CONTROLLED METHODS IN EQUIVARIANT TOPOLOGY, A SURVEY

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Abstract. The purpose of this paper is to discuss various applications of controlled methods to equivariant topology.

1. Introduction

In this survey we want to discuss various controlled methods in equivariant topology. We will start with an example from a paper written jointly with Steve Ferry[7] illustrating the power of controlled methods in an unequivariant setting. Next we discuss the simplest possible equivariant setting, namely semifree group actions on a sphere fixing a subsphere. The material in this section is taken from a paper written jointly with Douglas Anderson, [2]. In the third section we discuss how controlled methods sometimes can be used to settle the question of whether a group can act cocompactly freely on a sphere crossed with Euclidean space. The material in this section is taken from a joint work with Ian Hambleton, [9]. Finally we discuss how controlled methods are used to attack the problem of whether two representations can be equivariantly homeomorphic without being linearly isomorphic, also called the non–linear similarity problem. This problem is solved in a joint paper with Ian Hambleton, [8], in the sense of reducing to classical number theory, quite explicitly in the case when the group is cyclic of order 2^r.

2. Controlled classification of manifolds.

The surgery exact sequence provides a classification of smooth manifolds in dimensions bigger than 4 (and similarly PL and TOP manifolds) homotopy equivalent to a given compact manifold M

$$\cdots \to \mathcal{S}(M \times I, \partial(M \times I)) \to [\Sigma(M_+), F/O]_* \to \\ L^h_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(M) \to [M_+, F/O]_* \to L^h_n(\mathbb{Z}\pi)$$

and relative versions hereof. Here π is the fundamental group of M.

The classification is up to smooth h-cobordism. This means that an element in $\mathcal{S}(M)$ is a manifold N together with a homotopy equivalence of N to M. Two such elements, N_1 and N_2 are equivalent, if there is an h-cobordism between N_1 and N_2, and a map from the h-cobordism to M, extending the given maps on N_1 and N_2.

We may change the sequence by replacing the groups by $L^s_*(\mathbb{Z}\pi)$, requiring the homotopy equivalences to be simple homotopy equivalences, and replacing

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the h-cobordism above by s-cobordisms. Since a smooth s-cobordism is diffeomorphic to a product when the dimension of M is bigger than or equal to 5, we obtain a classification up to diffeomorphism in this case. In the topological and PL-categories there are similar results, basically by replacing F/O by F/PL and F/TOP respectively in the above sequence, and diffeomorphisms by PL-homeomorphisms and homeomorphisms respectively.

In the case of non-compact manifolds, it is less clear what kind of classification we should want. As an interesting example, let us consider $\mathbb{R}^n$. The most obvious choice, is classification up to proper homotopy equivalence. Such a theory was developed by Maumary and Taylor in [13, 19], and in the case of $\mathbb{R}^n$, the L-groups are trivial, so a proper homotopy equivalence of a manifold with $\mathbb{R}^n$ is proper homotopic to a homeomorphism. Note however that the higher structure sets, i.e. the classification of $\mathbb{R}^n \times D^m$ relative to the boundary, are isomorphic to the homotopy groups of F/O, hence non–trivial.

A less obvious way to try to study $\mathbb{R}^n$ is to note that it is also a metric space, and we can try to classify $\mathbb{R}^n$ up to bounded homotopy equivalence.

Definition 2.1. We say that $f : W \to \mathbb{R}^n$ is a *bounded homotopy equivalence* if there exist $g : \mathbb{R}^n \to W$ and homotopies $H : f \circ g \sim id$ and $K : g \circ f \sim id$ so that the sets $H(y \times I)$ and $f \circ K(x \times I)$ have uniformly bounded diameters .

Such a theory was developed in [7]. The basic ingredient is an algebraic criterion for bounded homotopy equivalence. In the case of compact manifolds one normally uses the criterion that the map induces an isomorphism on the fundamental group and a homology isomorphism on the universal cover. For our purposes it is better to think of this as saying that the map induces a chain homotopy equivalence of free $\mathbb{Z}\pi$-chain complexes, which is what leads to the $L^h_n(\mathbb{Z}\pi)$- and $L^s_n(\mathbb{Z}\pi)$-groups.

To get an algebraic criterion for bounded homotopy equivalence consider the following definition of a category $\mathcal{C}_n(\mathbb{Z})$.

Definition 2.2. An *object* A of $\mathcal{C}_n(\mathbb{Z})$ is a collection of finitely generated free abelian groups A_x, one for each $x \in \mathbb{R}^n$, such that for each ball $C \subset \mathbb{R}^n$ of finite radius, only finitely many A_x, $x \in C$, are nonzero. A *morphism* $\varphi : A \to B$ is a collection of morphisms $\varphi^x_y : A_x \to B_y$ such that there exists $k = k(\varphi)$ such that $\varphi^x_y = 0$ for $d(x, y) > k$.

The *composition* of $\varphi : A \to B$ and $\psi : B \to C$ is given by $(\psi \circ \varphi)^x_y = \sum_{z \in M} \psi^z_y \varphi^x_z$. The composition $(\psi \circ \varphi)$ satisfies the local finiteness and boundedness conditions whenever ψ and φ do.

In the definition above, it might be advantageous to think of finitely generated free abelian groups as finitely generated free $\mathbb{Z}$-modules, making it easy to replace $\mathbb{Z}$ by other rings in later sections.

Now let $f : W \to \mathbb{R}^n$ be a proper map. We may give W and $\mathbb{R}^n$ a cell decomposition in such a way, that there is a global bound on the diameter of the cells when measured in $\mathbb{R}^n$. The cellular chains may be considered chain complexes in the category $\mathcal{C}_n(\mathbb{Z})$, by assigning to each cell the point in $\mathbb{R}^n$ where the barycenter of the cell is mapped to, and remembering that the cellular chain complex has precisely one $\mathbb{Z}$ generator corresponding to each cell in the cell decomposition. The following is proved in [7].

