EMBEDDINGS OF TOPOLOGICAL MANIFOLDS

ERIK KJÆR PEDERSEN

It is the purpose of this paper to combine the methods of [8] and [9] to prove Haefliger type embedding theorems for topological manifolds. We prove the following embedding theorems:

THEOREM 1. *Let M be a topological manifold with boundary* $\dim(M) = m$, *$m \geq 6$, and let*

$$\alpha \in \pi_r(M, \partial M), \qquad r \leq m - 3.$$

Assume that $(M, \partial M)$ is $2r - m + 1$ connected. Then α can be represented by a locally flat embedded disc

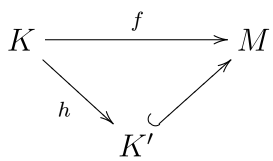
$$f : (D^r, S^{r-1}) \to (M, \partial M).$$

The proof is modeled on the similar PL proof in [3] using the above mentioned modifications of the techniques.

THEOREM 5. *Let K be a finite k-dimensional complex, M^n a topological manifold $m \geq 6$, $m - k \geq 3$, and f a continuous map, $f : K \to M$, so that*

$$\pi_i(f) = 0 \qquad \text{for} \quad i \geq 2k - m + 1.$$

Then there is a complex K' of dimension k', $k' \leq k$, such that K' is locally tamely embedded in M, and a simple homotopy equivalence $h : K \to K'$ so that the diagram



is homotopy commutative.

Finally we obtain the "Haefliger type" theorem:

THEOREM 7. *Let $f : M^p \to V^q$ be a map of topological manifolds, $q \geq 6$, $q - p \geq 3$, M closed, and assume that*

$$\pi_i(f) = 0 \qquad \text{for} \quad i \leq 2p - q - 1.$$

Then f is homotopic to a locally flat embedding.

Theorem 7 is proved as in the PL case using surgery and the normal theory due to Rourke and Sanderson classifying neighborhoods of topological manifolds [10] although we avoid the use of nonstable transversality. It generalizes the theorem of Lees [6]. Lees assumes the spaces are connected rather than the map.

For the proof of Theorem 1 we need a general position lemma.

Definition 2. Let P^p be a PL complex and M^m a topological manifold and $f : (P, P') \to (M, \partial M)$ a continuous map, P' a subcomplex of P. We say that f is in general position if $K = \mathrm{im}(f)$ has the structure of a complex such that f is PL, $K \subset M$ is locally tamely embedded and the double point set $S_2(f) \subset P$ is of dimension at most $2p - m$.

Lemma 3. *Let M be a topological manifold,* $\dim(M) \geq 6$ *and*

$$f : (D^r, S^{r-1}) \to (M, \partial M)$$

a continuous map, $m - r \geq 3$. Then f is homotopic to a map in general position.

Proof. By the relative version of Lees' immersion theorem (see [10]) the map

$$F : (D^r, S^{r-1}) \times \mathbb{R}^{m-r} \to (D^r, S^{r-1}) \to (M, \partial M)$$

is homotopic to an immersion, since any bundle over D^r is trivial. Therefore we may assume that F is an immersion, so we can cover D^r by finitely many open sets U_i and shrink fibers so that $F|U_i \times \mathbb{R}^{m-r}$, $i = 1, 2, \dots, s$ is an embedding. Lert $\{Z_i\}$ be a compact refinement of the covering $\{U_i\}$ such that Z_i is a subcomplex of D^r. The proof will be by induction ob the following statement. There is an isotopy h_i^t of $D^r \times \mathbb{R}^{m-r}$ and subcomplexes Q_i of D^r such that $\bigcup_{j \leq i} Z_j \subset \mathrm{int}(Q_i)$ and $F \circ h_i^1 | Q_i$ is in general position.

The induction starts trivially, so assume we have obtained the statement for $i - 1$. We may then as well assume ths true originally and denote $F \circ h_{i-1}^1$ by F and $f \circ h_{i-1}^1$ by f. Consider $V = F(U_i \times \mathbb{R}^{m-r}) \subset M$. V has a PL structure induced by F. Let $K_{i-1} = \mathrm{im}(Q_i)$. Then by the induction hypothesis K_{i-1} is a PL complex locally tamely embedded in M, and of codimension at least 3. Therefore by [2] (see [9, Theorem 2] we can change the PL structure of V so that a compact regular neighborhood of $K_{i-1} \cap V$ is PL embedded. Hence if we shrink U_i and the fibers, we may assume $K_{i-1} \cap V \subset V$ is a PL embedding. We shrink U_i so little that U_i still contains Z_i. Since $f : Q_{i-1} \to K_{i-1}$ is PL the restriction of f to $U_i \cap Q_{i-1}$ is PL, so as before by shrinking U_i and Q_{i-1} we can find an isotopy of $U_i \times \mathbb{R}^{m-r}$ that fixes everything outside a compact set and a neighborhood of $U_i \cap Q_{i-1}$, hence may be assumed to fix $Q_{i-1} \cap U_i \times \mathbb{R}^{m-r}$, and moves $f|U_i$ to a PL embedding. The restriction of f to $Q_{i-1} \cap U_i$ is already in general position in the PL sense, so by PL general position we may move U_i as above so that we end up having f in general position in a closed neighborhood Q_i of $\bigcup_{j \leq i} Z_j$. This ends the induction step. □

