

# K-THEORY HOMOLOGY OF SPACES

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ABSTRACT. Let  $KR$  be a nonconnective spectrum whose homotopy groups give the algebraic  $K$ -theory of the ring  $R$ . We give a description of the associated homology theory  $KR_*(X)$  associated to  $KR$ . We also show that the various constructions of  $KR$  in the literature are homotopy equivalent, and so give the same homology theory

## 0. INTRODUCTION

There is a generalized homology theory  $E_*$  associated to every spectrum  $E$ , namely

$$E_n(X) = \lim_{i \rightarrow \infty} \pi_{n+i}(E_i \wedge X)$$

In particular this is true if  $E$  is the nonconnective  $K$ -theory spectrum  $KR$  of a ring  $R$ . In this paper, we give a geometric interpretation of  $KR_n(X)$  for  $n \leq 0$  (and a new interpretation for  $n > 0$ )

Let  $X$  be a subcomplex of  $S^n$ , and form the open cone  $O(X)$  on  $X$  inside  $R^{n+1}$  (which is the open cone on  $S^n$ ). There is a category  $\mathcal{C} = \mathcal{C}_{O(X)}(R)$ , whose objects are based free  $R$ -modules parameterized in a locally finite way by  $O(X)$ , and whose morphisms are linear maps moving the bases a bounded amount. (Compare with Quinn's geometric  $R$ -modules in [13]). The group  $K_1(\mathcal{C})$  is generated by the automorphisms in  $\mathcal{C}$ , with well-known relations [1]; our main theorem yields the formula:

$$KR_0(X) \cong K_1(\mathcal{C})$$

When  $R = \mathbb{Z}\pi$ , these groups appear as obstruction groups of bounded (or thin)  $h$ -cobordisms parameterized by  $O(X)$  with constant fundamental group  $\pi$ . (See [8] for a relatively elementary proof of this. This, of course, is in accordance with the basic results of Chapman [2, 3] and Quinn [11, 12]).

The negative  $KR$ -homology groups of  $X$  can be obtained from the formula  $KR_{-n}(X) = KR_0(S^n X)$ . The positive  $KR$ -homology groups are Quillen's higher  $K$ -groups of  $\mathcal{C}$ :

$$KR_n(X) \cong K_{n+1}(\mathcal{C}) \quad \text{for } n \geq 0.$$

To make the proofs easier, it turns out to be better to generalize the above discussion, replacing the category  $\mathcal{F}_R$  of finitely generated based free  $R$ -modules by any additive category

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$\mathcal{A}$ . We impose the semisimple exact structure of [1], i. e., declaring every short exact sequence split, in order to compute the  $K$ -theory of  $\mathcal{A}$ ; this makes Bass' groups  $K_1(\mathcal{A})$  the same as Quillen's by [17].

This being said, we can generalize the spectrum  $K\mathcal{R}$  (for  $\mathcal{A} = \mathcal{F}_R$ ) to the nonconnective spectrum  $K\mathcal{A}$  constructed in [9] and ask about  $K\mathcal{A}$ -homology. There is a category  $\mathcal{C}_{O(X)}(\mathcal{A})$ , generalizing  $\mathcal{C}_{O(X)}(R)$  and described in §1 below. In §2 we make this construction functorial in  $X$ . Our Main Theorem is carefully stated in §3 and proved in §4:

**MAIN THEOREM.** *The  $K\mathcal{A}$ -homology of  $X$  is naturally isomorphic to the algebraic  $K$ -theory of the idempotent completion  $\mathcal{C}^\wedge$  of  $\mathcal{C} = \mathcal{C}_{O(X)}(\mathcal{A})$ , with a degree shift:*

$$K\mathcal{A}_{*-1}(X) \cong K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge).$$

Note that  $K_*(\mathcal{C}^\wedge) = K_*(\mathcal{C})$  for  $* \geq 1$  but that  $K_0(\mathcal{C}^\wedge) \neq K_0(\mathcal{C})$  in general. For example if  $\mathcal{C} = \mathcal{F}_R$ , then  $K_0(\mathcal{C})$  is  $\mathbb{Z}$  (or a quotient of  $\mathbb{Z}$ ) but  $K_0(\mathcal{C}^\wedge) = K_0(R)$ . If  $* < 0$  and  $\mathcal{C} \neq \mathcal{C}^\wedge$ , the groups  $K_*(\mathcal{C})$  are not even defined. As in [9], the key technical step is an application of Thomason's double mapping cylinder construction from [15].

The knowledgeable reader will wonder about the relationship between the spectrum  $K\mathcal{A}$  and other nonconnective spectra in the literature. We pin down this loose end in §6. When  $\mathcal{A} = \mathcal{F}_R$  we show that our spectrum  $K\mathcal{F}_R$  is homotopy equivalent to the Gersten-Wagoner spectrum of [4, 16]. For general  $\mathcal{A}$ , we prove that our spectrum  $K\mathcal{A}$  is homotopy equivalent to Karoubi's spectrum [6]. Actually, the discussion in [6] only mentions  $K_0$  and  $K_1$ , and not spectra, since it was written before higher  $K$ -theory emerged. We devote §5 to showing that Karoubi's prescription in [6] actually gives an infinite loop spectrum. The authors want to thank W. Vogell for pointing out an error in an earlier draft of this paper.

## 1. THE FUNCTOR $\mathcal{C}$

In this section we generalize the functor  $\mathcal{C}_i(\mathcal{A})$  considered in [9] to a functor in two variables. We have already described how  $\mathcal{C}_i(-)$  is an endofunctor of the category of filtered additive categories [9]. We remind the reader that a filtered additive category is an additive category  $\mathcal{A}$ , that comes with a filtration of the homsets  $\text{Hom}(A, B)$

$$0 \subseteq F_0 \text{Hom}(A, B) \subseteq F_1 \text{Hom}(A, B) \subseteq \dots \subseteq \text{Hom}(A, B)$$

such that

- a)  $F_i \text{Hom}(A, B)$  is a subgroup and  $\text{Hom}(A, B) = \cup F_i \text{Hom}(A, B)$ .
- b)  $F_0 \text{Hom}(A, A)$  contains  $0_A$  and  $1_A$  for each  $A$ , all coherence isomorphisms of  $\mathcal{A}$ , all projections  $A \oplus B \rightarrow A$  and all inclusions  $A \rightarrow A \oplus B$ .
- c) if  $f \in F_i \text{Hom}(A, B)$  and  $g \in F_j \text{Hom}(B, C)$  then  $g \circ f \in F_{i+j} \text{Hom}(A, C)$ .

Note that any additive category may be endowed with "discrete filtration", in which

$$F_0 \text{Hom}(A, B) = \text{Hom}(A, B)$$

for every  $A, B$ .

Thinking of the lower index  $i$  as the metric space  $\mathbb{Z}^i$  or  $\mathbb{R}^i$ , we shall now turn  $\mathcal{C}$  into a functor of that variable. (See [9, Remark 1,2,3]).

**Definition 1.1.** Let  $X$  be a metric space and  $\mathcal{A}$  a filtered additive category. We then define the filtered category  $\mathcal{C}_X(\mathcal{A})$  as follows:

- 1) An object  $A$  of  $\mathcal{C}_X(\mathcal{A})$  is a collection of objects  $A(x)$  of  $\mathcal{A}$ , one for each  $x \in X$ , satisfying the condition that for each ball  $B \subset X$ ,  $A(x) \neq 0$  for only finitely many  $x \in B$ .
- 2) A morphism  $\phi : A \rightarrow B$  is a collection of morphisms  $\phi_y^x : A(x) \rightarrow B(y)$  in  $\mathcal{A}$  such that there exists  $r$  depending only on  $\phi$  so that
  - (a)  $\phi_y^x = 0$  for  $d(x, y) > r$
  - (b) all  $\phi_y^x$  are in  $F_r \text{Hom}(A(x), B(y))$
 (We then say that  $\phi$  has filtration degree  $\leq r$ .)

