K_{-i}-INVARIANTS OF CHAIN COMPLEXES

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0. Introduction

In this paper we introduce a chain complex version of [1] and give some topological applications. We work in the categories $\mathcal{A}_i(R)$ and $\mathcal{C}_i(R)$ of $\mathbb{Z}^i$-graded R-modules and bounded homomorphism, respectively the full subcategory of finitely generated free $\mathbb{Z}^i$-graded R-modules (see §1 for precise definitions). The reason it takes some work to develope a chain complex theory, is that the categories $\mathcal{A}_i(R)$ and $\mathcal{C}_i(R)$ do not have kernels and cokernels. Thus the concept of a projective has to be replaced by a projection map, and throughout we have to work at the map level.

1. Reviewing the categories $\mathcal{A}_i(R)$ and $\mathcal{C}_i(R)$

Given a ring R with 1, $\mathcal{A}_i(R)$ dnotes the category of $\mathbb{Z}^i$-graded R-modules and bounded homomorphisms. An object A of $\mathcal{A}_i(R)$ is thus a direct sum $\oplus_{j_1,\ldots,j_i} A(j_1,\ldots,j_i)$ of R-modules $A(j_1,\ldots,j_i)$ and a morphism is an R-module morphism f such that there exists $k = k(f)$ satisfying

$$f(A(j_1,\ldots,j_i)) \subset \bigoplus_{\substack{h_s=-k \\ s=1,\ldots,i}}^{k} A(j_1+h_1,\ldots,j_i+h_i).$$

The category $\mathcal{C}_i(R)$ is the full subcategory with objects A satisfying $A(j_1,j_2,\ldots,j_i)$ are finitely generated free R-modules. The category $P\mathcal{C}_i(R)$ is defines to be the category of projections in $\mathcal{C}_i(R)$ i. e. an object is a morphism $p : A \to A$ with $p^2 = p$ and morphisms are commutative diagrams. In [1] we prove

Theorem 1.1. *$K_1(\mathcal{C}_{i+1}(R)) \cong K_0(P\mathcal{C}_i(R)) \cong K_{-i}(R)$.*

Remark. In $K_1(\mathcal{C}_{i+1}(R))$ we have generators (A,α) where α is an isomorphism of A and relations given by split exact sequences, and in $K_0(P\mathcal{C}_i)$ we divide out by $(A,0)$ and $(A,1_A)$ if $i > 0$ but only by $(A,0)$ if $i = 0$.

The basic ingredient in the proof of theorem 1.1 is the Bass-Heller-Swan homomorphisms which are described as follows: Let (A,α) represent an element of $K_{-i}(R)$, so A is an object of $\mathcal{C}_{i+1}(R)$. Adjoining indeterminates t, t^{-1} we obtain $A[t,t^{-1}]$ an object of $\mathcal{C}_{i+1}(R[t,t^{-1}])$

in the obvious way, and we may also think of α as an isomorphism of $A[t,t^{-1}]$. Define the isomorphism p_t^s of $A[t,t^{-1}]$ on homogeneous elements by

$$p_t^s(x) = \begin{cases} x & \text{if } s\text{-degree of } x \text{ is} \le 0 \\ t\cdot x & \text{if } s\text{-degree of } x \text{ is} > 0 \end{cases}$$

The commutator $[\alpha, p_t^s]$ will be the identity of $A[t,t^{-1}]$ except for a band $-k \le j_j \le k$ where k is a bound for α and we may think of this commutator as a $\mathbb{Z}^i$-graded isomorphism over $R[t,t^{-1}]$. In [1] we show this gioves a well-defined monomorphism $\lambda^s : K_{-i}(R) \to K_{i+1}(R[t,t^{-1}])$. In [1] we do not discuss the dependency on s, so it seems appropriate to do that here: Let $g \in Gl(i+1,\mathbb{Z})$. One easily sees that regrading $\mathbb{Z}^{i+1}$ by g sends bounded isomorphisms to bounded isomorphisms so $Gl(i+1,\mathbb{Z})$ acts on $K_{-i}(R)$

Proposition 1.2. *The action of $g \in Gl(i+1,\mathbb{Z})$ on $K_{-i}(R)$ is by multiplication by $\det g$.*

Corollary. The dependency of s in the Bass-Heller-Swan monomorphism is given by $\lambda^s = (-1)^{r-s}\cdot\lambda^r$.