Theorem 2.3. *The map $f : W \to \mathbb{R}^n$ is a bounded homotopy equivalence if and only if the induced map $f_\sharp : C_\sharp(W) \to C_\sharp(\mathbb{R}^n)$ is a chain homotopy equivalence in the category $\mathcal{C}_n(\mathbb{Z})$, and W is -1, 0 and 1-connected over $\mathbb{R}^n$.*

We refer the reader to the precise meaning of -1, 0, and 1-connected in the bounded sense. Intuitively -1-connected means the map is onto except for a uniformly bounded discrepancy, 0-connected means that point can be connected with a path within a certain diameter, and 1-connected means that a map from a circle can be extended to a disk without increasing the diameter of the image too much.

This leads to a surgery exact sequence

$$\cdots \to \mathcal{S}_b(\mathbb{R}^n \times I, \delta(\mathbb{R}^n \times I)) \to [\Sigma(\mathbb{R}^n_+), F/O]_* \to \\ L^h_{n+1}(\mathcal{C}_n(\mathbb{Z})) \to \mathcal{S}_b(\mathbb{R}^n) \to [\mathbb{R}^n_+, F/O]_* \to L^h_n(\mathcal{C}_n(\mathbb{Z}))$$

where the groups $L^h_{n+i}(\mathcal{C}_n(\mathbb{Z}))$ are defined completely algebraically by A. Ranicki in [18]. The groups $L^h_{n+i}(\mathcal{C}_n(\mathbb{Z}))$ are computable, computed by A. Ranicki in [18] and the result is

$$L^h_{n+i}(\mathcal{C}_n(\mathbb{Z})) \cong L^h_{n+i-1}(\mathcal{C}_{n-1}(\mathbb{Z})).$$

It follows that

$$L^h_{n+i}(\mathcal{C}_n(\mathbb{Z})) \cong L^h_i(\mathbb{Z}).$$

In particular we have that $L^h_{n+1}(\mathcal{C}_n(\mathbb{Z})) \cong L_1(\mathbb{Z}) = 0$. Since $\mathbb{R}^n$ is contractible and F/O is connected the term $[\mathbb{R}^n_+, F/O]_*$ is also trivial so we get

Theorem 2.4. *Let $f : W \to \mathbb{R}^n$ be a bounded homotopy equivalence, where W is a smooth manifold, $n \geq 5$. Then f is boundedly homotopic to a diffeomorphism.*

In this theorem we also use the bounded h-cobordism theorem [17] and the fact that $K_{-i}(\mathbb{Z}) = 0$ [3].

This result has some surprising immediate consequences. The first is a new proof of Kirby's annulus theorem.

Theorem 2.5. *An orientation preserving homeomorphism of $\mathbb{R}^n$ $n \geq 5$ is isotopic to the identity.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Then f is certainly a bounded homotopy equivalence (the tracks of the homotopies have diameter bounded by 0). Hence f is boundedly homotopic to a diffeomorphism g. Since f and g are boundedly homotopic, the distance $|g(x) - f(x)|$ is uniformly bounded, but then f and g are isotopic by an Alexander isotopy. It is easy to see that the orientation preserving diffeomorphism g is isotopic to the identity. Recall that an Alexander isotopy is constructed as follows: Identify $\mathbb{R}^n$ with the interior of a disk D^n by a radial homeomorphism. Under this identification $f \circ g^{-1}$ can be extended to the whole disk by using the identity on the boundary. We now construct a map

$$F : D^n \times I \to D^n \times I.$$

We start out letting the map be $f \circ g^{-1}$ on $D^n \times 0$, the identity on the rest of the boundary of $D^n \times I$ and sending $(0, \frac{1}{2})$ to $(0, \frac{1}{2})$. Since $D^n \times I$ can be thought of as the cone over its boundary with $(0, \frac{1}{2})$ as the cone point, we can extend the map to all of $D^n \times I$ in a conelike fashion. Noticing the I-levels are preserved by F, we have constructed an isotopy from $f \circ g^{-1}$ to the identity, a so-called Alexander isotopy. $\square$

So while the proper and bounded structure sets of $\mathbb{R}^n$ are both trivial, it is only in the bounded case we have the chance of using an Alexander isotopy.

Another immediate consequence is a new proof of Siebenmann's cell-like approximation theorem.

Theorem 2.6. *Let $f : M \to N$ be a cell-like map of manifolds of dimension ≥ 5. Then f can be arbitrarily closely approximated by a homeomorphism.*

Proof. Find embedded disks $D_\alpha \subset N$, in such a way that, denoting the interior of D_α by U_α, we have a locally finite covering $\{U_\alpha\}$ of N. Now identify U_α with $\mathbb{R}^n$ using a radial identification. The map $f : f^{-1}(U_\alpha) \to U_\alpha$ is cell-like so it is an arbitrarily small homotopy equivalence. That means that within a bounded distance there is a homeomorphism g. Replacing f by g on $f^{-1}(U_\alpha)$ and keeping f outside $f^{-1}(U_\alpha)$ is a continuous map because of the control conditions, so repeating this argument, we can replace f by a homeomorphism which can be made close to f by choosing the original covering from small D_α's. $\square$

This finishes the discussion of bounded control.

We shall now discuss a third way one might use to classify $\mathbb{R}^n$. Make a radial identification of $\mathbb{R}^n$ with the interior of D^n, $\mathbb{R}^n \subset D^n$. We now say that a homotopy equivalence $f : V \to \overset{\circ}{D}{}^n$ is "small at infinity" or "small near the boundary of D^n" if there is a homotopy inverse g, and homotopies G and H from $g \circ f$ to the identity and from $f \circ g$ to the identity, satisfying the condition that the tracks of the homotopies are small near the boundary of D^n, specifically (and technically) let $S = \{G(x \times I)\}_{x \in \overset{\circ}{D}{}^n} \cup \{f \circ H(y \times I)\}_{y \in V}$, then S has the property that for all x in the boundary of D^n and for every neighborhood U of x in D^n, there is a neighborhood V so that if $K \in S$ and $K \cap V \neq \emptyset$ then $K \subset U$. Such homotopy equivalences are said to be continuously controlled at the boundary of D^n. It is easy to see that this concept is independent of how $\mathbb{R}^n$ is identified with the interior of a disk, as long as it is done radially. An algebraic category testing this kind of homotopy equivalence is given as follows.