We now only need the following trivial lemma before the proof of Theorem 1.

LEMMA 4. *Let M be a manifold and assume $(M \times I, M \times 0, M \times 1)$ is written as the union of two triads V and W:*

$$(M \times I, M \times 0, M \times 1) = (V \bigcup_{\partial_+ V = \partial_- W} W, \partial_- V, \partial_+ W).$$

Then if $\pi_j(V, \partial_+ V) = 0$ for $j \leq r$, $r \geq 2$, then $\pi_j(W, \partial_+ W) = 0$ for $j \leq r-1$.

Proof. Van Kampen's Theorem applied to W and V shows that $\pi_1(W) = \pi_1(M \times I)$ since $r \geq 2$; hence $\pi_1(W) \cong \pi_1(M \times 1) = \pi_1(\partial_+ W)$. Let $\widetilde{M}$ denote the universal covering space of M and denote inverse images in $\widetilde{M} \times I$ by $\widetilde{\ }$. Then

$$H_{j+1}(\widetilde{V}, \widetilde{\partial_+ V}) = H_{j+1}(\widetilde{M} \times I, \widetilde{W}) = H_j(\widetilde{W}, \widetilde{\partial_+ W})$$

by excision and the long exact sequence for the triple $\widetilde{\partial_+ W} \subset \widetilde{W} \subset \widetilde{M} \times I$. □

Proof of Theorem 1. The proof is by induction on the following statement:

$$F : (D^r, S^{r-1}) \to (M, \partial M)$$

is homotopic to f_i, and ∂M has a collar in M that is decomposed as

$$(\partial M \times [0,1], \partial M \times 0, \partial M \times 1) = (V \bigcup_{\partial_+ V = \partial_- W} W, \partial_- V, \partial_+ W)$$

where V is obtained from $\partial_- V = \partial M \times 0$ by adjoining handles of dimension less than $2r - m + 1 - i$, $f_i^{-1}(V)$ is a collar of ∂D^r in D^r and

$$f_i | \overline{D^r - f_i^{-1}(V)} : \overline{D^r - f_0^{-1}(V)} \to \overline{M - V}$$

is a locally flat embedding.

We start the induction by homotoping f to f_0 a map in general position as in Lemma 3. Let $S_2(f_0)$ be the double point set of f_0. Then since the codimension of $S_2(f_0)$ in D^r is at least 3 we can find a complex Z_0 in D^r of dimension $\dim(S_2(f_0)) + 1$, i. e. of dimension less than $2r - m + 2$ so that $S_2(f_0) \subset Z_0$ and $\partial D^r \cup Z_0$ simplicially collapses to ∂D^r (see [3]). Let R_0 be the image of Z_0. Then R_0 is a subcomplex of $\mathrm{im}(f_0)$ of dimension at most $2r - m + 1$, and by Newman's engulfing theorem [7] we may assume that R_0 is contained in some collar of ∂M, $R_0 \subset \partial M \times [0,1]$. Let N_0 be a regular neighborhood of R_0 in $\partial M \times [0,1]$ that intersects $\mathrm{im}(f_0)$ in a regular neighborhood of R_0 in $\mathrm{im}(f_0)$ (see [9]). Then $f_0^{-1}(N_0)$ is a regular neigborhood of Z_0 so it collapses to ∂D^r. This picture is clearly homotopic to the picture where we have glued on a collar on ∂D^r and ∂M, so do that and put

$$V_0 = \partial M \times [0,1] \cup N_0, \quad W_0 = \overline{\partial M \times [-1,1] - V_0}.$$