Composition of  $\phi : A \rightarrow B$  with  $\psi : B \rightarrow C$  is given by  $(\psi\phi)_z^x = \sum_{y \in X} \psi_z^y \phi_y^x$ . Notice that the sum makes sense because the category is additive and because the sum will always be finite. The category  $\mathcal{C}_{\mathbb{Z}^i}(\mathcal{A})$  is the category  $\mathcal{C}_i(\mathcal{A})$  of [9].

We now introduce the category  $\mathcal{M}$  of metric spaces and proper, eventually Lipschitz maps.

**Definition 1.2.** The category  $\mathcal{M}$  has objects metric spaces  $X$ . A morphism  $f : X \rightarrow Y$  must be both proper and eventually Lipschitz. We remind the reader that a map  $f : X \rightarrow Y$  is Lipschitz if there exists a number  $k \in \mathbb{R}_+$  such that

$$d(f(x), f(y)) \leq kd(x, y)$$

We say that  $f$  is eventually Lipschitz if there exist  $r$  and  $k$ , only depending on  $f$ , so that

$$\forall x, y \in X, \forall s \in \mathbb{R}_+ : \quad \text{if } s > r \quad \text{and} \quad d(x, y) < s \quad \text{then} \quad d(f(x), f(y)) < k \cdot s.$$

Finally, we call  $f$  proper if the inverse image of a bounded set is bounded.

**Example 1.3.** One should note that maps in  $\mathcal{M}$  are not necessarily continuous, but any jumps allowed must be universally bounded. For example the map  $\mathbb{R} \rightarrow \mathbb{Z}$  sending a real number  $x$  to the greatest integer smaller than  $x$  is a map in  $\mathcal{M}$ .

Given a proper eventually Lipschitz map  $f : X \rightarrow Y$  we obtain a functor  $f_* : \mathcal{C}_X(\mathcal{A}) \rightarrow \mathcal{C}_Y(\mathcal{A})$  by defining  $(f_*(A))_y = \bigoplus_{z \in f^{-1}(y)} A(z)$  for objects  $A$  in  $\mathcal{C}_X(\mathcal{A})$ . Since the inverse of a bounded set is bounded, there are only finitely many nonzero modules in a ball in  $Y$ , and  $f_*(A)$  is well-defined. On morphisms  $f_*$  is induced by the identity. The eventually Lipschitz conditions on  $f$  ensures that we indeed do get morphisms in the category  $\mathcal{C}_Y(\mathcal{A})$ . Hence  $\mathcal{C}_-(\mathcal{A})$  is a functor from  $\mathcal{M}$  to (semisimple filtered) additive categories.

**Lemma 1.4.** Let  $X$  and  $Y$  be metric spaces and give  $X \times Y$  the max metric,

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)),$$

then  $\mathcal{C}_{X \times Y}(\mathcal{A}) = \mathcal{C}_X(\mathcal{C}_Y(\mathcal{A}))$ .

*Proof.* This is where we use that  $\mathcal{C}$  takes values in filtered categories. The internal filtration degree will control distances in the  $Y$  component, while the external filtration degree will control distances in the  $X$  component and thus the max distance will be controlled.  $\square$

*Remark.* Note that the isomorphism class in  $\mathcal{M}$  is not affected if we change the metric to a proper Lipschitz equivalent metric, i. e. a metric so that the identity map is a proper Lipschitz equivalence both ways. Therefore Lemma 1.4 remains true up to natural equivalence if the max metric is replaced by the usual product metric

$$d_{X \times Y}((x_1, x_2), (y_1, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

The concept homotopy is introduced in the category  $\mathcal{M}$  in a standard fashion using the inclusions  $X = X \times \{0\} \rightarrow X \times I$ ,  $X = X \times \{1\} \rightarrow X \times I$  and the projections  $X \times I \rightarrow X$ . Since  $X \times I$  has the max metric, these are maps in the category  $\mathcal{M}$ . Note that the inclusion  $\mathbb{Z} \rightarrow \mathbb{R}$  is a homotopy equivalence in  $\mathcal{M}$ , with homotopy inverse the greatest integer function of example 1.3.

**Lemma 1.5.** *A compact metric space  $X$  is homotopy equivalent to a point in  $\mathcal{M}$  and hence  $\mathcal{C}_X(\mathcal{A})$  is equivalent to the category  $\mathcal{A}$ .*

*Proof.* Since maps are not required to be continuous and a compact metric space is globally bounded, any map  $X \times I \rightarrow X$  which is constant on the top and the identity on the bottom will be a contracting homotopy. The second assertion follows from the evident fact that  $\mathcal{C}_X(\mathcal{A}) = \mathcal{A}$  when  $X$  is a point.  $\square$

**Proposition 1.6.** *The functor  $\mathcal{C}_-(\mathcal{A})$  is homotopy invariant. That is, for each metric space  $X$ , the inclusion  $X \subset X \times I$  as  $X \times 0$  and the projection  $X \times I \rightarrow X$  induce an equivalence of categories*

$$\mathcal{C}_X(\mathcal{A}) \rightarrow \mathcal{C}_{X \times I}(\mathcal{A}).$$

*Proof.* By 1.4 and 1.5,  $\mathcal{C}_{X \times I}(\mathcal{A}) = \mathcal{C}_I(\mathcal{C}_X(\mathcal{A}))$  is homotopic to  $\mathcal{C}_X(\mathcal{A})$ .  $\square$

## 2. OPEN CONES

In this section we construct an open cone functor  $O(X)$  from finite PL complexes to  $\mathcal{M}$ , so that  $\mathcal{C}_{O(X)}(\mathcal{A})$  depends functorially on  $X$  in a homotopy invariant way.

To fix notation let  $S^0 = \{-1, 1\} \subset \mathbb{R}$ . Then the  $n$ -sphere is the join  $S^n = S^0 * S^0 * \dots * S^0$  as a sub PL-complex of  $\mathbb{R}^{n+1}$ . We shall be considering the category of finite sub PL-complexes of  $S^\infty$  (PL complexes that are subcomplexes of some  $S^n$ ) and PL morphisms. We denote this category  $\mathcal{PL}$ . This is essentially the category of finite PL-complexes, since any such may be embedded in some  $S^n$ , but we need the way the complex sits in  $S^n$  as part of the structure. We think of  $\mathbb{R}^{n+1}$  as a metric space using the max metric

$$d(\underline{x}, \underline{y}) = \max |x_i - y_i|.$$

This induces a metric on the  $n$ -sphere and hence on any subcomplex.

We now construct a functor  $O$  from  $\mathcal{PL}$  to  $\mathcal{M}$ .

**Definition 2.1.**  $O$  sends an object  $X \subset S^n$  to  $O(X) = \{t \cdot x \in \mathbb{R}^{n+1} | t \in [0, \infty), x \in X\}$  with metric induced from  $\mathbb{R}^{n+1}$ . For example  $O(S^n) = \mathbb{R}^{n+1}$ . On morphisms  $f : X \rightarrow Y$  we extend to  $O(f) : O(X) \rightarrow O(Y)$  by linearity:  $f(t \cdot x) = t \cdot f(x)$ . One checks easily that  $O(f)$  is proper and Lipschitz (and therefore eventually Lipschitz) using the following well-known

**Lemma 2.2.** *A PL-map  $f : X \rightarrow Y$  between finite complexes is Lipschitz.*

*Proof.* Triangulate so that  $f$  is linear on each simplex. Since there are only finitely many simplices,  $f$  will be Lipschitz.  $\square$

*Remark 2.3.* For a PL complex  $X$ ,  $O(X)$  does not really depend on the embedding  $X \subset S^n$ , since a PL homeomorphism  $X_1 \subset S^n$  to  $X_2 \subset S^m$  will induce a proper Lipschitz homeomorphism from  $O(X_1) \rightarrow O(X_2)$ .

