A LOCAL APPROACH TO THE FINITENESS OBSTRUCTION

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0. Introduction

Given a finitely dominated CW-complex X, Wall [9] defined a finiteness obstruction

$$\sigma(X) = \sum_k (-1)^k [C_k(\widetilde{X})] \in \widetilde{K}_0(\mathbf{Z}\pi_1(X))$$

and showed that X has the homotopy type of a finite complex if and only if $\sigma(X) = 0$. For a fixed group π, it is well-known that any element of $\widetilde{K}_0(\mathbf{Z}\pi)$ can be realized as the finiteness obstruction for some finitely dominated complex with fundamental group π. If we restrict the class of spaces, however, then in general the set of elements in $\widetilde{K}_0(\mathbf{Z}\pi)$ that can be realized as finiteness obstructions is also restricted.

One natural class of spaces to consider is the class of complexes on which $\pi = \pi_1(X)$ acts trivially on rational homology. A special case of this is the class of nilpotent spaces with finite fundamental group – spaces for which $\pi_1(X)$ is nilpotent and acts nilpotently on homology. The question of which elements can be realized by nilpotent spaces has been extensively studied by Mislin [3, 4, 5, 6].

The purpose of this paper is to study the finiteness obstruction in such a way that one can begin to answer some of these questions in a systematic way. The primary tool is a decomposition of the finiteness obstruction into its p-parts for all primes p (Theorem 2.4). To do this, we use the techniques of localization and define p-local Reidemeister torsions whose images under a certain boundary map in K-theory provide the p-parts of the finiteness obstruction. In the case of nilpotent spaces, we are then able to describe (in principle, completely) the set of finiteness obstructions that can be realized.

1. Some algebraic preliminaries

We begin by considering the localization square

$$\begin{array}{ccc} \mathbf{Z} & \longrightarrow & \mathbf{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(1/p)} & \longrightarrow & \mathbf{Q} \end{array}$$

If we tensor with a ring R, which as an additive group is free abelian, we obtain

$$\begin{array}{ccc} R & \longrightarrow & R_{(p)} \\ \downarrow & & \downarrow \\ R_{(1/p)} & \longrightarrow & R_{(0)} \end{array}$$

In the usual way (see, for example Bass [1] or Wojtkowiak [10]) this yields an exact sequence in K-theory:

$$\begin{aligned} &\to K_1(R) \to K_1(R_{(p)}) \oplus K_1(R_{1/p)}) \to K_1(R_{(0)} \\ &\xrightarrow{\partial^p} K_0(R_{(p)}) \oplus K_0(R_{1/p)}) \to K_0(R_{(0)} \end{aligned}$$

Throughout this paper we will be concerned with the case where $R = \mathbf{Z}\pi$ for a finite group π and hence $R_{(0)} = \mathbf{Q}\pi$.

If π is a finite *nilpotent* group then $\mathbf{Q}\pi$ splits as follows. For any prime p, let π_p denote the p-Sylow subgroup of π. (We allow the case where af, is the trivial group!) Since π is nilpotent, it is isomorphic to the product of its nontrivial p-Sylow subgroups, and hence π_p has a complementary subgroup π'_p. Consider the central idempotent:

$$e_p = \frac{1}{|\pi_p|} \sum_{g \in \pi'_p} g \in \mathbf{Q}\pi.$$

Then $\mathbf{Q}\pi$ splits as:

$$\mathbf{Q}\pi \cong \mathbf{Q}\pi \cdot e_p \times \mathbf{Q}\pi \cdot (1 - e_p).$$

Note that the projection $\pi \to \pi_p$ induces a map $\mathbf{Q}\pi \to \mathbf{Q}\pi_p$, which in turn produces an isomorphism $\mathbf{Q}\pi \cdot e_p \cong \mathbf{Q}\pi_p$. We shall often simply identify the factor $\mathbf{Q}\pi \cdot e_p$ with $\mathbf{Q}\pi_p$ using this isomorphism. Of course, in case p does not divide $|\pi|$ we see that $\mathbf{Q}\pi \cdot e_p \cong \mathbf{Q}$

Definition 1.1. *For a finite nilpotent group and a prime p, let*

$$N_p(\pi) = \partial^p(K_1(\mathbf{Q}\pi \cdot e_p)) \subseteq K_0(\mathbf{Z}\pi).$$

We shall see in the sequel that the groups $N_p(\pi)$ play a central role in analyzing the finiteness obstructions that can be realized by nilpotent spaces.

2. Reidemeister torsions

In this section we give the definition of p-local Reidemeister torsion and prove the fundamental theorem relating such torsion to the finiteness obstruction. We then show that the p-local Reidemeister torsions of a nilpotent space are restricted.

We begin with some algebra. Let π be a finite group and consider a finitely dominated $\mathbf{Z}\pi$-chain complex C_* for which π acts trivially on rational homology. ("Finitely dominated" means that C_* is homotopy equivalent to a finitely generated projective $\mathbf{Z}\pi$-chain complex.)

Further assume that we are given a preferred basis $\{\underline{e}\}$ for $H_*(C_*;\mathbf{Q})$, up to simple isomorphism measured in $K_1(\mathbf{Q}) = \mathbf{Q}^*$.

We define the p-local Reidemeister torsion $RT_p(C_*\{\underline{e}\}) \in K_1(\mathbf{Z}_{(p)}\pi)$ as follows. Consider a finitely generated projective $\mathbf{Z}\pi$-chain complex P_* that is homotopy equivalent to C_*. Since the map $\widetilde{K}_0(\mathbf{Z}\pi) \to \widetilde{K}_0(\mathbf{Z}_{(p)}\pi)$ is trivial (see [8]), $P_{*_{(p)}} = P_* \otimes \mathbf{Z}_{(p)}$ is a finitely generated free $\mathbf{Z}_{(p)}\pi$- module. Choose a basis for $P_{*_{(p)}}$ and consider the induced basis for $P_{*_{(p)}} = P_* \otimes \mathbf{Q}$. We can use the idempotent

$$e = \frac{1}{|\pi|}\sum_{g\in\pi} g$$

to split the chain complex into two parts,

$$P_{*_{(p)}} = P_{*_{(0)}} \cdot e \times P_{*_{(0)}} \cdot (1-e).$$

Now $P_{*_{(0)}} \cdot (1-e)$ is contractible (since π acts trivially on rational homology) and based; hence it has a torsion defined in $K_1(\mathbf{Q}\pi\cdot(1-e))$. The complex $P_{*_{(0)}} \cdot e$ homology which is identified with $H_*(C_*;\mathbf{Q})$ and hence is based by assumption. But a based chain complex with based homology has a well-defined torsion (as in [2]) defined in $K_1(\mathbf{Q}\pi\cdot e)$. We collect the two torsions to define

$$RT_p(C_*,\{\underline{e}\}) \in K_1(\mathbf{Q}\pi) \cong K_1(\mathbf{Q}\pi\cdot e) \times K_1(\mathbf{Q}\pi\cdot(1-e)).$$

We must now consider the choices involved above in order to compute the indeterminacy. There were two choices: the choice of P_* and the choice of basis in $P_{*_{(0)}}$. Different choices, however, vary RT_p by an element in the image of $K_1(\mathbf{Z}_{(p)}\pi) \to K_1(\mathbf{Q}\pi)$. Hence $RT_p(C_*,\{\underline{e}\})$ is well-defined as an element in $K_1(\mathbf{Q}\pi)/K_1(\mathbf{Z}_{(p)}\pi)$.

