A FIBRATION FOR DIFFΣ^n

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0. Introduction

Let M^n be an oriented smooth manifold, and let $\operatorname{Diff} M^n$ be its group of orientation preserving diffeomorphisms. Let $\operatorname{Diff}(N^n, M^n_0)$ and $\operatorname{Diff}_* M^n$ be the two subgroups (with the C^∞ topology) which consist of diffeomorphisms fixing $M^n_0 = M^n - D^n$ (for some embedded disc D^n) and the base point $* \in D^n \subset M^n$ respectively.

It is a common procedure when studying $\pi_*(\operatorname{Diff} M^n)$ to restrict attention to either one of the above subgroups; it is therefore of some interest to study the homotopy braid of the triple $(\operatorname{Diff} M^n, \operatorname{Diff}_* M^n, \operatorname{Diff}(M^n, M^n_0))$.

In fact this is equivalent to considering the following 'comparison diagram' of principal fibrations, constructed by the usual techniques.

(0.1). Diagram.

$$
\begin{tikzcd}
 & \operatorname{Diff}_* M^n \arrow[rr, "d"] \arrow[dd, "j"] & & \mathrm{SO}(n) \arrow[dd, "j_P"] \arrow[dr] & \\
\operatorname{Diff}(M^n, M^n_0) \arrow[ur, "i'"] \arrow[dr, "i"'] & & & & B\operatorname{Diff}(M^n, M^n_0) \\
 & \operatorname{Diff} M^n \arrow[rr, "e"] \arrow[dr, "\pi"'] & & P\tau(M^n) \arrow[ur] \arrow[dl, "\pi"] & \\
 & & M^n & &
\end{tikzcd}
$$

Here d is the derivative map at $*$, $P\tau(M^n)$ is the total space of the oriented smooth principal tangent bundle to M^n, e maps $f \in \operatorname{Diff} M^n$ to the derivative at $*$ of the restriction $f|D^n$, and π is evaluation at $*$.

The only fibration on display above which is not well documented is the lower horizontal one, i. e.

$$\operatorname{Diff}(M^n, M^n_0) \underset{i}{\rightarrow} \operatorname{Diff} M^n \underset{e}{\rightarrow} P\tau(M^n) \tag{F}$$

Its existence follows from the fibration

$$\operatorname{Diff}(M^n, M^n_0) \underset{i}{\rightarrow} \operatorname{Diff} M^n \underset{e'}{\rightarrow} \mathcal{E}(D^n, M^n),$$

where $\mathcal{E}(\ ,\ )$ denotes the space of orientation preserving smooth embeddings on D^n. But $\mathcal{E}(D^n, M^n)$ may be identified with $P\tau(M^n)$, since

$$\mathcal{E}_*(D^n, M^n) \to \mathcal{E}(D^n, M^n) \to M^n$$

is a model for the principal tangent bundle of M^n via the derivative map.

The homotopy braid of (0.1) gives rise to two interlaced problems. Firstly, to what extent does the 'linearization' j_P determine the map j; and secondly, to what extent does d determine e?

We shall discuss the second of these questions in the special case of M^n an exotic sphere Σ^n. In this case, (F) generalizes an exact sequence described by R. Schultz [6]. In particular we study the difference between the boundary maps associated to d and to e, and reduce the detection of a certain class of 'stable' homotopy element so arising to an interesting, but apparently unsolved, problem in the homotopy groups of spheres.

Throughout, we shall write S^n_β for the exotic sphere given by an element $\beta \in \pi_n(\mathrm{Top}/O)$, $n \geq 7$. Such β arises from an isotopy class of diffeomorphisms $\beta \in \mathrm{Diff}\, S^{n-1}$, so any S^n_β can be represented as $D^n_0 \cup_\beta D^n_c$, where $0 \in D^n_0$ is the basepoint, and D^n_c is the complementary disc.

We are grateful to Dick Lashof for a helpful letter.

