CONTINUOUSLY CONTROLLED SURGERY THEORY

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0. Introduction

One of the basic questions in surgery theory is to determine whether a given homotopy equivalence of manifolds is homotopic to a homeomorphism. This can be determined by global algebraic topological invariants such as the normal invariant and the surgery obstruction (the Browder–Novikov–Sullivan–Wall theory). Another possibility is to impose extra geometric hypothesis on the homotopy equivalence. Such conditions are particularly useful when working in the topological category. Novikov's proof of the topological invariance of the rational Pontrjagin classes only used that a homeomorphism is a homotopy equivalence with contractible point inverses, as was first observed by Sullivan. Siebenmann [30] proved that every homotopy equivalence of manifolds in dimension bigger than five, with contractible point inverses is in fact homotopic to a homeomorphism by a small homotopy. Chapman and Ferry [10] generalized this to showing that ε-controlled homotopy equivalences are homotopic to homeomorphisms.

Controlled algebra was developed in order to guide geometric constructions maintaining control conditions, where the smallness is measured in some metric space. Such a theory was first proposed by Connell and Hollingsworth [11]. One of the aims was to prove the topological invariance of Whitehead torsion for homeomorphisms of polyhedra. In fact the first proof of topological invariance of Whitehead torsion [8] was developed without the use of controlled algebra. Such proofs have been developed later [14, 29]. Chapman developed a controlled Whitehead torsion theory using geometric methods [9]. Quinn [25, 26] developed the theory of Connell and Hollingsworth into a usable, computable tool. However there are technical difficulties in any kind of ε control because of the lack of functoriality. The composite of two ε maps is a 2ε map, so one needs to apply squeezing to regain control.

The basic idea in bounded topology and algebra is to keep control bounded, but let the metric space "go to infinity". This is obtained as follows. Assume $K \subset S^{n-1}$. We then define *the open cone*

$$O(K) = \{t \cdot x | t \in [0, \infty) \subset \mathbb{R}^n, \, x \in K\}.$$

The subset $t \cdot K \subset O(K)$ is a copy of K, but the metric has been enlarged by the factor t. This approach was developed in [21, 24, 15] and the controlled torsion and surgery obstructions live in the K and L-theory of additive categories (with involution). The basic fact being used

The author was partially supported by NSF grant DMS 9104026.

is that bounded + bounded is bounded. Similarly as ε goes to 0 we may use that $0 + 0 = 0$, and that is the basis of continuously controlled algebra and topology. Again the obstruction groups live in the K and L-theory of additive categories (with involution).

The object of this paper is to study the obstructions to deforming a homotopy equivalence of manifolds to a homeomorphism using continuously controlled algebra [1]. Suppose given a homotopy equivalence of manifolds $f : M \to N$ and a common closed subspace $K \subset M$, $K \subset N$ such that f restricted to K is the identity. We also assume $f : (M, K) \to (N, K)$ is a *strict map* i. e. that f sends $M - K$ to $N - K$. The basic question of continuously controlled surgery is whether it is possible to find a strict homotopy of f relative to K to a homeomorphism. A *strict homotopy* is a homotopy through strict maps. We shall also study an existence question corresponding to the uniqueness question above.

This type of question has typically been studied using bounded surgery [15], at least when K is compact, by methods as follows: Choose an embedding of K in a large dimensional sphere, and define $O(K) = \{t \cdot x \in \mathbb{R}^{N+1} | t \in [0, \infty), x \in K\}$. If K is a neighborhood retract, it is easy to produce a proper map from $N - K$ to $O(K)$ with the property that if we compactify $O(K)$ radially by adding a copy of K, the map extends continuously to N, using the identity on K. Thus when we approach K in N, the image goes to infinity in $O(K)$. If the map $M - K \to N - K$ can be homotoped to a homeomorphism which does not move any point more than a bounded amount when measured in $O(K)$, we may obviously complete the homotopy by the identity on K to obtain a strict homotopy of $f : M \to N$ relative to K, to a homeomorphism. This follows because the open cone construction blows up the metric near K, so bounded moves measured in $O(K)$ become arbitrarily small as we approach K in the original manifold. The method works well in the case when M and K are compact, and can be related to compact surgery theory via the torus. This is because homotopy equivalences parameterized by a torus become bounded homotopy equivalences parameterized by Euclidean space when passing to the universal cover of the torus, thus giving a way to use compact surgery theory for this kind of problem. The choice of the reference map to $O(K)$ however, is a bit unnatural, even though it can be shown not to matter, and in case M and K are not compact, this method does not work so well.

