\overline{E}_0$ satisfying (i)-(iii) above for $k=\emptyset$ (i. e. for $\overline{E}$), and set $\lambda(A,p)=U$. Suppose $V=\lambda\geq\lambda(A,p)=U$. Then V is a neighborhood of $\overline{\Sigma}-\overline{\Sigma}_0$ contained in U. We set $V_k=V\cap\overline{E}_k$ and let $A_{k\lambda}=A|_{V_k}\quad(k=1,2)$, $\overline{A}_{0\lambda}=A|\overline{E}-V, p_{k\lambda}=p|_{V_k}(k=1,2)$, and $\overline{p}_{0\lambda}=p|_{\overline{E}-V}$. Then $p_{k\lambda}=1(k=1,2)$ and the preferres decomposition of (A,p) is given by setting

$$(A,p)=(A_{1\lambda},1)\oplus(\overline{A}_{0\lambda},\overline{p}_{0\lambda}\oplus(A_{2\lambda},1).$$

Clearly $(A_{k\lambda},1)\in\widetilde{\mathcal{B}}(\overline{E}_k,\Sigma_k;\mathfrak{A})(k=1,2)$ and $(\overline{A}_{0\lambda},\overline{p}_{0\lambda})\in\hat{\mathcal{B}}(\overline{E}-V,\Sigma_0;\mathfrak{A})$. Since $\overline{E}_0,\Sigma_0)$ is an eventual neighborhood retract in $(\overline{E},\Sigma)$, there is a closed neighborhood N of Σ_0 in $\overline{E}$ and function $f:(N-\Sigma)\cup\overline{E}_0\to\overline{E}_0$ with $r^{-1}(\Sigma_0)$ and $r|_{\Sigma_0}=\mathrm{id}$ which has $r_W=r|;((N-\Sigma-W)\cup\overline{E}_0,\Sigma_0)\to(\overline{E}_0,\Sigma_0)$ a morphism in $\mathcal{E}$ with $r_W|_{\Sigma_0}$ for any neighborhood W of $\Sigma-\Sigma_0$ in $\overline{E}-\overline{E}_0$. Let V be one such neighborhood. We may extend r_V to a morphism $\rho:(\overline{E}-V,\Sigma_0)\to(\overline{E}_0,\Sigma_0)$, also in $\mathcal{E}$, by setting $\rho(y)=x$ for all $y\in E-(N\cup V\cup E_0)$ for some $x\in E_0$. Let $(A_{0\lambda},p_{0\lambda})=\rho_*(\overline{A}_{0\lambda},\rho_*\overline{A}_{0\lambda},\rho_*\overline{p}_{0\lambda})\in\hat{\mathcal{B}}(\overline{E}_0,\Sigma_0;\mathfrak{A})$. Since Corollary 1.5 shows that $\rho_*:\hat{\mathcal{B}}(\overline{E}-V,\Sigma_0;\mathfrak{A})\to\hat{\mathcal{B}}(\overline{E}_0,\Sigma_0;\mathfrak{A})$ is an equivalence of categories with inverse the inclusion $\hat{\mathcal{B}}(\overline{E}_0,\Sigma_0;\mathfrak{A})\to\hat{\mathcal{B}}(\overline{E}-V,\Sigma_0;\mathfrak{A})$, $(A_{0\lambda},p_{0\lambda}$ is isomorphic to $(\overline{A}_{0\lambda}\overline{p}_{0\lambda})$.

Suppose $f : A \to B$ is a morphism in $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ and that $U = \lambda \in \Lambda$. We define a subgroup $F_\lambda \operatorname{Hom}(A, B)$ of $\operatorname{Hom}(A, B)$ by letting $f \in F_\lambda \operatorname{Hom}(A, B)$ if

(i) f decomposes as

$$\begin{array}{ccccc} A_{1\lambda} & \oplus & \overline{A}_{0\lambda} & \oplus & A_{2\lambda} \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ B_1 & \oplus & B_0 & \oplus & B_2 \end{array}$$

relative to the decomposition of A given above and any decomposition $B = B_1 \oplus B_0 \oplus B_2$ with $B_k \in \mathcal{B}(\overline{E}_k \Sigma_k; \mathfrak{A})(k = 0, 1, 2)$; and

(ii) f decomposes as

$$\begin{array}{ccccc} A_1 & \oplus & \overline{A}_0 & \oplus & A_2 \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ B_{1\lambda} & \oplus & B_{0\lambda} & \oplus & B_{2\lambda} \end{array}$$

relative to any decomposition $A = A_1 \oplus A_0 \oplus A_2$ with $A_k \in \mathcal{B}(\overline{E}_k, \Sigma_k; \mathfrak{A})(k = 0, 1, 2)$ and the decomposition given above.

For (A, p), $(B, q) \in \widetilde{(\overline{E}, \Sigma; \mathfrak{A})}$ and $\lambda \in \Lambda$, we now let $F_\lambda \operatorname{Hom}((A, p), (B, q)) = \{f | f = qgp$ for some $g \in F_\lambda \operatorname{Hom}(A, B)\}$. Suppose $f \in \operatorname{Hom}((A, p), (B, q))$ in $\widetilde{\mathcal{B}}(\overline{E}, \Sigma; \mathfrak{A})$. Then $f : A \to B$ in $\mathcal{B}(\overline{E}, \Sigma; \mathfrak{A})$ and $f = qfp$. By Lemma 1.2 for $k = 1, 2$, there is a neighborhood U_k of $\Sigma_k - \Sigma_0$ with $U_k \subset \overline{E} - \overline{E}_{3-k}$ so that if either (x, y) or (y, x) is in $U_k \times \overline{E}_{3-k}$, then $f^x_y = 0$. Let $\lambda = U_1 \cup U_2$. Clearly $f \in F_\lambda \operatorname{Hom}(A, B)$ and since $f = qfp$, $f \in F_\lambda \operatorname{Hom}((A, p), (B, q))$. Thus $(\star\star)$ satisfies (2).