Proof of corollary. We only consider $s=1$ and $r=2$, the general case being obvious from this. Let $g \in Gl(i+1,\mathbb{Z})$ interchange the first 2 coordinates. Then $\det g = -1$ and $\lambda^2\circ g = \lambda^1$ and the result follows. □

Proof of proposition 1.2. First we show that if g is elementary, $g = E_{rs}(n)$ the action is trivial: If $A \in \mathcal{C}_{i+1}(R)$ is regraded by g we obtain A^g and we have (A,α) is sent to $A^g \xrightarrow{1_g} A \xrightarrow{\alpha} A \xrightarrow{1_g^{-1}} A^g$, where 1_g is the identity, if we forget grading. The problem is that 1_g is not bounded. Since $A^g(j_1,\dots,j_i) = A(g(j_1,\dots,j_i))$ we see that 1_g preserves all degree except the r'th degree so p_t^s commutes with 1_g and we get $\lambda^s((A^g, 1_g^{-1}\alpha 1_g, p_t^s)) = (A^g, 1_g^{-1}[\alpha,p_t^s]1_g)$, but 1_g is a bounded isomorphism when we restrict to a band around $j_s = 0$ so this last element is equivalent to $(A,[\alpha,p_t^s])$ which represents $\lambda^s(A,\alpha)$. Since λ^s is a monomorphism we are done. By repeated application of Bass-Heller-Swan monomorphism it now suffices to study the action of $\{-1\} \in Gl(1,\mathbb{Z})$ on a $\mathbb{Z}$-graded isomorphism.

The map $K_1(\mathcal{C}_1(R)) \to K_0(R)$ is given by

$$[(A,\alpha)] \to [\mathrm{Coker}(\bigoplus_{i=0}^{\infty} A(i) \to \bigoplus_{i=-k}^{\infty} A(i)] - [\bigoplus_{i=-k}^{-1}]$$

where k is a bound for α. To investigate the action of -1 it is enough to consider nice representatives. Let P be a finitely generated projective and $Q\oplus P$ a finitely generated free

R-module. Then the $\mathbb{Z}$-graded isomorphism

$$\begin{array}{ccccccccccccc}
\deg & \cdots & -2 & & -1 & & 0 & & 1 & & 2 & & \cdots \\
& & Q\oplus P & & Q\oplus P & & Q\oplus P & & Q\oplus P & & Q\oplus P & & \\
& & \downarrow & \searrow^{1_Q} & \downarrow & \searrow^{1_Q} & \downarrow 1_P & \searrow^{1_Q} & \downarrow & \searrow & \downarrow & & \\
& & Q\oplus P & & Q\oplus P & & Q\oplus P & & Q\oplus P & & Q\oplus P & &
\end{array}$$

by the above construction gives $[P]$. Changing the grading from i to $-i$ gives $[Q]-[P\oplus Q] = -[P]$. This finishes the proof of proposition 1.2 ☐

2. Torsion of chain complexes in $\mathcal{C}_{i+1}(R)$

In any additive category with 0 object there is a notion of chain complexes that is a sequence of morphisms $A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1}$ so that $\partial^2 = 0$. We shall also consider the usual notion of morphism, homotopy, contractions of chain complexes to be known, the only requirement being that all morphisms are required to be in the categroy. In this section we show how to define a torsion invariant of a chain complex in $\mathcal{C}_{i+1}(R)$. First we need a lemma.

Lemma 2.1. *Let A and B be objects of $\mathcal{C}_{i+1}(R)$. Then stably, there is a canonical isomorphism A to B, in the sense that any two choices $\phi_i : A\oplus D \to B\oplus D$ have $\phi_2^{-1}\phi_1$ representing the trivial element in $\widetilde{K}_{-i}(R)$.*

Remark 2.2. $\widetilde{K}_{-i}(R) = K_{-i}(R)$ for $i > 0$ and $\widetilde{K}_0(R) = K_0(P\mathcal{C}_0(R))$ with the extra relation $(A, 1) = 0$ (in other words the usual $\widetilde{K}_0(R)$).