We shall indicate how to develop a continuously controlled surgery theory, in the locally simply connected case, and in the non-simply connected case, in the special case of a group action. The algebra described here was developed in [18] in the case of finite isotropy groups at the singular set. The case developed here allowing infinite isotropy groups is new. This algebra is relevant to generalized assembly maps of the type considered in the Baum–Connes and Farrell–Jones conjectures, [2, 13] see [12]. This is discussed in §4 .

A lot of the arguments are very similar to the bounded surgery theory [15], and will not be repeated here. We shall choose to emphasize the points where there are essential differences, and try to state precise definitions.

Following the path initialized by Wall in [32] we need to develop algebra that determines when a strict map

$$(f, 1_K) : (M, K) \to (N, K)$$

is a strict homotopy equivalence relative to K. The problem will then be solved by establishing a continuously controlled surgery exact sequence.

1. The simply connected case

As a warmup let us consider the simply connected case. We assume M and N are manifolds, and by simply connected we shall mean $N - K$ is simply connected and N is locally connected and locally simply connected at K. Specifically for each point $x \in K$ and each neighborhood U in N, there must exist a neighborhood V so that every two points in $V - K$ can be connected in $V - K$, and every loop in $V - K$ bounds a disk in $U - K$. These conditions are satisfied if N is simply connected and K is of codimension at least three. Recall the definition of the continuously controlled category from [1] and [5].

1.1**. Definition.** Let R be a ring and $(\overline{X}, \partial X)$ a pair of topological spaces, $X = \overline{X} - \partial X$ with ∂X closed in $\overline{X}$ and X dense in $\overline{X}$. We define the category $\mathcal{B}(\overline{X}, \partial X; R)$ as follows: An object A is a collection $\{A_x\}_{x \in X}$ of finitely generated free R-modules so that $\{x | A_x \neq 0\}$ is locally finite in X. A morphism $\phi : A \to B$ is an R-module morphism $\oplus A_x \to \oplus B_y$, satisfying a continuously controlled condition: For every $z \in \partial X$ and for every neighborhood U of z in $\overline{X}$, there exists a neighborhood V of z in $\overline{X}$ such that $\phi_x^y = 0$ and $\phi_y^x = 0$ if $x \in V \cap X$ and $y \in X - U$.

The continuous control condition thus requires that non trivial components of a morphism must be "small" near ∂X.

In the case discussed above, we could put $(\overline{X}, \partial X) = (N, K)$. We may then triangulate $N - K$ in such a fashion that simplices become small near K. Specifically this means that for every $z \in K$ and every neighborhood U there exists a neighborhood V such that if a simplex σ intersects V, it must be contained in U. The cellular chain complex of $N - K$ can thus be thought of as a chain complex in $\mathcal{B}(N, K; \mathbb{Z})$ by associating each $\mathbb{Z}$- module generated by a simplex to the barycenter of that simplex. Since the boundary maps in a cellular chain complex are given by geometric intersection, boundary maps will indeed be continuously controlled in the sense of definition 1.1. The following condition gives an algebraic condition for strict homotopy equivalence and thus provides the key to a continuously controlled surgery theory.

1.2**. Proposition.** *Suppose M and N are manifolds, and that (M, K) and (N, K) are simply connected and locally simply connected at K. Given a strict map $f : (M, K) \to (N, K)$, which is the identity on K. Then f is a strict homotopy equivalence relative to K, if and only if $f_\sharp : C_\sharp(M - K) \to C_\sharp(N - K)$ is a chain homotopy equivalence in $\mathcal{B}(N, K; \mathbb{Z})$*

Remark. In the proposition above, we do not actually need M and N to be manifolds, it would suffice to have locally compact Hausdorff pairs (M,K) and (N,K) with a CW structure on $M-K$ and $N-K$ satisfying some extra conditions. We shall return to this.

The proof of this proposition is a straightforward handle argument. However translation to algebra depends very strongly on the pair (N,K). To remedy this situation and show that it really only depends on K we present the following lemma from [5]:

1.3. **Lemma.** *If $(\overline{X},\partial X)$ is a compact metrizable pair then, denoting the cone of ∂X by $C\partial X$ we have an equivalence of categories*

$$\mathcal{B}(\overline{X},\partial X;R)\cong\mathcal{B}(C\partial X,\partial X;R)$$

Proof. The isomorphism is given by moving the modules A_x, $x\in X$ to points in $C\partial X$, the same module. If two are put at the same place we take the direct sum. On morphisms the isomorphism is induced by the identity, so we have to ensure the continuously controlled condition is not violated. We proceed as follows: Choose a metric on $\overline{X}$ so that all distances are ≤ 1. Given $z\in X$, let y be a point in ∂X closest to z, and send z to $(1-d(z,y))y$. Clearly, as z approaches the boundary it is moved very little. In the other direction send $t\cdot y$ to a point in $B(y;1-t)$, the closed ball with center y and radius $1-t$, which is furthest away from ∂X. Again moves become small as t approaches 1 or equivalently as the point approaches ∂X. It is easy to see that we never take more than a finite direct sum, and that the local finiteness condition is preserved. □

This lemma shows that in the metrizable case the algebra only depends on K.