That $(\star\star)$ satisfies (3), (4) and (5) follows immediately from the definition of $F_\lambda \operatorname{Hom}((A, p), (B, q))$ and the choice of the preferred decompositions of (A, p) and (B, q) given above.

This completes the proof of Lemma 3.3 □

4. Some consequences of the Mayer-Vietories property

Some consequences of the Mayer-Vietories property are obtained in this section. They are used in the next section to prove the main results of this paper. For the remainder of this paper, we let $\mathcal{CM}$ be the category of compact metrizable spaces and continuous maps.

Theorem 4.1. *Let X be the pushout of the diagram $X_1 \xleftarrow{i_1} X_0 \xrightarrow{i_2} X_2$ of inclusions of spaces in $\mathcal{CM}$ and suppose that X_0 is a neighborhood retract in $X_i (i = 1, 2)$. Then the square*

$$\begin{array}{ccc} \mathbb{K}\hat{\mathcal{B}}(cX_0, X_0; \mathfrak{A}) & \xrightarrow{i_1} & \mathbb{K}\widetilde{\mathcal{B}}(cX_1, X_1; \mathfrak{A}) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathbb{K}\widetilde{\mathcal{B}}(cX_2, X_2; \mathfrak{A}) & \xrightarrow{j_2} & \mathbb{K}\widetilde{\mathcal{B}}(cX, X; \mathfrak{A}) \end{array}$$

is a pullback up to homotopy. hence there is a long exact sequence

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathcal{B}) \to \pi(\mathbb{K}\mathcal{B}_0) \to \pi_n(\mathbb{K}\mathcal{B}_1) \oplus \pi_n(\mathbb{K}\mathcal{B}_2) \to \pi_n(\mathbb{K}\mathcal{B}) \to \cdots$$

$(n \geq 0)$ *where* $\mathbb{K}\mathcal{B}_i = \mathbb{K}\widetilde{\mathcal{B}}(cX_i, X_i; \mathfrak{A})(i = 1, 2, \emptyset)$ *and* $\mathbb{K}\mathcal{B}_0 = \mathbb{K}\hat{\mathcal{B}}(cX_0, X_0; \mathfrak{A})$.

We recall that if $X_0 = \emptyset$, then $(cX_0, X_0) = (c, \emptyset)$.

Proof. Take $(\overline{E}_i, \Sigma_i) = (cX_i, X_i)(i = 0, 1, 2, \emptyset)$ in Theorem 3.1. Since X_0 is a neighborhood retract in $X_i(i = 1, 2)$, it is also a neighborhood retract in X. Hence (cX_0, X_0) is an eventual neighborhood retract of (cX, X) by Proposition 1.6 and Theorem 4.1 follows directly from Theorem 3.1 □

Corollary 4.2. *For any compact metrizable space X, there are homotopy equivalences*

$$\mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A}) \xleftarrow{f} \Omega\mathbb{K}\widetilde{\mathcal{B}}(cSX, SX; \mathfrak{A}) \xrightarrow{\Omega_i} \Omega\mathbb{K}\hat{\mathcal{B}}(cSX, SX; \mathfrak{A})$$

where SX is the suspension of X and i is induced by the inclusion $\widetilde{\mathcal{B}}(cSX, SX; \mathfrak{A}) \to \hat{\mathcal{B}}(cSX, SX; \mathfrak{A})$.

Proof. The homotopy equivalence f is obtained by applying Theorem 4.1 to the pushout diagram $vX \leftarrow X \to wX$ of spaces in $\mathcal{CM}$ and using the fact that $\mathbb{K}\widetilde{(cuX, uX; \mathfrak{A})}(u = v, w)$ is contractible. Here we think of SX as the join $\{v, w\}X$. Since $\widetilde{\mathcal{B}}(cSX, SX; \mathfrak{A})$ is cofinal in $\hat{\mathcal{B}}(cSX, SX; \mathfrak{A})$, $\mathbb{K}\hat{\mathcal{B}}(cSX, SX; \mathfrak{A})$ is a union of components of $\mathbb{K}\widetilde{\mathcal{B}}(cSX, SX; \mathfrak{A})$ and Ωi is actually a homeomorphism. □

Corollary 4.3. *For any additive category $\mathfrak{A}$, there are homotopy equivalences*

$$\mathbb{K}(\hat{\mathfrak{A}}) \xleftarrow{f} \Omega\mathbb{K}\hat{\mathcal{B}}(D^1, S^0; \mathfrak{A}) \xrightarrow{\Omega i} \Omega\mathbb{K}\hat{\mathcal{B}}(D^1, S^0; \mathfrak{A})$$

Proof. Let $X = \emptyset$. Then cX is a point, $\hat{\mathcal{B}}(cX, X; \mathfrak{A}) = \mathfrak{A}$, SX is the two-point space and $(cSX, X) = (D^1, S^0)$. Thus Corollary 4.3 is a special case of Corollary 4.2. □

