

# A FINITE LOOP SPACE NOT RATIONALLY EQUIVALENT TO A COMPACT LIE GROUP

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ABSTRACT. We construct a connected, finite loop space of rank 66 and dimension 1254 whose rational cohomology is not isomorphic as a graded vector space to the rational cohomology of any compact Lie group, hence providing a counterexample to a classical conjecture. Aided by machine calculation we verify that our counterexample is minimal, i.e., that any finite loop space of rank less than 66 is in fact rationally equivalent to a compact Lie group, extending the classical known bound of 5.

## 1. INTRODUCTION

Since the discovery of the Hilton-Roitberg ‘criminal’ [19, 20, 37] in 1968 it has been clear that not every finite loop space is homotopy equivalent to a compact Lie group. The conjecture emerged however, that this should hold rationally, i.e., that any finite loop space should be rationally equivalent to some compact Lie group (see [1], [24, Conj. A, p. 67]), and evidence for this has been accumulating over the years. In this paper we resolve this conjecture in the negative by exhibiting a concrete, finite loop space of rank 66 whose rational cohomology does not agree with that of any compact Lie group. To do this, we first use Sullivan’s arithmetic square [40, 41][6, VI.8.1] and the theory of  $p$ -compact groups [10, 13, 3] to translate the conjecture into a purely combinatorial statement. We then proceed to show that this statement is ‘generically’ false, with a counterexample appearing in rank 66. On the other hand we verify using a computer that our counterexample is in fact of minimal rank, i.e., that the conjecture is true for any finite loop space of rank less than 66. This extends earlier work of many authors which show the statement to be true when the rank is at most 5 (see [33, 36, 39, 22, 15] and also [2, 27, 29, 28]).

We now explain this in more detail. Recall that a finite, connected loop space is a triple  $(Y, BY, e)$  where  $Y$  is a finite, connected CW-complex,  $BY$  is a based CW-complex, and  $e : Y \rightarrow \Omega BY$  is a homotopy equivalence, where  $\Omega BY$  denotes the space of based loops in  $BY$ . (We usually refer to a loop space just as  $Y$  suppressing the rest of the structure.) It is an old theorem of Hopf [21, Satz I] that the rational cohomology of any connected, finite loop space is a graded exterior algebra  $H^*(Y; \mathbf{Q}) \cong \bigwedge_{\mathbf{Q}}(x_1, \dots, x_r)$ , where the generator  $x_i$  is in odd dimension  $2d_i - 1$ . The number  $r$  is called the *rank* of  $Y$  and the collection of  $d_i$ ’s are called the *degrees* (or if doubled the ‘type’) of  $Y$ . It is a classical result of Serre [34] that the collection  $\{d_1, \dots, d_r\}$  in fact uniquely determines the rational homotopy type of  $(Y, BY, e)$ . The  $p$ -completion  $(Y_{\hat{p}}, BY_{\hat{p}}, e_{\hat{p}})$  is a connected  $p$ -compact group, i.e.,  $H^*(Y_{\hat{p}}; \mathbf{F}_p)$  is finite dimensional and connected and  $BY_{\hat{p}}$  is  $p$ -complete (in the sense of Sullivan [40, 41] or Bousfield-Kan [6]; see e.g., [13], [10], or [3] for much more on  $p$ -compact groups.)

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1991 *Mathematics Subject Classification*. Primary: 55P35; Secondary: 55P15, 55R35.

The third named author was partially supported by NSF grant DMS-0104318.

The fourth named author thanks SFB 478 Münster for its hospitality and support.

An amazing result of Dwyer-Wilkerson [13], extending work of Dwyer-Miller-Wilkerson [11] and Adams-Wilkerson [1], says that any  $p$ -compact group  $X$  has a maximal torus, that is a loop map  $T \cong (S^1_p)^r \rightarrow X$  which is suitably maximal and an associated Weyl group  $W_X$ . If  $X$  is connected, then  $W_X$  acts faithfully on  $L_X = \pi_1(T)$ , in such a way that  $(W_X, L_X)$  becomes a finite  $\mathbf{Z}_p$ -reflection group and

$$H^*(BX; \mathbf{Z}_p) \otimes \mathbf{Q} \xrightarrow{\cong} (H^*(BT; \mathbf{Z}_p) \otimes \mathbf{Q})^{W_X}.$$