1. The fibration

Our fibration (F) of §0 can be further simplified when $M^n = S^n_\beta$. For it has long been known that, whatever β, $P\tau(S^n) = \mathrm{SO}(n+1)$. It is most convenient to describe this fact by means of

(1.1). Lemma. *There is a homeomorphism of degree 1, say $h : S^n_\beta \to S^n$, such that the diagram*

$$\begin{tikzcd} S^n \arrow[dr, "\tau(S^n)"] & \\ & B\,\mathrm{SO}(n) \\ S^n_\beta \arrow[uu, "h"] \arrow[ur, "\tau(S^n_\beta)"'] & \end{tikzcd}$$

homotopy commutes. Thus $P\tau(S^n_\beta)$ is homeomorphic to $\mathrm{SO}(n+1)$ via an $\mathrm{SO}(n)$ equivariant map,

Proof. Since S^n_β is stably parallelizable, $\tau(S^n_\beta)$ lifts to S^n. We can choose the lift h to have degree 1 by appealing to the euler characteristic if n is even, and the Kervaire semicharacteristic if n is odd $\square$

(1.2). Note. The resulting homeomorphism $\bar{h} : P\tau(S^n_\beta) \to \mathrm{SO}(n+1)$ is defined only up to alteration by any map $S^n_\beta \to \mathrm{SO}(n)$.

Proof. Since S^n_β is stably parallelizable, $\tau(S^n_\beta)$ lifts to S^n. We can choose the lift h to have degree 1 by appealing to the euler characteristic if n is even, and the Kervaire semi-characteristic if n is odd. $\square$

We can now construct our special version of (0.1) as follows:

(1.3). Diagram.

$$\begin{tikzcd}
 & \mathrm{Diff}_* S^n_\beta \arrow[rr, "d"] \arrow[dd, "j"] & & \mathrm{SO}(n) \arrow[dd, "j_P"] \arrow[dr, "\partial'"] & \\
\mathrm{Diff}(D^n, \partial) \arrow[ur, "i'"] \arrow[dr, "i"'] & & & & B\,\mathrm{Diff}(D^n, \partial) \\
 & \mathrm{Diff}\, S^n_\beta \arrow[rr, "\varepsilon"] \arrow[dr, "\pi"'] & & \mathrm{SO}(n+1) \arrow[ur, "g(\beta)"'] \arrow[dl, "\pi"] & \\
 & & S^n & &
\end{tikzcd}$$

Note that we have labeled $\bar{h} \circ e$ as ε, and the classifying map of ε as $g(\beta)$. Of course j_P is precisely the standard inclusion, and the homotopy commutativity of the central square is assured by construction.

(1.4). Definitions. (i) Let $W_*(S^n) \subset \pi_*(\mathrm{Diff}\, S^n)$ be the graded subgroup $\mathrm{Im}\, i'_* \cap \ker j_*$.
(ii) Let $X_*(S^n) \subset \pi_*(\mathrm{SO}(m+1))$ be the graded set of elements x with the property that $\pi_*(x) \neq 0 \neq g(\beta)_*(x)$.

Thus $W_*(S^n_\beta)$ is a measure of the extent to which j_P fails to determine j, and $X_*(S^n_\beta)$ is a measure of the extent to which ∂' fails to determine $g(\beta)$. Also $0 \neq x \in W_*(S^n_\beta)$ yields $(i')^{-1}(w) \in X_*(S^n_\beta)$.

We investigate $X_*(S^n_\beta)$ below. Note that, if $k < n$, then $W_k(S^n_\beta) = 0$ and $X_k(S^n_\beta) = \emptyset$. Also, $W_*(\ )$ and $X_*(\ )$ are defined for arbitrary M^n.

If S^n_β is the standard sphere S^n, then (1.3) 'collapses'. For the symmetry of S^n allows splitting $\mathrm{Diff}\, S^n \leftarrow \mathrm{SO}(n+1)$ of (F), which restricts to a splitting of the upper fibration. Thus $W_*(S^n) = 0$ and $X_*(S^n) = \emptyset$.

