EMBEDDINGS OF TOPOLOGICAL MANIFOLDS

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It is the purpose of this paper to combine the methods of [8] and [9] to prove Haefliger type embedding theorems for topological manifolds. We prove the following embedding theorems:

**Theorem 1.** Let $M$ be a topological manifold with boundary $\dim(M) = m$, $m \geq 6$, and let
\[ \alpha \in \pi_r(M, \partial M), \quad r \leq m - 3. \]
Assume that $(M, \partial M)$ is $2r - m + 1$ connected. Then $\alpha$ can be represented by a locally flat embedded disc
\[ f : (D^r, S^{r-1}) \to (M, \partial M). \]

The proof is modeled on the similar PL proof in [3] using the above mentioned modifications of the techniques.

**Theorem 5.** Let $K$ be a finite $k$-dimensional complex, $M$ a topological manifold $m \geq 6$, $m - k \geq 3$, and $f$ a continuous map, $f : K \to M$, so that
\[ \pi_i(f) = 0 \quad \text{for} \quad i \geq 2k - m + 1. \]
Then there is a complex $K'$ of dimension $k'$, $k' \leq k$, such that $K'$ is locally tamely embedded in $M$, and a simple homotopy equivalence $h : K \to K'$ so that the diagram
\[ \begin{array}{ccc}
K & \xrightarrow{f} & M \\
\downarrow{h} & & \downarrow{\text{homotopy}} \\
K' & & \\
\end{array} \]
is homotopy commutative.

Finally we obtain the “Haefliger type” theorem:

**Theorem 7.** Let $f : M^p \to V^q$ be a map of topological manifolds, $q \geq 6$, $q - p \geq 3$, $M$ closed, and assume that
\[ \pi_i(f) = 0 \quad \text{for} \quad i \leq 2p - q - 1. \]
Then $f$ is homotopic to a locally flat embedding.
Theorem 7 is proved as in the PL case using surgery and the normal theory due to Rourke and Sanderson classifying neighborhoods of topological manifolds [10] although we avoid the use of nonstable transversality. It generalizes the theorem of Lees [6]. Lees assumes the spaces are connected rather than the map.

For the proof of Theorem 1 we need a general position lemma.

**Definition 2.** Let $P^r$ be a PL complex and $M^m$ a topological manifold and $f : (P, P^r) \to (M, \partial M)$ a continuous map, $P^r$ a subcomplex of $P$. We say that $f$ is in general position if $K = \text{im}(f)$ has the structure of a complex such that $f$ is PL, $K \subset M$ is locally tamely embedded and the double point set $S_2(f) \subset P$ is of dimension at most $2p - m$.

**Lemma 3.** Let $M$ be a topological manifold, $\dim(M) \geq 6$ and $f : (D^r, S^{r-1}) \to (M, \partial M)$ a continuous map, $m - r \geq 3$. Then $f$ is homotopic to a map in general position.

**Proof.** By the relative version of Lees’ immersion theorem (see [10]) the map $F : (D^r, S^{r-1}) \times \mathbb{R}^{m-r} \to (D^r, S^{r-1}) \to (M, \partial M)$ is homotopic to an immersion, since any bundle over $D^r$ is trivial. Therefore we may assume that $F$ is an immersion, so we can cover $D^r$ by finitely many open sets $U_i$ and shrink fibers so that $F|U_i \times \mathbb{R}^{m-r}$, $i = 1, 2, \ldots, s$ is an embedding. Lert $\{Z_i\}$ be a compact refinement of the covering $\{U_i\}$ such that $Z_i$ is a subcomplex of $D^r$. The proof will be by induction on the following statement. There is an isotopy $h^i$ of $D^r \times \mathbb{R}^{m-r}$ and subcomplexes $Q_i$ of $D^r$ such that $\bigcup_{j \leq i} Z_j \subset \text{int}(Q_i)$ and $F \circ h^i$ is in general position.