The invariant ring  $(H^*(BT; \mathbf{Z}_p) \otimes \mathbf{Q})^{W_X}$  is a polynomial algebra with generators in dimensions  $2e_1, \dots, 2e_r$ , and the integers  $e_1, \dots, e_r$  are just the well known degrees of the  $\mathbf{Q}_p$ -reflection group  $(W_X, L_X \otimes \mathbf{Q})$  [5, Ch. 7],[17]. (In fact the harder classification states that  $(W_X, L_X)$  completely classifies  $X$  when  $p$  is odd [3].) If  $Y$  is a finite loop space then, for all primes  $p$ ,

$$H^*(BY; \mathbf{Q}) \otimes \mathbf{Q}_p \cong H^*(BY; \mathbf{Z}) \otimes \mathbf{Q}_p \cong H^*(BY_p^\wedge; \mathbf{Z}_p) \otimes \mathbf{Q}.$$

Furthermore by the Eilenberg-Moore spectral sequence  $H^*(BY; \mathbf{Q})$  is a polynomial algebra with generators in dimensions  $2d_1, \dots, 2d_r$ , where the  $d_i$ 's are the degrees of  $Y$  introduced earlier. Hence by the above we conclude that, for each prime  $p$ ,  $Y_p^\wedge$  has to be some  $p$ -compact group such that the degrees of  $Y$  match up with the degrees of the  $\mathbf{Q}_p$ -reflection group  $(W_{Y_p^\wedge}, L_{Y_p^\wedge} \otimes \mathbf{Q})$ . This puts severe restrictions on the possible degrees. Finite  $\mathbf{Q}_p$ -reflection groups have been classified by Clark-Ewing [9] building on the classification over  $\mathbf{C}$  by Shephard-Todd [35]. The classification divides into three infinite families along with 34 sporadic cases (the non-Lie ones only being realizable for certain primes). We denote by  $W_D$  the Weyl group coming from the Dynkin diagram  $D$  whereas the notation  $G(\cdot, \cdot, \cdot)$  means a given group from the infinite family 2, and  $G_n$  refers to one of the sporadic cases, in the standard notation listed e.g., in [9] or [17]. Historically, pioneering work of Clark [8] had already shown that if the maximal degree of  $Y$  is  $h$  then  $Y$  also has to have the degree  $m \leq h$  if  $m - 1$  and  $h$  are relatively prime, using arguments only involving large primes (compare also [23, 3.20]). This, as input to small rank calculations, served as original motivation for the conjecture. Adams-Wilkerson [1] much later found the restrictions imposed by reflection groups described above, but worked only at large primes since the technology of [13] was not available. They furthermore gave an example [1, Ex. 1.4] showing that the large prime information is not enough to settle the conjecture algebraically. What we show here is that, contrary to general expectation, the restrictions at all primes are not even sufficient.

**Theorem 1.1.** *There exists a connected, finite loop space  $Y$  of rank 66 such that  $Y_p^\wedge$  has Weyl group as a  $\mathbf{Q}_p$ -reflection group given by*

$$\begin{aligned} W_{A_4 B_4 B_5 B_8 B_8 E_8 A_{12} B_{14}} \times G_{24} & \quad \text{for } p = 2, \\ W_{A_4 D_4 B_5 B_8 B_8 E_8 A_{13} B_{14}} \times G_{12} & \quad \text{for } p \equiv 1, 3 \pmod{8}, \\ W_{G_2 A_4 B_4 B_4 B_7 D_{10} A_{13} B_{15}} \times G(4, 2, 7) & \quad \text{for } p \equiv 5 \pmod{8}, \\ W_{A_4 B_4 B_5 B_8 B_8 E_8 A_{13} B_{14}} \times G(6, 3, 2) & \quad \text{for } p \equiv 7 \pmod{24}, \\ W_{D_4 A_5 D_8 D_8 B_{10} A_{13} D_{16}} \times G(24, 24, 2) & \quad \text{for } p \equiv 23 \pmod{24}. \end{aligned}$$

The space  $Y$  has dimension 1254 and the degrees of  $Y$  are

$$\{2^8, 3^2, 4^8, 5^2, 6^7, 7, 8^7, 9, 10^5, 11, 12^5, 13, 14^5, 16^3, 18^2, 20^2, 22, 24^2, 26, 28, 30\}$$

(using exponent notation to denote repeated degrees) and these do not agree with the degrees of a finite  $\mathbf{Q}$ -reflection group, i.e., the graded vector space  $H^*(Y; \mathbf{Q})$  does not agree with  $H^*(G; \mathbf{Q})$  for any compact Lie group  $G$ .