Proof. Choose a basis in each $A(j_1, \ldots, j_n)$ and $B(j_1, j_2, \ldots, j_n)$ and send basis elements to basis elements in a way bounded by 1. If this is not possible then stabilize. To see that a basis change isomorphism is trivial in $\widetilde{K}_{-i}(R)$ note that after two applications of the Bass-Heller-Swan monomorphism such an isomorphism is the identity. ☐

Corollary 2.3. *An isomorphism $\alpha : A \to B$ in $\mathcal{C}_{i+1}(R)$ has a canonical torsion in $\widetilde{K}_{-i}(R)$.*

Remark 2.4. A different way to obtain this corollary is to define a projection $p^s : B \to B$ to be 0 in s-degree ≤ 0 and 1 in s-degree > 0. Then $\alpha^{-1}p^s\alpha$ is a projection of A which is 1 in s-degrees $> k$ and 0 in s-degrees $< -k$ where k is the bound of α. so restricting to some band and forgetting the s-degree gives a well defined prohection if we have set projections of the form $(A, 1) = 0$.

Let $A_* = \{0 \to A_n \xrightarrow{\partial} \cdots \to A_1 \xrightarrow{\partial} 0\}$ be a contractible chain complex in $\mathcal{C}_{i+1}(R)$. Choose a contraction s and consider

$$\bigoplus A_{\text{even}} \xrightarrow{s+\partial} \bigoplus A_{\text{odd}}.$$

Theorem 2.5. *This defines a bounded isomorphism with torsion in $\widetilde{K}_{-i}(R)$ independent of the choice of s.*

Proof. $s+\partial$ is clearly bounded, so we need to see the inverse is bounded. Note that $(s+\partial)^2 = 1+s^2$. Since the chain complex has finite length s^2 is nilpotent. Hence $\sum_{i=0}^{\infty}(-1)^i s^{2i}$ is an inverse of $1+s^2$ which is clearly bounded.

Note that we may replace s by $s\partial s$ since $s\partial s\partial + \partial s\partial s = 1$ and $(\partial + s) = (1+s^2\partial s)(\partial + s\partial s)$. This replacement does not change the torsion if $(1+s^2\partial s)$ has trivial torsion. This however is shown by a standard argument since $s^2\partial s$ is nilpotent. Since $(s\partial s)^2 = 0$ this means we may assume $s^2 = 0$.

Consider two contracting homotopies s and s_1 with $s^2 = s_1^2 = 0$. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\text{even}} & \xrightarrow{s\partial+\partial s} & \bigoplus A_{\text{even}} \\ \downarrow{\scriptstyle s_1+\partial} & & \downarrow{\scriptstyle s+\partial} \\ \bigoplus A_{\text{odd}} & \xrightarrow{s\partial+\partial s_1} & \bigoplus A_{\text{odd}} \end{array}$$

where $s\partial + \partial S_1 : A_i \to A_i$ is an isomorphism (with inverse $s_1\partial + \partial s$). However the two horizontal isomorphisms are not the same since they go between different objects so we need to show they both have trivial invariant in $K_{-i}(R)$. However we have the following isomorphism of (A_*, ∂)

$$\begin{array}{ccccccccccccccc} 0 & \to & A_n & \to & A_{n-1} & \to & \cdots & \to & A_{i+1} & \to & A_i & \to & A_{i-1} & \to & \cdots \to A_0 \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle s\partial+\partial s_1} & & \downarrow{\scriptstyle 1} & & \downarrow \\ 0 & \to & A_n & \to & A_{n-1} & \to & \cdots & \to & A_{i+1} & \to & A_i & \to & A_{i-1} & \to & \cdots \to A_0 \end{array}$$

and hence we will be done by the following lemma.

Lemma 2.6. *Let (A_*, ∂) be a contractible chain complex and α_* a self isomorphism. Then*

$$\sum_{i\ even} [A_i, \alpha_i] = \sum_{i=odd} [A_i, \alpha_i]$$

in $K_{-i}(R)$.

Proof. The proof is obtained by stabilization and conjugation. We indicate the first step. Choose a contracting homotopy s with $s^2 = 0$. Now stabilize the chain complex to get

$$0 \to A_n \to A_{n-1} \oplus A_n \xrightarrow{\partial\oplus 1} A_{n-2} \oplus A_n \to A_{n-3}$$

extending α_{n-1} and α_{n-2} by the identity.

