ON THE KAROUBI FILTRATION OF A CATEGORY

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ABSTRACT. Let $\mathcal{U}$ be an $\mathcal{A}$-filtered category in the sense of Karoubi. This is the categorical analogue of an ideal $\mathcal{A}$ in a ring $\mathcal{U}$. Pedersen and Weibel constructed a fibration of K-theory spectra associated with the sequence $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$. We present a new easier proof based on Waldhausen' generic fibration.

1. INTRODUCTION

In [5] Karoubi introduced the notion of an additive category $\mathcal{U}$ being filtered by a full subcategory $\mathcal{A}$ (The precise statement - definition 1.5 pages 115-116 of [5] - is recalled in definition 3.2). He then used this to give an axiomatic description of negative K-groups, including an exact sequence

$$K_0(\mathcal{A}) \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}) \to K_{-1}(\mathcal{A}) \to \dots.$$

This sequence was generalized to hold for higher K-groups in [7] where a fibration of spectra was obtained

$$K(\mathcal{A}^{\wedge K}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A}). \tag{1.0.1}$$

Here $\mathcal{A}^{\wedge K}$ is a certain subcategory of the idempotent completion of $\mathcal{A}$. In particular $K(\mathcal{A})$ and $K(\mathcal{A}^{\wedge K})$ only differ at K_0. This fibration was generalized in [3] to produce a fibration

$$K^{-\infty}(\mathcal{A}) \to K^{-\infty}(\mathcal{U}) \to K^{-\infty}(\mathcal{U}/\mathcal{A}) \tag{1.0.2}$$

where $K^{-\infty}$ is a non-connective spectrum whose negative homotopy groups are the negative K-groups of $\mathcal{A}$, and whose connective cover is the usual K-theory spectrum. These fibrations have been applied to produce excision results in controlled algebraic K-theory, see [2], [1], [3], [4]. Most of these excision results are easy consequences of the above fibrations. As an example we derive one of the excision results of [3] in the final section.

The proof in [7] was based on the double mapping cylinder construction of Thomason [11]. In recent years a number of results in algebraic K-theory have been given easier proofs by using Waldhausen's $S.$-construction see e. g. [6], and [9]. In this paper we give a proofs of 1.0.1 using that method. The basic idea is to consider finite chain complexes in $\mathcal{U}$ and

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two notions of weak equivalences, chain homotopy equivalence and chain maps inducing homotopy equivalence in $\mathcal{U}/\mathcal{A}$. The proof then is an application of Waldhausen's generic fibration lemma [13, Theorem 1.6.4], and identification of the terms. This identification uses results of Thomason and Trobaugh [10], which we recall in section 6. We also give a proof of 1.0.2 in the final section.

It is our aim to make this paper as self contained as possible. In the first sections we recall the basic notions and results we shall need in this paper.

2. Categories with cofibrations and weak equivalences

In this section we present a quick review of Waldhausen's K-theory of a small category with cofibrations and weak equivalences [13]. One example to keep in mind is an additive category where the cofibrations are inclusions of direct summands up to isomorphism, and the weak equivalences are the isomorphisms. Another example is the category of finite chain complexes in an additive category with cofibrations the degreewise inclusions of direct summands and weak equivalences the homotopy equivalences. If, in this example, we take the weak equivalences to be the isomorphisms, we get an example of an exact category (since exact sequences are only degreewise split exact). In addition we recall the basic tools which will allow us to decide when two categories have isomorphic K-theory.

Given any small category $\mathcal{C}$ with some extra structure described below, Waldhausen assigns functorially to $\mathcal{C}$ a topological space $K(\mathcal{C})$, which we call K-theory of $\mathcal{C}$. The homotopy groups are defined to be the K-groups of $\mathcal{C}$. This extends the classical definitions of K-groups of a ring R by taking $\mathcal{C}$ to be the additive category of finitely generated projective modules over R, with cofibrations inclusions of direct summands, and weak equivalences isomorphisms, see below for the meaning of these terms.

Definition 2.1. [13, Sections 1.1 and 1.2] A small category $\mathcal{C}$ with a zero object is said to be a category with cofibrations and weak equivalences if it has two distinguished subcategories, $\operatorname{co}\mathcal{C}$ and $w\mathcal{C}$, satisfying the following axioms:
a) $co\mathcal{C}$ axioms.

- *cof 1:* Isomorphisms in $\mathcal{C}$ are cofibrations.
- *cof 2:* For every $A \in \mathcal{C}$, $* \to A$ is a cofibration.
- *cof 3:* Cofibrations admit cobase change:
  - **a:** If $A \to B$ is a cofibration and $A \to C$ any map, then the push out exists in $\mathcal{C}$.
  - **b:** $C \to C \bigcup_A B$ is a cofibration.

b) $w\mathcal{C}$ axioms.

- *weq 1:* Isomorphisms in $\mathcal{C}$ are weak equivalences.

<u>*weq 2:*</u> (Gluing Lemma) If in the commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

the horizontal arrows on the left are cofibrations and all three vertical arrows are in $w\mathcal{C}$ then

$$B \bigcup_A C \to B' \bigcup_A C$$

is in $w\mathcal{C}$.

The two following axioms may, or may not, be satisfied by $\mathcal{C}$.

Saturation axiom: If a, b are composable maps in $\mathcal{C}$ and if two of a, b, ab are in $w\mathcal{C}$ then so is the third.

Extension axiom: Let

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & B'/A' \end{array}$$

be a map of cofibration sequences ($B/A = * \bigcup_A B$). If $A \to A'$ and $B/A \to B'/A'$ are in $w\mathcal{C}$ then $B \to B'$ is in $w\mathcal{C}$ as well.

Having fixed $co\mathcal{C}$ and $w\mathcal{C}$, we have a simplicial category:

$$\begin{aligned} S.\mathcal{C} : \Delta^{op} &\longrightarrow (cat) \\ [n] &\longmapsto S_n\mathcal{C} \end{aligned}$$

where $S_n\mathcal{C}$ is the category with objects composable cofibrations:

$$* \to A_1 \to A_2 \to \cdots \to A_n$$

with chosen quotients $A_{i,j} = A_i/A_j$, $1 \leq i \leq j \leq n$. We always have $A_{i,i} = *$. The morphisms in the category $S_n\mathcal{C}$ are maps $A_i \to B_i$ commuting with the cofibration sequences. $S.\mathcal{C}$ is a simplicial category as follows: the degeneracy maps are given by inserting identities, and the boundary maps d_i by omitting the index i, for $1 \leq i \leq n$. If d_0 were given by extending this recipe and omitting $*$, then the construction would give the nerve of the category co$\mathcal{C}$, which is contractible since there is an initial object. Instead, d_0 prescribes taking all the quotients by A_1, hence the necessity for including a choice of quotients from the beginning.

The category $S_n\mathcal{C}$ is a category with cofibrations and weak equivalences, by defining a map $A \to A'$ to be a cofibration if

$$A_j \longrightarrow A'_j \quad \text{and} \quad A'_j \bigcup_{A_j} A_{j+1} \longrightarrow A'_{j+1}$$

are cofibrations in $\mathcal{C}$ for all j. An arrow $A \to A'$ is defined to be a weak equivalence if the arrow $A_{i,j} \longrightarrow A'_{i,j}$ is a weak equivalence for each pair $i \leq j$. We thus have that $S.$ is a functor from categories with cofibrations and weak equivalences to simplicial categories with cofibrations and weak equivalences. For more details about this see sections 1.1, 1.2 and 1.3 in [13].

We can think of:

$$\begin{aligned} wS.\mathcal{C} : \Delta^{op} &\longrightarrow (cat) \\ [n] &\longmapsto wS_n\mathcal{C} \end{aligned}$$

as a bisimplicial set by taking the nerve of $wS_n\mathcal{C}$

Definition 2.2. [13, Section 1.3] The *Algebraic K-theory* of the category with cofibrations $\mathcal{C}$, with respect to the category of weak equivalences $w\mathcal{C}$ is given by the pointed space

$$K(\mathcal{C}) = \Omega|wS.\mathcal{C}|.$$

The K-groups of $\mathcal{C}$ are the homotopy groups of $K(\mathcal{C})$

$$K_*\mathcal{C} = \pi_*(\Omega|wS.\mathcal{C}|) \ (= \pi_{*+1}|wS.\mathcal{C}|).$$

Actually K-theory can be described as a spectrum rather than just a space. The $S.$-construction extends namely, by naturality, to simplicial categories with cofibrations and weak equivalences. In particular it thus applies to $S.\mathcal{C}$ to produce a bisimplicial category with cofibrations and weak equivalences, $S.S.\mathcal{C} = S.^{(2)}\mathcal{C}$. Again the construction extends to bisimplicial categories with cofibrations and weak equivalences and so on. Therefore we get a spectrum whose n'th space is $|wS.^{(n)}\mathcal{C}|$ The structural maps are defined as the adjoint of the map $\Sigma|w\mathcal{C}| \to |wS.\mathcal{C}|$ which is given as the inclusion of the 1-skeleton in the $S.$-construction, see [13, page 329].