Furthermore this counterexample is minimal in the sense that any connected, finite loop space of rank less than 66 is rationally equivalent to some compact Lie group  $G$ .

Clark-Ewing [9] constructed  $p$ -compact groups realizing each of the irreducible  $\mathbf{Q}_p$ -reflection groups above, in the cases where  $p$  does not divide the Weyl group order. The remaining important small prime cases which are not realized by Lie groups were constructed by Zabrodsky ( $G_{12}, p = 3$ ) [46], Dwyer-Wilkerson ( $G_{24}, p = 2$ ) [12], and Notbohm ( $G(\cdot, \cdot, \cdot), p$  small) [31]. Since  $G_{24}$  and  $G_{12}$  are the only finite, irreducible  $\mathbf{Q}_p$ -reflection groups which do not come from compact Lie groups, for  $p = 2$  and 3 respectively, any counterexample will have to involve these groups.

We note that by work of Bauer-Kitchloo-Notbohm-Pedersen [4] the loop space  $Y$  of Theorem 1.1 is in fact homotopy equivalent to a compact, smooth, parallelizable manifold.

Sullivan’s arithmetic square [40, 41] reduces the study of finite loop spaces to the study  $p$ -compact groups for all primes  $p$  with the same degrees together with the well understood concept of arithmetic square ‘mixing’. We restrict ourselves here to giving the following lemma which guarantees that an algebraic counterexample produces a topological counterexample.

**Lemma 1.2.** *Let  $\{d_1, \dots, d_r\}$  be a collection of positive integers (with repetitions allowed). Suppose that for each prime  $p$  we have a connected  $p$ -compact group  $X_p$  whose Weyl group  $(W_{X_p}, L_{X_p})$  has degrees  $\{d_1, \dots, d_r\}$ . Then there exists a (non-unique) connected, finite loop space  $Y$  such that  $Y_{\hat{p}} \cong X_p$  as  $p$ -compact groups.*

The next theorem guarantees that ‘generically’ there will be collections of degrees which are the degrees of a finite  $\mathbf{Q}_p$ -reflection group for all primes  $p$  without being the degrees of a finite  $\mathbf{Q}$ -reflection group. Together with Lemma 1.2 this shows why examples like the one in Theorem 1.1 exist. For the statement of the result we need to introduce some more notation. If  $\{d_1, d_2, \dots, d_r\}$  is a collection of degrees, then the associated *degree vector* equals  $(x_1, x_2, \dots) \in \mathbf{Z}^{(\infty)}$  where  $x_i$  is the number of degrees equal to  $i$ . We let  $K_{\text{Lie}} \subseteq \mathbf{Q}^{(\infty)}$  denote the positive, rational cone spanned by the degree vectors of the finite, irreducible  $\mathbf{Q}$ -reflection groups, i.e., the set of finite, non-negative, rational linear combinations of these vectors. Similarly we let  $K_p$  denote the positive, rational cone spanned by the degree vectors of the finite, irreducible  $\mathbf{Q}_p$ -reflection groups. Finally we let  $K_{\text{Lin}}$  denote the positive, rational cone spanned by the degree vectors of the finite, irreducible  $\mathbf{Q}$ -reflection groups and the degree vectors of the groups  $G(m, m, 2)$ ,  $m = 8, 12, 24$ , cf. [27, Thm. 1.1(b)].