Hence the cardinalities of $W_*()$ and $X_*()$ in some sense reflect the asymmetry of S^n_β. We develop below a detection procedure for 'stable' elements in $X_*(\ )$.

2. Detecting elements in $X_*()$

We first summarize some information from [6] concerning the map ∂' of (1.3). Before so doing, however, it is convenient to recall some familiar notation which will also be useful for the remainder of this section.

We shall write $\mathrm{Top}\, S^n$ for the group of orientation preserving homeomorphisms of S^n, so that we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{SO} \arrow[r] & S\,\mathrm{Top} \arrow[rr] & & \mathrm{Top}/O \arrow[dddd] \\
& & S\,\mathrm{Top}(n) \arrow[ul] \arrow[dl, "k"'] & \\
\mathrm{SO}(n+1) \arrow[uu] \arrow[dd] \arrow[r, "i"] & \mathrm{Top}\, S^n \arrow[dr, "j"] & & \\
& & \Omega_1^{n+1} S^{n+1} \arrow[dl] & \\
\mathrm{SO} \arrow[r, "J"] & \Omega_1^{\infty} S^{\infty} = SG \arrow[rr] & & G/O
\end{tikzcd}$$

Here i is the standard inclusion, j is the suspension, and k is one-point compactification.

Now according to [1], the composite $\pi_*(S\,\mathrm{Top}(n)) \to \pi_*(\mathrm{Top}/O)$ is epic, so we may suppose that S^n_β is represented by a map $S^n \to S\,\mathrm{Top}(n)$ (tantamount to choosing a framing for S^n_β); or, via k_*, a map $\bar\beta : S^n \to \mathrm{Top}\, S^n$. Continuing around the diagram, we also obtain $\tilde\beta \in \pi_n(G/O)$ as the image of both $\bar\beta$ and β. Moreover $\tilde\beta$ lies in the summand $\pi_n(G)/\operatorname{Im} J$.

We can now state

(2.1). Theorem. (R. Schultz). *Given $\alpha \in \pi_k(\mathrm{SO}(n))$, then $\partial'_*\alpha \neq 0$ in $\pi_k(B\,\mathrm{Diff}(D^n, \partial))$ whenever $\tilde\beta J(\alpha) \neq 0$ in $\pi^S_{n+k}/\operatorname{Im} J$.*

Our result concerning $X_*(\ )$ is in the same spirit, and can be stated thus:

(2.2). Theorem. *Given $x \in \pi(\mathrm{SO}(n+1))$, then $x \in X_k(S^n_\beta)$ whenever $\tilde\beta^2 \cdot H(x) \neq 0$ in $\pi^S_{n+k}/\operatorname{Im} J$.*

Here $H(x) \in \pi_k(S^n)$ is represented by the composite $S^k \xrightarrow{x} \mathrm{SO}(n+1) \xrightarrow{\pi} S^n$ (π being the usual projection), which is proven in [2] to be the Hopf invariant (up to sign and suspension) of the element $J(x) \in \pi_{k+n+1}(S^{n+1})$.

Elements so detectable constitute a stable subset $SX_*(S^n_\beta) \subset X_*(S^n_\beta)$. Unfortunately, we know of no non-zero classes of the form $y^2 \cdot H(x)$ in any stem mod $\operatorname{Im} J$. The experts seem to regard this as a potentially accessible, but unsolved , problem of homotopy theory. Certainly, all $x \in \pi_{n+t}(\mathrm{SO}(n+1))$ give $y^2 \cdot H(x) = 0$ for all y when t is small.

There are two main steps involved in establishing (2.2): these follow below as (2.3) and (2.4).

