THE SURGERY EXACT SEQUENCE REVISITED

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Abstract. The purpose of this paper is to discuss the four-periodicity of the topological surgery exact sequence from the point of view of controlled surgery.

Consider the classical surgery exact sequence [7], for determining the topological structure set for a given n-dimensional Poincaré duality space X

$$\ldots \to L_{n+1}(\mathbb{Z}\pi_1 X) \to \mathcal{S}(X) \to [X, G/\operatorname{Top}] \to L_n(\mathbb{Z}\pi_1 X).$$

This is a sequence of groups and homomorphisms except in the bottom of the sequence where interpretations are needed. The term $[X, G/\operatorname{Top}]$ needs a specific lift of $X \to BG$ to $X \to B\operatorname{Top}$ and it measures the variations in this lift. The map $\mathcal{S}(X) \to [X, G/\operatorname{Top}]$ compares the lift of $X \to B\operatorname{Top}$ given by a homotopy equivalence $M \to X$ to the one chosen above. The map $L_{n+1}(\mathbb{Z}\pi_1(X)) \to \mathcal{S}(X)$ is given as an action on a basepoint, so a specific manifold structure has to be chosen on X to define the map. Exactness of the sequence is in terms of based sets. One way to resolve these problems was given in Quinn's and Nicas' theses [3, 4]. The solution was to exhibit G/Top with a different infinite loop space structure and it turned out one could obtain a sequence of groups and homomorphisms. This solution is somewhat complicated involving a complicated definition of spectra to get to the very definition of the surgery exact sequence. It also provides a group structure on the structure set, that is not easily understood.

This issue is of some importance because of the applications. The Borel conjecture states that when X is a $K(\pi, 1)$ i. e. a space for which all higher homotopy groups vanish, then the structure set contains at most one point. In other words, given a homotopy equivalence of $K(\pi, 1)$- manifolds, it is homotopic to a homeomorphism. Farrell and Jones [1] confirm the Borel conjecture in many cases. The method however is always to prove that the higher structure sets are trivial

$$\mathcal{S}(X \times D^l, \partial(X \times D^l)) = 0, \qquad l \geq 1.$$

Farrell and Jones then use four periodicity to conclude $\mathcal{S}(X) = 0$. The four periodicity of the topological surgery exact sequence is a complicated

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argument using Sullivan's characteristic variety theorem. The main purpose of this note is to show how one can avoid using this four-periodicity.

We thus indicate a relatively direct proof that the structure set of a topological manifold M is precisely one point when $\mathcal{S}(M \times D^4, \partial(M \times D^4)) = 0$, using controlled techniques. This provides a different path to establishing $\mathcal{S}(X) = 0$ in the Farrell-Jones situation above.

Consider a compact topological manifold M without boundary, and a proper degree one map from a manifold pair

$$(W, \partial W) \to (M \times [0, 1), M \times 0)$$

which is a homotopy equivalence on the boundary $\partial W \to M \times 0$. We will call this map a continuously controlled homotopy equivalence if it is a proper homotopy equivalence of pairs, and the tracks of the homotopies become arbitrarily small when we approach 1, i. e. when the second coordinate is close to 1. The precise definition of "small" is that for every point x in $M \times 1$ and for every neighborhood U of x in $M \times I$ there exists a neighborhood V so that if the track of the homotopy at some point intersects V non trivially, then it is entirely contained in U.

We may now ask the question: Is it possible to do proper surgery to obtain a controlled homotopy equivalence of pairs? Using the techniques in [2] one sees it is possible to set up a surgery theory to determine this question, if we are able to construct an additive category with involution, which determines when a map is a continuously controlled homotopy equivalence. For the moment we shall assume such a category does exist, and we will give its definition later. Let us call such a category $\mathcal{B}$, and proceed to describe how it can be used to obtain our goal.