**Theorem 1.3.** *We have  $K_{\text{Lie}} \subsetneq \bigcap_p K_p$ , where the intersection is taken over all primes  $p$ . Moreover  $\bigcap_p K_p = K_2 \cap K_3 \cap K_5 \cap K_7 \cap K_{\text{Lin}}$ .*

**Acknowledgments.** We are grateful to Ib Madsen whose encouraging questions about integrality results in light of the  $p$ -complete theory led us to consider this conjecture. We thank Jørgen Tind for pointing us to Komei Fukuda’s program `cdd+` [16] for doing computations on polyhedral cones. The hospitality of University of Copenhagen, SFB 478 Münster, and Århus University helped make this collaboration possible.

## 2. PROOFS

Recall the following essentially classical result.

**Theorem 2.1.** *Let  $X$  be a simply connected  $p$ -compact group with degrees  $\{d_1, \dots, d_r\}$ . Then, as a space,  $X \simeq (S^{2d_1-1} \times \dots \times S^{2d_r-1})_{\hat{p}}$  if and only if  $p \geq \max\{d_1, \dots, d_r\}$ . If  $X$  is just assumed connected then under the stronger assumption  $p > \max\{d_1, \dots, d_r\}$ ,  $X$  still splits as a product of spheres.*

*Sketch of proof.* Set  $h = \max\{d_1, \dots, d_r\}$  and suppose first that  $X$  is simply connected. If  $p \geq h$  then by a Bockstein spectral sequence argument of Browder [7, Thm. 4.7],  $H^*(X; \mathbf{Z}_p)$  is torsion free and concentrated in odd degrees. Hence an easy argument of Serre [34, Ch. V Prop. 6] (see also [26]), using that  $\pi_n(S^{2d_i-1})$  has no  $p$ -torsion when  $n < 2d_i - 1 + 2p - 3$ , yields that  $X \simeq (S^{2d_1-1} \times \dots \times S^{2d_r-1})_{\hat{p}}$ . (In fact, this direction uses only that  $X$  is an  $H$ -space with  $H^*(X; \mathbf{F}_p)$  finite dimensional.) The other direction, which is more subtle and not needed here, was first established for compact Lie groups by Serre [34] and Kumpel [25] by case-by-case arguments, and later a general argument was given by Wilkerson [44, Thm. 4.1] using operations in  $K$ -theory.

Assume now that  $X$  is just connected and that  $p > h$ . If  $\pi_1(X)$  is torsion free, then as a space  $X \simeq \tilde{X} \times (S^1_{\hat{p}})^k$  where  $\tilde{X}$  is simply connected (cf. e.g. [24, p. 24]), which reduces us to the previous case. Hence we just have to justify that, with  $p$  as above,  $\pi_1(X)$  does not have torsion. By [30, Thm. 1.4] we have a fibration  $BK \rightarrow B\tilde{X} \times BT' \rightarrow BX$ , such that  $\tilde{X}$  is a simply connected  $p$ -compact group,  $T'$  is a  $p$ -completed torus,  $K$  is a finite, abelian  $p$ -group, and the projection map  $K \rightarrow \tilde{X}$  is a central monomorphism. But then by [14, Thm. 7.5], the image of  $K$  is contained in  $\check{T}^{W_{\tilde{X}}}$ , where  $\check{T}$  is a discrete approximation to a maximal torus in  $\tilde{X}$ . In particular if  $\check{T}^{W_{\tilde{X}}} = 0$ ,  $\pi_1(X)$  has to be torsion free. But if  $p > h$  then  $p \nmid |W_{\tilde{X}}|$ , so we have an exact sequence

$$\dots \rightarrow H^0(W_{\tilde{X}}; L_{\tilde{X}} \otimes \mathbf{Q}) \rightarrow H^0(W_{\tilde{X}}; \check{T}) \rightarrow H^1(W_{\tilde{X}}; L_{\tilde{X}}) \rightarrow \dots$$

where the first and third terms are zero so  $\check{T}^{W_{\tilde{X}}} = 0$  as wanted.  $\square$