We need duality in the category $\mathcal{B}(\overline{X},\partial X;R)$. The duality we need is that $\mathcal{B}(\overline{X},\partial X;R)$ is an additive category with involution in the sense of Ranicki [27]. The duality is given by $(A^*)_x=(A_x)^*$. This codifies the local nature of Poincaré duality. The dual cell of a cell sitting near x will also be near x. Given this duality the algebraic L-groups are defined in [27] using forms and formations. These will be the appropriate obstruction groups. Using this duality, a chain complex $C_\sharp$ in $\mathcal{B}(\overline{X},\partial X;R)$ has a "dual" chain complex $C^\sharp$ in $\mathcal{B}(\overline{X},\partial X;R)$.

Before continuing to develop the continuously controlled surgery theory, of which we have only touched upon the uniqueness aspects, notice that for the preceding discussion we do not really need that M and N are manifolds. All we needed was that $M-K$ and $N-K$ are manifolds, that allow a triangulation (or handle body decomposition) with small simplices near K. This is important in developing the existence aspects of a continuously controlled surgery theory which we shall proceed to do.

We need to codify a continuously controlled simply connected Poincaré duality space. We model this on a simply connected manifold M with a closed subset K of codimension at least three. Consider a pair $(\overline{X},\partial X)$ as above with ∂X closed in $\overline{X}$ and $X=\overline{X}-\partial X$ dense in $\overline{X}$.

1.4. **Definition.** The pair $(\overline{X},\partial X)$ is -1-*connected at* ∂X if for every point $z\in\partial X$ and every neighborhood U we have $U\cap X$ is nonempty. The pair $(\overline{X},\partial X)$ is 0-*connected at* ∂X, if for every $z \in \partial X$ and every neighborhood U of z in $\overline{X}$, there is a neighborhood V of z in $\overline{X}$ so that any two points in $V \cap Z$ can be connected by a path in $U \cap X$. The pair is 1-*connected at* ∂X if if for every $z \in \partial X$ and every neighborhood U of z in $\overline{X}$, there is a neighborhood V of z in $\overline{X}$ so that every loop in $V \cap X$ bounds a disc in $U \cap X$.

1.5**. Definition.** A *continuously controlled* CW-*structure* on the pair $(X, \partial X)$ is a CW structure on $(X - \partial X)$ such that the cells are small at ∂X i. e. such that for every $z \in \partial X$ and every neighborhood U of z in $\overline{X}$, there is a neighborhood V of z in $\overline{X}$ so that if a cell in the CW structure intersects V then the cell is contained in U. A *continuously controlled* CW-*complex* $(X, \partial X)$ is such a pair endowed with a continuously controlled CW-structure. We shall call the CW-structure *locally finite* if the CW-structure on $X - \partial X$ is locally finite. Similarly there is an obvious notion of locally finitely dominated in the continuously controlled sense.

Obviously a manifold M with a codimension at least three subcomplex K is -1-, 0- and 1-connected at K and can be given a CW- structure which is continuously controlled at K.

1.6**. Definition.** A *simply connected continuously controlled Poincaré Duality space at* ∂X is a continuously controlled, locally finite CW-complex $(\overline{X}, \partial X)$, $X = \overline{X} - \partial X$, such that X is simply connected, $(\overline{X}, \partial X)$ must be -1-, 0- and 1-connected at ∂X and the CW-structure must be continuously controlled at ∂X. Given this the cellular chains of X, define a chain complex in the category $\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})$ which we denote by $C_\sharp(X)$, with dual chain complex denoted $C^\sharp(X)$. We then further require the existence of a homology class $[X] \in H^{l.f.}(X; Z)$ such that cap product with $[X] \cap -$, which by its geometric nature defines a map of chain complexes in $\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})$, is a homotopy equivalence of chain complexes

$$C^\sharp(X; \mathbb{Z}) \to C_\sharp(X; \mathbb{Z})$$

as chain complexes in the category $\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})$.

Remark. The locally finite homology referred to above is singular homology based on locally finite chains

Once again it is clear that a simply connected manifold with a codimension three subcomplex can be triangulated to satisfy these conditions.