Definition 2.7. Consider the pair $(D^n, \partial D^n)$ of topological spaces. We define the category $\mathcal{B}(D^n, \partial D^n; \mathbb{Z})$ as follows: An *object* A is a collection $\{A_x\}_{x \in \overset{\circ}{D}{}^n}$ of finitely generated free abelian groups so that $\{x | A_x \neq 0\}$ is locally finite in $\overset{\circ}{D}{}^n$. A *morphism* $\phi : A \to B$ is a group homomorphism $\oplus A_x \to \oplus B_y$, satisfying a continuously controlled condition: For every $z \in \partial D^n$ and for every neighborhood U of z in D^n, there exists a neighborhood V of z in D^n such that $\phi_x^y = 0$ and $\phi_y^x = 0$ if $x \in V \cap \overset{\circ}{D}{}^n$ and $y \in \overset{\circ}{D}{}^n - U$.

Again it is not very difficult to see that this category determines whether a given map is a continuously controlled homotopy equivalence once certain conditions of a fundamental group nature are met. Similarly to the bounded surgery exact sequence, we obtain a continuously controlled surgery exact sequence.

$$\cdots \to \mathcal{S}_c(\mathbb{R}^n \times I, \delta(\mathbb{R}^n \times I)) \to [\Sigma(\mathbb{R}^n_+), F/O]_* \to \\ L^h_{n+1}(\mathcal{B}(D^n, \partial D^n; \mathbb{Z})) \to \mathcal{S}_c(\mathbb{R}^n) \to \\ [\mathbb{R}^n_+, F/O]_* \to L^h_n(\mathcal{B}(D^n, \partial D^n; \mathbb{Z}))$$

where the lower index c stands for continuous control. It is an unfortunate historical accident, for which the author of this survey is to blame, that $\mathcal{C}$ is used for bounded control, and $\mathcal{B}$ for continuous control.

We do not have any immediate applications for this simple kind of continuous control, except one could have made the applications above using continuous control rather than bounded control. The sequences are essentially the same, but it solves a number of technical problems to have both points of view at hand. To see that the sequences are essentially the same, notice that bounded control, where we do a radial compactification to a disk satisfies the continuous control condition, because sets that are bounded far out in $\mathbb{R}^n$ will certainly become small near the boundary. In this way radial compactification produces a map of surgery exact sequences

$$\begin{array}{ccccccc} L^h_{n+1}(\mathcal{C}_n(\mathbb{Z})) & \longrightarrow & \mathcal{S}_b(\mathbb{R}^n) & \to & [\mathbb{R}^n_+, F/O]_* & \longrightarrow & L^h_n(\mathcal{C}_n(\mathbb{Z})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L^h_{n+1}(\mathcal{B}(D^n, \partial D^n; \mathbb{Z})) & \to & \mathcal{S}_c(\mathbb{R}^n) & \to & [\mathbb{R}^n_+, F/O]_* & \to & L^h_n(\mathcal{B}(D^n, \partial D^n; \mathbb{Z})). \end{array}$$

Here the normal invariant term $[\mathbb{R}^n_+, F/O]_* \to [\mathbb{R}^n_+, F/O]_*$ is just the identity. The L-groups are computed in essentially the same way, [18], and going through the computation shows there is an isomorphism of L-groups, so hence the structure sets are also the same. This says in some roundabout way that it is always possible to parameterize a continuously controlled homotopy equivalence in such a way that it becomes a bounded homotopy equivalence. But that fact can be given a much easier and more direct proof. As we shall see in the next sections it is however very useful to pair off the continuously controlled and the boundedly controlled theories, since they are qualitatively different.

3. Semifree Group Action on a sphere

In this section we shall discuss a simple equivariant problem, where controlled methods are essential. It is also an example that nicely illustrates the interplay between bounded and continuously controlled techniques. The material in this section is mainly derived from [2], but in a somewhat more modern exposition. Since this is mostly interesting as a non-smooth and non-PL phenomenon, we shall consider everything in the topological category in this section.

Consider a group G acting semifreely on a sphere S^{n+k} with fixed set a standard subsphere S^{k-1}. The group G acts freely on $S^{n+k} - S^{k-1}$ which is homotopy equivalent to an n-sphere, so the space $X = (S^{n+k} - S^{k-1})/G$ has universal cover homotopy equivalent to an n-sphere. It follows from [6] that X is a finitely dominated complex, hence a so-called Swan complex. Thinking of

$$S^{n+k} = S^n * S^{k-1}$$

with S^{k-1} as the fixed sphere, we produce a map $\tilde{p}$ to $D^k = pt * S^{k-1}$, by sending S^n to a point. A gravity point construction

$$p(x) = \frac{1}{|G|}\sum_{g\in G}\tilde{p}(gx)$$

produces an equivariant map with respect to the trivial action on D^k. This of course uses the assumption that S^{k-1} is fixed under the action of G.

Denoting $(S^{n+k} - S^{k-1})/G$ by X as above, we have a map

$$f : (S^{n+k} - S^{k-1})/G \to X \times \overset{\circ}{D}{}^k$$

which is the identity on the first coordinate and p on the second coordinate. This map is a controlled homotopy equivalence in the following sense: f has a homotopy inverse g, and there are homotopies H from gf to the identity and K from fg to the identity satisfying a continuous control condition. The continuous control condition is measured in the k-disk. Specifically, let q be the projection on the second factor, $q : X \times \overset{\circ}{D}{}^k \to \overset{\circ}{D}{}^k$. We then require that $qfH(x \times I)$ and $qK(y \times I)$ are small near the boundary of the k-disk, in the same sense as in section one. The proof of this assertion is a bit complicated, and uses the full power of the work of Anderson and Munkholm, see [1].