Theorem 4.4. *Let X be the pushout of the diagram $X_1 \xleftarrow{i_1} X_0 \xrightarrow{i_2} X_2$ of inclusions of spaces in $\mathcal{CM}$ and suppose that X_0 is a neighborhood retract in $X_i(i = 1, 2)$. Then the square*

$$\begin{array}{ccc} \mathbb{K}\hat{\mathcal{B}}(cX_0, X_0; \mathfrak{A}) & \xrightarrow{i_1} & \mathbb{K}\hat{\mathcal{B}}(cX_1, X_1; \mathfrak{A}) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathbb{K}\hat{\mathcal{B}}(cX_2, X_2; \mathfrak{A}) & \xrightarrow{j_2} & \mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A}) \end{array}$$

is a pullback up to homotopy. hence there is a long exact sequence

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathcal{B}) \to \pi(\mathbb{K}\mathcal{B}_0) \to \pi_n(\mathbb{K}\mathcal{B}_1) \oplus \pi_n(\mathbb{K}\mathcal{B}_2) \to \pi_n(\mathbb{K}\mathcal{B}) \to \cdots$$

$(n \geq 0)$ *where* $\mathbb{K}\mathcal{B}_i = \mathbb{K}\hat{\mathcal{B}}(cX_i, X_i; \mathfrak{A})(i = 0, 1, 2, \emptyset)$.

The proof is given at the end of this section.

Corollary 4.5. *Let X be a compact metrizable space and $j_0 : X \to X \times I$ be the inclusion at level* 0*. Then*

$$\mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A}) \xrightarrow{j_{0*}} \mathbb{K}\hat{\mathcal{B}}(c(X \times I), X \times I; \mathfrak{A}).$$

is a homotopy equivalence. hence the correspondence $X \mapsto \mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A})$ is a homotopy functor.

Proof. Consider the inclusions $\leftarrow X \to X \times I$. Since X is a neigborhoos retract in each of these spaces, we may apply Theorem 4.4 to this diagram. Since the pushout of $vX \leftarrow X \to X \times I$ is homeomorphic to vX, the two spaces at the bottom of the resulting diagram are contractible by Corollary 2.3. Corllary 4.5 now follows from theorem 4.4. The last sentence is a well-known consequence of the first part. □

Corollary 4.6. *Let $;: A \to X$ be an inclusion of spaces in $\mathcal{CM}$. Then there is a fibration up to homotopy*

$$\mathbb{K}\hat{\mathcal{B}}(cA, A; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(c(X \cup vA), X \cup vA; \mathfrak{A})$$

Proof. Suppose first that A is a neighborhood retract in X and consider the inclusions $vA \leftarrow A \to X$. Since A is a neighborhood retract in each of these spaces, we may apply Theorem 4.4 to this diagram. Since $\mathbb{K}\hat{\mathcal{B}}(cvA, vA; \mathfrak{A})$ is contractible by Corollary 2.3, this case of Corollary 4.6 follows from Theorem 4.4.

Now consider the general case. Let (X, A) be a pair of spaces in $\mathcal{CM}$ and consider the space $X \cup vA$. Recall that $vA = A \times [0, 1]/A \times \{1\}$. Let $v_{1/2}A$ be the image of $A \times [1/2, 1]$ in vA, $Y = Cl[X \cup vA - v_{1/2}A)$ and consider the inclusion of A into Y that sends a to $(a, 1/2)$. We denote the image of this inclusion by $A_{1/2}$. Since $A_{1/2}$ is a neighborhood retract in Y, the preceding paragraph shows there is a fibration

$$\mathbb{K}\hat{\mathcal{B}}(cA_{1/2}; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cY, Y; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(c(X \cup vA), X \cup vA); \mathfrak{A}.$$

Since the inclusion $X \to Y$ is a homotopy equivalence, by Corollary 4.5 so is $\mathbb{K}\hat{\mathcal{B}}(cX, X\mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cY, Y; \mathfrak{A})$. Thus we may replace $\mathbb{K}\hat{\mathcal{B}}(cY, Y; \mathfrak{A})$ with $\mathbb{K}\hat{\mathcal{B}}(cX, X\mathfrak{A})$ to obtain the desired fibration up to homotopy. This completes the proof of Corollary 4.6. □

Proof of Theorem 4.4. Consider the pushout diagram $SX_1 \leftarrow SX_0 \to SX_2$ of inclusions of spaces in $\mathcal{CM}$. Since X_0 is a neighborhood retract in $X_i \quad (i = 1, 2)$, SX_0 is a neighborhood retract in $SX_i (i = 1, 2)$ by Proposition 1.7. By taking the loops on the spaces in the diagram obtained by applying Theorem 4.1 to these inclusions, we see that the diagram

$$\begin{array}{ccc} \Omega\mathbb{K}\hat{\mathcal{B}}(cSX_0, SX_0; \mathfrak{A}) & \longrightarrow & \Omega\mathbb{K}\widetilde{\mathcal{B}}(cSX_1, SX_1; \mathfrak{A} \\ \downarrow & & \downarrow \\ \Omega\mathbb{K}\widetilde{\mathcal{B}}(cSX_2, SX_2; \mathfrak{A} & \longrightarrow & \Omega\mathbb{K}\widetilde{\mathcal{B}}(cSX, SX; \mathfrak{A}) \end{array} \tag{3}$$