This then starts the induction. Assume the hypothesis for i. Denote

$$f_i(\overline{D^r - f_i^{-1}(V_i)}),$$

i. e. the image of D^r with a collar of the boundary deleted, by B^r. Then B^r is locally flatly embedded,

$$B^r \subset \overline{M - V_i},$$

and therefore extends to an embedding

$$B^r \times \mathbb{R}^{n-r} \subset \overline{M - V_i}$$

by [9, Lemma 3]. By Lemma 4 and [8, Theorem 1], W_i has a strong deformation retract, $\partial W_i \cup K_i$, where K_i is a locally tamely embedded complex of dimension at most $\max(2, 2r - m - i + 1)$. Using [2] we may change the PL structure on $B^r \times \mathbb{R}^{m-r}$ and shrink the fibers so that

$$K_i \cap B^r \times \mathbb{R}^{m-r} \subset B^r \times \mathbb{R}^{m-r}$$

is a PL embedding. We then isotop B^r so that B^r is PL embedded in $B^r \times \mathbb{R}^{m-r}$ and finally such that K_i and B^r are in general position. This isotoping B^r can obviously be done as a homotopy of f_i. Let $C_i = K_i \cap B^r$. Then, since B^r is of codimension at least 3, C_i is of dimnesion at most $2r - m - i - 2$ so we can find a complex Z_i in B^r of dimension one higher so that $Z_i \cup \partial B^r$ simplicially collapses to ∂B^r and Z_i contains C_i. Consider

$$K_i \cup Z_i \subset K_i \cup B^r \subset \overline{M - V_i}.$$

By Newman's engulfing theorem $K_i \cup Z_i \cup V$ is contained in a collar $\partial M \times I$ of the boundary, so we let N_i be a regular neighborhood of $K_i \cup Z_i$ that intersects $K_i \cup B^r$ in a regular neighborhood of $K_i \cup Z_i$ in $K_i \cup B^r$ (see [9]), hence intersects B^r in a regular neighborhood of Z_i. Define

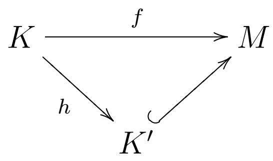
$$V_{i+1} = V_i \cup N_i, \quad W_{i+1} = \overline{\partial M \times [0,1] - V_{i+1}},$$

Then V_{i+1} is obtained from M by adjoining handles of dimension at most $2r - m - i - 1$ since that is the dimension of Z_i. $f_{i+1}^{-1}(V_{i+1})$ is a collar neighborhood of D^r since $f_{i+1}^{-1}(N_i)$ is a regular neighborhood of Z_i and Z_i simplicially collapses to ∂B^r.

Eventually we get the dimension of K_i to be 2, so when we put K_i in general position to B^r the intersection becomes empty and therefore when we have completed that step we have obtained that f is homotopic to a map f_i such that there is a collar of ∂M in M with $f_i^{-1}(\partial M \times I)$ a collar of ∂D^r in D^r and $f_i | f_i^{-1}(M - \partial M \times [0,1])$ is a locally flat embedding. It is now easy to homotop f so as to pinch off the collar where f is not yet an embedding. □

We now show how this can be used to embed complexes in topological manifolds up to homotopy type.

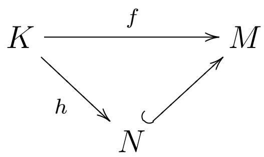
Theorem 5. *Let K^k be a finite complex, M^m a topological manifold, $m \geq 6$, $m - k \geq 3$, and f a continuous map, $f : K \to M$, such that $\pi_i(f) = 0$ for $i \leq 2k - m + 1$. Then there is a complex K' of dimension k', $k' \leq k$, such that K' is locally tamely embedded in M, and* *a simple homotopy equivalence* $h : K \to K'$ *such that the diagram*



is homotopy commutative.

Theorem 5 follows immediately from Theorem 1 in [8] and the following proposition.

Proposition 6. *Let* M^m *be a topological manifold of dimension* $m \geq 6$*, and let* K^k *be a* PL *complex,* $m - k \geq 3$*, and* f *a continuous map* $f : K \to M$ *such that* $\pi_i(f) = 0$ *for* $i \leq 2k - m + 1$*. Then there is a codimension* 0 *submanifold* N *of* M *and a simple homotopy equivalence* $h : K \to N$ *such that the diagram*



os homotopy commutative.

Proof. Filter K by simpleces of nondecreasing dimension

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_s = K$$

where K_{i+1} is obtained from K_i by adjoining a simplex. The proof will be by induction on the following statement: There is a codimension 0 submanifold N_j of M and a homotopy of f to f_j such that $f_j(K_j) \subset N_j$ and $f_j|K_j : K_j \to N_j$ is a simple homotopy equivalence.