Imitating the last section we see that this kind of homotopy equivalence is tested algebraically in the category $\mathcal{B}(D^k, \partial D^k; \mathbb{Z}G)$, and the same kind of arguments lead to a surgery exact sequence

$$L^h_{n+k+1}(\mathcal{B}(D^k, \partial D^k; \mathbb{Z}G)) \to \mathcal{S}_c(X \times \overset{\circ}{D}{}^k) \to [X_+, F/TOP]_* \to L^h_{n+k}(\mathcal{B}(D^k, \partial D^k; \mathbb{Z}G)).$$

The only difference from the continuously controlled sequence in the last section, is that the ring $\mathbb{Z}$ has been replaced by the group ring $\mathbb{Z}G$.

Theorem 3.1. *Let G act semifreely on S^{n+k} fixing a standard S^{k-1}, then*

$$(S^{n+k} - S^{k-1})/G \to X \times \overset{\circ}{D}{}^k$$

described above is an element in $\mathcal{S}_c(X \times \overset{\circ}{D}{}^k)$.

In particular this structure set is non-empty if there exists an action as described.

Just like in the last section there is a corresponding bounded theory. The bounded method was the one that was actually used in [2]. So consider a Swan complex X, i.e. a finitely dominated complex whose universal cover is homotopy equivalent to a sphere. We may ask the question: Is there a manifold W and a bounded homotopy equivalence $f : W \to X \times \mathbb{R}^k$, where the boundedness of the homotopy equivalence is measured in the $\mathbb{R}^k$ factor, more precisely, let q be the projection from $X \times \mathbb{R}^k$ to $\mathbb{R}^k$. We then say f is a bounded homotopy equivalence if f has a homotopy inverse g, and there are homotopies H from gf to the identity and K from fg to the identity, so that the diameters of $qfH(x \times I)$ and $qK(y \times I)$ are uniformly bounded. Algebraically this kind of bounded homotopy equivalence is detected by the map inducing a chain homotopy equivalence in the category

$\mathcal{C}_k(\mathbb{Z}G)$, the obvious variant of the category defined in the last section where we have free finitely generated $\mathbb{Z}G$-modules instead of finitely generated free abelian groups ($\mathbb{Z}$-modules). As usual we have a surgery exact sequence

$$\cdots \to \mathcal{S}_b(X \times \mathbb{R}^k \times I, X \times \delta(\mathbb{R}^k \times I)) \to$$
$$[\Sigma(X \times \mathbb{R}^k_+), F/TOP]_* \to L^h_{n+k+1}(\mathcal{C}_n(\mathbb{Z}G)) \to$$
$$\mathcal{S}_b(X \times \mathbb{R}^n) \to [X \times \mathbb{R}^n_+, F/TOP]_* \to L^h_{n+k}(\mathcal{C}_n(\mathbb{Z}G)).$$

The continuously controlled and the bounded surgery sequences are essentially the same, there is an obvious map from the bounded sequence to the continuously controlled sequence, just using the fact that something bounded in $\mathbb{R}^k$ when shrinked radially becomes small near the boundary. There is however a significant qualitative difference: There is an obvious suspension operation of semifree actions on spheres fixing a subsphere, but there is no obvious way to introduce a suspension in the continuously controlled surgery exact sequence. The problem is that the Poincaré complexes have to have some kind of given cell structure, and it is not clear how one would produce a functorial cell structure in the suspended version, but in the bounded version, crossing with $\mathbb{R}$ is a well defined operation, and working all this together, one may obtain the following result, which is essentially contained in [2].

Theorem 3.2. *Given a Swan complex X with fundamental group G, we then have that the simple bounded structure set $\mathcal{S}_b(X \times \mathbb{R}^k)$ is in one to one correspondence with homeomorphism classes of semifree G-actions on S^{n+k}, fixing S^{k-1}, where the correspondence is given by a radial completion. A semifree action is a suspension if and only if the corresponding element in the structure set is in the image of crossing with $\mathbb{R}$.*

Crossing with $\mathbb{R}$ kills torsion, so we get the following diagram of long exact sequences:

$$\begin{array}{ccccccc}
\to L^h(\mathcal{C}_{n+k+1}(\mathbb{Z}G)) & \longrightarrow & \mathcal{S}^h_b(X \times \mathbb{R}^k) & \longrightarrow & [X \times \mathbb{R}^k_+, F/TOP] \to \\
\downarrow & & \downarrow & & \downarrow \\
\to L^s(\mathcal{C}_{n+k+2}(\mathbb{Z}G)) & \longrightarrow & \mathcal{S}^s_b(X \times \mathbb{R}^{k+1}) & \longrightarrow & [X \times \mathbb{R}^k_+, F/TOP] \to
\end{array}$$

where the vertical arrows are induced by crossing with the reals. The map

$$[X \times \mathbb{R}^k, F/TOP] \to [X \times \mathbb{R}^{k+1}, F/TOP]$$

is obviously an isomorphism and so is (less obviously) the map

$$L^h(\mathcal{C}_{n+k+1}(\mathbb{Z}G)) \to L^s(\mathcal{C}_{n+k+2}(\mathbb{Z}G)).$$

It follows that the map $\mathcal{S}^h_b(X \times \mathbb{R}^k) \to \mathcal{S}^s_b(X \times \mathbb{R}^{k+1})$ is an isomorphism. From this it is fairly easy to conclude the following theorem

Theorem 3.3. *A semifree action of G on S^{n+k+1} fixing a standard subsphere S^k is a suspension if and only if the corresponding element in $\mathcal{S}^h_b(X \times \mathbb{R}^{k+1})$ actually lives in $\mathcal{S}^s_b(X \times \mathbb{R}^{k+1})$, i.e. if a certain torsion invariant vanishes.*

This torsion invariant lives in $K_1(\mathcal{C}_{k+1}(\mathbb{Z}G))$ which by [16] is isomorphic to $K_{-k}(\mathbb{Z}G)$. Since these groups are 0 for $k \geq 2$ it follows that a semifree group action of the type considered here is always equivariantly homeomorphic to a suspension of an action where the fixed set is a circle. There are obstructions to desuspending any further and these obstructions are realized in concrete examples.