**Lemma 2.4.** *Let  $X$  and  $Y$  be two PL complexes and let  $*$  denote the join. Then*

$$O(X * Y) \cong O(X) \times O(Y).$$

*In particular,  $O(\Sigma X) = O(X * S^0) \cong O(X) \times \mathbb{R}$  and  $O(CX) \cong O(X) \times [0, \infty)$ .*

*Proof.* Embed  $X \subset S^n$  and  $Y \subset S^m$  so  $X * Y \subset S^n * S^m = S^{n+m+1}$ . A point in  $O(X * Y)$  is  $s \cdot (t \cdot x + (1-t) \cdot y) = stx + s(1-t)y$  and  $stx$  lies in  $O(X) \subset \mathbb{R}^{n+1} \times 0$ , whereas  $s(1-t)y$  lies in  $O(Y) \subset 0 \times \mathbb{R}^{m+1}$ . The last sentence follows from the identities  $\Sigma X = X * S^0$ ,  $CX = X * (\text{point})$ ,  $O(S^0) = \mathbb{R}$  and  $O(\text{point}) = [0, \infty)$ .  $\square$

*Remark 2.5.* The functor  $O$  is not homotopy invariant. We only obtain homotopy invariance after passing to  $K$ -theory.

### 3. THE MAIN THEOREM

So far we have constructed functors

$$O : \mathcal{PL} \rightarrow \mathcal{M} \quad \text{and} \\ \mathcal{C}_-(\mathcal{A}) : \mathcal{M} \rightarrow (\text{semisimple}) \text{ additive categories}$$

For our main theorem, we need a functor  $K_*$  from additive categories to graded abelian groups. For  $n \geq 1$ , there is no problem: given an additive category  $\mathcal{A}$  we take  $K_n \mathcal{A} = \pi_n(\Omega BQ\mathcal{A})$  as in [10], using the semisimple exact structure (in which all exact sequences split). However, unless  $\mathcal{A}$  is idempotent complete, the groups  $K_0 \mathcal{A}$  may be wrong for our purposes, and  $\mathcal{A}$ 's negative  $K$ -groups will not even be defined.

For example consider the category  $\mathcal{F}_R$  of finitely generated based free  $R$ -modules. when  $R$  is a group ring, we know that the geometrically interesting group is not  $K_0(\mathcal{F}_R) = \mathbb{Z}$ , but rather  $K_0(R)$ , which measures projective modules.

To handle this problem, we pass to the idempotent completion  $\mathcal{A}^\wedge$  of  $\mathcal{A}$ . This provides the correct group  $K_0(\mathcal{A}^\wedge)$  and does not change the higher groups, since  $K_n(\mathcal{A}) = K_n(\mathcal{A}^\wedge)$

for  $n \geq 1$ . For example  $\mathcal{F}_R^\wedge$  is equivalent to the category of finitely generated projective  $R$ -modules.

*Scholium.*  $\mathcal{A}^\wedge$  inherits the structure of a filtered additive category from  $\mathcal{A}$ . The objects of  $\mathcal{A}^\wedge$  are pairs  $(A, p)$ , where  $p : A \rightarrow A$  is idempotent. A morphism  $\phi$  from  $(A_1, \phi_1)$  to  $(A_2, \phi_2)$  is an  $\mathcal{A}$ -morphism  $\phi : A_1 \rightarrow A_2$  with  $\phi = p_2 \phi p_1$ . The filtration degree of  $\phi$  is the smallest  $d$  such that  $\phi = p_2 f p_1$  for some  $f \in F_d \text{Hom}(A_1, A_2)$  satisfying  $f p_1 = p_2 f$ . This filtration should have been stated explicitly in [9, 1.4].

If  $\mathcal{A}$  is idempotent complete, the negative  $K$ -groups of  $\mathcal{A}$  were defined by Karoubi in [6], and agree with the definition in [9] (as we shall see in §6 below). If  $\mathcal{A} = \mathcal{F}_R$ , they agree with Bass' negative  $K$ -groups  $K_n(R)$ .

The construction in [9] actually gives us slightly more information. If  $\mathcal{A}$  is idempotent complete, it yields a nonconnective infinite loop spectrum  $K\mathcal{A}$ , and the homotopy groups of  $K\mathcal{A}$  are the groups  $K_*\mathcal{A}$  above.

Now associated to any spectrum such as  $K\mathcal{A}$  is a reduced homology theory  $K\mathcal{A}_*$ . It is defined by

$$K\mathcal{A}_n(X) = \lim \pi_{n+i}((K\mathcal{A})_n \wedge X).$$

The coefficients of this homology theory are the groups

$$K\mathcal{A}_n(S^0) = \pi_n(K\mathcal{A}) = K_n(\mathcal{A}).$$

We can now state our main theorem.

**Theorem 3.1.** *If  $\mathcal{A}$  is an idempotent complete additive category, the functor from  $\mathcal{P}\mathcal{L}$  to graded abelian groups sending  $X$  to  $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$  is the  $K\mathcal{A}_*$ -homology theory of  $X$ , with a degree shift:*

$$K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge) \cong K\mathcal{A}_*(X).$$

*Remark 3.2.* This theorem had previously been known for spheres. For  $X = S^i$ ,  $O(S^i) = \mathbb{R}^{i+1}$ , which is homotopy equivalent to  $\mathbb{Z}^{i+1}$  in  $\mathcal{M}$ , so therefore by 1.4

$$K_*(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})^\wedge) = K_*(\mathcal{C}_{\mathbb{Z}^i}(\mathcal{A})^\wedge),$$

which was shown in [9] to equal  $K_{*-i-1}(\mathcal{A}^\wedge)$ . The category  $\mathcal{C}_{\mathbb{Z}^{i+1}}(\mathcal{A})$  was first studied in [8], where  $\mathcal{A}$  was the category of finitely generated  $R$ -modules. There it was shown that

$$K_1(\mathcal{C}_{\mathbb{Z}^{i+1}}(R)) = K_{-i}(R)$$

which is equal to  $\text{KR}_i(S^0) = \text{KR}_0(S^i)$ , thus agreeing with our main theorem.

#### 4. PROOF OF MAIN THEOREM

Define the functor  $f$  as the composite

$$\mathcal{P}\mathcal{L} \xrightarrow{O} \mathcal{M} \xrightarrow{\mathcal{C}_-(\mathcal{A})} \text{filtered add. categ.} \xrightarrow{\square} \text{add. cat.} \xrightarrow{\wedge}$$

idempotent complete add. cat.  $\xrightarrow{\Omega^\infty K}$  Top. spaces

Here  $\square$  is the forgetful functor,  $\wedge$  is idempotent completion and  $\Omega^\infty K$  is the zeroth space of the infinite loop spectrum  $K$  giving the  $K$ -theory of the category, either  $\Omega BQ$  or the result of applying an infinite loop machine to the symmetric monoidal category of isomorphisms (see Thomason for a very functorial construction [15]). That is :

$$f(X) = \Omega^\infty K(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge).$$