Conjugating by $\left\{\begin{smallmatrix} 1-\partial s & \partial \\ s & 0 \end{smallmatrix}\right\}$ on $A_{n-1} \oplus A_n$ and by the identity in other dimensions we get (note the above matrix has square $= 1$) an isomorphism

$$
\begin{array}{ccccccc}
A_n & \underset{\{s\ 0\}}{\overset{\left\{\begin{smallmatrix} \partial \\ 0 \end{smallmatrix}\right\}}{\rightleftarrows}} & A_{n-1} \oplus A_n & \underset{\left\{\begin{smallmatrix} s & 0 \\ 0 & 1 \end{smallmatrix}\right\}}{\overset{\left\{\begin{smallmatrix} \partial & 0 \\ 0 & 1 \end{smallmatrix}\right\}}{\rightleftarrows}} & A_{n-2} \oplus A_n & \rightleftarrows & \cdots \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle \left\{\begin{smallmatrix} 1-\partial s & \partial \\ s & 0 \end{smallmatrix}\right\}} & & {\scriptstyle 1}\big\downarrow & & \\
A_n & \underset{\{0\ 1\}}{\overset{\left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\}}{\rightleftarrows}} & A_{n-1} \oplus A_n & \underset{\left\{\begin{smallmatrix} s & \partial \\ 0 & 0 \end{smallmatrix}\right\}}{\overset{\left\{\begin{smallmatrix} \partial & 0 \\ s & 0 \end{smallmatrix}\right\}}{\rightleftarrows}} & A_{n-2} \oplus A_n & \rightleftarrows & \cdots
\end{array}
\tag{2.7}
$$

to a chain complex which is the sum of the contractible chain complexes

$$0 \to A_{n-1} \to A_{n-2} \oplus A_n \to \cdot$$

and $0 \to A_n \xrightarrow{1} A_n \to 0$. Inductively we see it is enough to consider A of the form

$$0 \to A_n \xrightarrow{\{1\ 0\}} A_n \oplus A_{n-1} \xrightarrow{\left\{\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right\}} A_{n-1}A_{n-2} \xrightarrow{\left\{\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right\}} \cdots \to A_1 \oplus A_0 \xrightarrow{\{0\ \partial\}} A_0 \to 0$$

where ∂ is an isomorphism. This means that after this stabilization and conjugation procedure (which does not influence the class of α_i in $K_{-i}(R)$) the isomorphisms α_i are of the form

$$\alpha_n = \beta_n, \quad \alpha_{n-1} = \left\{\begin{smallmatrix} \beta_n & * \\ 0 & \beta_{n-1} \end{smallmatrix}\right\}, \quad \alpha_{n-2} = \left\{\begin{smallmatrix} \beta_{n-1} & * \\ 0 & \beta_{n-2} \end{smallmatrix}\right\} \quad \cdots \quad \alpha_0 = \partial\beta_0\partial^{-1}$$

and since $\left[\left\{\begin{smallmatrix} \beta_i & * \\ 0 & \beta_{i-1} \end{smallmatrix}\right\}\right] = [\beta_i] + [\beta_{i-1}]$ and $[\partial\beta_0\partial^{-1}] = [\beta_0]$ in $K_{-i}(R)$ we are done □

3. Finitely dominated chain complexes in $\mathcal{A}_i(R)$

Since the distinction between $\mathcal{A}_i(R)$ and $\mathcal{C}_i(R)$ is a finiteness condition it is natural to refer to a chain complex in $\mathcal{C}_i(R)$ of finite length as a finite chain complex.

Definition 3.1. A chain complex A_* in $\mathcal{A}_i(R)$ is finitely dominated if there is a chain complex of finite length in $\mathcal{C}_i(R)$ (i. e. a finite chain complex) and chain maps $A_* \xrightarrow{i} C_*$, $C_* \xrightarrow{r} A_*$ such that $r \cdot i$ is homotopic to the identity.

Remark 3.2. In case $i = 0$, the ungraded case, this definition degenerates to the usual definition of dominated chain complex.