4. Cocompact Group actions on a Sphere crossed with Euclidean Space

The material in this section is taken from [9]. In this section we want to discuss a question due to Wall. It follows from Milnor's theorem, [14] that a dihedral group does not act freely on a sphere. It obviously follows from this, that a group acting freely on a sphere can not have a dihedral subgroup. Wall asked whether it might be true, that a group acting freely cocompactly on a sphere crossed with Euclidean space can not have a dihedral subgroup. Without the cocompactness condition, there is no hope for this: take e.g. a Swan complex X with dihedral fundamental group. After crossing with a circle, it has O finiteness obstruction, and it is known that the Spivak fibration does lift to BO. Crossing with (D^3, S^2) removes any potential surgery obstruction, and this is seen to produce a free action of the dihedral group on a sphere crossed with $\mathbb{R}^4$.

T. Farrell suggested to the authors of [9] that an interesting case to study would be $D_p \times_\alpha \mathbb{Z}^k$, where α is some integral representation.

Theorem 4.1. *The group $D_p \times_\alpha \mathbb{Z}^k$ acts freely and properly discontinuously on $S^n \times \mathbb{R}^m$ for some n, m with compact quotient if and only if $n \equiv 3 \mod 4$, $m = k$ and α considered as a real representation has at least two $\mathbb{R}_-$-factors.*

It is relatively easy to see that n has to be equivalent to 3 mod 4 by a homology of groups argument, and also that m has to be equal to k. The existence part of this theorem is a classical surgery argument. It is done by doing the surgery in a kind of "blocked" way, to avoid having to deal with groups whose surgery groups are not known. The non-existence part is by far the most difficult part of the theorem. This is the part that uses controlled topology. We shall outline the general ideas translating the problem to a problem in controlled surgery here. So assume $G = D_p \times_\alpha \mathbb{Z}^k$ acts freely, cocompactly on $S^n \times \mathbb{R}^k$. Before we enter into the argument we need the concept of almost equivariant maps: Given a map f from a space to a metric space, where both spaces come exhibited with a group action. We say the map is "almost equivariant" if the distance between $f(gx)$ and $gf(x)$ is uniformly bounded.

The first step in the argument goes as follows:

Theorem 4.2. *Suppose that $D_p \times_\alpha \mathbb{Z}^k$ acts freely, cocompactly and properly discontinuously on $S^n \times \mathbb{R}^k$. Then there is a topological action of D_p on S^{n+k}, which is given by the representation α on a standardly embedded S^{k-1}, and away from this S^{k-1} may be equivariantly identified with the restriction of $D_p \times_\alpha \mathbb{Z}^k$ to D_p acting on $S^n \times \mathbb{R}^k$, hence is free off S^{k-1}.*

Proof. Consider the manifold $M = (S^n \times \mathbb{R}^k)/\mathbb{Z}^k$ and the map $M \longrightarrow T^k$ given by classification. Up to homotopy this is a spherical fibration. We replace this spherical fibration by a block fibration $\bar{M} \to T^k$, so $\bar{M}$ is compact. We have a homotopy commutative diagram

$$\begin{array}{ccc} \bar{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ T^k & \longrightarrow & T^k \end{array}$$

and $\bar{M} \to M$ is a homotopy equivalence. The universal cover of $\bar{M}$ is a block fibration over $\mathbb{R}^k$, so it is block (and hence boundedly) equivalent to the trivial block fibration $S^n \times \mathbb{R}^k \to \mathbb{R}^k$. We thus have a bounded homotopy equivalence $f : \widetilde{M} \to S^n \times \mathbb{R}^k$ with respect to the projection on $\mathbb{R}^k$. It is easy to see this map is almost $\mathbb{Z}^k$-equivariant with respect to the standard action on the second factor of $S^n \times \mathbb{R}^k$. Using the bounded surgery theory [7], it is easy to see that there is only one element in the bounded structure set of $S^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$. Hence there is a bounded homotopy of f to a homeomorphism h. On $\widetilde{M}$, we have the free action of $D_p \times_\alpha \mathbb{Z}^k$, and we want to consider the conjugate action by h on $S^n \times \mathbb{R}^k$. First we notice this action restricted to $\mathbb{Z}^k$ makes the projection p to $\mathbb{R}^k$ almost equivariant. To see this consider

$$d(phzh^{-1}x, zpx) \leq d(ph(zh^{-1}x), pf(zh^{-1}x)) + d(pfzh^{-1}x, zpx).$$

The first term is bounded since f is boundedly homotopic to h. The second term is a bounded distance from $d(zpfh^{-1}x, zpx)$ since f and p are almost equivariant with respect to the standard action on the second factor. Since z is an isometry of $\mathbb{R}^k$, this is equal to $d(pfh^{-1}x, px)$ which is bounded since f is boundedly homotopic to h. We now consider this conjugate action on $S^n \times \mathbb{R}^k$, and we want to show the projection to $\mathbb{R}^k$ is almost equivariant with respect to the $D_p \times_\alpha \mathbb{Z}^k$-action obtained by letting D_p act on $\mathbb{R}^k$ by the representation. Choose $U \subseteq S^n \times \mathbb{R}^k$, compact so that $\bigcup_{z \in \mathbb{Z}^k} z \cdot U = S^n \times \mathbb{R}^k$. Replacing U by $\bigcup_{g \in D_p} g \cdot U$ we may assume U to be D_p-invariant. Note that since D_p is finite and U is compact $\bigcup_{g \in D_p} g \cdot p(U)$ must have finite diameter. Consider $x \in S^n \times \mathbb{R}^k$ and $g \in D_p$. By the choice of U, there is $u \in U$ and $z \in \mathbb{Z}$, so that $x = z \cdot u$. Now

$$\begin{aligned} d(pgx, gpx) &= d(pgzu, gpzu) \\ &\leq d(pgzu, gzg^{-1}pgu) + d(gzg^{-1}pgu, gzpu) + d(gzpu, gpzu). \end{aligned}$$