**Lemma 4.1.** *If  $X$  is a cone,  $X = CK$ , then  $f(X)$  is contractible.*

*Proof.*  $\mathcal{C}_{O(CK)}(\mathcal{A}) = \mathcal{C}_{[0,\infty) \times O(K)}(\mathcal{A}) = \mathcal{C}_1(\mathcal{C}_{O(K)}(\mathcal{A}))$  in the notation of [9]. It was proven in [9, (3.1)] that  $\Omega^\infty K\mathcal{C}_+(\mathcal{A})$  is contractible for an arbitrary category,  $\mathcal{A}$ , and the argument given there applies verbatim to show that

$$f(X) = \Omega^\infty K((\mathcal{C}_+(\mathcal{C}_{O(K)}\mathcal{A})^\wedge))$$

is also contractible.  $\square$

**Proposition 4.2.** *For each  $X$ ,  $\Omega f(\Sigma X)$  is homotopy equivalent to  $f(X)$ . In particular  $f(X)$  is an infinite loop space.*

*Proof.* By 1.6 and 2.4 we have

$$\mathcal{C}_{O(\Sigma X)} = \mathcal{C}_{O(X) \times \mathbb{R}} = \mathcal{C}_{\mathbb{R}}(\mathcal{C}_{O(X)}(\mathcal{A})) = \mathcal{C}_1(\mathcal{C}_{O(X)}(\mathcal{A})).$$

By [9, Theorem (3.2)], applied to the filtered additive category  $\mathcal{C}_{O(X)}(\mathcal{A})^\wedge$ , we know that the loop space of  $\Omega^\infty K\mathcal{C}_{O(\Sigma X)}(\mathcal{A})$  is homotopy equivalent to  $f(X)$ . But by the cofinality theorem, the spaces  $\Omega^\infty K(\mathcal{C})$  and  $\Omega^\infty K(\mathcal{C}^\wedge)$  have homotopy equivalent connected components, and hence homotopy equivalent loop spaces, for any  $\mathcal{C}$ .  $\square$

**Theorem 4.3.** *The functor  $f$  sends cofibrations to fibrations.*

Before proving this result, let us draw a quick consequence. It follows from Lemma 4.1 that  $f$  is homotopy invariant, since  $X \rightarrow X \times I \rightarrow CX$  is a cofibration. The homotopy groups of the spectrum  $\{f(X), f(\Sigma X), f(\Sigma^2 X), \dots\}$  coincide with the groups  $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$ . These groups are homotopy invariants of  $X$  which vanish when  $X$  is contractible by 4.1. From 4.3 we immediately obtain

**Corollary 4.4.** *The functor from  $\mathcal{P}\mathcal{L}$  to graded abelian groups sending  $X$  to  $K_*(\mathcal{C}_{O(X)}(\mathcal{A})^\wedge)$  is a reduced homology theory.*

*Proof of Theorem 4.3.* The special case  $X \subset CX \subset \Sigma X$  follows from 4.1 and 4.2. To do the general case, we proceed in a manner very much like the proof of Theorem 3.4 in [9]. Consider a cofibration  $A \subset X \rightarrow X \cup CA$ . Then we get

$$O(A) \subset O(X) \subset O(X) \cup_{O(A)} O(CA)$$

and a diagram

$$\begin{array}{ccc} \mathcal{C}_{O(A)}(\mathcal{A}) & \longrightarrow & \mathcal{C}_{O(CA)}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathcal{C}_{O(X)}(\mathcal{A}) & \longrightarrow & \mathcal{C}_{O(X \cup CA)}(\mathcal{A}) \end{array}$$

By 1.6 and 2.4 we have  $O(CA) = O(A) \times [o, \infty)$  and

$$\mathcal{C}_{O(CA)}(\mathcal{A}) = \mathcal{C}_{[0, \infty)}(\mathcal{C}_{O(A)}(\mathcal{A})) = \mathcal{C}_+(\mathcal{C}_{O(A)}(\mathcal{A})).$$

To simplify notation, let us write  $\underline{\underline{C}}_X$  for the category of isomorphisms in  $\mathcal{C}_{O(X)}(\mathcal{A})^\wedge$ . Using the “double mapping cylinder” pushout construction  $\underline{\underline{P}}$  of Thomason [15, (5.1)], we get

$$\begin{array}{ccc} \underline{\underline{C}}_A & \longrightarrow & \underline{\underline{C}}_{CA} \\ \downarrow & & \downarrow \\ \underline{\underline{C}}_X & \longrightarrow & \underline{\underline{P}} \\ & & \searrow \Sigma \\ & & \underline{\underline{C}}_{X \cup CA} \end{array}$$

We wish to show that  $\Sigma$  induces a homotopy equivalence on certain components, i. e. that  $\Sigma$  induces a  $\pi_0$ -monomorphism and  $\pi_i$ -isomorphism. That  $\pi_0$  behaves correctly is then obtained by applying the argument to the suspension of this cofibration. It is therefore enough to show that for every object  $Y$  of  $\mathcal{C}_{O(X \cup CA)}(\mathcal{A})$ , considered as an object of  $\underline{\underline{C}}_{X \cup CA}$ , that the category  $Y \downarrow \Sigma$  is a contractible category. At this point we follow the proof of [9, Theorem (3.4)] very closely. We use the bound  $d$  to filter  $Y \downarrow \Sigma$  as the increasing union of subcategories  $\text{fil}_d$  and show each of these has an initial object  $*_d$ .

$\text{fil}_d$  is the full subcategory of all iso's  $\alpha : Y \rightarrow \Sigma(B^A, B^X, A^+)$ , where  $B^X$  is an object of  $\underline{\underline{C}}_X$ ,  $B^A$  an object of  $\underline{\underline{C}}_A$  and  $A^+$  an object of  $\underline{\underline{C}}_{CA}$ . We define  $Y_d^A$ ,  $Y_d^X$  and  $Y_d^+$  in  $\underline{\underline{C}}_A$ ,  $\underline{\underline{C}}_X$  and  $\underline{\underline{C}}_{CA}$  as follows: Let

$$N_d = \{x \in O(X \cup CA) \mid \exists y \in O(A) : d(x, y) \leq d\}.$$

Since  $A \subseteq$  is a cofibration it is easy to see that  $N_d$  is proper Lipschitz homotopy equivalent to  $O(A)$ . Choose some proper Lipschitz homotopy equivalence  $h : N_d \rightarrow O(A)$  and proceed as follows:



$$Y_d^X(x) = \begin{cases} Y(x) & \text{for } x \in O(X) - N_d \\ 0 & \text{otherwise} \end{cases}$$

$$Y_d^+(x) = \begin{cases} Y(x) & \text{for } x \in O(A) \times (d, \infty) = O(CA) - N_d \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_d^A = h_*(Y_d^A) \text{ where}$$

$$Y_d^A = \begin{cases} Y(x) & \text{for } x \in N_d \\ 0 & \text{otherwise} \end{cases}$$

There is now an obvious isomorphism

$$\sigma : Y \cong Y_d^X \oplus Y_d^A \oplus Y_d^+ = \Sigma(Y_d^A, Y_d^X, Y_d^+)$$

bounded by  $d$  (essentially the identity) and we may proceed to prove this is an initial object in  $\text{Fil}_d$  exactly as in the proof of [9, Theorem 3.4].  $\square$

We finish off the proof of the main theorem as follows:

*Proof of main theorem 3.1.* We need to identify the homology theory of 4.4 as the homology theory associated to the spectrum  $K(\mathcal{A})$ . It was proven by Thomas Gunnarson [5] that when a homotopy functor  $f$  sends cofibrations to fibrations and contractible spaces to contractible spaces, then the homology theory obtained by applying homotopy groups has as its representing spectrum  $\{f(S^0), f(S^1), f(S^2), \dots\}$ . This is a well-known fact when the homology theory is connective - see e. g. [18, theorem 1.14], [14] or [7] - but in the general case we need to use [5]. Now we have by Lemma 2.4 that

$$f(S^i) = \Omega^\infty K(\mathcal{C}_{i+1}(\mathcal{A})^\wedge).$$

The space  $f(S^i)$  is therefore the  $(i+1)^{\text{st}}$ -space of the spectrum  $K(\mathcal{A})$  constructed in Theorem B of [9]. The representing spectrum for the homology theory of 4.4 is therefore  $\Omega^{-1}K(\mathcal{A})$ , and the main theorem follows.  $\square$

We are finished, except we need to show that the delooping given in [9] agrees with the other deloopings in the literature. This is the subject of the final 2 sections.