Theorem 3.3. *Let A_* be a finitely dominated chain complex in $\mathcal{A}_i(R)$. Then there is an obstruction $\sigma(A_*) \in \widetilde{K}_{-i}(R)$, the vanishing of which is necessary and sufficient for A_* to be homotopy equivalent to a finite chain complex.*

For the proof we need some lemmas. First a lemma that defines the invariant.

Lemma 3.4. *Let A_* be a finitely dominated chain complex in $\mathcal{A}_i(R)$, then A_* is homotopy equivalent to an infinite length chain complex in $\mathcal{C}_i(R)$ of type*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F \xrightarrow{P} F \xrightarrow{1-P} F \to F \to \cdots$$

where $P : F \to F$ is a projection.

Definition 3.5. We define $\sigma(A_*) = [(F, P)] \in \widetilde{K}_{-i}(R)$.

We need to show that $\sigma(A_*)$ is well defined and that the vanishing ensures that A_* is homotopy equivalent to a fnite chain complex. Note that $\widetilde{K}_{-i}(R) = K_{-i}(R)$ for $i > 0$, but for $i = 0$ $\sigma(A_*)$ is only well defined in $\widetilde{K}_0(R)$.

Lemma 3.6. *If $\sigma(A_*) = 0 \in \widetilde{K}_{-i}(R)$ then A_* is homotopy equivalent to a finite complex.*

Proof. After stabilization $P : F \to F$ is a trivial projection so we have a commutative diagram

$$\begin{array}{ccc} F \oplus L_1 \oplus L_2 & \xrightarrow{\alpha \quad \cong} & L' \oplus L'' \\ \downarrow{\scriptstyle p\oplus 1\oplus 0} & & \downarrow{\scriptstyle 1\oplus 0} \\ F \oplus L_1 \oplus L_2 & \xrightarrow{\alpha \quad \cong} & L' \oplus L'' \end{array}$$

clearly

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 F \xrightarrow{P} F \xrightarrow{1-P} \to \cdots$$

is homotopy equivalent to

$$0 \to F_n \to \cdots \to F_1 \oplus L_1 \xrightarrow{(\partial \oplus 1, 0)} F \oplus L_1 \oplus L_2 \xrightarrow{P \oplus 0 \oplus 1}$$
$$F \oplus L_1 \oplus L_2 \xrightarrow{1-(P\oplus 0 \oplus 1)} F \oplus L_1 \oplus L_2 \to \cdots$$

which is isomorphic to

$$0 \to F_n \to \cdots \to F_1 \oplus L_1 \to L' \oplus L'' \xrightarrow{1 \oplus 0} L' \oplus L'' \xrightarrow{0 \oplus 1} \cdots$$

which is homotopy equivalent to

$$0 \to F_n \to \cdots \to F_1 \oplus L_1 \to L'' \to 0$$

□

To prove the well definedness we first need the following

Lemma 3.7. *Consider a commutative diagram in $\mathcal{C}_i(R)$*

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & F_1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} \\ F & \xrightarrow{\beta} & F_1 \end{array}$$

where P and P_1 are projections and α and β are isomorphisms. Then $[(F,P)]$ and $[(F_1,P_1)]$ represent the same element in $K_{-i}(R)$.

Proof. This is clear if $\alpha = \beta$. However by stabilization we get the following

$$\begin{array}{ccc} F \oplus F_1 & \xrightarrow{\begin{Bmatrix} 1-P & \beta^{-1}P_1 \\ \beta P & 1-P_1 \end{Bmatrix}} & F \oplus F_1 \\ \Big\downarrow {\scriptstyle \begin{Bmatrix} P & 0 \\ 0 & 0 \end{Bmatrix}} & & \Big\downarrow {\scriptstyle \begin{Bmatrix} 0 & 0 \\ 0 & P_1 \end{Bmatrix}} \\ F \oplus F_1 & \xrightarrow{\begin{Bmatrix} 1-P & \beta^{-1}P_1 \\ \beta P & 1-P_1 \end{Bmatrix}} & F \oplus F_1 \end{array}$$

which is easily shown to be commutative. Also one shows

$$\begin{Bmatrix} 1-P & \beta^{-1}P_1 \\ \beta P & 1-P_1 \end{Bmatrix}^2 = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

so the horizontal map is an isomorphism and the result follows. □

The following lemma shows that $\sigma(A_*)$ is well defined (if defined at all).