The first term is $d(pgzu, gzg^{-1}pgu) = d(p(gzg^{-1})gu, gzg^{-1}pgu)$, and this is already shown to be bounded since $gzg^{-1} \in \mathbb{Z}^k$. The second term is

$$d(gzg^{-1}pgu, gzpu) = d(gzg^{-1}pgu, gzg^{-1}gpu) = d(pgu, gpu)$$

since $gzg^{-1} \in \mathbb{Z}^k$ acts as isometry on $\mathbb{R}^k$, but gpu and pgu both belong to $\bigcup gp(U)$ which has finite diameter. Finally the third term is $d(gzpu, gpzu) = d(zpu, pzu)$ since g acts by isometry, and this term is bounded since $z \in \mathbb{Z}^k$. We are now ready to construct the action of D_p on S^{n+k}. Identifying $S^{n+k} - S^{k-1}$ with $S^n \times \mathbb{R}^k$ using the join lines, one sees that a homeomorphism bounded in the $\mathbb{R}^k$-factor will be small in S^{n+k} when we approach S^{k-1}, hence defining the action on S^{n+k} by the representation on S^{k-1} and using the above mentioned identification with $S^n \times \mathbb{R}^k$ away from S^{k-1} produces a continuous action on S^{n+k}. □

This essentially brings us to the continuously controlled situation of the last chapters. The gist of the argument is that D_p acts on the space pretty similarly to the way it acts on $\mathbb{Z}^k$ up to a compact discrepancy, and this allows the completion of the action with S^{k-1} being acted on by the representation. Radially identifying $\mathbb{R}^k$ with $\overset{\circ}{D}{}^k$ we see there is an element in the continuously controlled structure set $\mathcal{S}^h_c(S^n \times \overset{\circ}{D}{}^k)$ which has elements continuously controlled (with respect to the $\overset{\circ}{D}{}^k$-factor D_p-equivariant homotopy equivalence. The thing that is really different here is that the group is acting non-trivially on the space where we measure the control, and hence everything has to be expressed equivariantly rather than in the quotient space. As in the last section, we can also construct a bounded structure set $\mathcal{S}^h_b(S^n \times \mathbb{R}^k)$ where the elements are equivariant bounded homotopy equivalences of a manifold to $S^n \times \mathbb{R}^k$, and arguing with the surgery exact sequences as before we find that the continuously controlled and the bounded structure sets are isomorphic, by a radial identification. The appropriate definition of a category is a little bit more complicated. The main complication being, that in order to make the algebraic surgery theory work the way it is supposed to, we need an appropriate involution on the category to express Poincaré duality. This is no problem in the previous chapters, where we had a trivial action on the metric space. We need the following definition: Given a metric space E and a group acting on E by isometries, and a commutative ring with unit R, we define a category $\mathcal{C}^G_E(R)$ as follows:

Definition 4.3. An *object* A is a free left $R(G)$-module together with a map $f : A \longrightarrow F(M)$, where $F(M)$ is the set of finite subsets of M, satisfying

- (i) f is G-equivariant,
- (ii) $A_x = \{a \in A | f(a) \subseteq \{x\}\}$ is a finitely generated free sub R-module,
- (iii) as an R-module $A = \bigoplus_{x \in M} A_x$,
- (iv) $f(a + b) \subseteq f(a) \cup f(b)$, and
- (v) for each ball $B \subset M$, $\{x \in B | A_x \neq 0\}$ is finite.

A *morphism* $\phi : A \longrightarrow B$ is a morphism of RG-modules, satisfying the following condition: there exists k so that the components $\phi^m_n : A_m \longrightarrow B_n$ (which are R-module morphisms) are zero when $d(m, n) > k$. $\mathcal{C}^G_E(R)$ is an additive category in an obvious way.

In the case under consideration, when E is $\mathbb{R}^k$ with D_p acting, we shall denote this category by $\mathcal{C}^{D_p}_k(\mathbb{Z})$.

To summarize the discussion above this definition we have the following: If there is a free, cocompact action of $D_p \times_\alpha \mathbb{Z}^k$ on $S^n \times \mathbb{R}^k$, then the bounded equivariant structure set $\mathcal{S}^h_b(S^n \times \mathbb{R}^k)$ is non-empty. Still letting X denote the space $S^n \times \mathbb{R}^k/G$ consider the following diagram, where the upper row is the classical surgery exact sequence of the Swan complex X, and the lower row is the bounded equivariant surgery exact sequence for the product of $\tilde{X}$ and $\mathbb{R}^k$ with diagonal action. The vertical map is induced by crossing with $\mathbb{R}^k$ as a D_p

representation:

$$\begin{array}{ccccc} \to L^h(\mathbb{Z}D_p) & \longrightarrow & \mathcal{S}^h(X) & \longrightarrow & [X, F/TOP] \to \\ \downarrow & & \downarrow & & \downarrow \\ \to L^h(\mathcal{C}_k^{D_p}(\mathbb{Z})) & \longrightarrow & \mathcal{S}_b^{h,D_p}(\tilde{X} \times \mathbb{R}^k) & \longrightarrow & [(\tilde{X} \times \mathbb{R}^k)/D_p, F/TOP] \to \end{array} .$$

The structure of the argument now goes as follows: If the bounded equivariant structure set $\mathcal{S}_b^{h,D_p}(\tilde{X} \times \mathbb{R}^k)$ in the lower row is non-empty, then consider its normal invariant in $[(\tilde{X} \times \mathbb{R}^k)/D_p, F/TOP)]$. This normal invariant obviously comes uniquely from the upper row, $[X, F/TOP]$, since twisted product with $\mathbb{R}^k$ is a homotopy equivalence, and if that term maps to 0 in the Wall-group, it would have to come from $\mathcal{S}^h(X)$, which would mean that X is homotopy equivalent to a manifold, and thus that the dihedral group acts freely on a sphere contradicting Milnor's theorem. This means that the normal invariant goes non-trivially into the Wall group $L^h(\mathbb{Z}G)$, and the contradiction is now obtained by computing the transfer map at the Wall group level, and proving, that except for the case where α considered as a real representation has 2 one-dimensional nontrivial summands, this transfer is non-trivial, and hence contradicts the existence of an element in the bounded equivariant structure set. The precise argument is beyond the scope of this survey, and we refer the reader to [9].