Given a pair $(\overline{X}, \partial X)$ and a proper map $f : Y \to X$ we say *f is a continuously controlled homotopy equivalence at* ∂X if f induces a strict homotopy equivalence of pairs $(\overline{Y}, \partial X) \xrightarrow{(\overline{f},1)} (\overline{X}, \partial X)$ relative to ∂X. Here $\overline{Y}$ is a completion of Y by ∂X through the map f. As a set $\overline{Y}$ is the disjoint union of Y and ∂X, and the topology is given by the open sets being the open sets of Y and sets of the form $V \cap \partial X \cup f^{-1}(V)$, where V is open in $\overline{X}$. The aim of continuously controlled surgery is to turn a degree one normal map (in the proper sense) into a continuously controlled homotopy equivalence. Proper surgery [31] fits into this picture by completing an open Poincaré duality space by precisely one point for each end.

Since a continuously controlled Poincaré duality space is automatically a proper Poincaré duality space, we can use the theory of Spivak normal fibrations from the proper theory. Alternatively it is not very difficult to see that the inclusion of a the boundary of a regular neighborhood in Euclidean space produces a spherical fibration.

At this point we proceed exactly as in [15] to do surgery below the mid-dimension. The method introduced by Wall [32] for surgery below the mid-dimension is completely geometric: Given a surgery problem $M \to X$ we may replace the map by an inclusion, replacing X by the mapping cylinder. Doing a surgery for each of the cells in $X - M$ until we reach the mid-dimension produces a surgery problem $N \to X$ where X is obtained from N by attaching cells above the mid-dimension, see [15] for more details on this. This makes sense in the continuously controlled setting if we triangulate the manifold so that we obtain a continuously controlled CW-complex.

At this point we get obstructions to obtain continuously controlled homotopy equivalences with values in the algebraic L-theory of the additive categories with involution $\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})$. The proof of this once again follows [15] closely and is a translation of the arguments given in [32] avoiding homology at all points, i. e. working directly with the chain complexes in the additive category. Given a controlled simply connected Poincaré pair $(\overline{X}, \partial X)$ with $X = \overline{X} - \partial X$ with a reduction of the Spivak fibration to $B\,\mathrm{CAT}$ where $\mathrm{CAT} = \mathrm{Top}, \mathrm{PL}$ or O, we obtain a surgery exact sequence

$$\dots L_{n+1}(\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})) \to \mathcal{S}^{c.c.}(\overline{X}, \partial X) \to [X, G/\,\mathrm{CAT}] \to L_n(\mathcal{B}(\overline{X}, \partial X; \mathbb{Z}))$$

deciding topological, PL, and smooth, continuously controlled structures on X respectively. The continuously controlled structure set consists of CAT manifolds M together with a continuously controlled homotopy equivalence to $X = \overline{X} - \partial X$. Two structures $M_1 \to X$ and $M_2 \to X$ are equivalent, if there is a continuously controlled h-cobordism (in the CAT-category) between M_1, M_2 and a map extending the given maps.

Remark. There are all the usual possible modifications of a surgery theory. There is an obvious notion of a simple Poincaré complex in this context allowing the h-cobordism to be replaced by s-cobordism. Notice it is standard to prove a continuously controlled h- or s-cobordism theorem along the lines of of [22]. Similarly by allowing locally finitely dominated Poincaré complexes in the continuously controlled sense we would obtain a projective version of such a surgery theory along the lines of [23]. In the simply connected case these theories coincide, but later we shall indicate how to weaken the simply connected assumption.

2. Germ categories

One would obviously want to generalize this to a non simply connected situation, but before discussing that we shall consider a less obvious generalization involving germs, that turns out to be very useful for computations. Suppose U is an open subset of ∂X. We wish to develop a surgery theory where the aim is only to obtain a homotopy equivalence in a neighborhood of U. By this we mean that the "homotopy inverse" is only defined in a neighborhood of U, and that the compositions are homotopic by small homotopies to the identity in a neighborhood of U. We shall proceed to describe the algebra that describes this situation. Denote the complement of U in ∂X by Z. We need a definition

2.1. **Definition.** An object A in $\mathcal{B}(\overline{X}, \partial X; R)$ has support at infinity contained in Z if

$$\overline{\{x | A_x \neq 0\}} \cap \partial X \subset Z$$

We denote the full subcategory of $\mathcal{B}(\overline{X}, \partial X; R)$ on objects with support at infinity contained in Z by $\mathcal{B}(\overline{X}, \partial X; R)_Z$. This is a typical example of an additive category $\mathcal{U} = \mathcal{B}(\overline{X}, \partial X; R)$ which is $\mathcal{A} = \mathcal{B}(\overline{X}, \partial X; R)_Z$-filtered in the sense of Karoubi [20], see also [4]. We recall the notion of an $\mathcal{A}$-filtered additive category $\mathcal{U}$ in the following

2.2. **Definition.** We say $\mathcal{U}$ is $\mathcal{A}$-filtered if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$ (called a filtration of U) satisfying the following axioms: (We denote objects in $\mathcal{A}$ by $A, B, \dots$ and in $\mathcal{U}$ by $U, V, \dots$)

- F1: For each U, the decompositions form a filtered poset under the partial order $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$ whenever $U_\beta \subset U_\alpha$ and $E_\alpha \subset E_\beta$.
- F2: Every map $A \to U$ factors $A \to E_\alpha \to E_\alpha \oplus U_\alpha = U$ for some α.
- F3: Every map $U \to A$ factors $U = E_\alpha \oplus U_\alpha \to E_\alpha \to A$ for some α.
- F4: For each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$, i. e. to $\{U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)\}$.