## 5. KAROUBI'S NONCONNECTIVE SPECTRA

In this section, we follow Karoubi's ideas in [6] and construct nonconnective  $K$ -theory spectra, whose negative homotopy groups are the negative  $K$ -groups defined by Karoubi in op. cit. We then show that these agree with the spectra constructed in [9] (and in special cases, in [4], [16]). More explicitly, but also more technically, we show that  $\mathcal{C}_1(\mathcal{A}) \rightarrow \mathcal{C}_+(\mathcal{A})/\mathcal{A}$

induces an isomorphism on Quillen  $K$ -theory. We are indebted to Karoubi, who suggested this possibility to us in 1983.

We shall use Karoubi's delooping construction from [6], so we begin by recalling the construction. Let  $\mathcal{A}$  be a full subcategory of an additive (hence semisimple exact) category  $\mathcal{U}$ . We shall use the notation that letters  $A - F$  (resp  $U - Z$ ) denote the objects of  $\mathcal{A}$  (resp.  $\mathcal{U}$ ), and that  $U = E_\alpha \oplus U_\alpha$  means an internal direct sum decomposition of  $U$  with  $E_\alpha \in \mathcal{A}$ . We say that  $\mathcal{U}$  is  $\mathcal{A}$ -filtered if every object  $U$  has a family of decompositions  $\{U = E_\alpha \oplus U_\alpha\}$  (called a filtration) of  $U$ ) satisfying the following axioms (cf. [6, pp. 114 ff.]):

- (F1) For each  $U$ , the decompositions form a poset under the partial order that  $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$  whenever  $U_\beta \subseteq U_\alpha$  and  $E_\alpha \subseteq E_\beta$ .
- (F2) Every map  $A \rightarrow U$  factors  $A \rightarrow E_\alpha \rightarrow E_\alpha \oplus U_\alpha = U$  for some  $\alpha$ .
- (F3) Every map  $U \rightarrow A$  factors  $U = E_\alpha \oplus U_\alpha \rightarrow E_\alpha \rightarrow A$  for some  $\alpha$ .
- (F4) For each  $U, V$  the filtration on  $U \oplus V$  is equivalent to the sum of the filtrations  $\{U = E_\alpha \oplus U_\alpha\}$  and  $\{V = F_\beta \oplus V_\beta\}$  i. e., to  $\{U \oplus V = E_\alpha \oplus F_\beta\} \oplus \{U_\alpha \oplus V_\beta\}$ .

We shall assume each filtration is saturated in the sense that if  $U = E_\alpha \oplus U_\alpha$  is in the filtration and  $E_\alpha = A \oplus B$  in  $\mathcal{A}$ , then  $U = A \oplus (B \oplus U_\alpha)$  is also in the filtration. Finally, we say that  $\mathcal{U}$  is flasque if there is a functor  $\infty : \mathcal{U} \rightarrow \mathcal{U}$  and a natural transformation  $U^\infty \cong U \oplus U^\infty$  ([6, p. 147]).

Our favorite selection of  $\mathcal{U}$  is the following:

**Example 5.1.** The category  $\mathcal{C}_+(\mathcal{A})$  of [9, (1.2.4)] is flasque and  $\mathcal{A}$ -filtered. Objects of  $\mathcal{C}_+(\mathcal{A})$  are sequences  $(A_0, A_1, \dots)$  of objects in  $\mathcal{A}$ , and the morphisms are given by “bounded” matrices.  $\mathcal{A}$  is the full subcategory of objects  $(A_0, 0, 0, \dots)$ . The  $\mathcal{A}$ -filtration on an object  $U = (A_0, A_1, \dots)$  contains the decompositions

$$\begin{aligned} U &\cong \text{Fil}_n(U) \oplus (0, \dots, 0, A_{n+1}, \dots) \\ \text{Fil}_n(U) &= A_0 \oplus \dots \oplus A_n \text{ in } \mathcal{A}. \end{aligned}$$

We proved that  $\mathcal{C}_+(\mathcal{A})$  was flasque in [9, (1.3)], using the translation  $t(A_0, A_1, \dots) = (0, A_0, A_1, \dots)$  on  $\mathcal{C}_+(\mathcal{A})$ .

We now suppose given an  $\mathcal{A}$ -filtered category  $\mathcal{U}$ . Call a map  $U \rightarrow V$  completely continuous (cc) if it factors through an object of  $\mathcal{A}$ . Karoubi defined  $\mathcal{U}/\mathcal{A}$  to be the category with the same objects as  $\mathcal{U}$ , but with  $\text{Hom}_{\mathcal{U}/\mathcal{A}}(U, V) = \text{Hom}_{\mathcal{U}}(U, V)/\{\text{cc maps}\}$ .

**Lemma 5.2.** *Suppose  $\mathcal{A}$  is idempotent complete and that  $\bar{\phi} : U \rightarrow V$  is an isomorphism in  $\mathcal{U}/\mathcal{A}$ . Then there are decompositions  $U = E_\beta \oplus U_\alpha$  and  $V = F_\alpha \oplus V_\alpha$  in the (saturated) filtrations and a  $\mathcal{U}$ -isomorphism  $U_\alpha \cong V_\alpha$  such that  $\bar{\phi}$  is represented by  $U \rightarrow U_\alpha \cong V_\alpha \rightarrow V$ .*

*Proof.* Choose representatives  $\phi, \psi$  for  $\bar{\phi}, \bar{\phi}^{-1}$ . Since  $1_U - \psi\phi$  is cc, there is a decomposition  $U = E_\alpha \oplus U_\beta$  with  $1 = \psi\phi$  on  $U_\beta$ . Replace  $U$  by  $U_\beta$  to assume  $1_U = \psi\phi$ . Similarly, write  $V = F \oplus W$  with  $\phi\psi = 1_W$ . Write  $\phi = (\epsilon, i) : U \rightarrow F \oplus W$  and  $\psi = (\delta, j) : F \oplus W \rightarrow U$ .

Observe that  $ij = 1_W$  and that

$$(*) \quad \delta\epsilon + ji = 1_U$$

Multiplying  $(*)$  by  $i, j$  and by  $(*)$  yields the equations  $i\delta\epsilon = 0$ ,  $\delta\epsilon j = 0$  and  $(\delta\epsilon)^2 = \delta\epsilon$ . Replacing  $\epsilon$  by  $\epsilon\delta\epsilon$  makes  $(\epsilon\delta)^2 = \epsilon\delta$  without affecting  $\bar{\phi}$  or  $(*)$ . Set  $F_\alpha = \ker(\epsilon\delta)$  and  $V_\alpha = (\epsilon\delta F) \oplus W$ ;  $F_\alpha \in \mathcal{A}$  because  $\mathcal{A}$  is idempotent complete. The rest of the proof is straightforward, and left to the reader.  $\square$

*Remark.* The failure of this lemma when  $\mathcal{A}$  is not idempotent complete is the Bass-Heller-Swan phenomenon. See [8, (1.16)].