5. HOMEOMORPHIC REPRESENTATIONS

In this section again let us assume that G is a finite group. Let U_1 and U_2 be representations of G and assume there is an equivariant linear homeomorphism from U_1 to U_2. We want to consider the classical question of determining under what conditions U_1 and U_2 have to be linearly isomorphic. It is a celebrated result of Cappell and Shaneson that U_1 and U_2 do not always have to be linearly isomorphic. There is a very simple test for linear isomorphism coming from classical representation theory. The character of the representation is the map from the group to the reals given by taking the trace of the group element as a linear transformation of vector spaces. It is well known that representations are isomorphic if and only if they have the same characters. This obviously means that U_1 and U_2 are linearly isomorphic if they are linearly isomorphic when restricted to cyclic subgroups, so if there are to be homeomorphic representations that are not linearly isomorphic, then it will have to happen for cyclic groups as well. So from now on we shall assume that G is a cyclic group. The representations U_i may be written as the direct sum $V_i \oplus W_i$ where V_i is free and W_i has no free summands. If H is a proper subgroup of G, we have that the H-fixed set of U_i is the same as the H-fixed set of W_i. The given homeomorphism will induce a homeomorphism of W_1^H to W_2^H as a G/H representation, hence as a G-representation. Now determine W_2' so that $W_2' \oplus W_2^H \cong W_2$. We now have $V_1 \oplus W_1$ is equivariantly homeomorphic to $V_2 \oplus W_2' \oplus W_2^H$ which in turn is homeomorphic to $V_2 \oplus W_2' \oplus W_1^H$. Repeating this procedure we see that we have $V_1 \oplus W_1$ equivariantly homeomorphic to $V_2 \oplus W_1$ or in other words, we only need to consider the situation $W_1 = W_2$ when studying homeomorphic representations. So from now on we shall consider $V_1 \oplus W$ homeomorphic to $V_2 \oplus W$ where V_i are free representations, and W has no free summands. An easy general position argument shows, denoting the unit spheres of the representation V_i by $S(V_i)$, that $S(V_1)$ is equivariantly homotopy equivalent to $S(V_2)$. This thus represents the first necessary condition for homeomorphic representation. The homotopy theory of this situation is extremely simple, the homotopy type is given by one single k-invariant which has to be a unit in the cyclic group G.

Now assume that we have $S(V_1)$ equivariantly homotopy equivalent to $S(V_2)$. This means that $S(V_2)/G$ is an element in the homotopy structure set of $S(V_1)/G$. Now consider crossing the equivariant homotopy equivalence $S(V_2) \to S(V_1)$ by W. We are interested in deforming this in a bounded way, measured in W, to a homeomorphism. Once again, we have to find a suitable category $\mathcal{C}_W(\mathbb{Z}G)$ which measures bounded equivariant homotopy equivalence, and we can then set up an equivariant bounded surgery exact sequence for this situation. The category will consist of free $\mathbb{Z}G$ modules that are equivariantly parameterized by W in line with the earlier sections, and equivariant bounded morphisms.

As in the earlier sections, crossing with W produces a map from the structure set $\mathcal{S}^h(S(V_1)/G)$ to the bounded equivariant structure set $\mathcal{S}_b^{h,G}(S(V_1 \times W))$ where the boundedness is measured in the W factor. Assume this map is 0 on some given element, we may then cross with a copy of the reals to kill torsion, so we obtain that $S(V_1) \times W \times \mathbb{R}$ can be moved a bounded amount measured in the $W \times \mathbb{R}$ to an equivariant homeomorphism. Crossing with the reals will be an isomorphism on the normal invariant term, and on the L-group term as in the earlier sections.

The join of the homotopy equivalence from $S(V_2)$ to $S(V_1)$ and the identity produces an equivariant map from $S(V_2) * S(W \oplus \mathbb{R})$ to $S(V_1) * S(W \oplus \mathbb{R})$. Identifying $S(V_i) * S(W \oplus \mathbb{R}) - S(W \oplus \mathbb{R})$ with $S(V_i) \times W \times \mathbb{R}$ the above discussion shows that we can move the map a bounded amount measured in $W \oplus \mathbb{R}$ to obtain an equivariant homeomorphism, but that means that filling in with the identity on $S(W \oplus \mathbb{R})$ produces an equivariant homeomorphism of $S(V_1) * S(W \oplus \mathbb{R})$ with $S(V_2) * S(W \oplus \mathbb{R})$. Seeing that $S(V_i) * S(W \oplus \mathbb{R})$ is the same as $S(V_i \oplus W \oplus \mathbb{R})$ which in turn is the same as the one-point compactification of $V_i \oplus W$, and that the homeomorphism constructed has an obvious fixed point, we obtain that $V_2 \oplus W$ is equivariantly homeomorphic to $V_1 \oplus W$. It turns out this is the only way non-linear similarity can happen for cyclic groups, as expressed in the following theorem from [8].