is a pullback up to homotopy. On the other hand, if $j_1 : \mathbb{K}\hat{\mathcal{B}}(cSX_i, SX_i; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cSX_i, SX_i; \mathfrak{A})(i = 0, 1, 2, \emptyset)$ is induced by the inclusion, then Ωj_i is a homeomorphism since $\mathbb{K}\widetilde{\mathcal{B}}(cSX_i, SX_i; \mathfrak{A})$ is a union of components of $\mathbb{K}\hat{\mathcal{B}}(cSX_i, SX_i; \mathfrak{A})$. Hence (3) is homeomorphic to the diagram

$$\begin{array}{ccc} \Omega\mathbb{K}\hat{\mathcal{B}}(cSX_0, SX_0; \mathfrak{A}) & \longrightarrow & \Omega\mathbb{K}\hat{\mathcal{B}}(cSX_1, SX_1; \mathfrak{A} \\ \downarrow & & \downarrow \\ \Omega\mathbb{K}\hat{\mathcal{B}}(cSX_2, SX_2; \mathfrak{A} & \longrightarrow & \Omega\mathbb{K}\hat{\mathcal{B}}(cSX, SX; \mathfrak{A}) \end{array} \tag{4}$$

Let $g_i : \Omega\mathbb{K}\hat{\mathcal{B}}(cSX_i, SX_i; \mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cX_i, X_i; \mathfrak{A}) \quad (i = 0, 1, 2, \emptyset)$ be the inverse of the homotopy equivalence obtained by applying Corollary 4.2 to the space X_i. Consider the commutative cube obatined by using the maps g_i to map (4) into the diagram

$$\begin{array}{ccc} \Omega\mathbb{K}\hat{\mathcal{B}}(cX_0, X_0; \mathfrak{A}) & \longrightarrow & \Omega\mathbb{K}\hat{\mathcal{B}}(cX_1, X_1; \mathfrak{A} \\ \downarrow & & \downarrow \\ \Omega\mathbb{K}\hat{\mathcal{B}}(cX_2, X_2; \mathfrak{A} & \longrightarrow & \Omega\mathbb{K}\hat{\mathcal{B}}(cX, X; \mathfrak{A}) \end{array} \tag{5}$$

We regard (4) and (5) as being the front and back faces of this cube, respectively. A straighforward argument using Corollary 4.2 shows that each of the other faces commutes up to a pointed homotopy. Since (4) is a pullback up to homotopy, the same is true for (5). This completes the proof of Theorem 4.4 □

5. The proofs of the main theorems

Let $\mathfrak{A}$ be a fixed additive category. To simplify notation, we suppress mention of $\mathfrak{A}$ when it is clear from the context. Let $\Omega - \mathcal{SPEC}$ be the category in which an object is a Ω-spectrum $\{A_n; \varepsilon\} \to \{B_n; \eta_n\}$ is a sequence of maps $\{f_n : A_n \to B_n | n \geq 0\}$ for which $\eta_n f_n$ is pointed homotopic to $(\Omega f_{n+1})\varepsilon_n$.

Let $X \in \mathcal{CM}$ and define the Ω-spectrum $\widetilde{\mathbf{K}}(X; \mathfrak{A}) = \{\widetilde{\mathbf{K}}(X; \mathfrak{A})_n, \varepsilon_n\}$ as follows: Let

$$\widetilde{\mathbf{K}}(X; \mathfrak{A})_n = \mathbb{K}\hat{\mathcal{B}}(cS^nX, S^nX; \mathfrak{A}) \qquad (n \geq 0) \tag{6}$$

where S^nX is the nth suspension of X and $S^0X = X$. then Corollary 4.2 shows that there are homotopy equivalences

$$\mathbb{K}\hat{\mathcal{B}}(cY, Y; \mathfrak{A}) \xleftarrow{f} \Omega\mathbb{K}\widetilde{\mathcal{B}}(cSY, SY; \mathfrak{A}) \xrightarrow{\Omega_j} \Omega\mathbb{K}\hat{\mathcal{B}}(cSY, SY; \mathfrak{A})$$

form which we obtain a homotopy equivalence

$$\varepsilon_n = (\Omega j)f^1 : \mathbb{K}\hat{\mathcal{B}}(cS^nX, S^nX; \mathfrak{A}) \to \Omega\mathbb{K}\hat{\mathcal{B}}(cS^{n+1}X, S^{n+1}X; \mathfrak{A}) \tag{7}$$

by taking $Y = S^nX$ and choosing a homotopy inverse f^{-1} for f. Thus $\{\widetilde{\mathbf{K}}(X;\mathfrak{A})_n, \varepsilon_n\}$ is an Ω-spectrum and it is readily checked that the correspondance $X \mapsto \widetilde{\mathbf{K}}(X,\mathfrak{A})$ denfines a functor

$$\widetilde{\mathbf{K}}(-;\mathfrak{A}) : \mathcal{CM} \to \Omega - \mathcal{SPEC}.$$

Proposition 5.1. *The functor* $\widetilde{\mathbf{K}}(-;\mathfrak{A})$ *has the following properties: it is a homotopy functor;* $\tilde{\mathbf{K}}(pt;\mathfrak{A})$ *is contractible; and there is a natural isomorphism of* Ω*-spectra* $\Sigma\widetilde{\mathbf{K}}(X;\mathfrak{A}) \to \widetilde{\mathbf{K}}(SX;\mathfrak{A})$.

