A NONCONNECTIVE DELOOPING OF ALGEBRAIC K-THEORY

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ABSTRACT. Given a ring R, it is known that the topological space $BGl(R)^+$ is an infinite loop space. One way to construct an infinite loop structure is to consider the category $\underline{\underline{F}}$ of free R-modules, or rather its classifying space $B\underline{\underline{F}}$, as food for suitable infinite loop space machines. These machines produce connective spectra whose zeroth space is $(B\underline{\underline{F}})^+ = \mathbb{Z} \times BGl(R)^+$. In this paper we consider categories $\underline{\underline{C}}_1(\underline{\underline{F}}) = \underline{\underline{F}}, \underline{\underline{C}}_1(\underline{\underline{F}}), \dots$ of parameterized free modules and bounded homomorphisms and show that the spaces $(B\underline{\underline{C}}_0)^+ = (B\underline{\underline{F}})^+$, $(B\underline{\underline{C}}_1)^+, \dots$ are the connected components of a nonconnective Ω-spectrum $B\underline{\underline{C}}(F)$ with $K_i B\underline{\underline{C}}(F) = K_i(R)$ even for negative i.

0. INTRODUCTION

Given a ring R, let $\underline{\underline{F}}$ be the category of finitely generated R-modules and isomorphisms. Form the "group completion" category $\underline{\underline{F}}^{-1}\underline{\underline{F}}$ of $\underline{\underline{F}}$ (see [5]); it is known that its classifying space $B\underline{\underline{F}}^{-1}\underline{\underline{F}}$ is the algebraic K-theory space $BGl(R)^+ \times \mathbb{Z}$. The purpose of this paper is to produce a nonconnective delooping of $BGl(R)^+ \times K_0(R)$ by using a parameterized version $\underline{\underline{C}}_0(\underline{\underline{F}}) = \underline{\underline{F}}, \underline{\underline{C}}_1(\underline{\underline{F}}), \dots$ of $\underline{\underline{F}}$ given in [11]. Our main result is this:

Theorem A. Write B_i for the classifying space of the category $\underline{\underline{C}}^{-1}\underline{\underline{C}}$, except that $B_0 = BGl(R)^+$. Then the spaces B_i are connected, and for $i \geq 0$ we have

$$\Omega B_{i+1} = B_i \times K_{-i}(R).$$

Thus the sequence of spaces $\widehat{B}_i = B_i \times K_{-i}(R)$ forms a nonconnective Ω-spectrum $\widehat{\underline{\underline{B}}}$ with homotopy groups

$$\pi_i(\widehat{\underline{\underline{B}}}) = K_i(R), \qquad i \text{ any integer.}$$

In particular, the negative homotopy groups of $\widehat{\underline{\underline{B}}}$ are the negative K-groups of Bass [2].

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Actually, we work in the generality of a small additive category $\mathcal{A}$, rather than just the additive category $\mathcal{F}$ of finitely generated free R-modules. For example, one could take $\mathcal{P}$, the category of finitely generated projective R-modules. The category $\mathcal{P}$ is the idempotent completion of $\mathcal{F}$, and we recover the same spectrum $\widehat{\underline{\underline{B}}}$ if we replace $\mathcal{F}$ by $\mathcal{P}$. Note that $B\underline{\underline{P}}^{-1}\underline{\underline{P}}$ is $BGl(R)^+ \times K_0(R)$, where $\underline{\underline{P}}$ is the category of isomorphisms in $\mathcal{P}$.

Given $\mathcal{A}$, we consider the additive category $\mathcal{C}_i(\mathcal{A})$ of $\mathbb{Z}^i$-graded objects and bounded homomorphisms (see section 1 for details). If $\mathcal{A} = \mathcal{F}$ this definition specializes to the categories $\mathcal{C}_i$ of [11]. Let $\widehat{\mathcal{C}_i}$ be the idempotent completion of $\mathcal{C}_i(\mathcal{A})$, and let $\underline{\underline{A}}$, $\underline{\underline{C}}_i$, $\widehat{\underline{\underline{C}}}$ be the sub-categories of isomorphisms in $\mathcal{A}$, $\mathcal{C}_i$ and $\widehat{\mathcal{C}_i}$, respectively. Our second result is this

Theorem B. Write $\widehat{B}_i$ for the classifying space of the category $\widehat{\underline{\underline{C}}}_i^{-1}\widehat{\underline{\underline{C}}}_i$ and B_i for the classifying space of $\underline{\underline{C}}_i^{-1}\underline{\underline{C}}_i$. Then

$$\Omega\widehat{B}_{i+1} = \widehat{B}_i$$
$$\Omega^i\widehat{B}_i = \widehat{B}_0 = \text{“group completion” } (B\underline{\underline{A}})^+ \text{ of } B\underline{\underline{A}}.$$

The connected component of $\widehat{B}_i$ is B_i (except for $i = 0$), and the sequence of spaces $\widehat{B}_0, \widehat{B}_1, \ldots$ is a nonconnective Ω-spectrum. In particular, $\widehat{B}_i$ is an i-fold delooping of $(B\underline{\underline{A}})^+$.