This is a precise analogue in the category of small additive categories of an ideal in a ring. The quotient category $\mathcal{U}/\mathcal{A}$ is defined to have the same objects as $\mathcal{U}$, but two morphisms $\phi_1 : U \to V$ and $\phi_2 : U \to V$ are identified if the difference $\phi_1 - \phi_2$ factors through the category $\mathcal{A}$, i. e. if there exists an object A in $\mathcal{A}$ and a factorization $\phi_1 - \phi_2 : U \to A \to V$. The axioms ensure that $\mathcal{U}/\mathcal{A}$ is a category.

In the case we are considering $\mathcal{U} = \mathcal{B}(\overline{X}, \partial X; R)$ and $\mathcal{A} = \mathcal{B}(\overline{X}, \partial X; R)_Z$ it is easy to see the axioms above are satisfied: as indexing set we may use open neighborhoods of $\partial X - Z$ in X, and decompose an object $U = \{U_x\}$ in a part where U_x is replaced by 0 if x belongs to the given neighborhood, and another part where U_x is replaced by 0 if x does not belong to the given neighborhood. We denote the quotient category $\mathcal{U}/\mathcal{A}$ by $\mathcal{B}(\overline{X}, \partial X; R)^{\partial X - Z}$. Evidently two morphisms are identified if and only if they agree in a neighborhood of $U = \partial X - Z$, and it is not difficult to see that the category $\mathcal{B}(\overline{X}, \partial X; \mathbb{Z})^{\partial X - Z}$ measures morphisms that are homotopy equivalences in a neighborhood of U. To us however the real strength of these categories is in conjunction with Lemma 1.3, they allow computation of the L-groups, see Theorem 2.4 below. To prepare for this we need the idempotent completion of an additive category [16, p. 61]. The idempotent completion $\mathcal{A}^\wedge$ of $\mathcal{A}$ has objects (A, p) where A is an object of $\mathcal{A}$ and p is an idempotent morphism $p^2 = p$. A morphism $\phi : (A, p) \to (B, q)$ is an $\mathcal{A}$-morphism $\phi : A \to B$ such that $q\phi p = \phi$. Intuitively (A, p) represents the image of p, and the condition says that ϕ only depends on the image and lands in the image. The category $\mathcal{A}$ is embedded in its idempotent completion by sending A to $(A, 1)$. Given a subgroup

$k \subset K_0(\mathcal{A}^\wedge)$, we can perform a partial idempotent completion $\mathcal{A}^{\wedge k}$, the full subcategory of $\mathcal{A}^\wedge$ on objects (A,p) with $[(A,p)] \in k \subset K_0(\mathcal{A}^\wedge)$. The following theorem is proved in [5] based on ideas from [28].

2.3. **Theorem.** *Given an additive category with involution $\mathcal{U}$ which is $\mathcal{A}$-filtered by a $*$-invariant subcategory $\mathcal{A}$, there is a fibration of 4-periodic L-spectra*

$$\mathbb{L}(\mathcal{A}^{\wedge k}) \to \mathbb{L}(\mathcal{U}) \to \mathbb{L}(\mathcal{U}/\mathcal{A})$$

where k is the inverse image of $K_0(\mathcal{U})$ in $K_0(\mathcal{U}^\wedge)$

The proof of this theorem goes as follows: It follows from standard bordisms methods that there is a fibration of spectra $\mathbb{L}(\mathcal{A}) \to \mathbb{L}(\mathcal{U}) \to \mathbb{L}(\mathcal{U},\mathcal{A})$. Next one proves that a chain complex in $\mathcal{U}$ is dominated by a chain complex in $\mathcal{A}$ if and only if the chain complex induces a contractible chain complex in $\mathcal{U}/\mathcal{A}$. This means that an attempt to prove $\mathbb{L}(\mathcal{U},\mathcal{A})$ is homotopy equivalent to $\mathbb{L}(\mathcal{U}/\mathcal{A})$ is off by a finiteness obstruction, and adjusting with idempotent completion as in the statement above solves the problem. For the ultimate statement regarding the decorations on the L-groups see [17, Theorem 6.7]. It is now fairly straightforward to compute the L-groups of $\mathcal{B}(\overline{X},\partial X;\mathbb{Z})$ in the case $(\overline{X},\partial X)$ is metrizable using Lemma 1.3 as in [6]. The result is

2.4. **Theorem.** *If $(\overline{X},\partial X)$ is a compact metrizable pair, then*

$$L_*(\mathcal{B}(\overline{X},\partial X;\mathbb{Z}))$$

is isomorphic to the Steenrod homology theory (see axioms below) of ∂X associated to the spectrum $\Sigma\mathbb{L}(\mathbb{Z})$.