Theorem 1. *Let M be a topological manifold of dimension $n \geq 5$ Then there is a map*

$$\mathcal{S}(M \times D^l, \partial(M \times D^l)) \to L_{n+l+1}(\mathcal{B})$$

which is an isomorphism when $l \geq 1$. When $l = 0$, an element goes to zero if and only if the homotopy equivalence is h-cobordant to a homeomorphism.

Note that in the case $l = 0$, the function is from a pointed set to an abelian group.

Corollary 2. *If $\mathcal{S}(M \times D^4, \partial(M \times D^4)) = 0$ then $\mathcal{S}(M) = 0$*

Proof. Assume $N \to M$ is an element in $\mathcal{S}(M)$. The surgery obstruction lies in $L_{n+1}(\mathcal{B})$. But surgery groups are obviously 4-periodic, so $L_{n+1}(\mathcal{B}) = L_{n+5}(\mathcal{B})$ which is 0. We thus know that the homotopy equivalence is h-cobordant to a homeomorphism, and the element is 0 in the structure set. □

Proof of Theorem. We start out giving the proof in the case when $l = 0$ as follows. Consider an element in the structure set, i. e. a homotopy equivalence of compact manifolds

$$\phi : N \to M$$

Letting ψ be a homotopy inverse to ϕ we get a diagram

$$\begin{array}{ccc} \nu_N \times [0,1) & \longrightarrow & \psi^*(\nu_N) \times [0,1) \\ \downarrow & & \downarrow \\ N \times [0,1) & \xrightarrow{\phi \times 1} & M \times [0,1) \end{array}$$

Notice there is no choice in the bundle map, since ϕ is a homotopy equivalence. This may be considered a controlled surgery problem. The point is that even though ϕ is a homotopy equivalence, it is not continuously controlled. The tracks of the homotopy do not get small near 1. We now obtain an element in $L_{n+1}(\mathcal{B})$ by following the procedures in [2], first doing surgery below the mid-dimension, and finally getting forms and formations defining the surgery obstruction. The arguments in [2] adopt precisely to this situation. If the surgery obstruction vanishes, we do surgery relative to the boundary $N \to M \times 0$ to obtain a continuously controlled homotopy equivalence

$$(\partial_1 W, N) \to (M \times [0,1), M \times 0).$$

Consider the map $\partial_1 W \to M \times [0,1)$. The continuous control condition easily implies, that if we compose with the projection to M, $\partial_1 W \to M$, we get a tame end. Using Quinn's end theorem [5] we may add a boundary M_1 , to $\partial_1 W$ and extend the map to M_1, to get a map of triples

$$(\overline{\partial_1 W}; N, M_1) \to (M \times [0,1], M \times 0, M \times 1)$$

The map $M_1 \to M$ can be perturbed by a small homotopy to the composition

$$M_1 \to M_1 \times (1-\epsilon) \to M \times I \to M$$

where the first map is given by collaring the boundary, the second is the restriction of the map above, and the third map is the projection. The map $M_1 \to M$ is thus an arbitrarily small homotopy equivalence, hence a cell-like map. Using Siebenmann's approximation theorem [6] it can be moved by an arbitrarily small homotopy to a homeomorphism. The triple $(\overline{\partial_1 W}; N, M_1)$ is now an h-cobordism so the element in the structure set was the trivial element.

When $l > 0$ the structure set has elements $W \to M \times D^l$ which are homeomorphisms on the boundary. We have a group structure on the structure set in this case, by gluing elements along a piece of the boundary. The map we have described above becomes a homomorphism, because the way we obtain an element in the L-group is by doing surgery below the mid-dimension. When we have glued two elements as above, we can do the surgery below the mid-dimension on the pieces individually, and the chain complex of the result will just be the sum of the two chain complexes. The argument we gave for the case $l = 0$ did not prove the map was a monomorphism, since we do not have a group structure in that case. Carried over to the $l > 0$ case however, it does prove the map is a monomorphism. To see the map is an epimorphism we use Wall's realization argument. The argument is the same for all $l > 0$, so let us assume $l = 1$. We start with the identity map