The outline of this paper is as follows. In section 1 we give the definitions of the $\mathbb{Z}^i$-graded category $\mathcal{C}_i(\mathcal{A})$. In section 2, we recall the passage from categories to spectra, and review the main points of Thomason's paper [13] that we need. In section 3, we prove Theorems A and B.

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1. The categories $\mathcal{C}_i$

In this section we give the definition of the categories $\mathcal{C}_i(\mathcal{A})$ associated to a small additive category $\mathcal{A}$. We also review the notions of filtered additive categories and of the idempotent completion of $\mathcal{A}$ for the convenience of the reader.

1.1. **Definition.** An additive category is said to be filtered if there is an increasing filtration

$$F_0(A,B) \subseteq F_1(A,B) \subseteq \ldots \subseteq F_n(A,B) \subseteq \ldots$$

on $\hom(A,B)$ for every pair of objects A, B of $\mathcal{A}$. Each $F_n(A,B)$ is to be a subgroup of $\hom(A,B)$ and we must have $\cup F_n(A,B) = \hom(A,B)$. We require 0_A and 1_A to be in $F_0(A,B)$, and assume that the composition of morphisms in $F_m(A,B)$ and $F_n(A,B)$ belongs to $F_{m+n}(A,C)$. We also assume that the projections $A \oplus B \to A$, and inclusions $A \to A \oplus B$ and coherence isomorphisms all belong to F_0. If ϕ is in $F_d(A,B)$ we say that ϕ has filtration degree d.

The reason for concerning ourselves with filtered categories is that the categories $\mathcal{C}_i$ come with a natural filtration. Of course every additive category has a trivial filtration, obtained by setting $F_0(A,B) = \hom(A,B)$.

1.1.1. **Example.** Given a $\mathbb{Z}$-graded ring A such as $R[t,t^{-1}]$, let $\mathcal{A}$ be the category of graded A-modules. We can filter $\mathcal{A}$ by legislating that homogeneous maps of degree $\pm d$ have filtration degree d.

We now give our definition of the filtered category $\mathcal{C}_i$. Let the distance between points $J = (j_1, \dots, j_i)$ and $K = (k_1, \dots, k_i)$ in $\mathbb{Z}^i$ be given by

$$||J - K|| = \max_s |j_s - k_s|.$$

1.2. **Definition.** Let $\mathcal{A}$ be a (filtered) additive category. We define $\mathcal{C}_i(\mathcal{A})$ to be the category of $\mathbb{Z}^i$-graded objects and bounded homomorphisms. This means that an object A of $\mathcal{C}_i$ is a collection of objects $A(J)$ in $\mathcal{A}$, one for each J in $\mathbb{Z}^i$. A morphism $\phi : A \to B$ in $\mathcal{C}_i$ of filtration degree d is a collection

$$\phi(J,K) : A(J) \to B(K)$$

of $\mathcal{A}$-morphisms, where we require $\phi(J,K) = 0$ unless $||J - K|| \le d$. If $\mathcal{A}$ is filtered, we also require each $\phi(J,K)$ to have filtration $\le d$. Composition of $\phi : A \to B$ with $\psi : B \to C$ is defined by

$$(\psi \circ \phi)(J,L) = \sum_K \psi(K,L) \circ \phi(J,K).$$

Note that composition is well-defined because only finitely many elements in this sum are different from 0. It is easily seen that $\mathcal{C}_0(\mathcal{A}) = \mathcal{A}$.

1.2.1. **Example.** If $\mathcal{F}$ is the category of finitely generated free R-modules (with trivial filtration), the category $\mathcal{C}_i(\mathcal{F})$ is the same as the category $\mathcal{C}_i(R)$ constructed in [11]. In that paper it was proven that

$$K_1(\mathcal{C}_{i+1}(R)) = K_{-i}(R), \qquad i \ge 0.$$

This indicated that $\mathcal{C}_{i+1}$ might be a delooping of K-theory, and was the original motivation for this paper. That it cannot be exactly the case follows from (1.3.1) below.

1.2.2. **Example.** Since $\mathcal{C}_i(\mathcal{A})$ is filtered, we can iterate the construction. It is easy to see that

$$\mathcal{C}_i(\mathcal{C}_j(\mathcal{A})) = \mathcal{C}_{i+j}(\mathcal{A}).$$

However, if we forget the filtrations on $\mathcal{C}_j(\mathcal{A})$ this is no longer the case.