Remarks 2.1. The choice of basis for $H_*(C_*;\mathbf{Q})$ only affects the part of $RT_p(C_*;\{\underline{e}\})$ coming from $K_1(\mathbf{Q}\pi\cdot e)$. We shall see that this means that a different choice of basis changes $\partial^p(RT_p(C_*;\{\underline{e}\}))$ by an element in the image of the Swan homomorphism. (See Section 5 for the definition.) We will often suppress the specific choice of basis in what follows and simply write $RT_p(C_*)$. It is important to note, however, that the choice of basis is necessary in order to define the Reidemeister torsion.

We can now define the p-local Reidemeister torsion of a space.

Definition 2.2. *Let X be a finitely dominated space with universal cover $\widetilde{X}$. Assume that $\pi_1(X)$ acts trivially on the rational, singular homology of $\widetilde{X}$ and fix a basis $\{\underline{e}\}$ for $H_*(\widetilde{X};\mathbf{Q})$. Let C_* denote the singular chain complex of $\widetilde{X}$. Then*

$$RT_p(X) = RT_p(C_*,\{\underline{e}\}) \in K_1(\mathbf{Q}\pi_1(X))/K_1(\mathbf{Z}_{(p)}\pi_1(X)).$$

(Note that we have suppressed the dependence on the basis here.)

Lemma 2.3. *Given a finitely dominated space X with finite fundamental group $\pi = \pi_1(X)$ acting trivially on $H_*(\widetilde{X}; \mathbf{Q})$, and a basis $\{\overline{e}\}$, for $H_*(\widetilde{X}; \mathbf{Q})$, then $RT_p(X) = o$ for all but a finite number of primes.*

Proof. Suppose $p||\pi|$ Then the idempotent $e = (1/|\pi|)\sum_{g\in\pi} g$ is defined over $\mathbf{Z}_{(p)}\pi$ and there is a splitting

$$\mathbf{Z}_{(p)}\pi = \mathbf{Z}_{(p)}\pi \cdot e \times \mathbf{Z}_{(p)}\pi \cdot (1-e).$$

Now let P_* be a finitely generated projective chain complex homotopy equivalent to the singular chains of $\widetilde{X}$. Then the above splitting induces a splitting of $P_{*_{(p)}}$. If $H_*(\widetilde{X}; \mathbf{Z})$ has no p-torsion, then $P_{*(p)} \cdot (1-e)$ is contractible since $P_{*(0)} \cdot (1-e)$ is contractible.

The part of the Reidemeister torsion lying in $K_1(\mathbf{Q}\pi \cdot (1-e))$ thus lifts to $K_1(\mathbf{Z}_{(p)}\pi \cdot (1-e))$ and is consequently zero in the quotient. The remaining part of the Reidemeister torsion lies in $K_1(\mathbf{Q}\pi \cdot e) \cong K_1(\mathbf{Q})$ and is defined at once for *all* primes. Excluding the primes in this rational number, we see that $RT_p(X) = 0$ for all but a finite number of primes.

We can now state the fundamental theorem of this paper, which determines the finiteness obstruction in terms of the p-local Reidemeister torsions.

Theorem 2.4. *Let X be a finitely dominated space with finite fundamen- tal group $\pi = \pi_1(X)$ and universal cover $\widetilde{X}$. Assume that π acts trivially on $H_*(X; \mathbf{Q})$ and choose a basis for $H_*(\widetilde{X}; \mathbf{Q})$. Then the finiteness obstruction $\sigma(X)$ is given by*

$$\sigma(X) = \sum_{p \ prime} \partial^p(RT_p(X)).$$

(Note that by Lemma 2.3 this is a finite sum).

Notation. *: Given a space X as in Theorem 2.4, we write $\sigma_p(X) = \partial^p(RT_p(X))$ and call $\sigma_p(X)$ the p-part of the finiteness obstruction. (Of course, $\sigma_p(X)$ is only defined modulo a choice of basis for $H_*(\widetilde{X}; \mathbf{Q})$ We can then write*

$$\sigma(X) = \sum_{p \ prime} \sigma_p(X)$$

Before proving Theorem 2.4 we require an algebraic lemma.

Lemma 2.5. *Let π be a finite group and let $S = \{p_1, p_2, \ldots, p_k\}$ be a finite set of primes. Consider the $\mathbf{Z}\pi$-module A which is the pullback of the following diagram:*

$$
\begin{array}{ccccccc}
(\mathbf{Z}_{(p_1)}\pi)^n & & (\mathbf{Z}_{(p_2)}\pi)^n & \cdots & (\mathbf{Z}_{(p_3)}\pi)^n & & (\mathbf{Z}_{(p_4)}\pi)^n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(\mathbf{Q}_\pi)^n & & (\mathbf{Q}_\pi)^n & \cdots & (\mathbf{Q}_\pi)^n & & (\mathbf{Q}_\pi)^n \\
\cong \downarrow \alpha_1 & & \cong \downarrow \alpha_2 & & \cong \downarrow \alpha_k & & \cong \downarrow \alpha_{k+1} \\
(\mathbf{Q}_\pi)^n & = & (\mathbf{Q}_\pi)^n = & \cdots & = (\mathbf{Q}_\pi)^n & = & (\mathbf{Q}_\pi)^n
\end{array}
$$

Then A is a projective $\mathbf{Z}\pi$-module, and as an element in $K_0(\mathbf{Z}\pi)$,

$$A = \sum_{i=1}^{k} \partial^{p_i}[\alpha_i],$$

where $[\alpha_i]$ is the class of the isomorphism α_i in $K_1(\mathbf{Q}\pi)$.

Proof. Denote the pullback of the diagram

$$
\begin{array}{ccccccc}
(\mathbf{Z}_{(p_1)}\pi)^n & & (\mathbf{Z}_{(p_2)}\pi)^n & \cdots & (\mathbf{Z}_{(p_3)}\pi)^n & & (\mathbf{Z}_{(1/S)}\pi)^n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(\mathbf{Q}_\pi)^n & & (\mathbf{Q}_\pi)^n & \cdots & (\mathbf{Q}_\pi)^n & & (\mathbf{Q}_\pi)^n \\
\cong \downarrow \alpha_1 & & \cong \downarrow \alpha_2 & & \cong \downarrow \alpha_k & & \cong \downarrow \alpha_{k+1} \\
(\mathbf{Q}_\pi)^n & = & (\mathbf{Q}_\pi)^n = & \cdots & = (\mathbf{Q}_\pi)^n & = & (\mathbf{Q}_\pi)^n
\end{array}
$$

by $A(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1})$. We want to compute $A(\alpha_1, \alpha_2, \ldots, \alpha_k, 1)$.