The induction starts trivially, so assume we have obtained the statement for $i - 1$. We may then as well assume the true originally and denote $F \circ h^i_{i-1}$ by $F$ and $f \circ h_1^{i-1}$ by $f$. Consider $V = F(U_i \times \mathbb{R}^{m-r}) \subset M$. $V$ has a PL structure induced by $F$. Let $K_{i-1} = \text{im}(Q_{i-1})$. Then by the induction hypothesis $K_{i-1}$ is a PL complex locally tamely embedded in $M$, and of codimension at least 3. Therefore by [2] (see [9, Theorem 2]) we can change the PL structure of $V$ so that a compact regular neighborhood of $K_{i-1} \cap V$ is PL embedded. Hence if we shrink $U_i$ and the fibers, we may assume $K_{i-1} \cap V \subset V$ is a PL embedding. We shrink $U_i$ so little that $U_i$ still contains $Z_i$. Since $f : Q_{i-1} \to K_{i-1}$ is PL the restriction of $f$ to $U_i \cap Q_{i-1}$ is PL, so as before by shrinking $U_i$ and $Q_{i-1}$ we can find an isotopy of $U_i \times \mathbb{R}^{m-r}$ that fixes everything outside a compact set and a neighborhood of $U_i \cap Q_{i-1}$, hence may be assumed to fix $Q_{i-1} \cap U_i \times \mathbb{R}^{m-r}$, and moves $f|U_i$ to a PL embedding. The restriction of $f$ to $Q_{i-1} \cap U_i$ is already in general position in the PL sense, so by PL general position we may move $U_i$ as above so that we end up having $f$ in general position in a closed neighborhood $Q_i$ of $\bigcup_{j \leq i} Z_j$. This ends the induction step.

We now only need the following trivial lemma before the proof of Theorem 1.
Lemma 4. Let $M$ be a manifold and assume $(M \times I, M \times 0, M \times 1)$ is written as the union of two triads $V$ and $W:\]
(M \times I, M \times 0, M \times 1) = (V \bigcup_{\partial_+ V = \partial_+ W} W, \partial_- V, \partial_+ W).

Then if $\pi_j(V, \partial_+ V) = 0$ for $j \leq r$, $r \geq 2$, then $\pi_j(W, \partial_+ W) = 0$ for $j \leq r - 1$.

Proof. Van Kampen’s Theorem applied to $W$ and $V$ shows that $\pi_1(W) = \pi_1(M \times I)$ since $r \geq 2$; hence $\pi_1(W) \cong \pi_1(M \times 1) = \pi_1(\partial_+ W)$. Let $\tilde{M}$ denote the universal covering space of $M$ and denote inverse images in $\tilde{M} \times I$ by $\tilde{\cdot}$. Then

$$H_{j+1}(\tilde{V}, \partial_+ \tilde{V}) = H_{j+1}(\tilde{M} \times I, \tilde{W}) = H_j(\tilde{W}, \partial_+ \tilde{W})$$

by excision and the long exact sequence for the triple $\partial_+ \tilde{W} \subset \tilde{W} \subset \tilde{M} \times I$. □

Proof of Theorem 1. The proof is by induction on the following statement:

$$F : (D^r, S^{r-1}) \to (M, \partial M)$$

is homotopic to $f_i$, and $\partial M$ has a collar in $M$ that is decomposed as

$$(\partial M \times [0,1], \partial M \times 0, \partial M \times 1) = (V \bigcup_{\partial_+ V = \partial_+ W} W, \partial_- V, \partial_+ W)$$

where $V$ is obtained from $\partial_- V = \partial M \times 0$ by adjoining handles of dimension less than $2r - m + 1 - i$, $f_i^{-1}(V)$ is a collar of $\partial D^r$ in $D^r$ and

$$f_i|_{D^r - f_i^{-1}(V)} : D^r - f_i^{-1}(V) \to M - V$$

is a locally flat embedding.