In case ∂X is a CW-complex this is just the standard generalized homology theory associated with a spectrum. In the general case it satisfies the Steenrod axioms. The Steenrod axioms for a homology theory h [19] say

(i) given any sequence of compact metrizable spaces $A \subset B \to B/A$ there is a long exact sequence in homology

$$\ldots \to h_i(A) \to h_i(B) \to h_i(B/A) \to h_{i-1}(A) \ldots$$

(ii) Given a countable collection X_i of compact metrizable spaces letting $\bigvee X_i \subset \prod X_i$ be given the subset topology (the strong wedge) we have an isomorphism

$$h_*(\bigvee X_i) \cong \prod h_*(X_i)$$

It is also required that h is homotopy invariant.

3. The equivariant case

Finally we want to discuss to what extent we can avoid the simple connectedness assumption. We shall not try to deal with the most general case, even though something could be said using the germ methods mentioned above. We shall satisfy ourselves with the following situation where the local variation in the fundamental group is given by a global group action, in other words, we shall consider the situation where we have a group Γ acting freely cellularly on $X = \overline{X} - \partial X$, and $(\overline{X}, \partial X)$ is a simply connected, continuously controlled Poincaré duality space. We need to be able to decide when an equivariant proper map from M to X is a continuously controlled equivariant homotopy equivalence in order to setup a surgery theory as in the simply connected case.

The action of Γ is not assumed to be free on ∂X. In the case where we only have finite isotropy it is fairly easy to define a category which measures this kind of continuously controlled equivariant homotopy equivalence. This was done in [15] and [18]. Here we want to deal with the more general situation where we do not have an assumption about isotropy. This is of interest in connection with the kind of generalizations of the assembly map studied in the Baum–Connes and the Farrell–Jones conjectures [2, 13]. An extra complication is to be able to deal with duality on the category. Assume $(\overline{X}, \partial X)$ is a compact Hausdorff pair with a Γ-action (if the pair is only locally compact Hausdorff one point compactify). Choose once and for all a large $R[\Gamma]$ module say $U = R[\Gamma \times \mathbb{N}]$. We shall think of U as a universe. This will help make categories small and thus allow talking about group actions on categories. We shall call a subset $Z \subset \overline{X}$ *relatively* Γ- *compact* if $Z \cdot \Gamma/\Gamma \subset \overline{X}/\Gamma$ is contained in a compact subset of $\overline{X}/\Gamma$.

3.1. **Definition.** The category $\mathcal{D}(\overline{X}, \partial X; R)$ has objects A, an R-submodule of U, together with a map $f : A \longrightarrow F(X \times \Gamma)$, where $F(X \times \Gamma)$ is the set of finite subsets of $X \times \Gamma$, $X = \overline{X} - \partial X$, satisfying

- (i) $A_x = \{a \in A | f(a) \subseteq \{x\}\}$ is a finitely generated free sub R-module for each $x \in X \times \Gamma$.
- (ii) As an R-module $A = \bigoplus_{x \in X \times \Gamma} A_x$
- (iii) $f(a + b) \subseteq f(a) \cup f(b)$
- (iv) $\{x \in X \times \Gamma | A_x \neq 0\}$ is locally finite and relatively Γ-compact in $\overline{X} \times \Gamma$ with the diagonal Γ-action.

A morphism $\phi : A \longrightarrow B$ is a morphism of R- modules, satisfying the continuous control condition at ∂X when we forget the extra factor of Γ, i. e. for every point z in ∂X and for every neighborhood U of z in $\overline{X}$ there is a neighborhood V of z in $\overline{X}$ such that if $x \in (X - U) \times \Gamma$ and $y \in (V - \partial X) \times \Gamma$, then ϕ^x_y and ϕ^y_x are 0.

Combining the action of Γ on U and on $\overline{X}$ we get a Γ-action on $\mathcal{D}(\overline{X}, \partial X; R)$ by conjugation. An object A is fixed under this action if A is invariant under the Γ-action on U, thus inheriting an $R[\Gamma]$-module structure which has to be free, and the reference map is equivariant. A morphism is fixed under the Γ-action if it is Γ-equivariant. We denote the fixed category by $\mathcal{D}_\Gamma(\overline{X}, \partial X; R)$. These are the categories that determine the relevant equivariant continuously controlled homotopy equivalences.