The first includes generalizing a diagram of [6, p. 240]. Since in (1.3) we have set up a map $\varepsilon : \mathrm{Diff}\, S^n \to \mathrm{SO}(n+1)$ (which depends on the choice of $\bar{h}$ in (1.2)), it is important to relate ε with the standard inclusion of both $\mathrm{Diff}\, S^n$ and $\mathrm{SO}(n+1)$ in $\mathrm{Top}\, S^n$. This is done by

(2.3). Lemma. *The following diagram is homotopy commutative:*

$$\begin{tikzcd}
\mathrm{Diff}\, S^n_\beta \arrow[r, hook] \arrow[d, "e"'] \arrow[dd, bend right=60, "\varepsilon"'] & \mathrm{Top}\, S^n_\beta \arrow[d, "\chi_h"'] \\
P\tau(S^n) \arrow[d, "\bar{h}"'] & \mathrm{Top}\, S^n \arrow[d, "c_\beta"'] \\
\mathrm{SO}(n+1) \arrow[r, hook, "i"] & \mathrm{Top}\, S^n
\end{tikzcd}$$

Here χ_h is conjugation by the homeomorphism h, whereas c_β is the composition

$$\mathrm{Top}\, S^n \underset{\pi\times 1}{\longrightarrow} S^n \times \mathrm{Top}\, S^n \underset{\bar{\beta}\times 1}{\longrightarrow} \mathrm{Top}\, S^n \times \mathrm{Top}\, S^n \underset{\mu}{\to} \mathrm{Top}\, S^n$$

where π projects a homeomorphism onto its value at $0 \in S^n = \mathbb{R}^n \cup \{\infty\}$, and μ is composition of functions. Note that $\pi \circ_\beta (f) = \pi(f) \in S^n$ for all f, and that c_β is a homeomorphism with inverse $c_{-\beta}$.

The proof of (2.3) proceeds by passing between three equivalent versions of $P^{\mathrm{Top}}\tau(S^n)$, the oriented principal topological tangent bundle of S^n. These may be displayed by the commutative diagram of (principal) fibrations

$$\begin{tikzcd}
S\,\mathrm{Top}(n) \arrow[r, "\sigma_1"] \arrow[d] & \mathrm{Top}_*\, S^n \arrow[r, "\sigma_2"] \arrow[d] & \mathcal{E}^{\mathrm{Top}}_*(D^n, S^n) \arrow[d] \\
P^{\mathrm{Top}}\tau(S^n) \arrow[r, "\varphi_1"] \arrow[d, "\pi"'] & \mathrm{Top}\, S^n \arrow[r, "\varphi_2"] \arrow[d, "\pi"'] & \mathcal{E}^{\mathrm{Top}}(D^n, S^n) \arrow[d, "\pi'"] \\
S^n \arrow[r, "1"] & S^n \arrow[r, "1"] & S^n
\end{tikzcd}$$

The maps σ_1 and φ_1 are induced by compactification, and σ_2 and φ_2 be restricting a homeomorphism of $\mathbb{R}^n \cup \{\infty\}$ to D^n.

To introduce the second ingredient in the proof of (2.2), let us return to our fibration of (1.3). Suppose that $\alpha : Y \to \mathrm{SO}(n+1)$ is a map of some reasonable space into the base. If α does not factor through $\varepsilon : \mathrm{Diff}\, S^n \to \mathrm{SO}(n+1)$ and does not lift to $\mathrm{SO}(n)$, then it represents a class with the properties we are seeking for $X_*(S^n_\beta)$ (in case Y is a sphere). We thus wish to discuss the obstruction to lifting α to $\operatorname{Diff} S^n_\beta$.

We may assume without loss of generality that the suspension $S^1 \wedge Y$ is given as an open subset of some euclidean space $\mathbb{R}^n$; in other words as an open smooth manifold. Now let $\alpha^*\gamma_\beta$ be the topological S^n bundle over $S^1 \wedge Y$ which arises by adjoining the composite

$$\begin{tikzcd} Y \arrow[r, "\alpha"] & \mathrm{SO}(n+1) \arrow[r, "i"] \arrow[dr, "\gamma_\beta"'] & \operatorname{Top} S^n \arrow[d, "c_{-\beta}"] \\ & & \operatorname{Top} S^n \end{tikzcd}$$

Then if α lifts to $\operatorname{Diff} S^n_\beta$, (2.3) tells us that the total space $E(\alpha^*\gamma_\beta)$ admits a smoothing which restricts to β on each fibre.