*Proof of Lemma 1.2.* Set  $h = \max\{d_1, \dots, d_r\}$  and let  $BM = (\prod_p BX_p)_{\mathbf{Q}}$ . Since rationalization commutes with taking loop space and finite products we have that  $\Omega BM \simeq (\prod_{p < h} (X_p)_{\mathbf{Q}}) \times (\prod_{p \geq h} X_p)_{\mathbf{Q}}$ . By Theorem 2.1  $X_p \simeq (S^{2d_1-1} \times \dots \times S^{2d_r-1})_{\hat{p}}$  when  $p > h$  and by the same argument  $(X_p)_{\mathbf{Q}} \simeq ((S^{2d_1-1} \times \dots \times S^{2d_r-1})_{\hat{p}})_{\mathbf{Q}}$  for all primes  $p$ . Combined with the fact that  $(S^{2d_i-1})_{\hat{p}}_{\mathbf{Q}} \simeq K(\mathbf{Q}_p, 2d_i - 1)$  this implies that  $\pi_n(BM) = (\prod_n \pi_n(BX_p))_{\mathbf{Q}} \cong \bigoplus_{i, 2d_i=n} \mathbf{A}_f$ , where  $\mathbf{A}_f = (\prod_p \mathbf{Z}_p) \otimes \mathbf{Q}$  is the ring of finite adeles. In particular  $BM$  only has homotopy groups in even dimensions. But now a rational space which only has homotopy groups in even dimensions is necessarily a product of Eilenberg-Mac Lane spaces, as is easily seen by going up the Postnikov tower (cf. e.g., [43, Ch. IX]). Set  $BK = K(\mathbf{Q}, 2d_1) \times \dots \times K(\mathbf{Q}, 2d_r)$  and construct a map  $BK \rightarrow BM$  by levelwise taking the unit ring map  $\mathbf{Q} \rightarrow \mathbf{A}_f$ . Define  $BY$  as the homotopy pullback of the diagram  $BK \rightarrow BM \leftarrow \prod_p BX_p$ , and note that  $BY$  by construction is a CW-complex.

Since  $\mathbf{Q}$  and  $\hat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$  generate  $\mathbf{A}_f$  the Mayer-Vietoris sequence in homotopy groups corresponding to the homotopy pull-back in fact splits, so  $\pi_n(BY)$  is the pull-back in the category of groups of the diagram  $\pi_n(BK) \rightarrow \pi_n(BM) \leftarrow \prod_p \pi_n(BX_p)$ . Concretely, the homotopy groups of  $BY$  are given by

$$\pi_n(BY) = \left( \bigoplus_{i, 2d_i=n} \mathbf{Z} \right) \oplus \left( \bigoplus_p \mathrm{Tor}(\mathbf{Z}, \pi_n(BX_p)) \right).$$

In particular, this shows that  $\pi_n(\Omega BY)$  is finitely generated for all  $n$ . Hence also  $H^n(\Omega BY; \mathbf{Z})$  is finitely generated for all  $n$  (see [34][18, Thm. 2.16]).

By construction  $H^*(BY; \mathbf{Z}_p) \xrightarrow{\cong} H^*(BX_p; \mathbf{Z}_p)$  so, since the spaces involved are simply connected,  $H^*(\Omega BY; \mathbf{Z}_p) \xrightarrow{\cong} H^*(X_p; \mathbf{Z}_p)$  for all  $p$ . But since we have seen that each  $H^n(\Omega BY; \mathbf{Z}_p)$  is finitely generated and we know that  $X_p$  is homotopy equivalent to a product of spheres for all  $p > h$ , we conclude that in fact  $\bigoplus_n H^n(\Omega BY; \mathbf{Z})$  is finitely generated as an abelian group.

If  $\Omega BY$  is simply connected then it follows from the classical results of Wall [42, Thms. B and F] that  $\Omega BY$  is homotopy equivalent to a finite CW-complex  $Y$ . If  $\Omega BY$  is not simply connected then the conclusion still holds, now appealing to a more recent result of Notbohm [32] (see also [4]) which relies on  $\Omega BY$  being a loop space.  $\square$

*Proof of Theorem 1.1.* It follows directly from Lemma 1.2 that we can construct a connected, finite loop space  $Y$  with the listed properties. One can check directly by a finite search that the degrees of  $Y$  does not agree with those of a compact Lie group, but one can also argue more simply as follows. There are exactly 61 finite, irreducible  $\mathbf{Q}$ -reflection groups whose degrees are all at most 30. The inner product of the vector

$$(*) \quad (0, 2, -1, -1, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 1, -1, 0, 0, 0, 0, 0, 4, -1, -3, 1, -1, 0, 0, 0, 1)$$

and the degree vector of any of these is non-negative. Hence the same holds for any finite  $\mathbf{Q}$ -reflection group whose degrees are all at most 30. However the inner product of the vector  $(*)$  with the degree vector of  $Y$  equals  $-1$ , so the degrees of  $Y$  does not agree with those of a compact Lie group.