Theorem 5.1. *The representations $V_1 \oplus W$ and $V_2 \oplus W$ are equivariantly homeomorphic if and only if $S(V_2)/G$ and $S(V_1)/G$ are equivariantly homotopy equivalent and the element defined by a homotopy equivalence transfered by W to $\mathcal{S}_b^{h,G}(S(V_1) \times W$ is 0. This is the only way homeomorphic representations of cyclic groups can occur.*

We have pretty much given the proof of one direction of this theorem. The other direction consists of an equivariant engulfing argument, and we shall not give the argument here. Just like the previous section, we have a bounded surgery exact sequence parameterized equivariant by W with G acting, and this kind of homotopy equivalence is algebraically determined by $\mathcal{C}_W^G(\mathbb{Z})$ which was defined in the previous section. We once again get a surgery exact sequence. This sequence fits into the diagram below as the lower row:

$$\begin{array}{ccccc}
\to L^h(\mathbb{Z}G) & \longrightarrow & \mathcal{S}^h(S(V_1)/G) & \longrightarrow & [S(V_1)/G, F/TOP] \to \\
\downarrow & & \downarrow & & \downarrow \\
\to L^h(\mathcal{C}^G_W(\mathbb{Z})) & \longrightarrow & \mathcal{S}^{h,G}_b(S(V_1)\times W) & \longrightarrow & [S(V_1)\times W/G, F/TOP] \to \ .
\end{array}$$

The vertical maps are induced by crossing with W. On the normal invariant term $[S(V_1)/G, F/TOP] \to [S(V_1)\times W/G, F/TOP]$ we obviously have an isomorphism, and this gives rise to the next necessary condition for $V_1 \times W$ and $V_2 \times W$ to be equivariantly homeomorphic, namely that $S(V_1)/G$ and $S(V_2)/G$ not only be homotopy equivalent as discussed above, but the homotopy equivalence has to have trivial normal invariant. Once these necessary conditions are fulfilled, we see from this diagram that the question of homeomorphic representations of cyclic group representations has been translated completely to algebra, if somewhat unusual and complicated algebra. This algebra has been studied in [8], and as a sample result we mention one of the the main results from [8], which is somewhat elaborate even to state.

We need to first introduce some notation. There are no homeomorphic representations when the order of the group is odd or twice odd. This was shown in [11, 12], but it can also be shown using the methods indicated in this section [10]. We thus let G be a cyclic group of order $4q$, where $q > 1$, and let H denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t\rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbb{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbb{R}_-$.

A *free* G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$ denote the complex numbers $\mathbb{C}$ with action $t\cdot z = \zeta^a z$ for all $z \in \mathbb{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \mod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The *Reidemeister torsion invariant* of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k}(t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\}\ .$$

Let $V_2 = t^{b_1}+\cdots+t^{b_k}$ be another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent. This just means that we have the same k-invariant so the products of the weights are congruent: $\prod a_i \equiv \prod b_i \mod 4q$. Then the Whitehead torsion of any G-homotopy equivalence is determined by the element

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$$

since $\mathrm{Wh}(\mathbb{Z}G) \to \mathrm{Wh}(\mathbb{Q}G)$ is monic [15, p.14]. When there exists a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ which is normally cobordant to the identity map on $S(V_1)$, we say that $S(V_1)$ and $S(V_2)$ are *normally cobordant*. More generally, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant* if $S(V_1 \oplus U)/G$ and

$S(V_2 \oplus U)/G$ are normally cobordant for all free G–representations U. It is a necessary condition for non-linear similarity between $V_1 \oplus W$ and $V_2 \oplus W$ that that $S(V_1)$ and $S(V_2)$ are s–normally cobordant. It can decided by explicit congruences in the weights whether two free representations are s–normally cobordant [20, Thm. 1.2].

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. It represents a unit in the group ring $\mathbb{Z}G$, explicitly described for $G = C(2^r)$ by Cappell and Shaneson in [5, §1] using a pull-back square of rings. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbb{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbb{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbb{Z}/2; A)$ of $\mathbb{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbb{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbb{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod(t^{a_i}-1)\prod((-t)^{a_i}-1)}{\prod(t^{b_i}-1)\prod((-t)^{b_i}-1)} = \prod \frac{(t^2)^{a_i}-1}{((t^2)^{b_i}-1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbb{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbb{Z}G^-)/\mathrm{Wh}(\mathbb{Z}H)) .$$

This calculation takes place over the ring $\Lambda_{2q} = \mathbb{Z}[t]/(1 + t^2 + \cdots + t^{4q-2})$, but the result holds over $\mathbb{Z}G$ via the involution–invariant pull-back square

$$\begin{array}{ccc} \mathbb{Z}G & \rightarrow & \Lambda_{2q} \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}/2] & \rightarrow & \mathbb{Z}/2q[\mathbb{Z}/2]. \end{array}$$

Consider the exact sequence of modules with involution:

$$K_1(\mathbb{Z}H) \to K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}H \to \mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Z}G) \tag{1}$$

and define $\mathrm{Wh}(\mathbb{Z}H \to \mathbb{Z}G) = K_1(\mathbb{Z}H \to \mathbb{Z}G)/\{\pm G\}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbb{Z}G)/\mathrm{Wh}(\mathbb{Z}H) \to \mathrm{Wh}(\mathbb{Z}H \to \mathbb{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Z}G))$. Such an exact sequence of $\mathbb{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta \colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbb{Z}G^-)/\mathrm{Wh}(\mathbb{Z}H)) .$$

Our first result deals with isotropy groups of index 2, as is the case for the non–linear similarities constructed in [4].

Theorem 5.2. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, with $a_i \equiv b_i \equiv 1 \mod 4$. There exists a topological similarity $V_1 \oplus \mathbb{R}_- \sim_t V_2 \oplus \mathbb{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \mod 4q$,
(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*
(iii) *the element* $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbb{Z}G^-)/\mathrm{Wh}(\mathbb{Z}H))$ *is in the image of the coboundary* $\delta \colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbb{Z}G^-)/\mathrm{Wh}(\mathbb{Z}H))$.

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