Remark. In case Γ is the trivial group and $\overline{X}$ is compact we recover our old definition of the $\mathcal{B}$-categories. In case $\overline{X}$ is not compact we get a different category which corresponds to homology rather than homology with locally finite coefficients when applying K or L-theory.

The definition above is designed to make it easy to define a duality on the category. We define the dual of an object A to be the set of R-module homomorphisms from A to R that are locally finite i. e. only nontrivial on A_x for finitely many x. The reference map to $F(X \times \Gamma)$ is given by the set of x for which the homomorphism is non-zero. As usual we inherit a left Γ-action which we may turn into a right action, possibly using an orientation homomorphism in the process.

In the definition above we crossed with Γ. It is easy to see we get an equivalent category if instead we crossed with some other free Γ space. In particular we could cross with $E\Gamma$. This is relevant for developing this kind of theory in A-theory where the underlying homotopy type does play a role.

Suppose E is a locally compact Hausdorff Γ-space. Consider $E \subset E \times I$ included as $E \times 1$. The category $\mathcal{A} = \mathcal{D}_\Gamma(E \times I, E; R)_\emptyset$ is the full subcategory of $\mathcal{U} = \mathcal{D}_\Gamma(E \times I, E; R)$ on objects with empty support at infinity i.e. on objects A such that the closure of $\{x \in E \times [0,1) | \exists g \in \Gamma : A_{(x,g)} \neq 0\}$ intersects $E = E \times 1$ trivially. It is easy to see that $\mathcal{U}$ is $\mathcal{A}$- filtered. $\mathcal{D}_\Gamma(E \times I, E; R)_\emptyset$ is equivalent to the category of finitely generated free $R[\Gamma]$-modules. We denote the quotient category by $\mathcal{D}_\Gamma(E \times I, E; R)^E$. It is a germ category. Objects are the same as in $\mathcal{D}_\Gamma(E \times I, E; R)$, but morphisms are identified if they agree close to E.

When studying the L-theory we have to deal with the variations in the upper index. This is necessary in the geometric application. Here we choose to use $L^{-\infty}$, avoiding that problem. Similarly in algebraic K- theory we need to use $K^{-\infty}$, the K- theory functor that includes the negative K- theory groups. It was proved in [24], see [4] for a more modern proof, that we have a fibration of spectra

$$K^{-\infty}(\mathcal{A}) \to K^{-\infty}(\mathcal{U}) \to K^{-\infty}(\mathcal{U}/\mathcal{A})$$

whenever we have an $\mathcal{A}$- filtered category $\mathcal{U}$. This leads to the following

3.2. **Theorem.** *The functors from locally compact Hausdorff spaces with Γ-action and Γ-equivariant maps, sending E to*

$$K^{-\infty}(\mathcal{D}_\Gamma(E \times I, E; R)^E) \text{ and } L^{-\infty}(\mathcal{D}_\Gamma(E \times I, E; R)^E)$$

are homotopy invariant and excisive. If Γ acts transitively on S, the value on S is homotopy equivalent to $\Sigma K^{-\infty}(R[H])$ and $\Sigma L^{-\infty}(R[H])$ respectively, where H denotes the isotropy subgroup.

The proof of the first two statements follows the methods in [5] closely. It is an application of the basic $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ fibrations. In the last statement one should notice that the identification is not canonical, it depends on choosing a basepoint in S. To see the last statement notice that we are considering $R[\Gamma]$-modules parameterized by $S \times [0, 1)$, and germs of morphisms near $S \times 1$. The control conditions imply that near $S \times 1$, a morphism can not reach from one component of S to another. Hence this category is equivalent to the category of $R[H]$-modules parameterized by $[0, 1]$ with germs taken near 1, since the group action tells us what to do everywhere else. The K or L- theory of this category is a deloop of the K or L-theory of $R[H]$.

These methods are generalized to the C^*-situation in [7]. The advantage of this method is that we not only get the kind of description needed in the study of assembly maps in [12], but we also obtain the relevant spectra as fixed spectra under a Γ-action.

4. Assembly maps

Denote $K^{-\infty}$ or $L^{-\infty}$ applied to $\mathcal{D}_\Gamma(E \times I, E; R)^E$ by F. It is then easy to see, using the methods of [5], specifically [5, Theorem 1.28, Theorem 4.2] and their corollaries, that F is a homotopy invariant excisive functor from the category of Γ-spaces and Γ-maps to spectra, and thus fits precisely into the framework developed by Davis and Lück [12] for generalized assembly maps of the type considered by Quinn, Farrell-Jones, and Baum–Connes. That F is a functor on all Γ maps without any properness assumption uses the fact that we have a Γ-compactness assumption on the support of the modules considered.