It is clear that $A(1, 1, \ldots, 1, \alpha_i, 1, \ldots, 1)$ is projective and equal to $\partial^{p_i}[\alpha_i] \in K_0(\mathbf{Z}\pi)$, by definition of the boundary map ∂^{p_i}. It is also immediate from the definition of ∂^{p_i} that $\partial^{p_i}([\alpha_i] + [\alpha_i^{-1}]) = 0$. Thus, adding

$$
\begin{aligned}
A(\alpha_1^{-1}, 1, 1, \ldots, 1) \oplus & A(1, \alpha_2^{-1}, 1, 1, \ldots, 1) \oplus \cdots \oplus \\
& A(1, 1, \ldots, \alpha_k^{-1}, 1) \oplus A(\alpha_1, \alpha_2, \ldots, \alpha_k, 1),
\end{aligned}
$$

we are reduced to the case where $\partial^{p_i}[\alpha_i] = 0$; we need to show in this case that $A(\alpha_1, \alpha_2, \ldots, \alpha_k, 1)$ is stably free.

Given an isomorphism $\beta : (\mathbf{Q}\pi)^n \to (\mathbf{Q}\pi)^n$, one sees that

$$A(\beta\alpha_1, \ldots, \beta\alpha_k, \alpha_{k+1}) \cong A(\alpha_1, \ldots, \alpha_k, \beta^{-1}\alpha_{k+1})$$

Also if $\gamma : \mathbf{Z}_{(p_i)}\pi \to \mathbf{Z}_{(p_i)}\pi$ is an isomorphism inducing an isomorphism of $(\mathbf{Q}\pi)^n$ then

$$A(\alpha_1, \ldots, \alpha_i\gamma, \ldots, \alpha_{k+1}) \cong A(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{k+1})$$

Now consider one of the isomorphisms, say α_1. The fact that $\partial^{p_1}[\alpha_1] = 0$ means that after possible stabilization (which may be performed correspondingly on the pullback diagram), α_1 can be written as $\beta_1 \cdot \gamma_1$ where γ_1 comes from an isomorphism of $((\mathbf{Z}_{(p_1)}\pi)^n$ and β_1 is an isomorphism from $(\mathbf{Z}_{(1/p_1)}\pi)^n$. We thus have

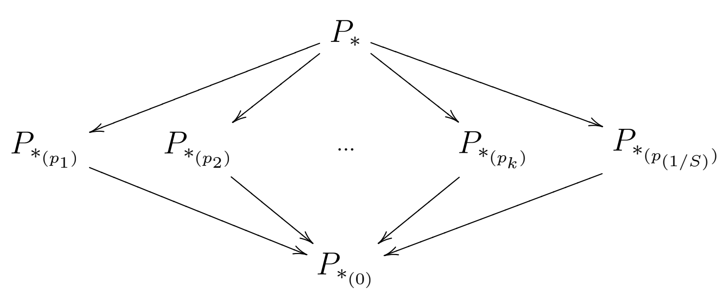
$$\begin{aligned} A(\alpha_1, \alpha_2, \ldots, \alpha_k, 1) &\cong A(\beta_1 \cdot \gamma_1, \alpha_2, \ldots, \alpha_k, 1) \\ &\cong A(\beta_1, \alpha_2, \ldots, \alpha_k, 1) \end{aligned}$$

by the above remark. But also

$$\begin{aligned} A(\beta_1, \alpha_2, \ldots, \alpha_k, 1) &\cong A(1, \alpha_2 \cdot \beta_1^{-1}, \ldots, \alpha_k \cdot \beta_1^{-1}, \beta_1^{-1}) \\ &\cong A(1, \alpha_2, \ldots, \alpha_k, 1) \end{aligned}$$

since β_1^{-1} lifts to an isomorphism of $(\mathbf{Z}_{(p)}\pi)^n$ for $p \neq p_1$. Repeating this argument for each p_i, we find that $A(\alpha_1, \ldots, \alpha_k, 1)$ is stably isomorphic to $A(1, 1, \ldots, 1)$, which is clearly free.

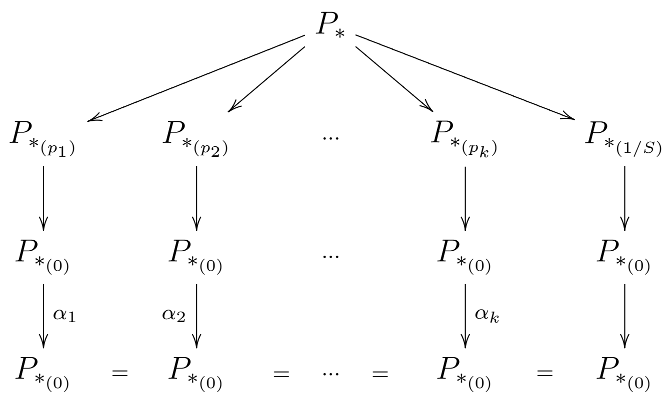
Proof of Theorem 2.4. Let P_* be a finitely generated, projective chain complex homotopy equivalent to the singular chains of $\widetilde{X}$. Consider the pullback diagram



where $S = \{p_1, \ldots, p_k\}$ is the set of primes p for which $RT_p(X) \neq 0$. Note that all chain complexes $P_{*_{(p_i)}}$ are free because $\widetilde{K}_0(\mathbf{Z}\pi) \to \widetilde{K}_0(\mathbf{Z}_{(p)}\pi)$ is the zero map. The complex $P_{*_{(1/S)}}$ is free because the Euler characteristic of X is zero (since we may assume π is nontrivial).

The fact that $RT_p(X) = 0$ for $p \notin S$ means that we can choose a basis for $P_{*_{1/S)}}$ so that for the induced basis of $P_{*_{(0)}}$ the torsion with respect to the given basis for rational homology is zero.

By choosing a basis for each chain complex $P_{*_{(p_i)}}$, and considering the induced basis for $P_{*_{(0)}}$, we may reinterpret the pullback diagram above follows:



Clearly the torsion of α_i is $RT_{p_i}(X)$ by definition. The theorem now follows by applying Lemma 2.5.

We are now in a position to show that for a finitely dominated, *nilpotent* space, the p-local Reidemeister torsions (and hence the p-parts of the finiteness obstruction) are restricted.