We start the induction by homotoping $f$ to $f_0$ a map in general position as in Lemma 3. Let $S_2(f_0)$ be the double point set of $f_0$. Then since the codimension of $S_2(f_0)$ in $D^r$ is at least 3 we can find a complex $Z_0$ in $D^r$ of dimension $\dim(S_2(f_0)) + 1$, i.e. of dimension less than $2r - m + 2$ so that $S_2(f_0) \subset Z_0$ and $\partial D^r \cup Z_0$ simplicially collapses to $\partial D^r$ (see [3]). Let $R_0$ be the image of $Z_0$. Then $R_0$ is a subcomplex of $\text{im}(f_0)$ of dimension at most $2r - m + 1$, and by Newman’s engulfing theorem [7] we may assume that $R_0$ is contained in some collar of $\partial M$, $R_0 \subset \partial M \times [0,1]$. Let $N_0$ be a regular neighborhood of $R_0$ in $\partial M \times [0,1]$ that intersects $\text{im}(f_0)$ in a regular neighborhood of $R_0$ in $\text{im}(f_0)$ (see [9]). Then $f_0^{-1}(N_0)$ is a regular neighborhood of $Z_0$ so it collapses to $\partial D^r$. This picture is clearly homotopic to the picture where we have glued on a collar on $\partial D^r$ and $\partial M$, so do that and put

$$V_0 = \partial M \times [0,1] \cup N_0, \quad W_0 = \overline{\partial M \times [-1,1]} - V_0.$$ 

This then starts the induction. Assume the hypothesis for $i$. Denote

$$f_i(D^r - f_i^{-1}(V_i)),$$
i. e. the image of $D^r$ with a collar of the boundary deleted, by $B^r$. Then $B^r$ is locally flatly embedded,

$$B^r \subset M - V_i,$$

and therefore extends to an embedding

$$B^r \times \mathbb{R}^{m-r} \subset M - V_i$$

by [9, Lemma 3]. By Lemma 4 and [8, Theorem 1], $W_i$ has a strong deformation retract, $\partial W_i \cup K_i$, where $K_i$ is a locally tamely embedded complex of dimension at most $\max(2, 2r - m - i + 1)$. Using [2] we may change the PL structure on $B^r \times \mathbb{R}^{m-r}$ and shrink the fibers so that

$$K_i \cap B^r \times \mathbb{R}^{m-r} \subset B^r \times \mathbb{R}^{m-r}$$

is a PL embedding. We then isotop $B^r$ so that $B^r$ is PL embedded in $B^r \times \mathbb{R}^{m-r}$ and finally such that $K_i$ and $B^r$ are in general position. This isotoping $B^r$ can obviously be done as a homotopy of $f_i$. Let $C_i = K_i \cap B^r$. Then, since $B^r$ is of codimension at least 3, $C_i$ is of dimension at most $2r - m - i - 2$ so we can find a complex $Z_i$ in $B^r$ of dimension one higher so that $Z_i \cup \partial B^r$ simplicially collapses to $\partial B^r$ and $Z_i$ contains $C_i$. Consider

$$K_i \cup Z_i \subset K_i \cup B^r \subset M - V_i.$$

By Newman’s engulfing theorem $K_i \cup Z_i \cup V$ is contained in a collar $\partial M \times I$ of the boundary, so we let $N_i$ be a regular neighborhood of $K_i \cup Z_i$ that intersects $K_i \cup B^r$ in a regular neighborhood of $K_i \cup Z_i$ in $K_i \cup B^r$ (see [9]), hence intersects $B^r$ in a regular neighborhood of $Z_i$. Define

$$V_{i+1} = V_i \cup N_i, \quad W_{i+1} = \overline{\partial M \times [0,1] - V_{i+1}}.$$

Then $V_{i+1}$ is obtained from $M$ by adjoining handles of dimension at most $2r - m - i - 1$ since that is the dimension of $Z_i$, $f^{-1}_{i+1}(V_{i+1})$ is a collar neighborhood of $D^r$ since $f^{-1}_{i+1}(N_i)$ is a regular neighborhood of $Z_i$ and $Z_i$ simplicially collapses to $\partial B^r$.