**Theorem 5.3.** *Let  $\mathcal{A}$  be semisimple and idempotent complete. Then for every  $\mathcal{A}$ -filtered category  $\mathcal{U}$  the sequence*

$$K^Q(\mathcal{A}) \rightarrow K^Q(\mathcal{U}) \rightarrow K^Q(\mathcal{U}/\mathcal{A})$$

*is a homotopy fibration, where  $K^Q(\mathcal{A})$  denotes the space whose homotopy groups give the algebraic K-theory of  $\mathcal{A}$ .*

Before proving this theorem, we draw our main conclusion. Suppose that in addition  $\mathcal{U}$  is flasque; from additivity and the equation  $\infty \cong 1 + \infty$  we conclude that  $K^Q(\mathcal{U}) \simeq *$ . Since  $\mathcal{U}/\mathcal{A}$  shares the same objects as  $\mathcal{U}$ , we conclude that  $K_0(\mathcal{U}/\mathcal{A}) = K_0(\mathcal{U}) = 0$ . Finally, applying 5.3 to the diagram  $\mathcal{U} \rightarrow \mathcal{C}_+(\mathcal{U}) \leftarrow \mathcal{C}_+(\mathcal{A})$  shows that  $K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{C}_+(\mathcal{A})/\mathcal{A})$ . This proves:

**Theorem/Definition 5.4.** Let  $\mathcal{A}$  be semisimple and idempotent complete. Choose a flasque,  $\mathcal{A}$ -filtered category  $\mathcal{U}$  and define  $S\mathcal{A}$  to be  $\mathcal{U}/\mathcal{A}$ . Then  $K^Q(S\mathcal{A})$  is a connected space with  $\Omega K^Q(S\mathcal{A}) \simeq K^Q(\mathcal{A})$ , and the homotopy type of  $K^Q(S\mathcal{A})$  is independent of the choice of  $\mathcal{U}$ .

Our proof of Theorem 5.3 follows the proof of [9, (3.4)]. Let  $\underline{\underline{A}}$ ,  $\underline{\underline{U}}$  and  $\underline{\underline{S}}$  denote the categories of isomorphisms of  $\mathcal{A}$ ,  $\mathcal{U}$  and  $\mathcal{U}/\mathcal{A}$ . It is well-known that  $K^Q(\mathcal{A})$  is the group completion of  $B\underline{\underline{A}}$ , and that  $K^Q(\mathcal{A})$ ,  $B\underline{\underline{A}}^{-1}\underline{\underline{A}}$  and  $\text{Spt}_0(\underline{\underline{A}})$  are homotopy equivalent. Let  $\underline{\underline{Q}}$  denote the double mapping cylinder of  $(0 \leftarrow \underline{\underline{A}} \rightarrow \underline{\underline{U}})$  given by Thomason in [15, (5.1)]. Thus objects of  $\underline{\underline{Q}}$  are pairs  $(A, U)$ , and a  $\underline{\underline{Q}}$ -map from  $(A, U)$  to  $(B, V)$  is an equivalence class of data  $(E, F, A \cong E \oplus B \oplus F, F \oplus U \cong V)$ . Thomason proves in [15, (5.2) and (5.5)] that  $\text{Spt}_0(\underline{\underline{A}}) \rightarrow \text{Spt}_0(\underline{\underline{U}}) \rightarrow \text{Spt}_0(\underline{\underline{Q}})$  is a homotopy fibration. Since  $\underline{\underline{U}} \rightarrow \underline{\underline{S}}$  factors through  $\underline{\underline{Q}}$ , we see that 5.3 follows from [15, (2.3)] and the following result:

**Proposition 5.5.** *The functor  $\Sigma : \underline{\underline{Q}} \rightarrow \underline{\underline{S}}$  given by  $\Sigma(A, U) = A \oplus U$  is a homotopy equivalence when  $\mathcal{A}$  is idempotent complete.*

*Proof.* Fix an object  $S$  of  $\underline{S}$ ; we will show that  $S \downarrow \Sigma$  is a contractible category. The desired result will follow from Quillen's Theorem A [10]. In order to do this, we need to thicken  $S \downarrow \Sigma$  up a bit. Let  $\mathcal{S}$  denote the category whose objects are tuples

$$\alpha = (A, U, S \cong D_\alpha \oplus S_\alpha, U \cong E_\alpha \oplus U_\alpha, f_\alpha : S_\alpha \cong U_\alpha)$$

where  $A \in \mathcal{A}$ ,  $U \in \mathcal{U}$ ,  $f_\alpha$  is a  $\mathcal{U}$ -isomorphism, and the direct sum decompositions belong to the  $\mathcal{A}$ -filtrations of  $S$  and  $U$ . A map from  $\alpha$  to

$$\beta = (B, V, S \cong D_\beta \oplus S_\beta, V \cong F_\beta \oplus V_\beta, f_\beta : S_\beta \cong V_\beta)$$

is just a map in  $\underline{Q}$  from  $(A, U)$  to  $(B, V)$ , say given by

$$e = (E, F, A \cong E \oplus B \oplus F, F \oplus U \cong V)$$

such that  $D_\alpha \oplus S_\alpha \geq D_\beta \oplus S_\beta$ ,  $(F \oplus E_\alpha) \oplus U_\alpha \geq F_\beta \oplus V_\beta$  in the filtrations of  $S$  and  $V$ , and such that there is a commutative square in  $\mathcal{U}$ :

$$\begin{array}{ccc} S_\beta & \cong & V_\beta \\ \uparrow & & \uparrow \\ S_\alpha & \cong & U_\alpha \end{array}$$

There is a natural functor  $\text{pr} : \mathcal{S} \rightarrow (S \downarrow \Sigma)$  sending  $\alpha$  to the object  $\text{pr}_\alpha : S \rightarrow S_\alpha \cong U_\alpha \rightarrow U \rightarrow A \oplus U = \Sigma(A, U)$  and  $e$  to itself. By Lemma 5.2,  $\text{pr}$  is onto. In fact  $\text{pr}$  is cofibered. We assert that  $\text{pr}$  is a homotopy equivalence, which follows from [10, p. 93] and the following:  $\square$

**Sublemma 5.6.** Given an isomorphism  $\bar{\phi} : S \rightarrow \Sigma(A, U)$  in  $\mathcal{U}/\mathcal{A}$ , the fiber category  $\text{pr}^{-1}(\bar{\phi})$  is a cofiltered poset, and hence contractible.

*Proof.* Set  $\Phi = \text{pr}^{-1}(\bar{\phi})$ ; it is clear from the definition of  $\mathcal{S}$  that  $\Phi$  is a poset. We need only show that for each  $\alpha, \beta$  in  $\Phi$  there is a diagram  $\alpha \leftarrow \gamma \rightarrow \beta$  in  $\Phi$ . To do this, we introduce some notation. Write  $\alpha = (A, U, S \cong D_\alpha \oplus S_\alpha, U \cong E_\alpha \oplus U_\alpha, f_\alpha)$  and  $\beta = (A, U, S \cong D_\beta \oplus S_\beta, U \cong E_\beta \oplus U_\beta, f_\beta)$ . If  $D_\alpha \oplus S_\alpha \leq D_\gamma \oplus S_\gamma$  we set  $D_{\gamma\alpha} = D_\gamma \cap S_\alpha$ ,  $E_{\gamma\alpha} = f_\alpha(D_{\gamma\alpha})$  and  $\alpha^1 = (A, U, S \cong D_\gamma \oplus S_\gamma, U \cong (E_\alpha \oplus E_{\gamma\alpha}) \oplus f_\alpha(S_\gamma), f_\alpha|_{S_\gamma} : S_\gamma \cong f_\alpha(S_\gamma))$ . We shall refer to  $\alpha^1$  as “ $\alpha$  cut down to  $D_\gamma \oplus S_\gamma$ ”; note that there is a map  $\alpha^1 \rightarrow \alpha$  in  $\Phi$ .