Let X be a finitely dominated nilpotent complex with finite fundamental group $\pi = \pi_1(X)$. Since π is nilpotent, it splits into the product of its p-Sylow subgroups π_p. As in Section 1, for a fixed prime p let π'_p denote the complementary group to π_p and let

$$e_p = \frac{1}{|\pi'_p|} \sum_{g \in \pi'_p} g$$

The idempotent e_p induces a splitting

$$\mathbf{Q}\pi = \mathbf{Q}\pi \cdot e_p \times \mathbf{Q}\pi \cdot (1 - e_p).$$

Proposition 2.6. *Let p be a prime. For a finitely dominated nilpotent space X with finite fundamental group π, the image of $RT_p(X) \in K_1(\mathbf{Q}\pi)/K_1(\mathbf{Z}_{(p)}\pi)$ is zero in $K_1(\mathbf{Q}\pi \cdot (1 - e_p))/K_1(\mathbf{Z}_{(p)}\pi \cdot (1 - e_p))$. Hence $\sigma(X) \in N_p(\pi)$. (See Definition 1.1.)*

Proof. Let P_* be a finitely generated, projective chain complex homotopy equivalent to the singular chains of the universal cover $\widetilde{X}$. The idempotent e_p induces a splitting

$$P_* = P_* \cdot e_p \times P_* \cdot (1 - e_p).$$

Since X is nilpotent, any element of π'_p must act trivially on the homology of $P_{*_{(p)}}$. (Such an element must act trivially on p-torsion and on rational homology.) It follows that an element of π'_p, acts trivially on the homology of $P_{*_{(p)}} \cdot (1 - e_p)$ and hence $P_{*_{(p)}} \cdot (1 - e_p)$ is a contractible chain complex.

Now e_p splits the chain complex $P_{*_{(0)}}$ similarly, and the argument above shows that the torsion of $P_{*_{(0)}} \cdot (1-e_p)$ comes from the torsion of $P_{*_{(p)}} \cdot (1-e_p)$. Hence the projection of $RT_p(X)$ into $K_1(\mathbf{Q}\pi \cdot (1-e_p))/K_1(\mathbf{Z}_{(p)}\pi \cdot (1-e_p))$ is zero.

We can summarize our results as follows. For any finitely dominated complex X with finite fundamental group π, for which π acts trivially on rational homology, the finiteness obstruction can be written as

$$\sigma(X) = \sum_{p \text{ prime}} \sigma_p(X)$$

where $\sigma_p(X) = \partial^p(RT_p(X))$. If X is nilpotent, then $\sigma(X) \in N_p(\pi) = \partial^p(K_1(\mathbf{Q}\pi \cdot e_p))$.

3. Realizing Reidemeister torsions

In the previous section we showed that the finiteness obstruction for a finitely dominated complex can be computed as the sum of its p-parts. For a *nilpotent* complex the p-part of the finiteness is restricted by Proposition 2.6 and must lie in $N_p(\pi)$.

Following Mislin [5], we define $N(\pi)$ to be the set of elements in $K_0(\mathbf{Z}\pi)$ that can be realized as finiteness obstructions of finitely dominated, nilpotent complexes with fundamental group π. From the preceding remarks, we conclude that

$$N(\pi) \subseteq \sum_{p \text{ prime}} N_p(\pi).$$

We now show that they are in fact equal.

Theorem 3.1. *For a finite nilpotent group π*

$$N(\pi) = \sum_{p \text{ prime}} N_p(\pi).$$

Proof. We need only show that every element of the right hand sum can be realized. We give a direct geometric proof of this, although an algebraic proof can be given following Mislin's work in [5].

First, one needs to show that $0 \in N(\pi)$. This is equivalent to the fact that π occurs as the fundamental group of a finite complex X. To see this, choose a faithful unitary representation of π. This induces a free action of π on $SU(n)$ for some n (by left multiplication), and we can take $X = SU(n)/\pi$. The space X is clearly a nilpotent complex.

Given the finite complex X with $\pi_1(X) = \pi$, we now need to show that we can modify X so that the p-local Reidemeister torsion is an arbitrary element in $K_1(\mathbf{Q}\pi \cdot e_p)$. (See Definition 1.1.)

The complex X is the homotopy pullback to the localization square:

$$\begin{array}{ccc} X & \longrightarrow & X_{(1/p)} \\ \downarrow & & \downarrow \\ X_{(p)} & \longrightarrow & X_{(0)} \end{array}$$

Let C_* denote the cellular chain complex of the universal cover $\widetilde{X}$. Applying a lemma of Wojtkoviac [10], we see that $C_{*_{(p)}}$ is homotopy equivalent to the singular chain complex of $X_{(p)} \times B\pi'_p$, the fiberwise p-localization of $X \to B\pi$. (As before, π'_p is the complement of the p-Sylow subgroup π_p in π.) However, we have seen that $C_{*(p)}$ splits into a $\mathbf{Z}_{(p)}(\pi_p)$-chain complex and a contractible $\mathbf{Z}_{(p)}(\pi'_p)$-chain complex via the idempotent e_p. The complex $C_{*_{(p)}}$ is stably free (since $\widetilde{K}_0(\mathbf{Z}\pi) \to \widetilde{K}_0(\mathbf{Z}_{(p)}\pi$ is the zero map) and so $C_{*_{(p)}}$ is homotopy equivalent to a free $\mathbf{Z}_{(p)}(\pi_p)$-chain complex, which as a $\mathbf{Z}_{(p)}(\pi)$-chain complex is projective. This is, however, exactly the chain complex of $X_{(p)}$ since the universal cover of $B\pi'_p$ is contractible.

We now modify $X_{(p)}$ so as to vary the p-local Reidemeister torsion. Suppose we are given an element $[\alpha] \in K_1(\mathbf{Q}\pi \cdot e_p) \cong K_1(\mathbf{Q}\pi_p)$. It is easy to see that α may be written as $\alpha = \beta \cdot \gamma^{-1}$, where β and γ lift to maps (but not necessarily *isomorphisms!*) of $(\mathbf{Z}_{(p)}\pi_p)^n$ to itself. Hence, we may assume that α itself lifts to a self-map of $(\mathbf{Z}_{(p)}\pi_p)^n$.

Replacing $X_{(p)}$ by

$$Z_{(p)} = X_{(p)} \vee \sum_{i=1}^{n} S^{2k}_{(p)} \cup_\alpha e^{2k+1},$$

adds

$$0 \to (\mathbf{Z}_{(p)}\pi_p)^n \xrightarrow{\alpha} (\mathbf{Z}_{(p)}\pi_p)^n$$

to the chain complex of $X_{(p)}$. The space $Z_{(p)}$ is again p-local, and since α is a rational homology isomorphism, $X_{(p)}$ and $Z_{(p)}$ are rationally equivalent. It is now clear that we have added $[\alpha]$ to the p-local Reidemeister torsion.