Eventually we get the dimension of $K_i$ to be 2, so when we put $K_i$ in general position to $B^r$ the intersection becomes empty and therefore when we have completed that step we have obtained that $f$ is homotopic to a map $f_i$ such that there is a collar of $\partial M$ in $M$ with $f_i^{-1}(\partial M \times I)$ a collar of $\partial D^r$ in $D^r$ and $f_i f^{-1}_i(M - \partial M \times [0,1])$ is a locally flat embedding. It is now easy to homotope $f$ so as to pinch off the collar where $f$ is not yet an embedding. \qed

We now show how this can be used to embed complexes in topological manifolds up to homotopy type.

**Theorem 5.** Let $K^k$ be a finite complex, $M^m$ a topological manifold, $m \geq 6$, $m - k \geq 3$, and $f$ a continuous map, $f : K \to M$, such that $\pi_i(f) = 0$ for $i \leq 2k - m + 1$. Then there is a complex $K'$ of dimension $k'$, $k' \leq k$, such that $K'$ is locally tamely embedded in $M$, and
a simple homotopy equivalence $h : K \to K'$ such that the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{f} & M \\
\downarrow h & & \downarrow \\
K' & \to & \text{N}
\end{array}
\]

is homotopy commutative.

Theorem 5 follows immediately from Theorem 1 in [8] and the following proposition.

**Proposition 6.** Let $M^m$ be a topological manifold of dimension $m \geq 6$, and let $K^k$ be a PL complex, $m - k \geq 3$, and $f$ a continuous map $f : K \to M$ such that $\pi_i(f) = 0$ for $i \leq 2k - m + 1$. Then there is a codimension 0 submanifold $N$ of $M$ and a simple homotopy equivalence $h : K \to N$ such that the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{f} & M \\
\downarrow h & & \downarrow \\
N & \to & \text{N}
\end{array}
\]

is homotopy commutative.

**Proof.** Filter $K$ by simplices of nondecreasing dimension

$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_s = K$

where $K_{i+1}$ is obtained from $K_i$ by adjoining a simplex. The proof will be by induction on the following statement: There is a codimension 0 submanifold $N_j$ of $M$ and a homotopy of $f$ to $f_j$ such that $f_j(K_j) \subset N_j$ and $f_j|K_j : K_j \to N_j$ is a simple homotopy equivalence.

It is trivial to start the induction so assume the statement for $j$. $K_{j+1} = K_j \bigcup_{s-r-1} D^r$ for some $r$, and $f_j|D^r$ determines an element in $\pi_r(M, N_j)$. We want to prove that the map

$\pi_r(M - N_j, \partial N_j) \to \pi_r(M, N_j)$

is onto. For $r < m - r - 1$ this is trivial since $N_j$ is obtained from $\partial N_j$ by adjoining handles of dimension $\geq m - r$, so

$\pi_s(\partial N_j) \to \pi_s(N_j)$

as well as

$\pi_s(M - N_j) \to \pi_s(M)$

is an isomorphism for $s \leq m - r - 2$ and onto for $s = m - r - 1$. For $r = m - r \geq 3$ we have as above

$\pi_s(N_j, \partial N_j) = 0$ for $s \leq r - 1$

and

$\pi_s(M - N_j, \partial N_j) = 0$ for $s \leq r - 1$
and since \( r - 1 \geq 2 \) this implies by Blakers-Massey [1] that

\[
\pi_s(M - N_j, \partial N_j) \to \pi_s(M, N_j)
\]

is an isomorphism for \( s \leq 2r - 3 \) and onto for \( s = 2r - 2 \).