Lemma 3.8. *Let F_* and G_* be homotopy equivalent chain complexes in $\mathcal{C}_i(R)$ of the form*

$$0 \to F_n \to \cdots \to F_1 \to F \xrightarrow{P} F \xrightarrow{1-P} F$$

$$0 \to G_n \to \cdots \to G_1 \to G \xrightarrow{Q} G \xrightarrow{1-Q} G$$

where P and Q are projections. Then

$$[(F,P)] = [(G,Q)] \in \widetilde{K}_{-i}(R).$$

Proof. Let $f : F \to G$ and $g : G \to F$ be the homotopy equivalences. It is easily seen that the mapping cylinder of f

$$0 \to F_n \xrightarrow{\begin{Bmatrix} (-1)^{n+1}f \\ \partial \end{Bmatrix}} F_{n-1} \oplus G_n \xrightarrow{\begin{Bmatrix} \partial & (-1)^n f \\ 0 & \partial \end{Bmatrix}} F_{n-2} \oplus G_{n-1} \to \cdots$$

$$\cdots \to F \oplus G_1 \xrightarrow{\begin{Bmatrix} P & -f \\ 0 & \partial \end{Bmatrix}} F \oplus G \xrightarrow{\begin{Bmatrix} 1-P & f \\ 0 & Q \end{Bmatrix}} F \oplus G \xrightarrow{\begin{Bmatrix} P & -f \\ 0 & 1-Q \end{Bmatrix}} F \oplus G$$

is contractible the contraction being

$$\begin{Bmatrix} s & 0 \\ (-1)^i g & s \end{Bmatrix} : F_i \oplus G_{i+1} \to F_{i+1} \oplus G_{i+2}.$$

The diagram

$$\begin{array}{ccc}
F \oplus G & \xrightarrow{\left\{\begin{smallmatrix}1 & f\\ 0 & 1\end{smallmatrix}\right\}} & F \oplus G \\
\Big\downarrow{\scriptstyle \left\{\begin{smallmatrix}1-P & F\\ 0 & Q\end{smallmatrix}\right\}} & & \Big\downarrow{\scriptstyle \left\{\begin{smallmatrix}1-P & 0\\ 0 & Q\end{smallmatrix}\right\}} \\
F \oplus G & \xrightarrow{\left\{\begin{smallmatrix}1 & -f\\ 0 & 1\end{smallmatrix}\right\}} & F \oplus G
\end{array}$$

shows, using lemma 3.7, that

$$\left[\left\{\begin{matrix}1-P & f\\ 0 & Q\end{matrix}\right\}\right] = \left[\left\{\begin{matrix}1-P & 0\\ 0 & Q\end{matrix}\right\}\right]$$

which is equal to $[Q]-[P]$ so the proof is finished by

Lemma 3.9. *Let F_* be a contractible chain complex in $\mathcal{C}_i(R)$ of the form*

$$0 \to F_n \xrightarrow{\partial} \cdots \to F_0 \xrightarrow{P} F \xrightarrow{1-P} F \to \cdots$$

where P is a projection then $[P] = 0 \in \widetilde{K}(R)$.

Proof. Using stabilization and conjugation 2.7 we may assume $n = 1$. Consider

$$\left\{\begin{matrix}sP\partial & 0 & \\ s & 0 & \partial\\ 0 & s & 0\end{matrix}\right\} : F \oplus F_0 \oplus F_1 \to F \oplus F_0 \oplus F_1.$$

One checks the square is 1 so it is an isomorphism. The diagram

$$\begin{array}{ccc}
F \oplus F_0 \oplus F_1 & \xrightarrow{\left\{\begin{smallmatrix}sP & \partial & 0\\ s & 0 & \partial\\ 0 & s & 0\end{smallmatrix}\right\}} & F \oplus F_0 \oplus F_1 \\
\Big\downarrow{\scriptstyle \left\{\begin{smallmatrix}P & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{smallmatrix}\right\}} & & \Big\downarrow{\scriptstyle \left\{\begin{smallmatrix}1 & 0 & 0\\ sP & 0 & 0\\ 0 & 0 & 1\end{smallmatrix}\right\}} \\
F \oplus F_0 \oplus F_1 & \xrightarrow{\left\{\begin{smallmatrix}sP & \partial & 0\\ s & 0 & \partial\\ 0 & s & 0\end{smallmatrix}\right\}} & F \oplus F_0 \oplus F_1
\end{array}$$

shows that $[P]$ is equivalent to

$$\left[\left\{\begin{matrix}1 & 0 & 0\\ sP & 0 & 0\\ 0 & 0 & 1\end{matrix}\right\}\right]$$

which by an elementary matrix operation (using lemma 3.7) is equivalent to $\left[\begin{Bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{Bmatrix}\right]$ representing $0 \in \widetilde{K}_{-i}(R)$. □