The pullback

$$\begin{array}{ccc} Z & \longrightarrow & X_{(1/p)} \\ \downarrow & & \downarrow \\ Z_{(p)} & \longrightarrow & X_0 \end{array}$$

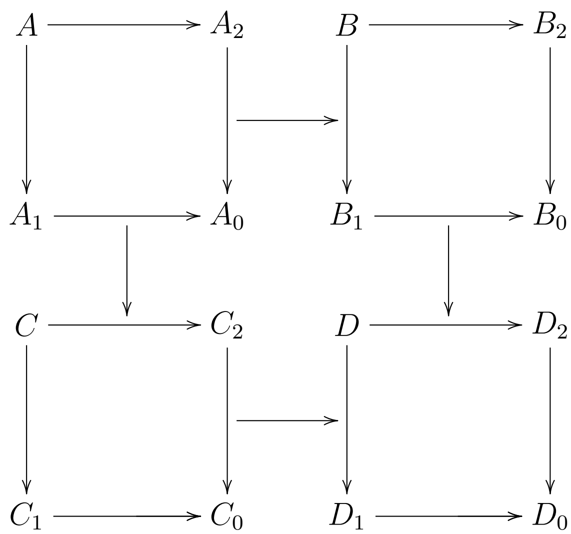
will be nilpotent, since nilpotent spaces are characterized by the fact that the fundamental group is nilpotent and its p-Sylow subgroups can act non-trivially only on p-torsion. Away from the prime p we have not changed anything there; at the prime p we have only introduced p-torsion.

Finally, Z is finitely dominated since it is nilpotent and $H_*(Z; \mathbf{Z})$ is a finitely generated abelian group. (See [4])

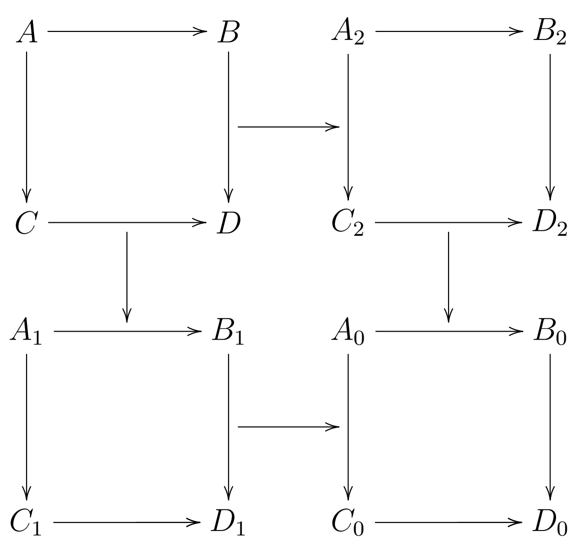
4. Pullbacks of pullbacks

In this section we present some preliminary algebra that will be used to analyze the groups $N(\pi)$ and $N_p(\pi)$ in the next section.

Consider a pullback diagram of pullback diagrams of rings:



Such a pullback can also be written as the pullback:



We will assume throughout this section that the following two conditions hold:

(i) The two pullback diagrams

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

have Milnor exact sequences in K-theory.

(ii) $D_0 = O$.

Remarks 4.1. It is standard that condition (i) holds if one of the maps to the lower corner is onto. The condition holds, however, in some cases where neither map is onto; for example, in localization squares. The second condition implies that $C_o \times B_0 = A_0$ and $D_1 \times D_2 = D$.

Theorem 4.2. *Given a pullback of pullbacks as above, satisfying conditions (i) and (ii) above, the following diagram commutes:*

$$\begin{array}{ccccc} [K_1(C_1) \times K_1(C_2)] \times [K_1(B_1) \times K_1(B_2)] & & \xrightarrow{\mu} & & [K_1(C_1) \times K_1(B_1)] \times [K_1(C_1) \times K_1(B_2)] \\ \downarrow{\scriptstyle j_C \times j_B} & & & & \downarrow{\scriptstyle j_1 \times j_2} \\ K_1(C_0) \times K_1(B_0) & & & & K_1(D_1) \times K_1(D_2) \\ \| & & & & \| \\ K_1(A_0) & \xrightarrow{\partial_N} & K_0(A) & \xrightarrow{\partial_L} & K_1(D) \end{array}$$

where $\mu = \left(1, \left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right], 1\right)$ and where j_C, j_B, j_1, j_2,∂_N,∂_L all are the appropriate maps from the Milnor exact sequence. (The letters N and L here stand for "number" and "letter".)

Remarks 4.3. Such pullbacks occur more often than one might think at first. In fact, most Milnor squares

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

can be thought of as the result of applying a functor to the identity map $A \xrightarrow{1} A$ in the category of "rings under A"; that is, in the category of rings R together with a homomorphism

$A \to R$. In such a situation, it is enough to specify the two pullback squares:

$$\begin{array}{ccc} A & \longrightarrow & A_2 \\ \big\downarrow & & \big\downarrow \\ A_1 & \longrightarrow & A_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

The rest of the diagram can be obtained from these by applying the functors. (See the examples in Section 5.)

Proof of Theorem 4.2. By symmetry it is sufficient to consider an element $[\alpha] \in K_1(C_1)$ where α is an isomorphism of C_1^n. Following the diagram along the left side, let $[\beta] \in K_1(A_0)$ denote the image of $[\alpha]$; the representative β is an isomorphism of A_0^n which under the identification $A_0 = B_0 \times C_0$ is the identity on B_0^n.

Now $\partial_N[\beta]$ is represented by a projective module P given by the pullback:

$$\begin{array}{ccc} P & \longrightarrow & A_2^n \\ \Big\downarrow & & \big\downarrow \\ & & A_0^n \\ & & \big\downarrow \\ A_1^n & \longrightarrow & A_0^n \end{array}$$

We wish to show that P can be described as well by following the diagram around the right side.

To do this, we first note that $P \otimes_A A = P$, and so P can also be described as the pullback:

$$\begin{array}{ccc} P & \longrightarrow & P \otimes_B B \\ \big\downarrow & & \big\downarrow \\ P \otimes_C C & \longrightarrow & P \otimes_D D \end{array} \qquad (*)$$

(To see that this *is* a pullback, note that it is when P is free. For an arbitrary projective P, add a projective to obtain a free module. Now use the fact that summands of pullback diagrams are pullback diagrams themselves.)