It turns out that this spectrum is an Ω-spectrum beyond the first term (the additivity theorem 2.8 below is needed to prove this). As the spectrum is connective (the n-th term is (n-1)-connected) an equivalent assertion is that in the sequence

$$|w\mathcal{C}| \to \Omega|wS.\mathcal{C}| \to \Omega^2|wS.S.\mathcal{C}| \to \cdots$$

all maps except the first are homotopy equivalences. Hence K-theory of $\mathcal{C}$ could equivalently be defined as the infinite loop space

$$\Omega^\infty|wS.^{(\infty)}\mathcal{C}| = \varinjlim_n \Omega^n|wS.^{(n)}\mathcal{C}|$$

We will refer to any of the three versions as the K-theory of $\mathcal{C}$ and denote it as $K(\mathcal{C})$. If it is necessary to emphasize the category of weak equivalences $w\mathcal{C}$ used to define the K-theory of $\mathcal{C}$, we will write $K(w\mathcal{C})$ instead of $K(\mathcal{C})$, by a slight abuse of notation.

Now we recall criteria that determine when two categories have homotopy equivalent K-theories. Some extra structure is required on the category. It is necessary to have a notion of cylinder in order to define some kind of homotopy theory.

Definition 2.3. [13, section 1.1] A functor $F : C \to C'$, between categories with cofibrations and weak equivalences is said to be *exact* if F preserves all relevant structures. Such a functor induces in a natural way a map

$$wS.F : wS.\mathcal{C} \to wS.\mathcal{C}'$$

and therefore a map between the K-theories.

2.4. The properties of the product and the realization functor ensure that, given a map (simplicial homotopy)

$$H : X \times I \to Y$$

where X and Y are simplicial sets, there is an induced homotopy

$$H : |X| \times I \to |Y|$$

between $|F| = |H_{|X|\times\{0\}}|$ and $|G| = |H_{|X|\times\{1\}}|$. This applies, in particular, to our case when X and Y are the $S.$-constructions of categories $\mathcal{C}$ and $\mathcal{C}'$.

Therefore we have a notion of homotopy between functors. To see more about this we refer the reader to [12, Section 5, Notions of homotopy theory].

Definition 2.5. [13, Section 1.6] A category $\mathcal{C}$ with cofibrations and weak equivalences has a cylinder functor if there is a functor

$$T : \operatorname{Ar}\mathcal{C} \to \operatorname{Diag}\mathcal{C}$$

where $\operatorname{Ar}\mathcal{C}$ is the category of arrows of $\mathcal{C}$ and $\operatorname{Diag}\mathcal{C}$ is the category of diagrams in $\mathcal{C}$.

$$T(f : A \to B) \quad \equiv \quad \begin{array}{ccccc} A & \xrightarrow{i_1} & T(f) & \xleftarrow{i_2} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & /\!/ {\scriptstyle id} & \\ & & B & & \end{array}$$

satisfying:

<u>*Cyl 1:*</u> Front and back inclusion assemble to an exact functor

$$\begin{aligned} \operatorname{Ar}\mathcal{C} &\to F_1\mathcal{C} \\ (f : A \to B) &\to (A \vee B \rightarrowtail T(f)) \end{aligned}$$

where $F_1\mathcal{C}$ is the full subcategory of $\operatorname{Ar}\mathcal{C}$ whose objects are the cofibrations in $\mathcal{C}$.

Cyl 2: $T(* \to A) = A$, for every $A \in C$ and projection and back inclusion are the identity on A.

There is an additional axiom that is often satisfied:

Cylinder axiom: The projection $T(f) \to B$ is in $w\mathcal{C}$ for every $f : A \to B$.

Definition 2.6. [13, section 1.3] A *cofibration sequence* of exact functors $\mathcal{C} \to \mathcal{C}'$ is a sequence of natural transformations $F' \to F \to F''$ having the property that for every $A \in \mathcal{C}$ $F'(A) \to F(A) \to F''(A)$ is a cofibration sequence in $\mathcal{C}'$.

One of the basic tools is the additivity theorem [13, Theorem 1.4.2 and Proposition 1.3.2], see also [6]. To state it we need a definition.

Definition 2.7. [13, section 1.1] Given a category with cofibrations and weak equivalences $\mathcal{C}$ and subcategories with cofibrations and weak equivalences $\mathcal{A}$ and $\mathcal{B}$, we define the extension category $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to be the category with objects cofibrations $A \to C \to B$ in $\mathcal{C}$ where A is an object of $\mathcal{A}$, B an object of $\mathcal{B}$ and C an object of $\mathcal{C}$. This is a category with cofibrations and weak equivalences in an obvious manner as a subcategory of $S_2\mathcal{C}$. We shall denote $E(\mathcal{C}, \mathcal{C}, \mathcal{C})$ as $E(\mathcal{C})$.

We can now state the additivity theorem.

Theorem 2.8. *The maps*

$$|wS.F| \text{ and } |wS.(F' \vee F'')| \tag{2.8.1}$$

are homotopic.

This statement is equivalent to either of the following statements:

(i) *The map*

$$\begin{aligned} wS.E\,(\mathcal{A}, \mathcal{C}, \mathcal{B}) &\longrightarrow wS.\mathcal{A} \times wS.\mathcal{B} \\ A \to C \to B &\longmapsto (A, B) \end{aligned} \tag{2.8.2}$$

is a homotopy equivalence.

(ii) *The map*

$$\begin{aligned} wS.E\,(\mathcal{C}) &\longrightarrow wS.\mathcal{C} \times wS.\mathcal{C} \\ A \to C \to B &\longmapsto (A, B) \end{aligned} \tag{2.8.3}$$

is a homotopy equivalence.

(iii) *The two maps*

$$\begin{gathered} wS.E\,(\mathcal{C}) \longrightarrow wS.\mathcal{C} \\ (A \to C \to B) \mapsto C \text{ and } (A \to C \to B) \mapsto A \vee B \end{gathered} \tag{2.8.4}$$

are homotopic.

Let us see how the K-theories of a category and a subcategory relate to each other.

Definition 2.9. Let $\mathcal{A}$ be an exact subcategory of the exact category $\mathcal{B}$. $\mathcal{A}$ is said to be cofinal in $\mathcal{B}$ if $0 \to A' \to B \to A'' \to 0$ is exact in $\mathcal{B}$ with A' and A'' in $\mathcal{A}$, then so is B, and if for each B in $\mathcal{B}$ there is a B' in $\mathcal{B}$ so that $B \oplus B'$ is isomorphic to an object in $\mathcal{A}$. (For simplicity we will assume $\mathcal{A}$ is isomorphism closed in $\mathcal{B}$. This does not change the K-theory of $\mathcal{A}$).

The next theorem is known as the cofinality theorem.

Theorem 2.10. [9, Theorem 2.1] *Let $\mathcal{A}$ be cofinal in $\mathcal{B}$ and $G = K_0(\mathcal{B})/K_0(\mathcal{A})$. Then there is a fibration sequence up to homotopy*

$$iS.\mathcal{A} \to iS.\mathcal{B} \to BG.$$

where the weak equivalences are chosen to be the isomorphisms, the minimal possible choice.

In general, given a category $\mathcal{C}$ we will fix the cofibrations and then look at the interplay of the two K-theories defined by two different notions of weak equivalences. Let $\mathcal{C}$ be a category with cofibrations equipped with two categories of weak equivalences, one finer than the other, $v\mathcal{C} \subset w\mathcal{C}$. Let $\mathcal{C}^w$ denote the full subcategory with cofibrations of $\mathcal{C}$ given by the objects A in $\mathcal{C}$ having the property $* \to A$ is in $w\mathcal{C}$. It inherits weak equivalences:

$$v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C} \qquad w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$$

Now recall the generic fibration lemma.

Lemma 2.11. [13, Theorem 1.6.4] *If $\mathcal{C}$ has a cylinder functor, and the coarse category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom, saturation axiom and extension axiom, then the square:*

$$\begin{array}{ccc} vS.\mathcal{C}^w & \longrightarrow & wS.\mathcal{C}^w \ (\simeq *) \\ \downarrow & & \downarrow \\ vS.\mathcal{C} & \longrightarrow & wS.\mathcal{C} \end{array}$$

is homotopy cartesian, and the upper right term is contractible.

Next we recall the approximation theorem, a sufficient condition for an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ to induce a homotopy equivalence $wS.\mathcal{A} \to wS.\mathcal{B}$.