In fact it is most useful to work universally, and to consider the case of $Y = \mathrm{SO}(n+1)$ and α the identity map. Then $E(\gamma_\beta$ can be constructed by first choosing the 'core' plus a single fibre, i. e. $S^n \cup C\,\mathrm{SO}(n+1)$, where the attaching map is $\pi \circ \gamma_\beta = \pi$. To this we must further attach a cone on the join $S^{n-1} * \mathrm{SO}(n+1)$ by a suitable map η. We therefore have a cofibre sequence

$$S^n \wedge \mathrm{SO}(n+1) \xrightarrow{\eta} S^n \cup_\pi C\,\mathrm{SO}(n+1) \xrightarrow{\theta} E(\gamma_\beta) \tag{C}$$

But $E(\gamma_\beta)$ is a topological manifold, fibered by S^n's and over a smooth base. As such it admits a $S\operatorname{Top}(n)$ bundle of tangents along the fibres, say

$$\operatorname{Top}_{\tau_F} : E(\gamma_\beta) \to BS\operatorname{Top}(n).$$

Our aim is to determine the extent to which $E(\gamma_\beta)$ admits a smoothing fibred by S^n_β's, or equivalently to which it carries an n-plane bundle τ_F, agreeing with $\operatorname{Top}_{\tau_F}$ topologically and restricting to $\tau(S^n_\beta)$ on each fibre.

Now from §1, $\tau_\beta : S^n_\beta \to B\,\mathrm{SO}(n)$ extends to some bundle $\bar\tau_\beta$ over $S^n_\beta \cup_\pi C\,\mathrm{SO}(n+1)$. Thus we may construct τ_F at least over $S^n \cup_\pi C\,\mathrm{SO}(n+1)$, by composing $\bar\tau_\beta$ with $\bar h^{-1}$.

Returning to our cofibration (C), we can consider $\eta^*\tau_F$ over $S^n \wedge \mathrm{SO}(n+1)$. This is topologically trivialized by the existence of $\operatorname{Top}_{\tau_F}$, so we have a map $\sigma(\beta) : S^n \wedge \mathrm{SO}(n+1) \to \operatorname{Top}(n)/O(n)$ which fits into the following homotopy commutative diagram

$$\begin{tikzcd} S^n \wedge \mathrm{SO}(n+1) \arrow[r, "\eta"] \arrow[d, "\sigma(\beta)"] & S^n \cup_\pi C\,\mathrm{SO}(n+1) \arrow[r, "\theta"] \arrow[d, "\tau_F"] & E(\gamma_\beta) \arrow[d, "\operatorname{Top}_{\tau_F}"] \\ \operatorname{Top}(n)/O(n) \arrow[r] & B\,\mathrm{SO}(n) \arrow[r] & BS\operatorname{Top}(n) \end{tikzcd}$$

So $\sigma(\beta)$. which we shall confuse with its adjoint $\mathrm{SO}(n+1) \to \Omega(\operatorname{Top}(n)/O(n))$, is the obstruction to extending τ_F over the whole of $E(\gamma_\beta)$.

Thus in terms of our original $\alpha^*\gamma_\beta$ we deduce that if $\alpha : Y \to \mathrm{SO}(n+1)$ lifts to Diff S^n_β then the composite

$$Y \xrightarrow{\alpha} \mathrm{SO}(n+1) \xrightarrow{\sigma(\beta)} \Omega(\mathrm{Top}(n)/O(n))$$

is null-homotopic.

We have shown only that this map is a necessary obstruction to lifting α. In the light of the celebrated Morlet equivalence $B\,\mathrm{Diff}(D^n, \partial) \simeq \Omega(\mathrm{Top}(n)/O(n))$ (e. g. see [3]) it seems highly likely that $\sigma(\beta)$ and $g(\beta)$ of (1.3) are the same map. Note that on homotopy groups $\sigma(\beta)$ induces a non-bilinear extension to $\pi_k(\mathrm{SO}(n+1))$ of the Milnor pairing $(\ ;\beta) : \pi_k(\mathrm{SO}(n)) \to \pi_{k+n}(\mathrm{Top}/O)$.