In case \( r > m - r \) we have, since \( r \leq k \), that \( 2k - m + 1 \geq 2 \) so \( \pi_1(K) \cong \pi_1(M) \); since \( r \geq 3 \), \( \pi_1(K_j) \cong \pi_1(K) \) and

\[
\pi_1(N_j) \cong \pi_1(K) \cong \pi_1(N_j) \cong \pi_1(M - N_j)
\]

so all fundamental groups are the same. We denote universal covering spaces by \( \tilde{\ \ } \) and we have

\[
H_s(\tilde{M}, \tilde{N}_j) \cong H_s((M - N_j), \partial N_j)
\]

by excision, but

\[
H_s(\tilde{M}, \tilde{N}_j) = 0 \quad \text{for} \quad s \leq \min(r - 1, 2k - m + 1)
\]

so

\[
\pi_s(M - N_j, \partial N_j) = 0 \quad \text{for} \quad s \leq \min(r - 1, 2k - m + 1)
\]

and as before \( \pi_s(N_j, \partial N_j) = 0 \) for \( s \leq m - r - 1 \) so by Blakers-Massey [1]

\[
\pi_s(M - N_j, \partial N_j) \to \pi_s(M, N_j)
\]

is onto for \( s \leq \min(r - 1, 2k - m + 1) + m - r - 1 = 2k - r \). Hence we can choose a map

\[
\alpha : (D^r, S^{r-1}) \to (M - N_j, \partial N_j)
\]

representing \( f_j|D^r \). Since \( \pi_s(M - N_j, N_j) = 0 \) for \( s \leq \min(r - 1, 2k - m + 1) \), \( \alpha \) can, by Proposition 6 be chosen to be a locally flat embedding which can be extended to an embedding

\[
A : (D^r, S^{r-1}) \times \mathbb{R}^{m-r} \to (M - N_j, \partial N_j)
\]

(see [9, Lemma 3]). We now define \( N_{j+1} = N_j \cup A(D^r \times B^{m-r}) \) where \( B^{m-r} \) is the unit ball in \( \mathbb{R}^{m-r} \). Since \( K_j \subset K_{j+1} \subset K \) are cofibrations we can homotop \( f_j \) to \( f_{j+1} \) such that we get a Mayer-Vietoris diagram

\[
\begin{array}{ccc}
K_{j+1} & \xrightarrow{f_{j+1}} & N_{j+1} \\
\downarrow & & \downarrow \\
K_j & \xrightarrow{f_j} & N_j \\
\downarrow & & \downarrow \\
S^{r-1} & \xrightarrow{A(S^{r-1} \times B^{m-r})} & A(D^r \times B^{m-r})
\end{array}
\]

where \( f_{j+1} \) is a simple homotopy equivalence on the 3 small terms hence on \( K_{j+1} \). This ends the induction step \( \square \).
**Theorem 7.** Let $f : M^p \to V^q$ be a map of topological manifolds, $q \leq 6$, $q - p \geq 3$, $M$ closed, and assume that $\pi_i(f) = 0$ for $i \leq 2p - q + 1$. Then $f$ is homotopic to a locally flat embedding.

To prove Theorem 7 we need some lemmas.

**Lemma 8.** With the assumptions of Theorem 7 there is a codimension 0 submanifolds $N$ of $V$ such that $\pi_1(\partial N) \cong \pi_1(N)$ and a simple homotopy equivalence $h : M \to N$ such that the diagram

$$
\begin{array}{ccc}
M & \to & V \\
\downarrow & & \downarrow \\
N & \to & \\
\end{array}
$$

is homotopy commutative.