Definition 2.12. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of categories with cofibrations and weak equivalences. We say it has the approximation property if it satisfies:

<u>*App 1:*</u> An arrow in $\mathcal{A}$ is a weak equivalence in $\mathcal{A}$ if and only if its image in $\mathcal{B}$ is a weak equivalence in $\mathcal{B}$.

App 2: Given any object A in $\mathcal{A}$ and any map $x : F(A) \to B$ in $\mathcal{B}$ there exists a cofibration $a\colon A \to A'$ in $\mathcal{A}$ and a weak equivalence $x' : F(A') \to B$ in $\mathcal{B}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{x} & B \\ {\scriptstyle F(a)}\downarrow & \nearrow_{x'} & \\ F(A') & & \end{array}$$

commutes.

The approximation theorem says:

Theorem 2.13. [13, Theorem 1.6.7] *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and weak equivalences. Assume $w\mathcal{A}$ and $w\mathcal{B}$ satisfy the saturation axiom. Suppose $\mathcal{A}$ has a cylinder functor that satisfies the cylinder axiom. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor having the approximation properties. Then $w\mathcal{A} \to w\mathcal{B}$ and $wS.\mathcal{A} \to wS.\mathcal{B}$ induce homotopy equivalences.*

3. Additive categories. Filtrations.

In this and the next section, we recall Karoubi's notion of filtration of an additive category [5] and present the natural structures of categories of cofibrations and weak equivalences that this concepts lead to.

Definition 3.1. An additive category is a small category with a zero object 0, where $\mathrm{Hom}(U, V)$, the group of morphisms between objects U and V, is abelian. Moreover, the composition is bilinear with respect to this operation. Finite products and coproducts exist in such a category and are isomorphic.

All definitions that follow in this section are taken from [7, Section 5].

Let $\mathcal{A}$ be a full subcategory of the additive category $\mathcal{U}$. We shall use the letters A through F (resp. U through Z) to denote objects of $\mathcal{A}$ (resp. $\mathcal{U}$).

Definition 3.2. We say $\mathcal{U}$ is $\mathcal{A}$-filtered if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$ (called a filtration of U) satisfying the following axioms:

F1: For each U, the decompositions form a filtered poset under the partial order $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$ whenever $U_\beta \subset U_\alpha$ and $E_\alpha \subset E_\beta$.

F2: Every map $A \to U$ factors $A \to E_\alpha \to E_\alpha \oplus U_\alpha = U$ for some α.

F3: Every map $U \to A$ factors $U = E_\alpha \oplus U_\alpha \to E_\alpha \to A$ for some α.

F4: For each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$, i. e. to $\{U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)\}$.

Definition 3.3. We now suppose given an $\mathcal{A}$-filtered category $\mathcal{U}$. Call a map $U \to V$ completely continuous, (cc), if it factors through an object in $\mathcal{A}$. $\mathcal{U}/\mathcal{A}$ is defined to be the category with the same objects as $\mathcal{U}$ but with

$$\mathrm{Hom}_{\mathcal{U}/\mathcal{A}}(U, V) = \mathrm{Hom}_{\mathcal{U}}(U, V)/\,\{cc\ maps\}$$

i. e. two maps are the same if their difference factors through an object in $\mathcal{A}$.

The additive categories $\mathcal{U}$ and $\mathcal{U}/\mathcal{A}$ have compatible natural structures as categories of cofibrations and weak equivalences where the cofibrations are the morphisms that are isomorphic to split monomorphisms into direct summands and the weak equivalences are the isomorphisms.

Given the $\mathcal{A}$-filtration of $\mathcal{U}$ we can endow $\mathcal{U}$ with another, larger, category of weak equivalences than the isomorphisms of $\mathcal{U}$. This new one, w, will be those morphisms whose classes in $\mathcal{U}/\mathcal{A}$ are isomorphisms. We retain the same category of cofibrations as in $\mathcal{U}$. The category $\mathcal{U}$ with this choice of cofibrations and weak equivalences will be denoted $\mathcal{U}(\mathcal{A})$.

The objective is to apply the generic fibration lemma 2.11, to the identity functor

$$\mathcal{U} \longrightarrow \mathcal{U}\,(\mathcal{A}) \tag{3.3.1}$$

hoping to obtain as fiber the category $\mathcal{A}$. We can not use these categories directly since neither has a cylinder functor as the generic fibration lemma requires. Therefore we need to consider the corresponding categories of finite chain complexes.

4. The category $C(\mathcal{U})$ and its structures

Given an additive category $\mathcal{U}$, we can define the category of finite chain complexes in $\mathcal{U}$, where objects are:

$$C_\# : 0 \to C_r \xrightarrow{d} C_{r-1} \xrightarrow{d} \cdots \to C_l \to 0$$

such that $d^2 = 0$, i. e. d^2 factors through the zero object. A chain map $f : C_\# \to D_\#$ is a collection of morphisms $f = \{f_r \colon C_r \to D_r\}$ such that $d_D f = f d_C$. A chain homotopy in $\mathcal{U}$

$$e\colon f \simeq f' : C \to D$$

is a collection of morphisms $\{e\colon C_r \to D_{r+1}\}$ such that $d_D e + e d_C = f' - f : C_r \to D_r$. A chain equivalence is a chain map $f : C \to D$ which admits a chain homotopy inverse, that is, a chain map $g\colon D \to C$ such that

$$\exists h : gf \simeq 1 : C \to C \text{ and} \exists k : fg \simeq 1 : D \to D\ .$$

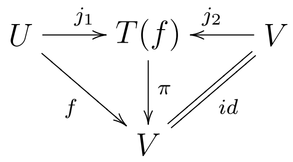
The cofibrations will be those chain maps which degree-wise are inclusions into direct summands. The weak equivalences will be the chain homotopy equivalences. We shall denote this category $C(\mathcal{U})$.

4.1. $C(\mathcal{U})$ has a cylinder functor.

Given $f : U \to V$ a morphism, let $T(f)$ be the chain complex $(T(f))_p = U_p \oplus U_{p-1} \oplus V_p$ with boundary

$$d_p \equiv \begin{pmatrix} d_U & -1 & 0 \\ 0 & -d_U & 0 \\ 0 & f & d_V \end{pmatrix}$$

We have the following diagram:



where j_1 and j_2 are the obvious inclusions as direct summands. Degree-wise π is defined as:

$$\pi_p \equiv (f, 0,\ 1)$$

It is easy to check that *Cyl 1* and *Cyl 2* are satisfied. The cylinder axiom also holds. To see this, we need to show that π is a weak equivalence, i. e. a chain homotopy equivalence. The homotopy inverse is the natural inclusion

$$i_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Degree-wise, all is given by the following matrices:

$$\pi_p = (f, 0,\ 1) \quad d_p = \begin{pmatrix} d & -1 & 0 \\ 0 & -d & 0 \\ 0 & f & d \end{pmatrix} \quad \Gamma_p = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to check now that $\Gamma_p d_{p+1} + d_{p+2}\Gamma_{p+1} = i_2\pi_{p+1} - 1$ and $\pi i_2 = 1$.

4.2. $C(\mathcal{U})$ satisfies the saturation and the extension axiom as well.

This is proved by elementary chain complex manipulations involving iterated mapping cones.

When we take weak equivalences in $\mathcal{C}(\mathcal{U})$ to be the chain maps that induce homotopy equivalences in $\mathcal{U}/\mathcal{A}$, we use the notation $C(\mathcal{U}(\mathcal{A}))$ for the resulting category with cofibrations and weak equivalences. Obviously the subcategory of chain complexes concentrated in degree 0 is exactly $\mathcal{U}(\mathcal{A})$. $C(\mathcal{U}/\mathcal{A})$ has a cylinder functor inherited from $C(\mathcal{U})$ satisfying the cylinder axiom, saturation axiom and extension axiom. This follows by working in $C(\mathcal{U}/\mathcal{A})$.

At this point, we can apply the generic fibration lemma 2.11 to the functor induced by the identity $C(\mathcal{U}) \longrightarrow C(\mathcal{U}(\mathcal{A}))$ and obtain a fibration up to homotopy

$$K(C(\mathcal{U})^u) \longrightarrow K(C(\mathcal{U})) \longrightarrow K(C(\mathcal{U}(\mathcal{A})))$$

where u denote the weak equivalences in $C(\mathcal{U}(\mathcal{A}))$.

We intend to show this is the fibration we have been aiming for. So it remains to identify the terms of this fibration.

5. Idempotent completions.

Definition 5.1. An additive category satisfies property (P) if given maps $f : E \to F$ and $s : F \to E$ such that $fs = 1_F$, then there is an object G and an isomorphism $E \cong F \oplus G$ under which f becomes projection on the first factor.

We do not wish to assume our categories satisfy property (P), and one of the aims of this section is to be able to replace an additive category by an additive category which does satisfy property (P) without changing its K-theory. This is obtained by considering suitable subcategories of the idempotent completion of an additive category.