By assumption $B_0^n = A_0^n \otimes_{A_o} B_0 = B_0^n$ is the identity, thus producing an isomorphism $P \otimes_A B \simeq B^n$. The isomorphism $C_0^n = A_0^n \otimes_{A_0} C_0 \xrightarrow{\beta \otimes 1} A_0^n \otimes_{A_0} C_0 = C_0^n$ lifts to $\alpha : C_1^n \to C_1^n$, producing an isomorphism $P \otimes_A C \simeq C^n$

The pullback diagram $(*)$ can therefore be written as:

$$\begin{array}{ccc} P & \longrightarrow & C^n \\ \Big\downarrow & & \Big\downarrow \\ & & D^n \\ & & \Big\downarrow \\ B^n & \longrightarrow & D^n \end{array}$$

It is now easy to verify that the compatibility isomorphism

$$\begin{aligned} D^n = C^n \otimes_C D \simeq P \otimes_A C \otimes_C D = P \otimes_A D \\ \simeq P \otimes_A B \otimes_B D \simeq B^n \otimes_B D = D^n \end{aligned}$$

is induced from the given isomorphism α of C_1^n. Hence P is the image of $[\alpha]$ around the right side of the diagram.

Starting with an element $[\alpha] \in K_1(C_2)$ or $K_1(B_1)$, the same argument applies, but the compatibility isomorphisms have opposite directions, thus introducing a minus sign.

5. More about $N(\pi)$ and $N_p(\pi)$

In this section we complete the general discussion about the groups $N(\pi)$ and $N_p(\pi)$. We show that $N(\pi) = D(\pi)$ for a p-group, and recover as a corollary the result of Mislin and Varadarajan [6] that $N(\pi) \subseteq D(\pi)$ for any finite nilpotent group. (Recall that $D(\pi)$ is the kernel of the map $K_0(\mathbf{Z}\pi) \to K_0(\mathcal{M})$ where $\mathcal{M}$ is a maximal order in $\mathbf{Q}\pi$. We also fit $N_p(\pi)$ into an exact sequence.

We begin by relating the ∂^p homomorphisms of Section 1 to the Swan homomorphism SW, which can be described as the boundary homomorphism in the Milnor sequence associated to the pullback:

$$\begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z} \\ \Big\downarrow & & \Big\downarrow \\ \mathbf{Z}\pi/\Sigma & \longrightarrow & \mathbf{Z}/|\pi| \end{array}$$

where $\Sigma = \sum_{g \in \pi} g$. To do this, we need to compare the boundary homomorphisms in two Milnor sequences.

To be precise, let π be a finite group and consider the idempotent.

$$e = (1/|\pi|) \sum_{g \in \pi} g.$$

As usual $\mathbf{Q}\pi$ splits as $\mathbf{Q}\pi \cdot e \times \mathbf{Q}\pi \cdot (1-e)$. We can define an embedding $\phi : \mathbf{Q}^* \to \mathbf{Q}\pi^*$ by $\phi(x) = x \cdot e + (1-e)$.

We wish to compute ∂^p on the image of ϕ. Since ϕ is multiplicative, it clearly suffices to compute ∂^p on the image of each prime.

Proposition 5.1. *Let π be a finite group of order $p^n \cdot q^m \cdot r$, where p and q are distinct primes not dividing r. (We allow $n = 0$ or $m = 0$ here!) Let $\phi : \mathbf{Q}^* \to (\mathbf{Q}\pi)^*$ be the embedding defined above. Then*

$$\partial^p(\phi(p)) = -Sw(x), \qquad \textit{where} \quad x = \begin{cases} 1 & \mod \; p^n \\ p & \mod \; q^m r \end{cases}$$

and

$$\partial^p(\phi(p)) = -Sw(x), \qquad \textit{where} \quad x = \begin{cases} 1 & \mod \; q^m \\ q & \mod \; p^n r \end{cases}$$

Remarks 5.2. Since we allow $n = 0$ or $m = 0$, this provides a complete determination of ∂^p on the image of ϕ in terms of the Swan homomorphism.

Proof of Proposition 1.5. According to Remark 4.3, the two pullback diagrams

$$\begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(1-p)}\pi & \longrightarrow & A_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}\pi/\Sigma & \longrightarrow & D \end{array}$$

determine a pullback of pullback diagrams. Applying Theorem 4.2 to this situation, one obtains a commutative diagram:

$$\begin{array}{ccccc} & & K_1(\mathbf{Z}_{(1/p)}) \times K_1(\mathbf{Z}_{(p)}) & & \\ & \swarrow^{i \times i} & & \searrow^{j_\alpha - j} & \\ K_1(\mathbf{Q}) & & & & K_1((\mathbf{Z}/|\pi|)_{(1/p)}) \times K_1((\mathbf{Z}/|\pi|)_{(p)}) \\ \downarrow \phi & & & & \| \\ K_1(\mathbf{Q}\pi) & & & & K_1(\mathbf{Z}/|\pi|) \\ & \searrow^{\partial^p} & & \swarrow^{Sw} & \\ & & K_0(\mathbf{Z}\pi) & & \end{array}$$

where i and j are the natural inclusions of units. One now immediately reads off the answer from this diagram, starting with the element in $K_1(\mathbf{Q})$ and tracing both ways around.

Theorem 5.3. *Let π be a finite p-group. Then $N(\pi) = D(\pi)$, where as usual $D(\pi) = \ker\{K_0(\mathbf{Z}\pi) \to K_0(\mathcal{M})\}$ for a maximal order $\mathcal{M}$ in $\mathbf{Q}\pi$.*

Proof. For $q \neq p$, $N_p(\pi)$ is contained in the image of the Swan homomorphism by Proposition 5.1. However, the same proposition now shows that the image of the Swan homomorphism is contained in $N_p(\pi)$. Hence $N(\pi) = N_p(\pi)$.

Now the image of ∂^p contains $D(\pi)$. To see this, let $\mathcal{M}$ be a maximal order in $\mathbf{Q}\pi$ and note that $\mathcal{M}_{(1/p)} = \mathbf{Z}_{(1/p)}\pi$ since π is a p-group. The sequence

$$K_1(\mathbf{Q}\pi) \to K_0(\mathbf{Z}\pi) \xrightarrow{i} K_0(\mathbf{Z}_{(1/p)}\pi)$$

is exact, and the map factors as:

$$\begin{array}{ccc} K_0(\mathbf{Z}\pi) & \xrightarrow{i_*} & K_0(\mathbf{Z}_{(1/p)}\pi) \\ \downarrow & & \| \\ K_0(\mathcal{M}) & \longrightarrow & K_0(\mathcal{M}_{(1/p)}) \end{array}$$

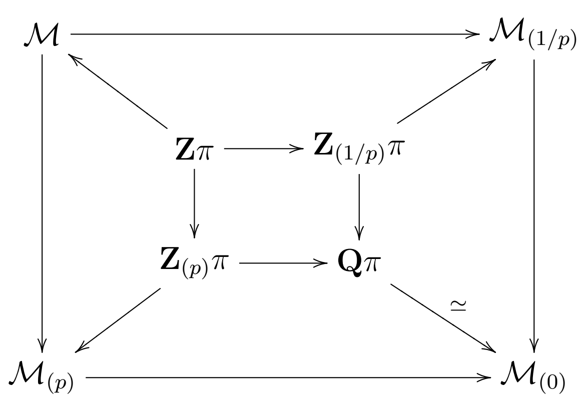
Hence $(N(\pi) \subseteq D(\pi)$.