It is trivial to start the induction so assume the statement for j. $K_{j+1} = K_j \bigcup_{S^{r-1}} D^r$ for some r, and $f_j|D^r$ determines an element in $\pi_r(M, N_j)$. We want to prove that the map

$$\pi_r(\overline{M - N_j}, \partial N_j) \to \pi_r(M, N_j)$$

is onto. For $r \leq m - r - 1$ this is trivial since N_j is obtained from ∂N_j by adjoining handles of dimension $\geq m - r$, so

$$\pi_s(\partial N_j) \to \pi_s(N_j)$$

as well as

$$\pi_s(\overline{M - N_j}) \to \pi_s(M)$$

is an isomorphism for $s \leq m - r - 2$ and onto for $s = m - r - 1$. For $r = m - r \geq 3$ we have as above

$$\pi_s(N_j, \partial N_j) = 0 \qquad \text{for} \quad s \leq r - 1$$

and

$$\pi_s(\overline{M - N_j}, \partial N_j) = 0 \qquad \text{for} \quad s \leq r - 1$$

and since $r-1 \geq 2$ this implies by Blakers-Massey [1] that

$$\pi_s(\overline{M-N_j}, \partial N_j) \to \pi_s(M, N_j)$$

is an isomorphism for $s \leq 2r-3$ and onto for $s = 2r-2$.

In case $r > m-r$ we have, since $r \leq k$, that $2k-m+1 \geq 2$ so $\pi_1(K) \cong \pi_1(M)$; since $r \geq 3$, $\pi_1(K_j) \cong \pi_1(K)$ and

$$\pi_1(N_j) \cong \pi_1(K) \cong \pi_1(N_j) \cong \pi_1(\overline{M-N_j})$$

so all fundamental groups are the same. We denote universal covering spaces by $\widetilde{\ }$ and we have

$$H_s(\widetilde{M}, \widetilde{N}_j) \cong H_s((\overline{M-N_j})\widetilde{\ }, \widetilde{\partial N_j})$$

by excision, but

$$H_s(\widetilde{M}, \widetilde{N}_j) = 0 \qquad \text{for} \quad s \leq \min(r-1, 2k-m+1)$$

so

$$\pi_s(\overline{M-N_j}, \partial N_j) = 0 \qquad \text{for} \quad s \leq \min(r-1, 2k-m+1)$$

and as before $\pi_s(N_j, \partial N_j) = 0$ for $s \leq m-r-1$ so by Blakers-Massey [1]

$$\pi_s(\overline{M-N_j}, \partial N_j) \to \pi_s(M, N_j)$$

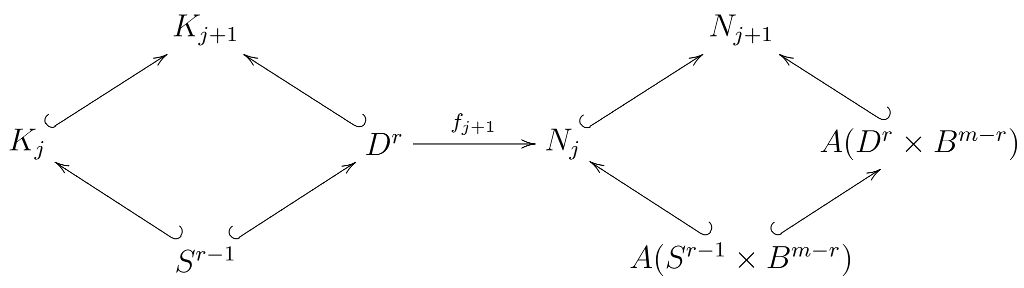
is onto for $s \leq \min(r-1, 2k-m+1) + m-r-1 = 2k-r$. Hence we can choose a map

$$\alpha : (D^r, S^{r-1}) \to (\overline{M-N_j}, \partial N_j)$$

representing $f_j|D^r$. Since $\pi_s(\overline{M-N_j}, N_j) = 0$ for $s \leq \min(r-1, 2k-m+1)$, α can, by Proposition 6 be chosen to be a locally flat embedding which can be extended to an embedding

$$A : (D^r S^{r-1}) \times \mathbb{R}^{m-r} \to (\overline{M-N_j}, \partial N_j)$$

(see [9, Lemma 3]). We now define $N_{j+1} = N_j \cup A(D^r \times B^{m-r})$ where B^{m-r} is the unit ball in $\mathbb{R}^{m-r}$. Sinve $K_j \subset K_{j+1} \subset K$ are cofibrations we can homotop f_j to f_{j+1} such that we get a Mayor-Vietoris diagram

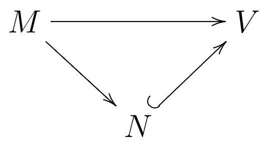


where f_{j+1} is a simple homotopy equivalence on the 3 small terms hence on K_{j+1}. This ends the induction step □

Theorem 7. *Let $f : M^p \to V^q$ be a map of topological manifolds, $q \leq 6$, $q - p \geq 3$, M closed, and assume that $\pi_i(f) =$ for $i \leq 2p - q + 1$. Then f is homotopic to a locally flat embedding.*

To prove Theorem 7 we need some lemmas.