The idempotent completion of an additive category $\mathcal{U}$, denoted $\mathcal{U}^\wedge$, is the additive category with objects (U, p) with $p = p^2 : U \to U$ and morphisms $f : (U, p) \to (V, q)$ satisfying $f = qfp \colon U \to V$. The identity morphism of (U, p) in $\mathcal{U}^\wedge$ is represented by p. We get an embedding of additive categories:

$$\mathcal{U} \hookrightarrow \mathcal{U}^\wedge$$

sending U to $(U, 1)$ which is full and cofinal. The morphisms $f \colon (U, 1) \to (V, 1)$ in $\mathcal{U}^\wedge$ are precisely those in $\mathcal{U}$, and for every (U, p) in $\mathcal{U}^\wedge$

$$(U, p) \oplus (U, 1-p) \underset{\binom{p}{1-p}}{\overset{(p,1-p)}{\rightleftarrows}} (U, 1)$$

are isomorphisms expressing (U, p) as a direct summand of $(U, 1)$. By the cofinality theorem 2.10 we have a fibration up to homotopy:

$$K(\mathcal{U}) \to K(\mathcal{U}^\wedge) \to B\pi$$

where $\pi = K_0(\mathcal{U}^\wedge)\,/K_0(\mathcal{U})$. In particular, this implies

$$K_0(\mathcal{U}) \rightarrowtail K_0(\mathcal{U}^\wedge).$$

Lemma 5.2. *Property (P) holds for $\mathcal{U}^\wedge$.*

Proof. Let

$$(U, p) \underset{s}{\overset{r}{\rightleftarrows}} (V, q) \tag{5.3}$$

be such that $rs = q$. We have also

$$qrp = r \quad p^2 = p \quad psq = s \quad s^2 = s.$$

Now $(sr)(sr) = s(rs)r = sqr = (psq)qr = psq^2r = psqr = (psq)r = sr$ so (U, sr) makes sense in $\mathcal{U}^\wedge$ and moreover it is an idempotent for (U, p). Since $\mathcal{U}^\wedge$ is complete by definition we have

$$(U,p) \cong (U, p - sr) \oplus (U, sr). \tag{5.3.1}$$

Moreover

$$(U, sr) \underset{s}{\overset{r}{\rightleftarrows}} (V, q)$$

are isomorphic by those morphisms and therefore r in 5.3 is an admissible epimorphism. □

5.4. The isomorphism 5.3.1 is true by the following argument.

If $q\colon (U,p) \to (U,p)$ is such that $q^2 = q$, also $pqp = q$, then

$$(pqp)(pqp) = (pqp)$$

thus by the properties of the idempotent completion we have

$$(U,p) \cong (U, pqp) \oplus (U, p - pqp)$$

where the isomorphisms are given by the matrices

$$\begin{pmatrix} pqp \\ p - pqp \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} pqp, & p - pqp \end{pmatrix}.$$

It is easy to see that the category $\mathcal{U}^\wedge$ satisfies a stronger property than property (P). $\mathcal{U}^\wedge$ is Karoubian or equivalently idempotent complete. We say that an additive category E is Karoubian if whenever $p\colon E \to E$ such that $p^2 = p$ then there is an isomorphism $E \cong E' \oplus E''$ under which p corresponds to the endomorphism $1 \oplus 0$.

Now let $\mathcal{A}$ be a full subcategory of the additive category $\mathcal{U}$.

Definition 5.5. Let $K \subset K_0(\mathcal{A}^\wedge)$ be the inverse image of $K_0(\mathcal{U})$ under the map $K_0(\mathcal{A}^\wedge) \to K_0(\mathcal{U}^\wedge)$. We shall denote the full subcategory of $\mathcal{U}^\wedge$ with objects $U \oplus (A,p)$, where $[(A,p)] \in K$ by $\mathcal{U}^{\wedge K}$. Notice that $\mathcal{A}^\wedge$ is embedded in $\mathcal{U}^\wedge$, $\mathcal{A}^\wedge \hookrightarrow \mathcal{U}^\wedge$.

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathcal{U}^\wedge \\ & \searrow & \uparrow \\ & & \mathcal{U}^{\wedge K} \end{array}$$

$\mathcal{U}$ is cofinal in $\mathcal{U}^{\wedge K}$ and in $\mathcal{U}^\wedge$ hence $\mathcal{U}^{\wedge K}$ is cofinal in $\mathcal{U}^\wedge$. We thus obtain a diagram of monomorphisms

$$\begin{array}{ccc} K_0(\mathcal{U}) & \rightarrowtail & K_0(\mathcal{U}^\wedge) \\ & \searrow & \uparrow \\ & & K_0(\mathcal{U}^{\wedge K}) \end{array}$$

where the images of $K_0(\mathcal{U})$ and $K_0(\mathcal{U}^{\wedge K})$ in $K_0(\mathcal{U}^{\wedge})$ are the same. Hence

$$K_0(\mathcal{U}) \to K_0(\mathcal{U}^{\wedge K})$$

is an isomorphism and therefore $\mathcal{U}$ and $\mathcal{U}^{\wedge K}$ have homotopy equivalent K-theories, by the cofinality theorem.

In a more general setting we give the following definition.

Definition 5.6. Given $\mathcal{U}$ an additive category and K a subgroup of $K_0(\mathcal{U})$. Let $\mathcal{U}^{\wedge K}$ be the full subcategory of $\mathcal{U}^{\wedge}$ with objects (U, p) so that its stable isomorphism class lies in K. When $K = K_0(\mathcal{U})$ we denote $\mathcal{U}^{\wedge K_0(\mathcal{U})}$ by $\overline{\mathcal{U}}$.

Example 5.7. If $\mathcal{U}$ is the category of finitely generated free R-modules for some ring R, then $\mathcal{U}^{\wedge}$ is equivalent to the category of finitely generated projective R-modules, and $\overline{\mathcal{U}}$ is equivalent to the category of finitely generated stably free R-modules.

Remark 5.8. Notice the following

(i) The category $\overline{\mathcal{U}}$ can be seen in terms of the first definition as $\mathcal{U}^{\wedge K_0(\mathcal{U})}$ by taking the trivial filtration $\mathcal{A} = \mathcal{U}$.

(ii) Using the same notation for $\mathcal{U}^{\wedge K}$ in the two definitions above will not cause confusion since in one situation $K \subset K_0(\mathcal{A}^{\wedge})$ and in the other $K \subset K_0(\mathcal{U}^{\wedge})$.

Lemma 5.9. *The inclusion $\mathcal{U} \subset \overline{\mathcal{U}}$ induces an isomorphism in K-theory.*

Proof. The category $\mathcal{U}$ is cofinal in $\overline{\mathcal{U}}$ and therefore

$$\begin{array}{ccc} K_0(\mathcal{U}) & \rightarrowtail & K_0(\mathcal{U}^{\wedge}) \\ & \searrow & \uparrow \\ & & K_0(\overline{\mathcal{U}}) \end{array}$$

is a commutative diagram where all arrows are monomorphisms. By the same argument as above, $K_0(\mathcal{U}) \cong K_0(\overline{\mathcal{U}})$ is an isomorphism. Again, by the cofinality theorem 2.10, $\mathcal{U}$ and $\overline{\mathcal{U}}$ have homotopy equivalent K-theories. □

Lemma 5.10. *The category $\overline{\mathcal{U}}$ satisfies property (P).*

Proof. We can use an argument similar to the one used above for $\mathcal{U}^{\wedge}$. If we have the diagram in $\overline{\mathcal{U}}$

$$(U, p) \underset{s}{\overset{r}{\rightleftarrows}} (V, q)$$

with $rs = q$, it is also a map in $\mathcal{U}^{\wedge}$ and thus

$$(U, p) \cong (U, p - sr) \oplus (U, sr)$$

and

$$(U, sr) \cong (V, q).$$

But (U, p) and (V, q) are in $\overline{\mathcal{U}}$ so by the properties of K_0 and the definition of $\overline{\mathcal{U}}$ we conclude (U, sr) and $(U, p - sr)$ are in $\overline{\mathcal{U}}$. Hence r is an admissible epimorphism and $\overline{\mathcal{U}}$ satisfies the property (P). □

Before we can state and prove the Main Theorem, we need some results on chain complexes.

6. Chain complex categories

In this section we show that an additive category and its associated category of chain complexes have the same K-theory. A precise statement of the result is proposition 6.1 below. Basically, we recall the proof of this statement due to Thomason and Trobaugh.

As we have already mentioned, the purpose is to be able to use the generic fibration lemma, which we can not use it directly because the additive categories do not have cylinder functors. Therefore we replace the categories by their corresponding categories of finite chain complexes, and use the following.