Proof. To see $\widetilde{\mathbf{K}}(-;\mathfrak{A})$ is a homotopy functor, let $j_0 : X \to X \times I$ be the inclusion at level 0 and recall that

$$\mathbb{K}\hat{\mathcal{B}}(cX, X;\mathfrak{A}) \xrightarrow{j_{0*}} \mathbb{K}\hat{\mathcal{B}}(c(X \times I), X \times I;\mathfrak{A})$$

is a homotopy equivalence by Corollary 4.5. A standard argument (cf. [2, p.575]) now easily adapts to prove that j_0 induces a homotopy equivalence of spectra and that $\widetilde{\mathbf{K}}(-;\mathfrak{A})$ is a homotopy functor. The second statement follows from the definition of $\widetilde{\mathbf{K}}(pt;\mathfrak{A})$ and Corollary 2.3. The last statement is immediate from the definition of $\widetilde{\mathbf{K}}(X)$. □

Let $\widetilde{K}_*(-;\mathfrak{A}) = \pi_*^S\widetilde{\mathbf{K}}(-;\mathfrak{A}) : \mathcal{CM} \to \mathcal{GA}$ where π_*^S is the stable homotopy functor aand $\mathcal{GA}$ is the category of graded abelian groups. Theorem I is a consequence of the following more complete result:

Theorem 5.2. *The functor* $\widetilde{K}_*(-;\mathfrak{A}) = \pi_*^S\widetilde{\mathbf{K}}(-;\mathfrak{A}) : \mathcal{CM} \to \mathcal{GA}$ *is a reduced homology theory on* $\mathcal{CM}$*; that is,*

(i) $\widetilde{K}_*(-;\mathfrak{A})$ *is a homotopy functor.*
(ii) *There is a natural isomorphism* $\Sigma : \widetilde{K}_*(X;\mathfrak{A}) \to \widetilde{K}_{*+1}(SX;\mathfrak{A})$.
(iii) *For any pair* (X, A) *of spaces* $\mathcal{CM}$*, there is an exact sequence*

$$\cdots \to \widetilde{K}_*(A;\mathfrak{A}) \to \widetilde{K}_*(X;\mathfrak{A}) \to \widetilde{K}_*(X \cup vA;\mathfrak{A}) \to \widetilde{K}_{*-1}(A;\mathfrak{A}) \to \cdots .$$

(iv) *widetildeK*$_*(pt;\mathfrak{A}) = 0$.

Proof. Part (i) follows directly from Proposition 5.1;(ii) follows from the definition of $\widetilde{\mathbf{K}}(X;\mathfrak{A})$; while (iv) follows from proposition 5.1. Finally for every $n \geq 0$, there is a fibration up to homotopy

$$\mathbb{K}\hat{\mathcal{B}}(cS^nA, S^n;\mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cS^nX, S^nX;\mathfrak{A}) \to \mathbb{K}\hat{\mathcal{B}}(cS^n(X \cup vA), S^n(X \cup vA);\mathfrak{A})$$

by Corollary 4.6. The associated homotopy exact sequence may now be spliced together using (ii) to obtain the exact sequence of (iii). □

Fro any pair (X, A) of spaces in $\mathcal{CM}$, let $K_*(X, A;\mathfrak{A}) = \widetilde{K}_*(X \cup vA)$ where the union is over A and vA is a point if $A = \emptyset$. In particular, $X \cup v\emptyset = X^+$ is X with a disjoint basepoint attaches and $K_*(X, \emptyset;\mathfrak{A} = \widetilde{K}_*(X^+)$. The following result is Theorem II:

Theorem 5.3. *The functor $K_*(X, A; \mathfrak{A})$ is a homology theory on the category of pairs of spaces in $\mathcal{CM}$; that is,*

(i) *$K_*(-, -; \mathfrak{A})$ is a homotopy functor.*
(ii) *For any pair (X, A) of spaces in $\mathcal{CM}$, there is an exact sequence*

$$\cdots to K_*(A^+; \mathfrak{A}) \to K_*(X^+; \mathfrak{A}) \to K_*(X, A; \mathfrak{A}; \mathfrak{A}) \to K_{*-1}(A^+; \mathfrak{A}) \to \cdots .$$

(iii) *If U is an open set with $Cl(U) \subset \int(A)$, then the inclusion map $(X - U, A - U) \to (X, A)$ induces an isomorphism $K_*(X - U, A - U) \to K_*(X, A)$.*

Furthermore, if we restrict this homology theory to the category of finite CW *complexes, the representing spectrum $\mathbf{K}(pt; \mathfrak{A}) = \widetilde{K}(S^0; \mathfrak{A})$ has $\Omega(\mathbf{K}(pt; \mathfrak{A}) = \mathbf{K}(\mathfrak{A})$.*

In this theorem $\mathbf{K}(\mathfrak{A})$ is the non-connective spectrum that [8] associates with the additive category $\mathfrak{A}$. The reader will recall that $\mathbf{K}(\mathfrak{A}) = \mathbf{K}(\widetilde{\mathfrak{A}})$.