To finish off the proof of theorem 3.3 we only need

Proof of lemma 3.4. The proof of Ranicki [4] of the ungraded case works verbatim. For convenience we give the proof here.

Let $C_* : 0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$ be a dominating complex so we have $A_* \xrightarrow{i} C_*$ and $C_* \xrightarrow{r} A_*$ and a chain homotopy s of A_* so $s\partial + \partial s = 1 - r \circ i$. Define $F = C_0 \oplus C_1 \oplus \cdots \oplus C_n$. Then

$$P = \begin{Bmatrix} ir & \partial & 0 & 0 & 0 & \cdots \\ isr & 1-ir & -\partial & 0 & 0 & \cdots \\ is^2r & -isr & ir & \partial & 0 & \cdots \\ is^3r & -is^2r & isr & 1-ir & -\partial & \cdots \\ is^4r & -is^3r & is^2r & -isr & ir & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{Bmatrix}$$

has $P^2 = P$. Defining

$$F_r = C_r \oplus C_{r+1} \oplus \cdots \oplus C_n$$

and $\partial : F_i \to F_{i-1}$ by

$$\begin{Bmatrix} \partial & 0 & 0 & \cdots \\ 1-ir & -\partial & 0 & \cdots \\ -isr & ir & \partial & \cdots \\ is^2r & isr & 1-ir & \cdots \\ \vdots & \vdots & \vdots & \end{Bmatrix}$$

we get a chain complex $F_* \quad 0 \to F_n \to \cdots \to F_1 \to F \xrightarrow{P} F \xrightarrow{1-P} F \cdots$ which one may check is homotopy equivalent to A_* by

$$I : A_* \to F_* \qquad \text{and} \qquad R : F_* \to A_*$$

where

$$I = \begin{Bmatrix} i \\ is \\ is^2 \\ \vdots \end{Bmatrix} : A_m \to F_m$$

and $R = (r, 0, 0, \ldots) : F_m \to A_m$. Since $R \circ I = r \circ i \sim 1_A$ and the projection $F_r = C_r \oplus C_{r-1} \oplus \cdots \oplus C_n \to F_{r+1} = C_{r+1} \oplus \cdots \oplus C_n$ is a homotopy from IR to the identity, we are done. □

4. Geometrical applications

In this section we briefly indicate some geometrical applications. Consider a space X with a reference map $X \xrightarrow{p_X} \mathbb{R}^i$. We form a category with such objects and morphisms

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbb{R}^i & & \mathbb{R}^i \end{array}$$

satisfying $|p_X(x) - p_Y(f(x))|$ is uniformly bounded. We say that $X \to \mathbb{R}^i$ is a CW complex if X has a CW decomposition with uniformly bounded cells (when measured in $\mathbb{R}^i$). Similarly we define simplicial complex. Two maps $f, g : (X \to \mathbb{R}^i) \to (Y \to \mathbb{R}^i)$ are homotopic if they are homotopic in a uniformly bounded way (i. e. $\exists K$ such that the homotopy H satisfies $|p_Y H(x,t_1) - p_Y H(x,t_2)| \leq K$ for all $t_1, t_2 \in I$. This gives rise to an obvious notion of homotopy equivalence.

This is a type of category that has been studied by Chapman, Ferry, and Quinn. Our aim is to give a more direct definition of the invariants encountered in [2, 3]. In the process we show these invariants are defined in a more general setting.