Proposition 6.1. [10, Theorem 1.11.7] *Given $\mathcal{U}$ an additive category, let $C(\mathcal{U})$ be its category of finite chain complexes. Assume $\mathcal{U}$ and $C(\mathcal{U})$ are given the usual 'structures' of categories with cofibrations and weak equivalences as explained in sections 3 and 4. Then, the embedding $\mathcal{U} \hookrightarrow C(\mathcal{U})$, as chain complexes of length 1, induces a homotopy equivalence of K-theory spectra.*

Proof. First let us assume that $\mathcal{U}$ satisfies property (P) (see definition 5.1). It has been shown that $C(\mathcal{U})$ has a cylinder functor and satisfies the saturation axiom, the extension axiom and the cylinder axiom, see section 4. Recall that the weak equivalences in $C(\mathcal{U})$ are the chain homotopy equivalences. We will denote this by $wC(\mathcal{U})$. Without changing the subcategory of cofibrations, we can regard $C(\mathcal{U})$ as having as its weak equivalences just the isomorphisms of chain complexes. This 'new structure' on $C(\mathcal{U})$ will be denoted as $iC(\mathcal{U})$. The category $C(\mathcal{U})$ can be thought of as $\lim\limits_{\substack{a \to -\infty \\ b \to +\infty}} C_a^b$, where C_a^b is the full subcategory of complexes in $C(\mathcal{U})$ with $C_i = 0$ whenever $i < a$ or $i > b$. For any $a, b \in \mathbb{Z}$, $w\, C_a^b$ is a category with cofibrations and two notions of weak equivalences inherited from $wC(\mathcal{U})$ and $iC(\mathcal{U})$ respectively. We shall denote these by $w\, C_a^b$ and $i\, C_a^b$. We identify $\mathcal{U}$ with C_0^0. It is clear that $wC_0^0 = i\, C_0^0$.

In this context, we may consider $iC(\mathcal{U})^w$, which is the full subcategory of $iC(\mathcal{U})$ whose objects are those chain complexes that are w-contractible, i. e. chain homotopy equivalent to the 0-chain complex. By the generic fibration lemma:

$$K(iC(\mathcal{U})^w) \to K(iC(\mathcal{U})) \to K(wC(\mathcal{U}))$$

is a fibration, up to homotopy.

We shall show that $K(\mathcal{U})$ $(= K(w\mathcal{U}) = K(i\mathcal{U})$ considering $\mathcal{U}$ embedded in $C(\mathcal{U}))$, is the cofiber of the map

$$K(iC(\mathcal{U})^w) \to K(iC(\mathcal{U}))$$

Consider the exact functor

$$\begin{aligned} i\,C_a^b &\longrightarrow \prod^{b-a+1} \mathcal{U} \\ C_\# &\mapsto (C_a, \ldots, C_b) \end{aligned} \tag{6.1.1}$$

We claim this functor induces a homotopy equivalence of K-theories.

For $a = b$, it is clear. Now, by induction, we will show

$$\begin{gathered} i\,C_a^b \longrightarrow i\,C_{a+1}^b \times \mathcal{U} \\ (C_b \to C_{b-1} \to \cdots \to C_a) \mapsto ((C_b \to \cdots \to C_{a+1}), (0 \to \cdots \to 0 \to C_a)) \end{gathered} \tag{6.1.2}$$

induces homotopy equivalence in K-theory. This however is clear by the additivity theorem 2.8, since

$$i\,C_a^b = E(\ i\,C_{a+1}^b,\ C_a^b,\ \mathcal{U}).$$

We now claim that $i\,C_a^{b^w}$ is homotopy equivalent in K-theory to $\prod^{b-a} \mathcal{U}$. We do this by induction on $b - a$.

For $b = a$, $i\,C_a^{a^w} = i(\mathcal{U})^w$ which is equivalent to the 0-category.

For $b = a + 1$, it is also clear that

$$i\,C_a^{a+1^w} \equiv \{ \text{ category of complexes } C_{a+1} \xrightarrow{\partial} C_a \text{ where } \partial \text{ is an isomorphism}\}$$

so it is equivalent to $\mathcal{U}$.

We continue by induction on $b - a$. We shall produce a homotopy equivalence:

$$K(i\,C_a^{b^w}) \xrightarrow{\simeq} K(i\,C_{a+1}^{b^w}) \times K(i\,C_a^{a+1^w} \cong \mathcal{U})$$

This is obtained by applying the additivity theorem 2.8 to the equivalence of categories

$$i\,C_a^{b^w} \cong E\left(i\ C_{a+1}^{b^w},\ i\,C_a^{b^w},\ i\ C_a^{a+1^w}\right). \tag{6.1.3}$$

We need to show this equivalence. Given a chain complex $C_\#$ in $i\,C_a^{b^w}$ we need to produce an associated extension with a chain complex of length $b - a - 1$, $\tau^{\leq b-a-1}(C_\#)$, and other one of length 1, $\tau^1(C_\#)$. The inverse equivalence of categories takes the total complex $C_\#$ and forgets the extensions. It is easy to check that both the equivalence of categories and its inverse are exact functors.

Since

$$C_\# \equiv \left\{\ 0 \to C_b \to C_{b-1} \to \cdots \to C_{a+2} \xrightarrow{d_{a+2}} C_{a+1} \xrightarrow{d_{a+1}} C_a \to 0\right\}$$

is contractible, we have a chain map s such that $sd+ds=1$. In degree $a+1$, s is a splitting. Therefore, d_{a+1} is a splitting epimorphism in $\mathcal{U}$. But $\mathcal{U}$ satisfies (P), so there exist Z_{a+1} and an isomorphism such that $C_{a+1} \cong Z_{a+1} \oplus C_a$. Moreover, through this isomorphism, d_{a+1} becomes a projection onto C_a. The maps d_{a+2} and s factor through Z_{a+1}. In this way, we obtain shorter contractible chain complexes:

$$\tau^{\leq b-a-1}(C_\#) \equiv (0 \to C_b \to C_{b-1} \to \cdots \to C_{a+2} \to Z_{a+1} \to 0)$$
$$\tau^1(C_\#) \equiv (0 \to C_a \xrightarrow{\text{id}} C_a \to 0)$$

Now, $C_\#$ fits into the sequence:

$$\tau^{\leq b-a-1}(C_\#) \rightarrowtail C_\# \twoheadrightarrow \tau^1(C_\#)$$

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C_b & = & C_b & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C_{b-1} & = & C_{b-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C_{a+2} & = & C_{a+2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Z_{a+1} & \rightarrowtail & C_{a+1} & \twoheadrightarrow & C_a \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_a & = & C_a \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

We have the equality of the formula 6.1.3 as we wanted. The additivity theorem 2.8 can be applied obtaining the homotopy equivalence of K-theories we wanted.

Unraveling the induction, we have shown that for $C_\#$ in $C_a^{b^w}$, there are kernels Z_i $a+1 \leq i \leq b$ in $\mathcal{U}$ and the functor $C_\# \longrightarrow Z_i$ is an exact functor, for each i. In fact, the homotopy equivalence is induced by

(6.1.4)
$$\begin{aligned} iC_a^{b^w} &\longrightarrow \prod^{b-a} \mathcal{U} \\ C_\# &\longrightarrow (Z_{a+1}, \ldots, Z_b) \end{aligned}$$

Now, let us consider the exact inclusion

$$iC_a^{b^w} \rightarrowtail iC_a^b$$

and the induced maps of K-theory spectra:

$$\begin{array}{ccc} K(iC_a^{b^w}) & \longrightarrow & K(iC_a^b) \\ \downarrow & & \downarrow \\ \prod^{b-a} K(\mathcal{U}) & \longrightarrow & \prod^{b-a+1} K(\mathcal{U}) \end{array}$$

Given a chain complex $C_\#$ in $iC_a^{b^w}$ the term in $\prod^{b-a} \mathcal{U}$ is $(Z_{a+1}, \ldots, Z_b)$ and in $\prod^{b-a+1} \mathcal{U}$ is $(C_a, C_{a+1}, \ldots, C_b)$. Since in $C_\#$ we can identify, for each dimension, the exact sequence

$$Z_k \rightarrowtail C_k \twoheadrightarrow Z_{k+1}$$

it can be shown, using the additivity theorem 2.8, that the map, once passing to K-theory, sending $C_\#$ to C_k is homotopic to the 'sum' of the maps sending $C_\#$ to Z_k and $C_\#$ to Z_{k+1}.