To obtain the other inclusion, note that by [7, Chap. 22] the map $K_0(\mathcal{M}) \to K_0(\mathcal{M}_{(1/p)})$ is monic. Hence any element in $D(\pi)$ is contained in the kernel of i_*, showing that $D(\pi) \subseteq N(\pi)$.

As a corollary of the preceding result, we recover the following result of Mislin and Varadarajan [6].

Corollary 5.4. *For any finite nilpotent group π, $N(\pi) \subseteq D(\pi)$.*

Proof. It is enough to show that $N_p(\pi) \subseteq D(\pi)$ for any prime p. Consider the diagram:



where $\mathcal{M}$ is a maximal order in $\mathbf{Q}\pi$.

Now $N_p(\pi)$ is the image of ∂^p restricted to $K_1(\mathbf{Q}\pi \cdot e_p)$, where e_p is the idempotent defined in Section 1. We need to show that this image maps to zero in $K_0(\mathcal{M})$. That is equivalent to showing that the boundary map in the Milnor sequence of the outer diagram maps the $\mathbf{Q}\pi \cdot e_p$ component of $\mathcal{M}_{(0)} = \mathbf{Q}\pi$ to zero.

The outer diagram splits, however, and the component we must consider is precisely the component corresponding to the maximal order of π_p, the p-Sylow subgroup of π. This case was considered in Theorem 5.3.

For a finite nilpotent group π of *composite* order, the group $N(\pi)$ is in general not equal to all of $D(\pi)$. To obtain further information about $N(\pi)$ it is necessary to study the groups $N_p(\pi)$ individually.

Consider the nilpotent group π and a prime p dividing the order of π. As usual we write $\pi = \pi_p \times \pi'_p$ where π_p is the p-Sylow subgroup and π'_p is the complementary subgroup. There is then a Milnor square:

$$\begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z}\pi_p \\ \downarrow & & \downarrow \\ \mathbf{Z}\pi/\Sigma & \longrightarrow & \mathbf{Z}\pi_p/|\pi'_p| \end{array}$$

where $\Sigma' = \sum_{g \in \pi'_p} g$. We denote the boundary map in the Milnor sequence for this square by ∂^{f_p}. (The "f"" stands for "finite" here.)

Theorem 5.5. *For any finite nilpotent group π there is an exact sequence:*

$$K_1(\mathbf{Z}_{1/p)}\pi_p) \xrightarrow{\psi_p} N_p(\pi) \xrightarrow{j_p} D(\pi_p) \to 0$$

where j_p is induced from the projection $\pi \to \pi_p$, and ψ_p is the composition

$$K_1(\mathbf{Z}_{(1/p)}\pi_p) \to K_1(\mathbf{Z}_{(1/p)}\pi_p/|\pi'_p|) = K_1(\mathbf{Z}\pi_p/|\pi'_p|) \xrightarrow{\partial^{f_p}} K_0(\mathbf{Z}\pi)$$

Proof. According to Remark 4.3, the two pullback squares

$$\begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z}_{(1/p)} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(p)}\pi & \longrightarrow & \mathbf{Q}\pi \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z}\pi_p \\ \downarrow & & \downarrow \\ \mathbf{Z}\pi/\Sigma & \longrightarrow & \mathbf{Z}\pi_p/|\pi'_p| \end{array}$$

determine a pullback of pullback diagrams. Applying Theorem 4.2 to this situation, we obtain a commutative diagram:

$$
\begin{array}{ccccc}
 & & & & K_1(\mathbf{Z}_{(1/p)}\pi_p) \\
 & \overset{(i,0)}{\swarrow} & & & \downarrow \\
K_1(\mathbf{Q}\pi \cdot e_p) \times K_1(\mathbf{Q}\pi \cdot (1-e_p)) & & K_1(\mathbf{Z}\pi_p/|\pi'_p| & = & K_1(\mathbf{Z}_{(1/p)}\pi_p/|\pi'_p|) \\
\| & & & & \downarrow \\
K_1(\mathbf{Q}\pi) & & \xrightarrow{\partial} & & K_0(\mathbf{Z}\pi) \\
\downarrow & & & & \downarrow \\
K_1(\mathbf{Q}\pi_p) & & \xrightarrow{\partial^p} & & K_0(\mathbf{Z}\pi_p)
\end{array}
$$

The lower square shows that projection induces an epimorphism $N_p(\pi) \to N_p(\pi_p)$, and by Theorem 5.3 we know that $N_p(\pi_p) = D(\pi_p)$.

Commutativity of the diagram shows that the image of ψ_p is contained in $N_p(\pi)$. Since

$$K_1(\mathbf{Z}_{(p)}\pi_p) \oplus K_1(\mathbf{Z}_{(1/p)}\pi_p) \to K_1(\mathbf{Q}\pi_p)$$

is part of a Milnor exact sequence, we also conclude that the image of ψ_p is contained in the kernel of j_p.

We need only show, therefore, that if $x \in N_p(\pi)$ and $j_p(x) = 0$ then x is in the image of ψ_p. Since $x \in N_p(\pi)$ we know that $x = \partial^p y$ for some $y \in K_1(\mathbf{Q}\pi \cdot e_p) = K_1(\mathbf{Q}\pi_p)$. But in the exact sequence for π_p, the boundary of y is zero in $K_0(\mathbf{Z}\pi_p)$. Hence there are elements $z_1 \in K_1(\mathbf{Z}_{(1/p)}\pi_p)$ and $z_2 \in K_1(\mathbf{Z}_{(p)}\pi_p)$ such that (z_1, z_2) maps to y.

Now the idempotent e_p is also an idempotent in $\mathbf{Z}_{(p)}\pi$, since the denominators involve only primes different from p. Hence $\mathbf{Z}_{(p)}\pi$ splits as

$$\mathbf{Z}_{(p)}\pi = \mathbf{Z}_{(p)}\pi \cdot e_p \times \mathbf{Z}_{(p)}\pi \cdot (1 - e_p)$$

and the element $(z_2, 1)$ defines an element in $K_1(\mathbf{Z}_{(p)}\pi)$ that maps to z_2 under projection. Let y_2 denote the image of z_2 in $K_1(\mathbf{Q}\pi)$. Then $y - y_2$ lies in $K_1(\mathbf{Q}\pi \cdot e_p)$ and maps to x under the boundary map, and is also the image of $z_1 \in K_1(\mathbf{Z}_{(1/p)}\pi_p)$. By commutativity of the diagram we conclude that x is in the image of ψ_p.