By axiom (F1), there is a decomposition  $S \cong D_0 \oplus S_0$  larger than both  $D_\alpha \oplus S_\alpha$  and  $D_\beta \oplus S_\beta$ . Cutting  $\alpha$  and  $\beta$  down, we can assume that  $D_\alpha = D_\beta = D_0$  and  $S_\alpha = S_\beta = S_0$ . Now  $f_\alpha - f_\beta : S_0 \rightarrow D$  is completely continuous, so after cutting  $\alpha$  and  $\beta$  down further we can assume that  $f_\alpha = f_\beta$ . Note that we still may have  $E_\alpha \neq E_\beta$ . Consider the maps

$$E_\alpha \rightarrow U \rightarrow U_\beta \cong S_0 \quad \text{and} \quad E_\beta \rightarrow U \rightarrow U_\alpha \cong S_0.$$

By axiom (F2) there is a decomposition  $S \cong D_\gamma \oplus S_\gamma$  for which these two maps factor through  $D_\gamma$ . Let  $\gamma$  be  $\alpha$  cut down to  $D_\gamma \oplus S_\gamma$ ; evidently  $\gamma$  is also  $\beta$  cut down to  $D_\gamma \oplus S_\gamma$ .

The resulting maps  $\alpha \leftarrow \gamma \rightarrow \beta$  in  $\Phi$  were what we needed to prove sublemma 5.6, so we are done.  $\square$

Resuming the proof of 5.5 we let  $e : \alpha \rightarrow \beta$  be the map described at the proof's outset. Let  $D_{\alpha\beta} = D_\alpha \cap S_\beta$  and  $E_{\alpha\beta} = f_\beta(D_{\alpha\beta})$ , so that  $D_{\alpha\beta} \oplus D_\beta = D_\alpha$  and  $D_{\alpha\beta} \oplus S_\alpha = S_\beta$ . We first observe that from the definition of the map  $e$  there is a natural identification of subobjects of  $V$ :

$$(*) \quad F_\beta \oplus E_{\alpha\beta} = F \oplus E_\alpha.$$

Using this, there is a natural isomorphism in  $\mathcal{A}$ :

$$s_e : D_\alpha \oplus A \oplus E_\alpha \cong (D_{\alpha\beta} \oplus D_\beta) \oplus (E \oplus B \oplus F) \oplus E_\alpha \cong (D_{\alpha\beta} \oplus E) \oplus (D_\beta \oplus B \oplus F_\beta) \oplus E_{\alpha\beta}.$$

Now define  $q_\alpha$  to be the object  $S \rightarrow S_\alpha \rightarrow \Sigma(D_\alpha \oplus A \oplus E_\alpha, S_\alpha)$  of  $S \downarrow \Sigma$  defined naturally by  $\alpha$ , and let  $q_e : q_\alpha \rightarrow q_\beta$  be the map

$$(D_{\alpha\beta} \oplus E, E_{\alpha\beta}, s_e, E_{\alpha\beta} \oplus S_\alpha \cong S_\beta).$$

It is easy to see that  $q$  is a functor from  $\mathcal{S}$  to  $S \downarrow \Sigma$ .

Now let  $z$  denote the object  $1 : S \rightarrow \Sigma(0, S)$  of  $S \downarrow \Sigma$ . There is a map  $\theta_\alpha : q_\alpha \rightarrow z$  given by the data

$$(A \oplus E_\alpha, D_\alpha, D_\alpha \oplus A \oplus E_\alpha \cong (A \oplus E_\alpha) \oplus 0 \oplus D_\alpha, D_\alpha \oplus S_\alpha \cong S)$$

and a map  $\eta_\alpha : q_\alpha \rightarrow \text{pr}_\alpha$  given by the data

$$(D_\alpha, E_\alpha, D_\alpha \oplus A \oplus E_\alpha = D_\alpha \oplus A \oplus E_\alpha, E_\alpha \oplus S_\alpha \cong E_\alpha \oplus U_\alpha \cong U).$$

Using (\*), it is a straightforward matter to check that  $\theta$  and  $\eta$  are natural transformations. This proves that the maps  $z$ ,  $q$  and  $\text{pr}$  from  $B\mathcal{S}$  to  $B(S \downarrow \Sigma)$  are homotopic. Since  $\text{pr}$  is a homotopy equivalence, this shows that  $B(S \downarrow \Sigma)$  is contractible. This finishes the proof of Proposition 5.5, and hence of Theorem 5.3.

Next we show how to remove the hypothesis that  $\mathcal{A}$  is idempotent complete from Theorem 5.4. Let  $\mathcal{U}$  be a flasque  $\mathcal{A}$ -filtered category. Let  $\mathcal{A}^\wedge$ , and let  $\mathcal{U}^\wedge$  be the full subcategory of the idempotent completion of  $\mathcal{U}$  on objects  $P \oplus U$ ,  $P$  in  $\mathcal{A}^\wedge$ , and  $U$  in  $\mathcal{U}$ . Then  $\mathcal{U}^\wedge$  is  $\mathcal{A}^\wedge$ -filtered but not flasque. However it is easy to see that  $K^Q(\mathcal{U}^\wedge)$  is contractible, and Theorem 5.3 applies to show that  $\Omega K^Q(\mathcal{U}^\wedge/\mathcal{A}^\wedge) \cong K^Q(\mathcal{A}^\wedge)$ . On the other hand,  $\mathcal{U}^\wedge/\mathcal{A}^\wedge \cong \mathcal{U}/\mathcal{A}$ , so we have proven:

**Corollary 5.7.** *If  $\mathcal{A}^\wedge$  denotes the idempotent completion of  $\mathcal{A}$ , and  $\mathcal{U}$  is a flasque  $\mathcal{A}$ -filtered category, then*

- (i)  $K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{U}^\wedge/\mathcal{A}^\wedge)$
- (ii)  $\Omega K^Q(\mathcal{U}/\mathcal{A}) \simeq K^Q(\mathcal{A}^\wedge)$
- (iii)  $K^Q(\mathcal{A}) \rightarrow K^Q(\mathcal{U}) \rightarrow K^Q(\mathcal{U}/\mathcal{A})$  is a homotopy fibration if and only if  $K_0(\mathcal{A}) \cong K_0(\mathcal{A}^\wedge)$

**Definition 5.8.** (Karoubi [6]) Given a semisimple exact category  $\mathcal{A}$ , we define  $K_{-n}(\mathcal{A})$  to be  $K_1(S^{n+1}\mathcal{A})$ . Note that  $K_{-0}(\mathcal{A}) = K_0(\mathcal{A}^\wedge)$  by 5.7 (ii). This is well-defined because by 5.4 the homotopy type of  $S^{n+1}\mathcal{A}$  is independent of the choice of flasque category  $\mathcal{U}$  used to construct  $S\mathcal{A} = \mathcal{U}/\mathcal{A}$ . Note that  $K_0(S^n\mathcal{A}) = 0$  for  $n \geq 1$  because  $S^n\mathcal{A}$  need not be idempotent complete. In fact  $K_0((S^n\mathcal{A})^\wedge) = K_{-n}(\mathcal{A})$ . (Cf. [6, p.151])

**Definition 5.9.** By 5.4, 5.7 and 5.8, there is an  $\Omega$ -spectrum

$$|K_{-n}(\mathcal{A}) \times K^Q(S^n\mathcal{A})| \simeq |K^Q((S^n\mathcal{A})^\wedge)|$$

We shall call it Karoubi's non-connective  $K$ -theory spectrum for  $\mathcal{A}$ , since Karoubi gave the prescription for this spectrum in [6].