Therefore, we can assume the map in the above square

$$\prod^{b-a} K(\mathcal{U}) \longrightarrow \prod^{b-a+1} K(\mathcal{U})$$

is induced by

$$(Z_{a+1}, \ldots, Z_b) \longrightarrow (Z_{a+1}, Z_{a+1} \oplus Z_{a+2}, \ldots, Z_{b-1} \oplus Z_b, Z_b)$$

The homotopy cofiber of this map is $K(\mathcal{U})$. It is induced by

$$\prod^{b-a+1} K(\mathcal{U}) \to K(\mathcal{U})$$

$$(x_a, \ldots, x_b) \to \sum_{k=a}^{b} (-1)^k x_k$$

Taking direct limits ($a \to -\infty$, $b \to +\infty$), we get the cofiber homotopy sequence

(6.1.5)
$$\begin{aligned} K(iC^w) \longrightarrow K(iC) &\longrightarrow K(\mathcal{U}) \\ C_\# &\longrightarrow \sum (-1)^k C_k \end{aligned}$$

By the generic fibration lemma 2.11, the homotopy cofiber spectrum is, up to homotopy, $K(wC)$. Thus, there is a homotopy equivalence

$$K(\mathcal{U}) \xrightarrow{\simeq} K(wC(\mathcal{U}))$$

induced by the exact functor $\mathcal{U} \to C(\mathcal{U})$. This completes the proof when $\mathcal{U}$ satisfies property (P).

If $\mathcal{U}$ does not satisfy the property (P) then $\overline{\mathcal{U}}$ does, see section 5. Moreover, $\mathcal{U}$ is cofinal in $\overline{\mathcal{U}}$ and $K(\mathcal{U}) \to K(\overline{\mathcal{U}})$ is a homotopy equivalence by the cofinality theorem since it is an isomorphism on K_0, see section 5. Similarly $iC(\mathcal{U}) \to iC(\overline{\mathcal{U}})$ and $iC(\mathcal{U})^w \to iC(\overline{\mathcal{U}})^w$ are cofinal inclusions which are easily seen to induce isomorphisms on K_0, hence induce homotopy equivalences in K-theory by the cofinality theorem.

Consider the diagram

$$\begin{array}{ccccccccc} K(iC(\mathcal{U})^w) & \longrightarrow & K(iC(\mathcal{U})) & \longrightarrow & K(wC(\mathcal{U})) & & & \nwarrow & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ K(iC(\overline{\mathcal{U}})^w) & \longrightarrow & K(iC(\overline{\mathcal{U}})) & \longrightarrow & K(wC(\overline{\mathcal{U}})) & \simeq & K(\overline{\mathcal{U}}) & \simeq & K(\mathcal{U}). \end{array}$$

The top and bottom row are fibrations by the generic fibration lemma 2.11. The two homotopy equivalences at the bottom are consequences of proposition 6.1 and cofinality 2.10. We have just argued that the vertical arrows on the left and in the middle are homotopy equivalences , hence we can conclude that the vertical arrow on the right is a homotopy equivalence. The right hand side diagram commutes and therefore the theorem holds for arbitrary additive categories. □

7. Proof of the main theorem

In this section we will prove

Theorem 7.1. [7, Theorem 5.3] *Given $\mathcal{U}$, an additive $\mathcal{A}$-filtered category, then*

$$K(\mathcal{A}^{\wedge K}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A})$$

is a fibration, up to homotopy. Here K is the inverse image of $K_0(\mathcal{U}) \subset K(\mathcal{U}^\wedge)$ under the induced map $K(\mathcal{A}^\wedge) \to K(\mathcal{U}^\wedge)$, and the notation $\mathcal{A}^{\wedge K}$ is explained in definition 5.5.

We will apply the generic fibration lemma to the functor induced by the identity.

$$C(\mathcal{U}) \longrightarrow C\left(\mathcal{U}\left(\mathcal{A}\right)\right)$$

and furthermore show that $C\left(\mathcal{U}\left(\mathcal{A}\right)\right)$ and $C\left(\mathcal{U}/\mathcal{A}\right)$ are homotopy equivalent in K-theory. By proposition 6.1 we thus have models for $K(\mathcal{U})$ and $K(\mathcal{U}/\mathcal{A})$. To avoid notational confusion we shall denote the weak equivalences in $C\left(\mathcal{U}(\mathcal{A})\right)$ by $uC\left(\mathcal{U}(\mathcal{A})\right)$. Recall these are the morphism that induce homotopy equivalences in $C\left(\mathcal{U}/\mathcal{A}\right)$.

Proposition 7.2. *Let $\mathcal{U}$ be an additive $\mathcal{A}$-filtered category. The functor*

$$F\colon C(\mathcal{U}(A)) \to C\left(\mathcal{U}/\mathcal{A}\right)$$

which is the identity on the objects and takes classes on the morphisms mod{cc maps}, induces a homotopy equivalence of K-theories.

Proof. We will use the approximation theorem 2.13. The weak equivalences $uC(\mathcal{U}(A))$ and $wC\left(\mathcal{U}/\mathcal{A}\right)$ satisfy the saturation axiom. $C(\mathcal{U}(A))$ has a natural cylinder functor inherited from $C(\mathcal{U})$ satisfying the cylinder axiom, see section 4.

App 1 is satisfied trivially, by definition of $uC(\mathcal{U}(A))$.

App 2 will be easy after the following remark:

By the properties of the $\mathcal{A}$-filtration of $\mathcal{U}$ any $D_\#$ in $C\left(\mathcal{U}/\mathcal{A}\right)$ is isomorphic in $\mathcal{U}/\mathcal{A}$ to a chain complex from $C(\mathcal{U})$ [4, Proof of theorem 4.1].

Let $C_\#$ be in $C(\mathcal{U}(A))$ and $F(C_\#) \xrightarrow{x} D_\#$ in $C\left(\mathcal{U}/\mathcal{A}\right)$. We are assuming, by the remark, that $D_\#$ is isomorphic by φ to a chain complex $D'_\#$ which is from $C(\mathcal{U})$. We can apply the cylinder functor to φx, obtaining an object, $T(\varphi x)$ in $C(\mathcal{U})$. The diagram is

$$\begin{array}{ccccc} F(C_\#) & \xrightarrow{x} & D_\# & \underset{\sim}{\xrightarrow{\varphi}} & D'_\# \\ \downarrow & & & \nearrow_{p} & \\ T(\varphi x) & & & & \end{array}$$

where φ is an isomorphism and hence a weak equivalence. So is p by the cylinder axiom. Therefore $\varphi^{-1}p$ is a weak equivalence. All of this only needs to commute mod $\mathcal{A}$ because the ambient category is $\mathcal{U}/\mathcal{A}$.

Therefore F verifies the approximation properties, and by 2.13, it induces a homotopy equivalence of K-theories. □

This last result has told us we are on the right track. Therefore our next step is to investigate $K(C(\mathcal{U})^u)$, the fiber of $K(C(\mathcal{U})) \longrightarrow K(C\left(\mathcal{U}\left(\mathcal{A}\right)\right))$. We need the following two results from [8] (see also [4]) in order to continue the argument. Recall that a chain complex $U_\#$ in $\mathcal{U}$ is $\mathcal{A}$-dominated if there is a chain complex $A_\#$ in $\mathcal{A}$ and chain maps $i : U_\# \to A_\#$ and $r : A_\# \to u_\#$ so that ri is chain homotopic to the identity.

Proposition 7.3. [4, Proposition 4.7] *Let $\mathcal{U}$ be an $\mathcal{A}$-filtered category. A chain complex $U_\#$ in $\mathcal{U}$ is $\mathcal{A}$-dominated iff the induced $\mathcal{U}/\mathcal{A}$-chain complex is contractible.*

Lemma 7.4. [4, Lemma 4.8] *Let $\mathcal{A}$ be a full subcategory of $\mathcal{U}$, $U_\#$ an $\mathcal{A}$-dominated chain complex in $\mathcal{U}$. Let K be the inverse image of $K_0(\mathcal{U})$ under the induced map $K_0(\mathcal{A}^\wedge) \longrightarrow K_0(\mathcal{U}^\wedge)$, and let $\mathcal{U}^{\wedge K}$ be the full subcategory with objects $U \oplus (A,p)$, $[(A,p)] \in K$.*

Then the induced chain complex in $\mathcal{U}^{\wedge K}$ under the inclusion $\mathcal{U} \to \mathcal{U}^{\wedge K}$ is chain homotopy equivalent to a chain complex in $\mathcal{A}^{\wedge K}$.

In order to apply 7.3 we restate it in the following way.