Lemma 8. *With the assumptions of Theorem 7 there is a codimension* 0 *submanifolds N of V such that $\pi_1(\partial N) \cong \pi_1(N)$ and a simple homotopy equivalence $h : M \to N$ such that the diagram*



is homotopy commutative.

Proof. There is a p-dimensional complex K which is simple homotopy equivalent to M. K is obtained as follows. Let C be the total space of the normal discbundle of M. Then C is a PL manifold and determines by definition the simple homotopy type of M. We can then let K be a PL spine of D. The codimension 0 submanifold N of V is now constructed by Proposition 6. □

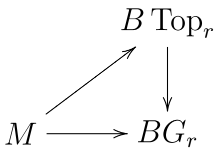
The proof of Theorem 7 is completed by the following lemma.

Lemma 9. *Let $f : M^p \to V^q$ be a simple homotopy equivalence, M and V topological manifolds, M closed, $q \geq 6$, $q - p \geq 3$, and $\pi_1(\partial V) \to \pi_1(V)$ an isomorphism. Then f is homotopic to a locally flat embedding.*

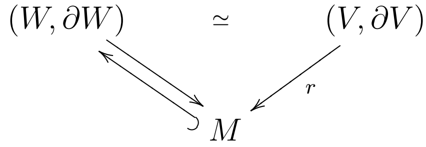
Proof. It is proved in [10] that there is a classifying space $B\,\mathrm{Top}_r$ for topological neighborhoods of codimension r. The classification goes via microbundles: to a topological neighborhood $P \subset Q$ one assigns the stable microbundle pair $(\tau_Q, i^*\tau_Q)$ which is then classified by $B\,\mathrm{Top}_r$. There is a map $B\,\mathrm{Top}_r \to B\mathbf{G}_r$ assigning to the microbundle pair the corresponding spherical fibration pair. Since spherical fibration pairs split uniquely $B\mathbf{G}_r$ is identified with BG_r the classifying space for spherical fibrations of dimension r.

If $M \subset W$ is a locally flat inclusion of topological manifolds of codimension ≥ 3 we can take a regular neighborhood N of M in W (see [4]). By [11] the map $\partial N \subset N \to M$ is equivalent to a spherical fibration ν, and $\nu_M = \nu \oplus i^*\nu_W$ as spherical fiber spaces. Therefore the map $B\,\mathrm{Top}_r \to BG_r$ is described as follows: Take a closed regular neighborhood N, and turn $\partial N \to M$ into an r-spherical fibration ν. The classifying map for ν is then $M \to B\,\mathrm{Top}_r\, BG_r$.

We now return to our problem. Consider $r : V \to M$, a homotopy inverse to f, and the restriction of r to $\partial V \to M$. This map is by Spivak [11] equivalent to a spherical fibration $\xi : E \to M$. Let $\overline{E}(\xi)$ be the mapping cone of ξ; then $(\overline{E}(\xi), E(\xi))$ is naturally homotopy equivalenbt to $(V, \partial V)$. Assume that the map $\xi \to BG_r$ lifts to $B\,\mathrm{Top}_r$:



We then have M locally flatly embedded in a manifold and if we take a regular neighborhood W of M we get a fiber homotopy equivalence



and if the homotopy equivalence is homotopic to a homeomorphism we have proved that f is homotopic to an embedding. By Sullivan theory this is determined by the normal obstruction in $[W, G/\mathrm{Top}] = [M, G/\mathrm{Top}]$ but we have freedom in the choice of the lifting of ξ to $B\,\mathrm{Top}_r$ corresponding to G_r/Top_r. But since $r \geq 3$ this space is homotopy equivalent to G/Top by [10]. Hence we can choose our lifting so as to make the normal obstruction 0. This then ends the proof once we show the existence of one lifting. However $\xi \oplus f^*\nu_V = \nu_M$ as spherical fibrations, so stably $\xi = \nu_M \oplus (f^*\nu_V)^{-1}$, hence

$$M \xrightarrow{\xi} BG_r \to BG$$

lifts to $B\,\mathrm{Top}$. and since $G_r/\mathrm{Top}_r = G/\mathrm{Top}$, $\xi : M \to BG_r$, lifts to $B\,\mathrm{Top}_r$. □

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Odense University, Odense, Denmark