Definition 4.1. Consider $X \to \mathbb{R}^i$. We define $S(X)$, a chain complex in $\mathcal{A}_i(\mathbb{Z}\pi_1 X)$ as follows: $\widetilde{X}$ the universal cover of X has a reference map $\widetilde{X} \to X \to \mathbb{R}^i$. Consider singular chains of $\widetilde{X}$ with image in $\mathbb{R}^i$ contained in $\prod_{j=1}^{i}]n_j - 1, n_j + 1[$. To such a simplex we assign degree $(n_1, \ldots, n_i)$. This may be ambiguous but then we just choose one. This defines $S(X)$ which has a natural structure as a $\mathbb{Z}\pi$-module (as usual) up to canonical bounded isomorphism (the isomorphism being the identity which will be bounded by 1.

If X is a bounded CW complex we may also consider the cellular chains.

Definition 4.2. Let X be a bounded CW complex. We define CX to be the usual cellular chain complex of $\widetilde{X}$, and we give it a $\mathbb{Z}^i$-graded structure as in definition 4.1 above. By a standard argument [5] $S(X)$ is homotopy equivalent to $C(X)$.

Definition 4.3. An object $X \to \mathbb{R}^i$ is finite if $X \to \mathbb{R}^i$ is proper.

Note that CX is a chain complex in $\mathcal{C}_i(\mathbb{Z}\pi_1 X)$ if $X \to \mathbb{R}^i$ is a finite CW complex.

Proposition 4.4. *Let $f : (X \to \mathbb{R}^i) \to (Y \to \mathbb{R}^i)$ be a bounded homotopy equivalence then there is an associated invariant $\sigma(f) \in \widetilde{K}_{-i+1}(\mathbb{Z}\pi_1 X)$.*

Proof. f induces by bounded cellular approximation a chain homotopy equivalence which according to 2.5 has an invariant in $\widetilde{K}_{-i+1}(\mathbb{Z}\pi 1 X)$. □

As usual it is easy to see.

Proposition 4.5. *If $f : (X \to \mathbb{R}^i) \to (Y \to \mathbb{R}^i)$ is a bounded* PL *homeomorphism of simplicial complexes the $\sigma(f) = 0$.*

Proof. First one shows that subdivision gives rise to s homotopy equivalence of chain complexes with 0-torsion of the homotopy equivalence, so one may assume f is a simplicial homeomorphism thus $C(f) : C(X) \to C(Y)$ is an isomorphism sending a basis to a basis and for such it is easy to see the torsion is trivial. □

Proposition 4.6. *If $f : (X \to \mathbb{R}^i) \to (Y \to \mathbb{R}^i)$ is a bounded homotopy equivalence, then $f \times 1_{\mathbb{R}} : (X \times \mathbb{R} \to \mathbb{R}^i \times \mathbb{R})$ has $\sigma(f \times 1) = 0 \in \widetilde{K}_{-i}(\mathbb{Z}\pi)$.*

Proof. Crossing f with S^1 has the effect on CX of tensoring with $\mathbb{Z}[t, t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t, t^{-1}]$ so $\sigma(f \times 1_{S^1}) = 0 \in \widetilde{K}_{-i+1}(\mathbb{Z}(\pi \times \mathbb{Z}))$. Going to the infinite cyclic cover corresponds to introducing t as a grading which is exactly the left inverse of the Bass-Heller-Swan monomorphism, so we are done. □

Using proposition 4.5 and 4.6 one sees that $\sigma(f)$ is an obstruction to splitting off an $\mathbb{R}$-factor when there is no obstruction to doing so up to bounded homotopy.

Definition 4.7. A space $X \to \mathbb{R}^i$ is finitely dominated if there is a finite CW complex $K \to \mathbb{R}^i$ and bounded maps $(X \to \mathbb{R}^i) \xrightarrow{I} (K \to \mathbb{R}^i) \xrightarrow{R} (X \to \mathbb{R}^i)$ such that $R \circ I$ is homotopic to the identity.

Theorem 4.8. *If $(X \to \mathbb{R}^i)$ is finitely dominated then there is an obstruction in $\widetilde{K}_{-i}(\mathbb{Z}\pi_1 X)$ for X being boundedly homotopy equivalent to a finite* CW *complex.*

Proof. Follows immediately from theorem 3.3 □

We do not claim any converse. To do so needs stronger assumptions on $\pi_1(X)$ as in [2, 3]. The point we are making is a more direct definition of the invariant which turns out not to need special π_1-assumptions.

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