Proposition 7.5. *Let $\mathcal{U}$ be an $\mathcal{A}$-filtered additive category and $C(\mathcal{U})$ its category of finite chain complexes. Let $C(\mathcal{U})^u$ be the full subcategory of chain complexes in $\mathcal{U}$ that are contractible in $\mathcal{U}/\mathcal{A}$ and let $C(\mathcal{U})^{\mathcal{A}}$ be the full subcategory of chain complexes in $\mathcal{U}$ that are $\mathcal{A}$-dominated. Then $C(\mathcal{U})^u = C(\mathcal{U})^{\mathcal{A}}$.*

The category $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}$ is clearly isomorphic to $\mathcal{U}/\mathcal{A}$ and $\mathcal{U}$ is cofinal in $\mathcal{U}^{\wedge K}$, see section 5. Therefore the functor induced by this cofinality

$$C\left(\mathcal{U}/\mathcal{A}\right) \longrightarrow C\left(\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}\right)$$

induces a homotopy equivalence of K-theories. Also the functor

$$C(\mathcal{U}) \longrightarrow C\left(\mathcal{U}^{\wedge K}\right)$$

induces a homotopy equivalence of K-theories and both K-theories are homotopy equivalent to that of $\mathcal{U}$ through the respective inclusions by proposition 6.1.

Let us denote the weak equivalences in $C\left(\mathcal{U}^{\wedge K}\left(\mathcal{A}^{\wedge K}\right)\right)$ by u, so the u-weak equivalences are chain maps inducing homotopy equivalence in $C\left(\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}\right)$. If we apply the generic fibration lemma 2.11, to

$$C\left(\mathcal{U}^{\wedge K}\right) \longrightarrow C\left(\mathcal{U}^{\wedge K}\left(\mathcal{A}^{\wedge K}\right)\right),$$

we obtain the fibration, up to homotopy:

$$K(C\left(\mathcal{U}^{\wedge K}\right)^u) \longrightarrow K(C\left(\mathcal{U}^{\wedge K}\right)) \longrightarrow K(C\left(\mathcal{U}^{\wedge K}\left(\mathcal{A}^{\wedge K}\right)\right)) \tag{7.5.1}$$

as we did for $C(\mathcal{U})$ in the proof of proposition 6.1. On the other hand, applying the generic fibration lemma 2.11 to

$$C(\mathcal{U}) \longrightarrow C\left(\mathcal{U}\left(\mathcal{A}\right)\right).$$

we obtain the fibration

$$K(C(\mathcal{U})^u) \longrightarrow K(C(\mathcal{U})) \longrightarrow K(C\left(\mathcal{U}\left(\mathcal{A}\right)\right))$$

Consider the following diagram with horizontal maps induced by inclusions

$$\begin{array}{ccc}
K(C\left(\mathcal{U}^{\wedge K}\right)^u) & \xleftarrow{\ \varphi\ } & K(C(\mathcal{U})^u) \\
\downarrow & & \downarrow \\
K(C\left(\mathcal{U}^{\wedge K}\right)) & \xleftarrow{\ \simeq\ } & K(C(\mathcal{U})) \\
\downarrow & & \downarrow \\
K(C\left(\mathcal{U}^{\wedge K}\left(\mathcal{A}^{\wedge K}\right)\right)) & \xleftarrow{\ \simeq\ } & K(C\left(\mathcal{U}\left(\mathcal{A}\right)\right)) \\
\simeq\downarrow & & \simeq\downarrow \\
K(C\left(\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}\right)) & \xleftarrow{\ \simeq\ } & K(C\left(\mathcal{U}/\mathcal{A}\right))
\end{array}$$

where we have used 7.2. By the long exact sequence of homotopy groups we can conclude that

Proposition 7.6. *The functor* $\varphi\colon C(\mathcal{U})^u \longrightarrow C\left(\mathcal{U}^{\wedge K}\right)^u$ *induces a homotopy equivalence of* K*-theories.*

We have the natural inclusion $F : C\left(\mathcal{A}^{\wedge K}\right) \longrightarrow C\left(\mathcal{U}^{\wedge K}\right)^u$ and by proposition 7.5 (or more exactly [4, Proposition 4.7]) we have $C(\mathcal{U})^{\mathcal{A}} = C(\mathcal{U})^u$ and $C\left(\mathcal{U}^{\wedge K}\right)^{\mathcal{A}^{\wedge K}} = C\left(\mathcal{U}^{\wedge K}\right)^u$. The diagram above then becomes

$$
\begin{array}{ccccccc}
 & & K(C(\mathcal{A}^{\wedge K})) & & & & \\
 & & \downarrow & & & & \\
K(C(\mathcal{U}^{\wedge K})^{\mathcal{A}^{\wedge K}}) & \xrightarrow{\simeq} & K(C(\mathcal{U}^{\wedge K})^u) & \xleftarrow[\varphi]{\simeq} & K(C(\mathcal{U})^u) & = & K(C(\mathcal{U})^{\mathcal{A}}) \\
 & & \downarrow & & \downarrow & & \\
 & & K(C(\mathcal{U}^{\wedge K})) & \xleftarrow{\simeq} & K(C(\mathcal{U})) & & \\
 & & \downarrow & & \downarrow & & \\
 & & K(C(\mathcal{U}^{\wedge K}(\mathcal{A}^{\wedge K}))) & \xleftarrow{\simeq} & K(C(\mathcal{U}(\mathcal{A}))) & & \\
 & & \simeq\downarrow & & \simeq\downarrow & & \\
 & & K(C(\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K})) & \xleftarrow{\simeq} & K(C(\mathcal{U}/\mathcal{A})) & &
\end{array}
$$

Proposition 7.7. *The functor* $F\colon C\left(\mathcal{A}^{\wedge K}\right) \longrightarrow C\left(\mathcal{U}^{\wedge K}\right)^u$ *induces a homotopy equivalence of* K*-theories.*

Proof. We want to apply the approximation theorem 2.13 to F. The categories $C\left(\mathcal{A}^{\wedge K}\right)$ and $C\left(\mathcal{U}^{\wedge K}\right)^u$ satisfy the saturation axiom. $C\left(\mathcal{A}^{\wedge K}\right)$ satisfies the cylinder axiom as well. Let us check *App 1* and *App 2*.

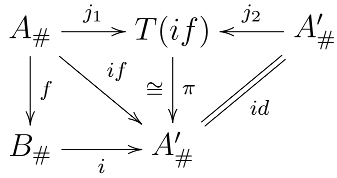
App 1 holds trivially since $C\left(\mathcal{A}^{\wedge K}\right)$ is a full subcategory of $C\left(\mathcal{U}^{\wedge K}\right)^u$ and therefore inherits weak equivalences from $C\left(\mathcal{U}^{\wedge K}\right)$.

App 2 follows easily after the following remark:

Given $B_\#$ in $C\left(\mathcal{U}^{\wedge K}\right)^u$ then $B_\#$ is $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}$-contractible. But any chain complex in $C\left(\mathcal{U}^{\wedge K}\right)$ is homotopy equivalent to one in $C(\mathcal{U})$. See the proof of [4, Theorem 4.1]. Therefore $B_\#$ is homotopy equivalent to $B'_\#$ in $C(\mathcal{U}) \subset C\left(\mathcal{U}^{\wedge K}\right)$. But $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K} = \mathcal{U}/\mathcal{A}$. This means that $B'_\#$ is $\mathcal{U}/\mathcal{A}$-contractible. Then by 7.5 $B'_\#$ is $\mathcal{A}$-dominated hence by lemma 7.4 $B'_\#$ is homotopy equivalent, in $\mathcal{U}^{\wedge K}$, to a chain complex $A'_\#$ in $\mathcal{A}^{\wedge K}$. We conclude that $B_\#$ is homotopy equivalent to an object $A'_\#$ in $\mathcal{A}^{\wedge K}$.

Now, let us verify *App 2*.

Let $A_\# \xrightarrow{f} B_\#$ be a morphism from an object in $C\left(\mathcal{A}^{\wedge K}\right)$ to an object in $C\left(\mathcal{U}^{\wedge K}\right)^u$. By the remark above, we have the homotopy equivalence $i\colon B_\# \xrightarrow{\sim} A'_\#$ with inverse r such that $\partial\Gamma + \Gamma\partial = r\,i - 1$ where Γ is a chain homotopy. The composite $A_\# \xrightarrow{f} B_\# \xrightarrow{i} A'_\#$ lies in $C\left(\mathcal{A}^{\wedge K}\right)$. We can apply the cylinder functor to it.