For calculational purposes, and given the current state of the art, unstable results such as we have obtained are not especially helpful. We must therefore show

(2.4). Lemma. *The stabilization*

$$s\sigma(\beta) : \mathrm{SO}(n+1) \xrightarrow{\sigma(\beta)} \Omega^n(\mathrm{Top}(n)/O(n) \to \Omega^n(\mathrm{Top}/O)$$

may be described as

$$\mathrm{SO}(n+1) \underset{\gamma_\beta}{\longrightarrow} \mathrm{Top}\, S^n \underset{J}{\to} \Omega^{n+1}S^{n+1} \underset{\Omega^{n+1}\beta}{\longrightarrow} \Omega^{n+1}B(\mathrm{Top}/O)$$

This formula follows simply from stabilizing the bundles in our discussion above.

To complete the proof of (2.2), we must choose $Y = S^k$ and α to represent a class $x \in \pi_k(\mathrm{SO}(n+1))$ such that $\pi_*(x) \neq 0$ in $\pi_k(S^n)$. Then by (2.4), $x \in X_k(S^n)$ if

$$S^k \underset{x}{\to} \mathrm{SO}(n+1) \underset{\sigma(\beta)}{\longrightarrow} \Omega^n(\mathrm{Top}/O)$$

is not null-homotopic. The usual detection procedure for such a map is then to pass to $\Omega^n(G/O)$, and to compute its value in the summand $\pi^S_{n+k}/\operatorname{Im} J \subset \pi_{k+n}(G/O)$.

In our case the maps involved can be unraveled to give $s\sigma(\beta)_*x$ module $\operatorname{Im} J$ as

$$S^k \xrightarrow{x} \mathrm{SO}(n+1) \xrightarrow{\pi\times J} S^n \times G \xrightarrow{-\tilde\beta\times 1} G \times G \xrightarrow{\circ} G \xrightarrow{\circ\tilde\beta} \Omega^n G.$$

This represents $\tilde\beta \circ (J(x) - \tilde\beta(\pi\circ x))$ in $\pi^S_{n+k}/\operatorname{Im} J$. But $\pi \circ x = H(x)$, whilst Novikov [5] and Kosinski [4] have shown that $\tilde\beta \circ J(x) \in \operatorname{Im} J$ whenever $k > \frac{1}{2}n + 1$ (which is certainly the case here).

We can now deduce our detection formula (2.2) in the form

$$s\sigma(\beta)_*x = \pm\tilde\beta^2 \cdot H(x) \quad \text{in} \quad \pi^S_{n+k}/\operatorname{Im} J.$$

Note that if S^n_β bounds a parallelizable manifold, then $\tilde\beta = 0$ by definition. So $SX_*(S^n_\beta) = \emptyset$. We conclude with a result which is a more subtle version of this same fact

(2.5). Proposition. *Let $S'X_*(S^n_\beta)$ be the intermediate set $SX_*(S^n_\beta) \subset S'X_*(S^n_\beta) \subset X_*(S^n_\beta)$ of elements detected by $\sigma(\beta)_*x \in \pi_{n+k}(\mathrm{Top}/O)$. Then $S'X_*(S^n_\beta) = \emptyset$ if S^n_β bounds a parallelizable manifold.*

Proof. By choice, $\beta \in \pi_{n+k}(B\,\mathrm{Top}/O))$ lifts to $\pi_n(G/\mathrm{Top})$. But localized at 2, G/Top is a product of Eilenberg-MacLane spaces, and at odd primes is equivalent to BO. In either case $\beta \circ f = 0$ for any $f \in \pi_{k+n+1}(S^{n+1})$. □

This may be one more way of saying that such S^n_β's are the most symmetric of exotic spheres.

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