$$M \times [0,1) \to M \times [0,1)$$

and an element in $L_{n+2}(\mathcal{B})$ where n is the dimension of M. If n is even, this element is given by an intersection form. We embed a system of trivial $[\frac{n}{2}]$-dimensional spheres in $M \times [0,1)$ corresponding to a basis of the object in $\mathcal{B}$ defining the element, increasingly smaller as we approach 1. We now follow Wall's realization procedure. First cross with I, then change the embedding of the spheres cross I so that the intersections mirror the intersections given by the element in L-group. At the end of this process we have a differently embedded system of spheres at level 1 in I, and after doing surgery on this system of spheres we get a map at level one which is a chain homotopy equivalence when measured in $\mathcal{B}$, hence a continuously controlled homotopy equivalence. We now proceed as in the $l = 0$ case to put an end on the manifold. Using a small collar of the added end, we have a homeomorphism at the 1-level, and the trace of the surgery is easily seen to have the given surgery obstruction. When n is odd, we follow Wall's procedure for odd dimensional realization, and proceed as above. □

We finish off by giving a description of the categories $\mathcal{B}$ above. Let E be a free Γ-space and R a ring with involution. For the application above choose $\mathbb{Z}$ with the trivial involution. The category $\mathcal{B}^\Gamma(E \times [0,1], E \times 1; R)$ has objects based free R-modules with a free Γ-action on the basis, together with a proper reference map from the basis to $E \times [0,1)$. This means that we can think of an object A as a direct sum of finitely generated free based R-modules A_x being generated by the basis-elements mapping to $x \in E \times [0,1)$. The properness condition ensures that A_x is finitely generated and that the set $\{x \in E \times [0,1) | A_x \neq 0\}$ is locally finite in $E \times [0,1)$.

The objects are free $R\Gamma$-modules in an obvious way. As morphisms we allow $R\Gamma$-module morphisms ϕ that satisfy a continuous control condition near 1: Write $\phi = \{\phi^x_y\}$ where $\phi : A \to B$, $A = \oplus A_x$, and $B = \oplus B_y$, $\phi^x_y : A_x \to B_y$ so ϕ^x_y is precisely the component of ϕ mapping the summand A_x to B_y. The control condition now states: For every $z \in E \times 1$ and for every neighborhood U of z in $E \times [0,1]$ there must exist a neighborhood V of z in $E \times [0,1]$ so that ϕ^x_y must be the zero homomorphism if $x \in V$ and $y \notin U$ or $y \in V$ and $x \notin U$. Visually if we draw an arrow from x to y whenever ϕ^x_y is nonzero, the arrows have to become small as we approach points in $E \times 1$.

Now let $\mathcal{B} = \mathcal{B}^\pi(\tilde{M} \times [0,1], \tilde{M} \times 1; \mathbb{Z})$, where $\pi = \pi_1(M)$. It is relatively easy to see that we can test whether a proper map $f : W \to M \times [0,1)$ is a controlled homotopy equivalence as follows: First we check some conditions on the fundamental group of W, that small loops near 1 bounding a disc in W also bound a small disk, and that all elements in $\pi_1(W)$ can be realized by small loops near 1. Secondly we triangulate W and $M \times [0, 1)$ in such a way that all simplexes become small near 1. We may then think of the chain complex of the universal cover as a chain complex in $\mathcal{B}$. This is done by associating to each simplex in $\tilde{M} \times [0, 1)$ its barycenter, and to each simplex in $\tilde{W}$ the image of its barycenter. Then f will be a controlled homotopy equivalence if and only if $f_\#$ is a chain homotopy equivalence in the category $\mathcal{B}$. The argument is a variant of the standard argument giving algebraic conditions for homotopy equivalence.

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