To check that our counterexample has minimal rank we proceed as follows. For any prime  $p$ , there are only finitely many finite  $\mathbf{Q}_p$ -reflection groups of a given rank, cf. [9]. Hence one can go through the list of (say) finite  $\mathbf{Q}_2$ -reflection groups and check which of these have degrees not matching those of a finite  $\mathbf{Q}$ -reflection group, but matching those of a finite  $\mathbf{Q}_p$ -reflection group for  $p = 3, 5, \dots$ . We have written a C++ program which implements this algorithm, and used it to check all finite  $\mathbf{Q}_2$ -reflection groups of rank less than 66.  $\square$

**Remark 2.2.** Note that Theorem 1.1 in particular tells us that there exists a loop space whose  $p$ -completion is not homotopy equivalent to the  $p$ -completion of a compact Lie group for any prime  $p$ . If we only want this to hold for a single prime  $p$  we can find much simpler examples. For instance using Lemma 1.2 one can construct a finite loop space which 2-completed is homotopy equivalent to  $X_2 = DI(4) \times \mathrm{Sp}(1)\hat{2} \times \mathrm{Sp}(6)\hat{2}$  and which  $p$ -completed for  $p \neq 2$  is homotopy equivalent to  $X_p = \mathrm{Sp}(3)\hat{p} \times \mathrm{Sp}(7)\hat{p}$ , where  $DI(4)$  denotes the 2-compact group constructed by Dwyer-Wilkerson [12] realizing the  $\mathbf{Q}_2$ -reflection group  $G_{24}$ . However,  $X_2$  is not homotopy equivalent to the 2-completion of a compact Lie group, since the only simply connected, compact Lie groups with the right rational degrees are  $\mathrm{Spin}(7) \times \mathrm{Spin}(15)$ ,  $\mathrm{Spin}(7) \times \mathrm{Sp}(7)$ ,  $\mathrm{Sp}(3) \times \mathrm{Spin}(15)$  and  $\mathrm{Sp}(3) \times \mathrm{Sp}(7)$ , but these do not have the same mod 2 Poincaré series as  $X_2$ , as can be obtained from [12]. In fact an easy check reveals that the mod 2 Poincaré series of  $X_2$  does not agree with that of any compact Lie group. (Compare [45, Conj. 2], [38, Prob. 9], [24, Conjs. B and C, p. 67].)

*Proof of Theorem 1.3.* The first claim follows from the proof of Theorem 1.1, see also Remark 2.3 below. To show the second claim note that the proof of [27, Thm. 1.1(b)] shows

that  $\bigcap_p K_p \subseteq K_{\text{Lin}}$  and hence

$$\bigcap_p K_p \subseteq K_2 \cap K_3 \cap K_5 \cap K_7 \cap K_{\text{Lin}}.$$

To prove the reverse inclusion, note that the only finite, irreducible  $\mathbf{Q}_3$ -reflection group which is not a  $\mathbf{Q}$ -reflection group is the group  $G_{12}$ . Since this is a  $\mathbf{Q}_p$ -reflection group for all primes  $p$  satisfying  $p \equiv 1, 3 \pmod{8}$  we get  $K_3 \subseteq K_p$  for these primes. Similarly the finite, irreducible  $\mathbf{Q}_5$ -reflection groups which are not  $\mathbf{Q}$ -reflection groups are precisely the groups  $G(4, 1, n), G(4, 2, n)$  for  $n \geq 2$ ,  $G(4, 4, n)$  for  $n \geq 3$ ,  $\mathbf{Z}/4$  from family 3,  $G_8, G_{29}$  and  $G_{31}$ . This shows that  $K_5 \subseteq K_p$  when  $p \equiv 1 \pmod{4}$ . In the same way we see that  $K_7 \subseteq K_p$  when  $p$  satisfies  $p \equiv 1 \pmod{6}$  and  $p \equiv \pm 1 \pmod{8}$ , i.e. when  $p \equiv 1, 7 \pmod{24}$ . Finally all the groups  $G(m, m, 2)$ ,  $m = 8, 12, 24$  are  $\mathbf{Q}_p$ -reflection groups when  $p \equiv \pm 1 \pmod{24}$ , so  $K_{\text{Lin}} \subseteq K_p$  for these primes. This proves the result since any prime  $p$  satisfies  $p = 2$ ,  $p \equiv 1, 3 \pmod{8}$ ,  $p \equiv 1 \pmod{4}$ ,  $p \equiv 1, 7 \pmod{24}$  or  $p \equiv \pm 1 \pmod{24}$ .  $\square$