Davis and Lück describe assembly maps of this type as the induced map $F(E) \to F(*)$, where E is a Γ-space. The Farrell-Jones conjecture is then the statement that $F(E) \to F(*)$ is an isomorphism when E is the universal space for Γ actions with isotropy groups virtually cyclic groups, and the Baum-Connes conjecture is a similar statement where F is defined using topological K-theory and E is the universal space for Γ-actions with finite isotropy. Consider

$$\mathcal{D}_\Gamma(E \times I, E; R)_\emptyset \to \mathcal{D}_\Gamma(E \times I, E; R) \to \mathcal{D}_\Gamma(E \times I, E; R)^E$$

Applying $K^{-\infty}$ or $L^{-\infty}$ we get a fibration of spectra (by [5, Theorem 1.28 and Theorem 4.2])

4.1. **Theorem.** *The generalized assembly map is the connecting homomorphism in the above mentioned fibration.*

Proof. The category $\mathcal{D}_\Gamma(E \times I, E; R)_\emptyset$ is equivalent to the category of finitely generated $R\Gamma$-modules since the support conditions make the modules finitely generated $R\Gamma$-modules and the control conditions vacuous. Consider the diagram:

$$\begin{array}{ccccc}
\mathcal{D}_\Gamma(E \times I, E; R)_\emptyset & \longrightarrow & \mathcal{D}_\Gamma(E \times I, E; R) & \longrightarrow & \mathcal{D}_\Gamma(E \times I, E; R)^E \\
\downarrow a & & \downarrow b & & \downarrow c \\
\mathcal{D}_\Gamma(* \times I, *; R)_\emptyset & \longrightarrow & \mathcal{D}_\Gamma(* \times I, *; R) & \longrightarrow & \mathcal{D}_\Gamma(* \times I, *; R)^*
\end{array}$$

Here a is an equivalence of categories, since both categories are equivalent to the category of finitely generated free $R\Gamma$-modules. The K- and L-theory of $\mathcal{D}_\Gamma(* \times I, *; R)$ is trivial since the category admits a flasque structure by shifting modules towards 1. The map c is the induced map of $E \to *$. The result now follows.

□

We finish by mentioning a result which is essentially contained in [5] but not explicitly stated.

4.2. **Theorem.** *Let M be an n-dimensional topological manifold. Then the (4-periodical) structure set of M is isomorphic to*

$$L^h_{n+1}(\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z}))$$

where Γ is the fundamental group of M acting on the universal cover $\widetilde{M}$.

Remark. The surgery exact sequence for topological manifolds is 4-periodic except the periodicity breaks down in the bottom since the normal invariant is given by $[M, G/\operatorname{Top}]$, not $[M, G/\operatorname{Top} \times \mathbb{Z}]$. The result above identifies the L-group with a periodical structure set. See [3] for a further discussion of this phenomenon.

Proof. Consider

$$\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})_\emptyset \to \mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z}) \to \mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})^{\widetilde{M}}$$

which, in the following we will discuss as

$$\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$$

According to [5, Theorem 4.1] we get a fibration of $\mathbb{L}$-spectra if we choose appropriate decorations. We can, as mentioned above, always use the $-\infty$ decoration. but we need to improve on that a bit. we have

$$\mathcal{U}/\mathcal{A} = \mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})^{\widetilde{M}} = \mathcal{B}(M \times I, M; \mathbb{Z})^M$$

and by [1] $K_i(\mathcal{B}(M \times I, M; \mathbb{Z}) = h_{i-1}(M_+; K\mathbb{Z})$ where $K\mathbb{Z}$ is the K-theory spectrum for the integers. Hence $K_0(\mathcal{U}/\mathcal{A}) = \mathbb{Z}$ and the boundary map (which is the K-theory assembly map) hits the subgroup of $K_0(\mathbb{Z}\Gamma)$ given by the free modules. This means we get a fibration of spectra

$$\mathbb{L}^h(\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})_\emptyset) \to \mathbb{L}^h(\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})) \to \mathbb{L}^h(\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})^{\widetilde{M}})$$

which we identify with

$$\mathbb{L}^h(\mathbb{Z}\Gamma) \to \mathbb{L}^h(\mathcal{D}_\Gamma(\widetilde{M} \times I, \widetilde{M}; \mathbb{Z})) \to \Sigma M_+ \wedge \mathbb{L}(\mathbb{Z})$$

as in [5]. The classifying map $\Sigma M_+ \wedge \mathbb{L}(\mathbb{Z}) \to \Sigma \mathbb{L}^h(\mathbb{Z}\Gamma)$ was identified with the (suspension of) the assembly map in [5]. Hence the fibre represents the structure set, and the result follows. □

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