Proof. That K_* satisfies the homotopy and exactness condition axioms follows directly from the corresponding statements for $\widetilde{K}_*$. That K_* satisfies the excision axiom follows from the arguments given in [6, p. 19]. □

It remains to identify the representing spectrum $\mathbf{K}(pt; \mathfrak{A})$ for the restriction of K_* to the category of finite CW complexes. Let $\widetilde{K}'(S^0; \mathfrak{A}) = \{\widetilde{K}'(S^0; \mathfrak{A})_n, \varepsilon'_n\}$ be the spectrum with

$$\widetilde{\mathbf{K}}'(S^0; \mathfrak{A})_n = \mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^{n+1}; \mathfrak{A}) \xrightarrow{\varepsilon'_n} \Omega\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^{n+2}; \mathfrak{A}) = \Omega\widetilde{\mathbf{K}}'(S^0; \mathfrak{A})_{n+1}$$

constructed from the homotopy equivalences of [8]

$$\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^{n+1}; \mathfrak{A}) \xleftarrow{f'} \Omega\mathbb{K}\widetilde{\mathcal{C}}(\mathbb{R}^{n+2}; \mathfrak{A}) \xrightarrow{\Omega'_j} \Omega\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^{n+2}; \mathfrak{A})$$

by choosing a homotopy inverse for f'. Recall that these are obtained by a Mayer-Vietories argument similar to the proof of Corollary 4.2 that uses the categories $\mathcal{C}(-; \mathfrak{A})$ instead of $\mathcal{B}(-; \mathfrak{A})$. Then $\widetilde{\mathbf{K}}'(S^0; \mathfrak{A})$ is the spectrum that [8] associtGes with S^0.

We now identify $\mathbb{R}^{n+1} = \mathcal{O}(S^n)$ with $oS^n = \int D^{n+1} = \{x \in \mathbb{R}^{n+1} | \parallel x \parallel < t\}$ as above and write $cS^n = D^{n+1}$. Since every bounded map of $\mathbb{R}^{n+1}$ extends via the identity on S^n to a map of D^{n+1}, there is an inclusion of additive categories

$$i : \hat{\mathcal{C}}(\mathbb{R}^{n+1}; \mathfrak{A}) \to \hat{\mathcal{B}}(D^{n+1}, S^n; \mathfrak{A}).$$

It is easily checked that these inclusions for $n \geq 0$ induce a map of spectra $i_* : \widetilde{\mathbf{K}}'(S^0; \mathfrak{A}) \to \widetilde{\mathbf{K}}(S^0; \mathfrak{A})$. The proof of Theorem 5.3 is completed by proving the following lemma:

Lemma 5.4. *The map $i_* : \widetilde{\mathbf{K}}'(S^0; \mathfrak{A}) \to \widetilde{\mathbf{K}}(S^0; \mathfrak{A})$ is a homotopy equivalence of spectra. In particular, $\Omega\widetilde{\mathbf{K}}(S^0; \mathfrak{A}) = \mathbf{K}(\mathfrak{A})$.*

Proof. Since [8] and [9] show that $\Omega\widetilde{\mathbf{K}}'(S^0;\mathfrak{A}) = \mathbf{K}(\mathfrak{A})$, the second sentence follows from the first. To show the first part, it suffices to show that $i_* : \widetilde{\mathbf{K}}'_*(S^0;\mathfrak{A}) \to \widetilde{\mathbf{K}}_*(S^0;\mathfrak{A})$ is an isomorphism. Notice that by the definition of $\widetilde{\mathbf{K}}(S^0;\mathfrak{A})$ and Corollary 4.2,

$$\widetilde{\mathbf{K}}_n(S^0;\mathfrak{A}) = K_n\hat{\mathcal{B}}(D^1, S^0;\mathfrak{A}) \qquad \text{if } n \geq 0.$$

Similarly

$$\widetilde{K}'_n(S^0;\mathfrak{A}) = K_n\hat{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) \qquad \text{if } n \geq 1,$$

and

$$\widetilde{K}'_{-n}(S^0;\mathfrak{A}) = K_1\hat{\mathcal{C}}(\mathbb{R}^{n+2};\mathfrak{A}) \qquad \text{if } n \geq 0.$$

We examine

$$i_* : K_n\hat{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) \to K_n\hat{\mathcal{B}}(D^1, S^0;\mathfrak{A})$$

for $n \geq 1$ in Lemma 5.5 and

$$i_* : K_1\hat{\mathcal{C}}(\mathbb{R}^{n+2};\mathfrak{A}) \to K_0\hat{\mathcal{B}}(D^{n+2}, S^{n+1};\mathfrak{A})$$

for $n \geq 0$ in Lemma 5.6. □

Lemma 5.5. *For any additive category $\mathfrak{A}$, there is a homotopy commutative diagram of homotopy equivalences*

$$\begin{array}{ccc} & \mathbb{K}(\hat{\mathcal{A}}) & \\ {}^{\varepsilon'}\swarrow & & \searrow^{\varepsilon} \\ \Omega\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) & \xrightarrow{\Omega_i} & \Omega\mathbb{K}\hat{\mathcal{B}}(D^1, S^0;\mathfrak{A}) \end{array}$$

Hence for all $n \geq 1$, $i_* : K_n\hat{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) \to K_n\hat{\mathcal{B}}(D^1, S^0;\mathfrak{A})$ is an isomorphism.