## 6. AGREEMENT OF SPECTRA

Our task is now to show that Karoubi's spectrum agrees with the other spectra in the literature. We first recall Wagoner's construction in [16]. Given a ring  $R$ , let  $lR$  denote the ring of locally finite  $\mathbb{N}$ -indexed matrices over  $R$ , i. e. matrices  $(r_{ij})$  with  $1 \leq i, j < \infty$  such that each row and each column has only finitely many nonzero entries. The finite matrices form an ideal  $mR$  of  $lR$ , and we let  $\mu R = lR/mR$ . Wagoner's spectrum is

$$\{K_0(\mu^n R) \times BGL^+(\mu^n R)\}.$$

We shall show that this spectrum is the same as Karoubi's spectrum for the category  $\mathcal{F}_R$  of (based) finitely generated free  $R$ -modules, Note that  $\mathcal{F}^\wedge(R)$  is equivalent to the category of finitely generated projective  $R$ -modules, so  $K(\mathcal{F}^\wedge(R))$  is the usual space  $K_0^Q(R) \times BGL^+(R)$ . The following argument was shown to us by H. J. Munkholm and A. A. Ranicki.

**Proposition 6.1.** *Let  $\mathcal{U}$  denote the category of countably (but not necessarily infinitely) generated based free  $R$ -modules and locally finite matrices over  $R$ . Then  $\mathcal{U}/\mathcal{F}(R)$  is equivalent to the category  $\mathcal{F}(\mu R)$ . Consequently, Wagoner's spectrum for  $R$  is homotopy equivalent to Karoubi's  $K$ -theory spectrum for  $\mathcal{F}(R)$ .*

*Proof.* (Munkholm-Ranicki) Choose an infinitely generated based  $R$ -module  $R^\infty$  in  $\mathcal{U}$ , and observe that  $\text{End}_{\mathcal{U}}(R^\infty) = lR$ . Now  $\mathcal{U}$  is  $\mathcal{F}(R)$ -filtered, and the completely continuous endomorphisms of  $R^\infty$  form the ideal  $mR$ . Thus  $\text{End}_{\mathcal{U}/\mathcal{F}}(R^\infty) \cong \mu R$ . The additive functor  $\mathcal{F}(\mu R) \rightarrow \mathcal{U}/\mathcal{F}(R)$  which sends  $1_{\mu R}$  to  $1_{R^\infty}$  is therefore full and faithful. But every object of  $\mathcal{U}/\mathcal{F}(R)$  is either isomorphic to 0 or to  $R^\infty$ , so this functor is also an equivalence. Done.  $\square$

Gersten has also constructed a nonconnective spectrum for the  $K$ -theory of a ring in [4]. Since Wagoner showed in [16] that it agreed with Wagoner's spectrum, Gersten's spectrum is also homotopy equivalent to Karoubi's spectrum.

Finally we must compare Karoubi's  $K$ -theory spectrum with the spectrum  $\{f(S^n)\}$  of section 4 above, constructed in [9] using the categories  $\mathcal{C}_n(\mathcal{A})$ . We assume that  $\mathcal{A}$  is filtered in the sense of [9, (1.1)], or §2 above, so that in the notation of op. cit. we have  $\mathcal{C}_{n+1}(\mathcal{A}) =$

$\mathcal{C}_1(\mathcal{C}_n(\mathcal{A}))$ . The choice of the filtration affects the morphisms allowed in  $\mathcal{C}_n(\mathcal{A})$  and  $\mathcal{C}_+(\mathcal{A})$ , but not the homotopy type of  $K^Q(\mathcal{C}_n(\mathcal{A}))$ , as [9, (3.2)] shows. The real point of the filtration on  $\mathcal{A}$  is to reduce the discussion to the case  $n = 1$ .

Recall the objects of  $\mathcal{C}_1(\mathcal{A}) = \mathcal{C}_{\mathbb{Z}}(\mathcal{A})$  are  $\mathbb{Z}$ -graded sequences  $A = (\dots, A_{-1}, A_0, A_1, \dots)$  in  $\mathcal{A}$ . A map  $\phi : A \rightarrow B$  is a matrix of maps  $\phi_{ij} : A_i \rightarrow B_j$  such that for some bound  $b = b(\phi)$  we have  $\phi_{ij} = 0$  whenever  $|i - j| > b$ . Composition is given by matrix multiplication. Define  $\text{trunc}(A)$  to be the object  $(A_0, A_1, \dots)$  of  $\mathcal{C}_+(\mathcal{A})$  and  $\text{trunc}(\phi)$  to be the submatrix of  $\phi_{ij}$  with  $i, j > 0$ . If  $\psi : B \rightarrow C$ , then  $\text{trunc}(\psi\phi) - \text{trunc}(\psi)\text{trunc}(\phi)$  is completely continuous being bounded by  $b(\psi) + b(\phi)$ . Hence  $\text{trunc}$  defines a functor from  $\mathcal{C}_1(\mathcal{A}) \rightarrow \mathcal{C}_+(\mathcal{A})/\mathcal{A}$ .

**Theorem 6.2.** *The functor  $\text{trunc}$  induces a homotopy equivalence*

$$K^Q(\mathcal{C}_1(\mathcal{A})) \xrightarrow{\sim} K^Q(\mathcal{C}_+(\mathcal{A})/\mathcal{A}) \simeq K^Q(S\mathcal{A})$$

Assuming this result, it follows directly from the above remarks that we have  $K^Q(\mathcal{C}_{n+1}(\mathcal{A})) \xrightarrow{\sim} K^Q(\mathcal{C}_1(\mathcal{C}_n(\mathcal{A}))) \simeq K^Q(SC_n(\mathcal{A})) \simeq K^Q(S^{n+1}\mathcal{A})$ . Hence the spaces  $\hat{B}_n$  of [9] are  $K_{-n}(\mathcal{A}) \times K^Q(\mathcal{C}_n(\mathcal{A}))$ , and we have

**Corollary 6.3.** *The nonconnective spectra of [9] agree with Karoubi's. In fact,  $\text{trunc}$  induces a homotopy equivalence of spectra:*

$$\{K_{-n}(\mathcal{A}) \times K^Q(\mathcal{C}_n\mathcal{A})\} \xrightarrow{\sim} \{K_{-n}(\mathcal{A}) \times K^Q(S^n\mathcal{A})\}$$

*Proof of Theorem 6.2.* We shall use the notation of [9], only remarking that  $\underline{\underline{C}}_{\epsilon}$  is the category of isomorphisms of  $\mathcal{C}_{\epsilon}(\mathcal{A})$ , and that  $\text{Spt}_0(\underline{\underline{C}}_{\epsilon}) \simeq K^Q(\mathcal{C}_{\epsilon}(\mathcal{A}))$ . By 5.7 and [9, §3], we can assume that  $\mathcal{A}$  is idempotent complete. There is a map of squares

$$\begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_{+} \\ \downarrow & & \downarrow \\ \underline{\underline{C}}_{-} & \longrightarrow & \underline{\underline{C}}_{1} \end{array} \quad \rightarrow \quad \begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_{+} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{\underline{S}} \end{array}$$

By Theorem 5.3 and [9, §3], applying  $\text{Spt}_0$  yields homotopy cartesian squares with  $\text{Spt}_0(\underline{\underline{C}}_{1}) \simeq K^Q(\mathcal{C}_1(\mathcal{A}))$  and  $\text{Spt}_0(\underline{\underline{S}}) \simeq K^Q(S\mathcal{A})$  connected. Since  $\text{Spt}(\underline{\underline{C}}_{-})$  and  $\text{Spt}(\underline{\underline{C}}_{+})$  are contractible, it follows that  $K^Q(\mathcal{C}_1(\mathcal{A})) \rightarrow K^Q(S\mathcal{A})$  is a weak homotopy equivalence, hence a homotopy equivalence.  $\square$

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