It is left to define $f'\colon T(if) \longrightarrow B_\#$ such that $f'j_1 = f$ and f' is a weak equivalence.
We define f' as follows.

$$(T(i\,f))_p = A_p \oplus A_{p-1} \oplus A'_p \xrightarrow{f'_p} B_p \qquad f'_p = (f,\ \Gamma f,\ r)$$

Let us check f' is a chain map.
Since

$$r\partial = \partial r,\ \ f\partial = \partial f \quad \text{and} \quad \partial\Gamma + \Gamma\partial = r\,i - 1$$

then

$$\partial\Gamma = -1 - \Gamma\partial + ri$$

and hence

$$\begin{aligned} f'\,d = (f,\ \Gamma f,\ r) \cdot \begin{pmatrix} \partial & -1 & 0 \\ 0 & -\partial & 0 \\ 0 & i\,f & \partial \end{pmatrix} &= (f\partial,\ -f - \Gamma f\partial + rif,\ r\partial) \\ &= (\partial f,\ \partial\Gamma f,\ \partial r) = \partial\, f'. \end{aligned}$$

A chain homotopy inverse for f' is $\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} = j_2\, i.$

Now,

$$\left(f,\ \Gamma f,\ r\right) \cdot \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} = r\,i$$

$\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$ is a weak equivalence, since j_2 and i are. Then, by saturation, since ri is a weak equivalence, so is f'. Clearly $f' j_1 = f$. So we have

$$\begin{array}{ccc} A_{\#} & \xrightarrow{j_1} & T(if) \\ {\scriptstyle f}\downarrow & \overset{f'}{\swarrow} \cong & \downarrow{\scriptstyle p}\ \cong \\ B_{\#} & \underset{r}{\overset{i}{\rightleftarrows}} & A'_{\#} \end{array}$$

We have verified App 2, and get the result. □

Corollary 7.8. *$K(C\left(\mathcal{A}^{\wedge K}\right))$ is homotopy equivalent to $K(C(\mathcal{U})^{\mathcal{A}})$.*

Proof. By proposition 7.7 $K(C\left(\mathcal{A}^{\wedge K}\right))$ is homotopy equivalent to $K(C\left(\mathcal{U}^{\wedge K}\right)^u)$, but this is homotopy equivalent to $K(C(\mathcal{U})^u)$ by proposition 7.6, which by proposition 7.5 is equal to $K(C(\mathcal{U})^{\mathcal{A}})$. □

Proof of Main Theorem. Let us condense all of the above as follows. Applying the generic fibration lemma 2.11 to

$$C(\mathcal{U}) \longrightarrow C\left(\mathcal{U}\left(\mathcal{A}\right)\right)$$

we obtain the fibration

$$K(C(\mathcal{U})^u) \longrightarrow K(C(\mathcal{U})) \longrightarrow K(C\left(\mathcal{U}\left(\mathcal{A}\right)\right)). \tag{7.8.1}$$

But by proposition 7.2, $K(C\left(\mathcal{U}\left(\mathcal{A}\right)\right)) \simeq K(C\left(\mathcal{U}/\mathcal{A}\right))$ and by proposition 7.5, $C(\mathcal{U})^u = C(\mathcal{U})^{\mathcal{A}}$. So, 7.8.1 now looks like

$$K(C(\mathcal{U})^{\mathcal{A}}) \longrightarrow K(C(\mathcal{U})) \longrightarrow K(C\left(\mathcal{U}/\mathcal{A}\right)). \tag{7.8.2}$$

But by corollary 7.8, $K(C(\mathcal{U})^{\mathcal{A}}) \cong K(C\left(\mathcal{A}^{\wedge K}\right))$, therefore 7.8.2 becomes

$$K(C\left(\mathcal{A}^{\wedge K}\right)) \longrightarrow K(C(\mathcal{U})) \longrightarrow K(C\left(\mathcal{U}/\mathcal{A}\right)). \tag{7.8.3}$$

Finally, applying proposition 6.1 to the three terms, we obtain that

$$K(\mathcal{A}^{\wedge K}) \longrightarrow K(\mathcal{U}) \longrightarrow K(\mathcal{U}/\mathcal{A})$$

is a fibration, up to homotopy. □

Corollary 7.9. *If $\mathcal{A}$ is idempotent complete*

$$K(\mathcal{A}) \rightarrow K(\mathcal{U}) \rightarrow K(\mathcal{U}/\mathcal{A})$$

is a fibration, up to homotopy.

8. An application

We finally show how to use the theorem to obtain excision as in [3] for bounded K-theory. Let $M = M_1 \cup M_2$ be a metric space decomposed as two metric subspaces. Let $\mathcal{U}_1 = \mathcal{C}(M;R)$ the category of finitely generated free R-modules parameterized by M and bounded morphisms, as in [7]. Let $\mathcal{A}_1 = \mathcal{C}(M;R)_{M_1}$, the full subcategory with objects having support in a bounded neighborhood of M_1. Then clearly $\mathcal{U}_1$ is $\mathcal{A}_1$ filtered, and $\mathcal{C}(M;R)_{M_1} \cong \mathcal{C}(M_1;R)$. Similarly let $\mathcal{U}_2 = \mathcal{C}(M;R)_{M_2} \cong \mathcal{C}(M_2;R)$ and $\mathcal{A}_2$ the full subcategory with objects support in a bounded neighborhood of M_1 intersected with a bounded neighborhood of M_2. It is easy to see that

$$\mathcal{U}_1/\mathcal{A}_1 \cong \mathcal{U}_2/\mathcal{A}_2 \ ,$$

and we obtain excision from the diagram

$$\begin{array}{ccccc} \mathcal{A}_1 & \longrightarrow & \mathcal{U}_1 & \longrightarrow & \mathcal{U}_1/\mathcal{A}_1 \\ \uparrow & & \uparrow & & \uparrow \cong \\ \mathcal{A}_2 & \longrightarrow & \mathcal{U}_2 & \longrightarrow & \mathcal{U}_2/\mathcal{A}_2. \end{array}$$

To give a proof of 1.0.2 we need to recall the definition of $K^{-\infty}$. Let $\mathcal{A}$ be an additive category, M a proper metric space.

Definition 8.1. The *bounded category* $\mathcal{C}(M;\mathcal{A})$ has objects $A = \{A_x\}_{x\in M}$, a collection of objects from $\mathcal{A}$ indexed by points of M, satisfying $\{x|A_x \neq 0\}$ is locally finite in M. A morphism $\phi : A \to B$ is a collection of morphisms $\phi^x_y : A_x \to B_y$ so that there exists $k = k(\phi)$ so $\phi^x_y = 0$ if $d_M(x,y) > k$.

Composition is defined as matrix multiplication. Given a subspace $N \subset M$, we denote the full subcategory with objects A so that $\{x|A_x \neq 0\}$ is contained in a bounded neighborhood of N by $\mathcal{C}(M;\mathcal{A})_N$. It is easy to see that $\mathcal{C}(M;\mathcal{A})$ is $\mathcal{C}(M,\mathcal{A})_N$-filtered . We denote the quotient category by $\mathcal{C}(M,\mathcal{A})^{>N}$. We shall need this in the particular case when M is euclidean space $\mathbb{R}^i$. Consider $\mathcal{C}(\mathbb{R}^i;\mathcal{A}) \to \mathcal{C}(\mathbb{R}^{i+1};\mathcal{A})$ induced by the standard inclusion. This inclusion factors through $H^{i+1}_{\pm}$ where H^{i+1}_{+} and H^{i+1}_{-} are the two halfspaces intersecting in $\mathbb{R}^i$. Clearly $\mathcal{C}(H^{i+1}_{\pm};\mathcal{A})$ has an Eilenberg swindle shifting modules by 1 in the direction of the last coordinate. Hence these categories have trivial K-theory so the map

$$K(\mathcal{C}(\mathbb{R}^i;\mathcal{A})) \to K(\mathcal{C}(\mathbb{R}^{i+1};\mathcal{A}))$$

is canonically null homotopic in two ways thus giving a functorial map

$$\Sigma K(\mathcal{C}(\mathbb{R}^i;\mathcal{A})) \to K(\mathcal{C}(\mathbb{R}^{i+1};\mathcal{A}))$$

or by adjointness

$$K(\mathcal{C}(\mathbb{R}^i;\mathcal{A})) \to \Omega K(\mathcal{C}(\mathbb{R}^{i+1};\mathcal{A})).$$

It follows from 6.1 that this is an isomorphism in homotopy groups in dimensions bigger than 0. We define

$$K^{-\infty}(\mathcal{A}) = \operatorname{hocolim} \Omega^i K(\mathcal{C}(\mathbb{R}^i; \mathcal{A})).$$

It is easy to see that if $\mathcal{U}$ is $\mathcal{A}$-filtered, then $\mathcal{C}(\mathbb{R}^i;\mathcal{U})$ is $\mathcal{C}(\mathbb{R}^i;\mathcal{A})$-filtered and we thus recover the fibration of spectra

$$K^{-\infty}(\mathcal{A}) \to K^{-\infty}(\mathcal{U}) \to K^{-\infty}(\mathcal{U}/\mathcal{A})$$

by taking the homotopy colimit of the fibrations

$$\Omega^i K(\mathcal{C}(\mathbb{R}^i;\mathcal{A})^{K_i}) \to \Omega^i K(\mathcal{C}(\mathbb{R}^i;\mathcal{U})) \to \Omega^i K(\mathcal{C}(\mathbb{R}^i;\mathcal{U}/\mathcal{A}))$$

where K_i is the appropriate subgroup of $K_0(\mathcal{C}(\mathbb{R}^i;\mathcal{A})^\wedge)$.

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