Proof. We claim that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathbb{K}\hat{\mathfrak{A}} & \xleftarrow{f'} & \Omega\mathbb{K}\widetilde{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) & \xrightarrow{\Omega j'} & \Omega\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^1;\mathfrak{A}) \\ \downarrow & & \downarrow{\scriptstyle \Omega i} & & \downarrow{\scriptstyle \Omega i} \\ \mathbb{K}\hat{\mathfrak{A}} & \xleftarrow{f} & \Omega\mathbb{K}\widetilde{\mathcal{B}}(D^1, S^0;\mathfrak{A}) & \xrightarrow{\Omega j} & \Omega\mathbb{K}\hat{\mathcal{B}}(D^1, S^0;\mathfrak{A}) \end{array}$$

in which all the horizontal maps are homotopy equivalences. Letting $\varepsilon' = j'f'^{-1}$ and $\varepsilon = jf^{-1}$. where f'^{-1} and f^{-1} are homotopy inverses for f' and f will complete the proof. To obtainn the diagram, recall the homotopy equivalences f was constructed in Corollary 4.3 using a Mayer-Voetories argument. A corresponding Mayer-Vietories argument using $\mathcal{C}$, etc. instead of $\mathcal{B}$, etc. gives the homotopy equivalence f'. Since the Mayer-Vietories diagram using $\mathcal{C}$ maps to the corresponding diagram using $\mathcal{B}$, the left-hand square homotopy commutes. The right-hand square commutes since the underlying diagram of categories commutes. □

Lemma 5.6. *Let $n \geq 0$. Then $i_* : K_1\hat{\mathcal{C}}(\mathbb{R}^{n+2}, \mathfrak{A}) to K_1\hat{\mathcal{B}}(D^{n+2}, S^{n+1}; \mathfrak{A})$ is an isomorphism.*

Proof. Since the map $\Omega_i : \Omega\mathbb{K}\mathcal{C}(\mathbb{R}^{n+2}; \mathfrak{A}) \to \Omega\mathbb{K}\hat{\mathcal{C}}(\mathbb{R}^{n+2}; \mathfrak{A})$ induced by the inclusion is a homeomorphism, we may replace $\hat{\mathcal{C}}$ with $\mathcal{C}$ in this lemma. Similarly, we may replace $\hat{\mathcal{B}}$ with $\mathcal{B}$.

To see that i_* is onto, let $x = [A, f] \in K_1\mathcal{B}(D^{n+2}, S^{n+1}; \mathfrak{A})$ where $f : A \to A$ is an isomorphism in $\mathcal{B}(D^{n+1}, S^n; \mathfrak{A})$. By Theorem 2.1(i) and Lemma 2.7, there is a homeomorphism h of $D^{n+2} = \mathbb{R}^{n+2} \cup S^{n+1}(\infty)$ onto itself with $h|_{S^{n+1}}(\infty) = 1$ for which $h_*(f)$ is bounded. Since h_* is the identity by Lemma 1.3, $x = [A, f] = [h_*(A), h_*(f)]$ and i_* is onto.

For any additive category $\mathfrak{A}$, let $\mathfrak{A}[t, t^{-1}$ be the "polynomial extension" categoryu of $\mathfrak{A}$ introduced by Ranicki [10]. This category has two descriptions. In one description, the objects of $\mathfrak{A}[t, t^{-1}$ are the same as the objects of $\mathfrak{A}$ and $\mathrm{Hom}(A, B)$ is the set of all formal sums $\sum_{-\infty}^{\infty} t^n f_n$ with $f_n = 0$ for all n with $|n|$ large enough. In the other description, the objects of $\mathfrak{A}[t, t^{-1}$ are the objects $\{A_n | n \in \mathbb{Z}\}$ of $\mathcal{C}(\mathbb{Z}, \mathfrak{A})$ for which $A_n = A_0$ for all n and the morphisms $f : A \to B$ are the $\mathbb{Z}$-equivariant morphisms $f = \{f^n_m : A_n \to B_m | n, m \in \mathbb{Z}\}$ in $\mathcal{C}(\mathbb{Z}; \mathfrak{A})$. In either case, we let $A[t, t^{-1}]$ denote the object of $\mathfrak{A}[t, t^{-1}]$ determined by $A \in \mathfrak{A}$.

The arguments given in Section 2 of [7] extend easily to define a "Bass-Heller-Swan" homomorphism $\lambda_{\mathcal{C}} : K_1\mathcal{C}(\mathbb{R}^{n+1}; \mathfrak{A}) \to K_1\mathcal{C}(\mathbb{R}^n; \mathfrak{A}[t, t^{-1}])$ for any $n \geq 0$ and to show that $\lambda_{\mathcal{C}}$ is injective. Since $K_1\hat{\mathfrak{A}} = K_1\mathfrak{A}$ for nay additive category, the facxt that i_* is one to one follows fron the next lemma by an obvious inductive argument. This completes the proof of Lemma 5.6. □

Lemma 5.7. *Let $n \geq 0$. there is a homomorphism $\lambda_{\mathcal{B}}$ making the following diagram commute:*

$$\begin{array}{ccc} K_1\mathcal{C}(\mathbb{R}^{n+1}; \mathfrak{A}) & \xrightarrow{\lambda_{\mathcal{C}}} & K_1\mathcal{C}(\mathbb{R}^n; \mathfrak{A}[t, t^{-1}]) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K_1\mathcal{B}(D^{n+1}, S^n; \mathfrak{A}) & \xrightarrow{\lambda_{\mathcal{B}}} & K_1\mathcal{B}(D^n, S^{n-1}; \mathfrak{A}[t, t^{-1}]) \end{array}$$

Proof. The construction of $\lambda_{\mathcal{B}}$ will follow Pedersen's construction [7] of $\lambda_{\mathcal{C}}$ quite closely.