**Proof.** There is a $p$-dimensional complex $K$ which is simple homotopy equivalent to $M$. $K$ is obtained as follows. Let $C$ be the total space of the normal discbundle of $M$. Then $C$ is a PL manifold and determines by definition the simple homotopy type of $M$. We can then let $K$ be a PL spine of $D$. The codimension 0 submanifold $N$ of $V$ is now constructed by Proposition 6. □

The proof of Theorem 7 is completed by the following lemma.

**Lemma 9.** Let $f : M^p \to V^q$ be a simple homotopy equivalence, $M$ and $V$ topological manifolds, $M$ closed, $q \geq 6$, $q - p \geq 3$, and $\pi_1(\partial V) \to \pi_1(V)$ an isomorphism. Then $f$ is homotopic to a locally flat embedding.

**Proof.** It is proved in [10] that there is a classifying space $B\text{Top}_r$ for topological neighborhoods of codimension $r$. The classification goes via microbundles: to a topological neighborhood $P \subset Q$ one assigns the stable microbundle pair $(\tau_Q, i^*\tau_Q)$ which is then classified by $B\text{Top}_r$. There is a map $B\text{Top}_r \to BG_r$ assigning to the microbundle pair the corresponding spherical fibration pair. Since spherical fibration pairs split uniquely $BG_r$ is identified with $BG_r$ the classifying space for spherical fibrations of dimension $r$.

If $M \subset W$ is a locally flat inclusion of topological manifolds of codimension $\geq 3$ we can take a regular neighborhood $N$ of $M$ in $W$ (see [4]). By [11] the map $\partial N \subset N \to M$ is equivalent to a spherical fibration $\nu$, and $\nu_M = \nu \oplus i^*\nu_W$ as spherical fiber spaces. Therefore the map $B\text{Top}_r \to BG_r$ is described as follows: Take a closed regular neighborhood $N$, and turn $\partial N \to M$ into an $r$-spherical fibration $\nu$. The classifying map for $\nu$ is then $M \to B\text{Top}_r BG_r$.

We now return to our problem. Consider $r : V \to M$, a homotopy inverse to $f$, and the restriction of $r$ to $\partial V \to M$. This map is by Spivak [11] equivalent to a spherical fibration $\xi : E \to M$. Let $\overline{E}(\xi)$ be the mapping cone of $\xi$; then $(\overline{E}(\xi), E(\xi))$ is naturally homotopy
equivalent to \((V, \partial V)\). Assume that the map \(\xi \to BG_r\) lifts to \(B\text{Top}_r\):

\[
\begin{array}{c}
B\text{Top}_r \\
\downarrow \\
M \\
\rightarrow \\
BG_r
\end{array}
\]

We then have \(M\) locally flatly embedded in a manifold and if we take a regular neighborhood \(W\) of \(M\) we get a fiber homotopy equivalence

\[
\begin{array}{c}
(W, \partial W) \\
\downarrow \\
M \\
r \\
\rightarrow \\
(V, \partial V)
\end{array}
\]

and if the homotopy equivalence is homotopic to a homeomorphism we have proved that \(f\) is homotopic to an embedding. By Sullivan theory this is determined by the normal obstruction in \([W, G/\text{Top}] = [M, G/\text{Top}]\) but we have freedom in the choice of the lifting of \(\xi\) to \(B\text{Top}_r\) corresponding to \(G_r/\text{Top}_r\). But since \(r \geq 3\) this space is homotopy equivalent to \(G/\text{Top}\) by [10]. Hence we can choose our lifting so as to make the normal obstruction 0. This then ends the proof once we show the existence of one lifting. However \(\xi \oplus f^*\nu_V = \nu_M\) as spherical fibrations, so stably \(\xi = \nu_M \oplus (f^*\nu_V)^{-1}\), hence

\[
M \xrightarrow{\xi} BG_r \rightarrow BG
\]

lifts to \(B\text{Top}\) and since \(G_r/\text{Top}_r = G/\text{Top}\), \(\xi : M \to BG_r\), lifts to \(B\text{Top}_r\). \(\square\)

References


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