**Remark 2.3.** A few remarks about how we found the counterexample in Theorem 1.1 might be in order, since this is not really clear from the proof. First we used Fukuda's cdd+ program [16] to establish Theorem 1.3 by showing that  $K_{\text{Lie}}$  truncated at degree say 30 does not agree with the intersections of the similarly truncated versions of  $K_2, K_3, K_5, K_7$ , and  $K_{\text{Lin}}$ . From this it is a linear programming problem to obtain a concrete point in the difference, and by solving the associated integer programming problem one gets a point in the difference which is minimal with respect to rank (or dimension). Note however that being minimal in this sense is slightly weaker than being a minimal counterexample, which is why we had to finish off our proof of minimality in Theorem 1.1 with a brute force check, which required rather massive computer calculations. Using the geometric picture we have found a counterexample of smaller dimension but larger rank. Namely, there exists a connected, finite loop space with rank 74 and dimension 1250 and degrees

$$\{2^9, 3^2, 4^7, 5^3, 6^8, 7^3, 8^8, 9^3, 10^6, 11^2, 12^6, 13^2, 14^5, 15, 16^3, 18^2, 20^2, 22, 24\},$$

which is not rationally equivalent to a compact Lie group. This is seen by considering the  $\mathbf{Q}_p$ -reflection groups

$$\begin{aligned} W_{A_1 A_1 E_6 D_7 D_8 A_9 D_{11} D_{13} A_{15}} \times G_{24} & \quad \text{for } p = 2, \\ W_{A_1 A_1 D_5 D_7 D_9 B_{10} B_{12} A_{13} A_{14}} \times G_{12} & \quad \text{for } p \equiv 1, 3 \pmod{8}, \\ W_{A_1 A_1 D_5 D_6 E_7 D_9 B_{11} A_{13} A_{14}} \times G(4, 4, 7) & \quad \text{for } p \equiv 5 \pmod{8}, \\ W_{G_2 D_5 D_7 D_9 B_{10} B_{12} A_{13} A_{14}} \times G(8, 8, 2) & \quad \text{for } p \equiv 7 \pmod{8}. \end{aligned}$$

It is also possible to construct a counterexample where all degrees are even, cf. [29]. For instance there is an example of rank 68, dimension 1468 and degrees

$$\{2^8, 4^8, 6^8, 8^8, 10^6, 12^5, 14^6, 16^4, 18^3, 20^3, 22^2, 24^3, 26^2, 28, 30\}.$$

Here one can use the  $\mathbf{Q}_p$ -reflection groups

$$\begin{aligned} W_{B_4 B_5 B_5 B_8 B_8 E_8 B_{13} B_{14}} \times G_{24} & \quad \text{for } p = 2, \\ W_{D_4 B_5 B_5 B_8 B_8 E_8 B_{14} D_{14}} \times G_{12} & \quad \text{for } p \equiv 1, 3 \pmod{8}, \\ W_{G_2 B_4 B_5 B_5 B_7 B_9 D_{14} B_{15}} \times G(4, 2, 7) & \quad \text{for } p \equiv 5 \pmod{8}, \\ W_{B_4 B_5 B_5 B_8 B_8 E_8 B_{14} D_{14}} \times G(6, 3, 2) & \quad \text{for } p \equiv 7 \pmod{24}, \\ W_{D_4 D_6 D_8 D_8 B_{10} D_{14} D_{16}} \times G(24, 24, 2) & \quad \text{for } p \equiv 23 \pmod{24}. \end{aligned}$$

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