Let $\mathcal{D}(\mathfrak{A})$ be either $\mathcal{C}(\mathbb{R}^{n+1}; \mathfrak{A})$ or $\mathcal{B}(D^{n+1}, S^n; \mathfrak{A})$, $A = \{A_x | x \in X\} \in \mathcal{D}(\mathfrak{A})$, and $A[t, t^{-1}] = \{A_x[t, t^{-1}] | x \in X\}$ be the object of $\mathcal{D}(\mathfrak{A}[t, t^{-1}])$ determined by A. Here X is either $\mathbb{R}^{n+1}$ or $D^n n + 1$. Let $p_- : A \to A$ be the projection that is the identity on A_x if $x = (x_1, \ldots, x_{n+1})$ has $x_{n+1} < 0$ and is zero otherwise. Let p_t be the morphism of $A[t, t^{-1}] = \{A_x[t, t^{-1}] | x \in X\}$ given by $p_t = tp_- + (1 - p_-)$ and notice that p_t is an automorphims with inverse $t^{-1}p_- + (1 - p_-)$. Finally, if $a : A \to A$ is an automorphims, we let $a_t = a[t, t^{-1}]$ be the automorphims of $A[t, t^{-1}]$ induced from a.

Let $[A, a]$ represent an element of $K_1\mathcal{D}(\mathfrak{A})$ and consider the commutator $[a_t, p_t]$. If $\mathcal{D}(\mathfrak{A}) = \mathcal{C}(\mathbb{R}^{n+1}; \mathfrak{A})$ and a has bound d, [7, p. 469] shows that $[a_t, p_t | \mathbb{R}^{n+1} - B = \mathrm{id}$ where B is the band $\{x = (x_1, \ldots, x_{n+1} || x_{n+1} \leq 2d\}$. It also shows that $[a_t, p_t]$ restricts to an automorphims of $A[t, t^{-1}]|_B$ and lets $\lambda_{\mathcal{C}}([A, a])$ be represented by $[\overline{A}[t, t^{-1}], [a_t, p_t]]$. here if $y = (y_1, \ldots, y_n)$,

$\overline{A}[t,t^{-1}] \in \mathcal{C}(\mathbb{R}^n;\mathfrak{A}[t,t^{-1}])$ has $\overline{A}[t,t^{-1}]_y = \sum\{A_x[t,t^{-1}]|x = (y_1,\ldots,y_n,x_{n+1})$ with $||x_{n+1} \leq 2d\}$.

If $\mathcal{D}(\mathfrak{A}) = \mathcal{B}(D^{n+1},S^n;\mathfrak{A})$ and a is continuously controlled at infinity, then by appl;ying Lemma 1.2 with $K = H_\pm = \{x = (x_a,\ldots,x_{n+1}| \pm x_{n+1} \geq 0\}$ and $W = \int D_\mp$ where $D_\mp = H_\pm \cap S^n$, there are neighborhoods $U_\pm$ of $\int D_\pm$ so that if $a^{-1} = b = \{b^x_y|x,y \in \int D^{n+1}\}$, then $b^x_y = 0$ for $(x,y) \in (U_\pm \times H_\mp) \cup (H_\pm \times U_\mp)$. It now follows that $c = [a_t,p_t]$ has

$$c|_{U+\pm} = \mathrm{id} : A[t,t^{-1}]|_{U_\pm} \to A[t,t^{-1}]|_{U_\pm}.$$

Consider $f = c| : A[t,t^{-1}]|_M \to A[t,t^{-1}]|$ where $M = D^{n+1} - (U_+ \cup U_-)$. Applying Lemma 1.2 a second time with $K = M$ and $W = \int D_+ \cup D_[$, there is a neighborhood V of W with $f^x_y = 0$ if $(x,y) \in V \times K \cup K \times V$. Let $W_\pm = U_\pm \cap V$. It is thenb easy to check that $c|_{W_\pm} = \mathrm{id} : A[t,t^{-1}]_{M_\pm} \to A[t,t^{-1}|_{W_\pm}$ and that the rstriction of c maps $A[t,t^{-1}]|_{D^{n+1}-(W_+\cup W_-)}$ isomorphically onto itself. Let $\lambda_{\mathcal{B}}([A,a])$ be represented by $[\overline{A}[t,t^{-1}],[a_t,p_t]]$ where if $y = (y_1,\ldots,y_n)$ then $\overline{A}[t,t^{-1}] \in \mathcal{B}(D^n,S^{n-1};\mathfrak{A}[t,t^{-1}])$ had $\overline{A}[t,t^{-1}]_y = \sum A_x[t,t^{-1}]$ where the sum runs over those $x \in im\{z \times \mathbb{R}\} \cap \{D^{n+1} - (W_+ \cup W_-)\}$. Here $z = (1- \parallel y \parallel^{-1} y$ and $im\{z \times \mathbb{R}\}$ is the image of this line in $\mathbb{R}^{n+1}$ under the identification of $\mathbb{R}^{n+1}$ with $\int D^{n+1}$ given above. The proof that $\lambda_{\mathcal{C}}$ is well defined given in [7, Theorem 2.3] carries over, almost without change, to show that $\lambda_{\mathcal{B}}$ is a well-defined homomorphism.

Since it follows immediately from the definitions of $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{B}}$ that the diagram above commutes, this completes teh proof of Lemma 5.7. □

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