6. A computation

Let C_n denote the cyclic group of order n. In [5] Mislin has shown that $N(C_{21})$ is either 0 or $\mathbf{Z}/2$. To illustrate Proposition 5.4 we will show that $N(C_{21})$ is in fact $\mathbf{Z}/2$ by showing that $N_3(C_{21}) \neq 0$. Since $D(C_3) = 0$, we must show that the map ψ_3 has nontrivial image. This involves a small computation in number theory.

The Milnor square defining ∂^{f_3} is now

$$\begin{array}{ccc} \mathbf{Z}[C_{21}] & \longrightarrow & \mathbf{Z}[\zeta][C_3] \\ \downarrow & & \downarrow \\ \mathbf{Z}[C_3] & \longrightarrow & F_7[C_3] \end{array}$$

where $\zeta = e^{2\pi i/7}$ and F_7 is the field of 7 elements (a less cumbersome notation that $\mathbf{Z}/7$.) The portion of the K-theory exact sequence we are interested in is

$$\mathbf{Z}[C_3]^* \times \mathbf{Z}[\zeta][C_3]^* \xrightarrow{j} F_7[C_3]^* \xrightarrow{\partial^{f_3}} K_0(\mathbf{Z}[C_{21}]).$$

Since we have a splitting $F_7[C_3]^* = F_7^* \times F_7^*[\omega]^*$, where $\omega = e^{2\pi i/3}$, we will write our units accordingly.

Now we claim that the unit $(1, \omega - 1) \in F_7[C_3]^*$ is not in the image of j and hence $\partial^{f_3}(1, \omega - 1) \neq 0$. Since $(x-1) + \frac{1}{3}(1 + x + x^2) \in \mathbf{Z}_{(1/3)}[C_3]^*$ maps to $(1, \omega - 1)$, we can conclude that $N_3(C_{21}) \neq 0$. To prove the claim we need to do a little work.

First, note that since $\mathbf{Z}[C_3]^* \subseteq \mathbf{Z}[\zeta][C_3]^*$, it is enough to show that $(1, \omega - 1)$ is not in the image of $\mathbf{Z}[\zeta][C_3]^*$. To analyze this group we use the standard pullback

$$\begin{array}{ccc} \mathbf{Z}[\zeta][C_3] & \longrightarrow & \mathbf{Z}[\zeta] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\zeta\omega] & \longrightarrow & F_3[\zeta] \end{array}$$

yielding the Milnor sequence:

$$1 \to \mathbf{Z}[\zeta][C_3]^* \to \mathbf{Z}[\zeta\omega]^* \times \mathbf{Z}[\zeta]^* \to F_3[\zeta]^*$$

We now require two lemmas.

Lemma 6.1. *The image of $\mathbf{Z}[\zeta]^* \to F_3[\zeta]^*$ is generated by $(\zeta - 1)^2$.*

Proof. Fortunately, one knows that the units of $\mathbf{Z}[\zeta]^*$ are generated by the cyclotomic units $-\zeta^{-1}$,$(\zeta^2 - 1)/\zeta - 1)$, and $(\zeta^3 - 1)/(\zeta - 1)$.

Repeated cubing shows that

$$(\zeta - 1)^3 = \zeta^3 - 1 \mod 3$$
$$(\zeta - 1)^3 = \zeta^9 - 1 \mod 3$$

and

$$(\zeta)^{27} = \zeta^6 - 1 \mod 3$$

Hence,

$$(\zeta^3 - 1)/(\zeta - 1) = (\zeta - 1)^2 \mod 3$$
$$(\zeta^2 - 1)/(\zeta - 1) = (\zeta - 1)^8 \mod 3$$

and

$$(\zeta^6 - 1)/(\zeta - 1) = -\zeta^{-1} = (\zeta - 1)^{26} \mod 3$$

Lemma 6.2. *Given a unit* $u \in \mathbf{Z}[\zeta\omega]^*$ *such that* $u \equiv 1 \mod (\zeta - 1)$*, the projection of* u *in* $F_3[\zeta]$ *is an even power of* $\zeta - 1$*).*

Proof. We consider the commutative square

$$\begin{array}{ccc} \mathbf{Z}[\zeta\omega]^* & \xrightarrow{\phi_1} & F_3[\zeta]^* \\ \downarrow & & \downarrow \\ \mathbf{Z}[\zeta]^* & \xrightarrow{\phi_2} & F_3[\zeta]^* \end{array}$$

where ϕ_1 is reduction mod $(\omega - 1)$, ϕ_2 is reduction mod 3, N is the norm map, and Sq is the squaring map.

Since $u \equiv 1 \mod (\zeta - 1)$, we see that $N(u) \equiv 1 \mod (\zeta - 1)$ as well. But as mentioned above, we know the units of $\mathbf{Z}[\zeta]^*$ and hence

$$N(u) = (-\zeta^{-1})^a \left(\frac{\zeta^2 - 1}{\zeta - 1}\right)^b \left(\frac{\zeta^3}{\zeta - 1}\right)^c$$

for some integers, a, b, and c. Reducing mod $(\zeta - 1)$ we have

$$(-1)^a (2)^b (3)^c \equiv 1 \mod 7.$$

Now write $2 \equiv 3^2 \mod 7$ and $-1 \equiv 3^3 \mod 7$. We then obtain

$$3^{3a} \cdot 3^{2b} \cdot 3^c \equiv 1 \mod 7$$

showing that $3a + 2b + c \equiv 0 \mod 6$.

From the proof of Lemma 6.1 we also know that

$$\phi_2 N(u) = (\zeta - 1)^{26a} (\zeta - 1)^{8b} (\zeta - 1)^{2c} = (\zeta - 1)^{26a + 8b + 2c}.$$

Since $c = -3a - 2b + 6q$ for some integer q, we can substitute to see that

$$\phi_2 N(u) = (\zeta - 1)^{20a + 4b + 12q} = Sq\phi_1(u).$$

We can conclude that $\phi_1(u)$ is $(\zeta - 1)$ to an even power.

To complete the proof of the claim, suppose $(1, \omega - 1)$ is the image of $u \in \mathbf{Z}[\zeta][C_3]^*$. Let $(u_1, u_2) \in \mathbf{Z}[\zeta]^* \times \mathbf{Z}[\zeta\omega]^*$ be the projection of u; we know, of course, that $u_1 = u_2$ in $F_3[\zeta]$. Now $u_2(\zeta\omega - 1)^{-1} \equiv 1 \mod (\zeta - 1)$. Projecting into $F_3[\zeta]$ and applying Lemma 6.1, we see that $u_2 = (\zeta - 1)^{2k+1}$ in $F_3[\zeta]$ for some k. Hence $u_1 = (\zeta - 1)^{2k+1}$ in $F_3[\zeta]$ contradicting Lemma 6.2.

With a small additional effort it is possible to show that $N_3(C_{21})$ is precisely $\mathbf{Z}/2$ (although this follows from Mislin's result). The type of argument employed here can be used more generally for cyclic groups. The number theoretic problems, however, rapidly become more difficult.

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