1.2.3. *Remark.* If V is any metric space, we can define a category $\mathcal{C}_V(\mathcal{A})$ in a way generalizing the case $V = \mathbb{Z}^i$. An object A of $\mathcal{C}_V$ is a collection of objects $A(v)$, one for each v in V, subject to the following constraint: for every $d > 0$ and v, $A(w) \neq 0$ for only finitely many w of distance less than d from v. Morphisms are defined as for $\mathcal{C}_i$. It is easy to see that if $V = \mathbb{R}^i$ then $\mathcal{C}_V$ is naturally equivalent to its subcategory $\mathcal{C}_i$. This shows that the difference between $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ is the rate of growth of the number $n(d, J)$ of points K within a distance of d from J.

1.2.4. **Example.** If we take $V = (0, 1, 2, \ldots)$ then we will let $\mathcal{C}_+(\mathcal{A})$ denote $\mathcal{C}_V(\mathcal{A})$. This is the full subcategory of $\mathcal{C}_1(\mathcal{A})$ whose objects satisfy $A(j) = 0$ for $j < 0$. Similarly, if we take $V = (0, -1, -2, \ldots)$, we will write $\mathcal{C}_-(\mathcal{A})$ for $\mathcal{C}_V(\mathcal{A})$. We can identify $\mathcal{C}_+(\mathcal{A}) \cap \mathcal{C}_-(\mathcal{A})$ with $\mathcal{A}$ in the obvious way.

There is a shift functor $T : \mathcal{C}_1(\mathcal{A}) \to \mathcal{C}_1(\mathcal{A})$ sending A to TA with $TA(j) = A(j-1)$, and T restricts to an endofunctor of $\mathcal{C}_+(\mathcal{A})$. There is an obvious natural isomorphism t from A to TA in both $\mathcal{C}_1$ and $\mathcal{C}_+$. We include the following result here for expositional purposes, and will generalize it in section 3 below.

1.3. **Lemma.** *Every object in $\mathcal{C}_+(\mathcal{A})$ is stably isomorphic to 0. In particular, the Grothendieck group $K_0(\mathcal{C}_+)$ is zero.*

Proof. Given A in $\mathcal{C}_+$, let $B = \sum T^n A$. That is, $B(j) = A(j) \oplus A(j-1) \oplus \ldots \oplus A(0)$. It is clear that $A \oplus TA = B$. The result follows from the observation that $t : B \cong TB$ is an isomorphisms in $\mathcal{C}_+(\mathcal{A})$. □

1.3.1. **Corollary.** If $i \neq 0$ then every object of $\mathcal{C}_i(\mathcal{A})$ is stably isomorphic to 0. In particular, $K_0(\mathcal{C}_i) = 0$.

Proof. By (1.2.2) we can assume that $i = 1$. But every object of $\mathcal{C}_1$ can be written $A_+ \oplus A_-$ with A_+ in $\mathcal{C}_+$ and A_- in $\mathcal{C}_-$. Hence $K_0(\mathcal{C}_1)$ is a quotient of $K_0(\mathcal{C}_+) \oplus K_0(\mathcal{C}_-) = 0$. □

1.4. **Definition.** (see, e. g., [3, p. 61]). Let $\mathcal{A}$ be an additive category. The <u>idempotent completion</u> $\widehat{\mathcal{A}}$ of $\mathcal{A}$ has as objects all morphisms $p : A \to A$ from $\mathcal{A}$ satisfying $p^2 = p$. An $\widehat{\mathcal{A}}$-morphism from p_1 to p_2 is an $\mathcal{A}$-morphism ϕ from the domain A_1 of p_1 to the domain A_2 of p_2 satisfying that $\phi = p_2 \phi p_1$. It is easily seen that $\widehat{\mathcal{A}}$ is an additive category and that $\hom(p_1, p_2)$ is a subgroup of $\hom(A_1, A_2)$. Hence $\widehat{\mathcal{A}}$ inherits any filtered structure that $\mathcal{A}$ might have. There is a full embedding $\mathcal{A}$ in $\widehat{\mathcal{A}}$ sending A to 1_A; if this is an equivalence of categories, we say that $\mathcal{A}$ is <u>idempotent complete</u>.

1.4.1. **Example.** The idempotent completion of the category $\mathcal{F}$ of free R-modules is equivalent to the category $\mathcal{P}$ of projective R-modules.

1.4.2. **Lemma.** The categories $\mathcal{A}$ and $\mathcal{C}_i(\mathcal{A})$ are cofinal in their idempotent completions $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{C}_i}(\mathcal{A})$. Moreover, $\mathcal{C}_i(\mathcal{A})$ is cofinal in $\mathcal{C}_i(\widehat{\mathcal{A}})$.

Proof. This is an easy computation. For example, if p is an object of $\mathcal{C}_i(\widehat{\mathcal{A}})$, define q by $q(J) = 1 - p(J)$. Then $p \oplus q$ belongs to $\mathcal{C}_i(\mathcal{A})$. □

To compute the K-theory of $\mathcal{A}$, we need to know which sequences are "exact": a different embedding of $\mathcal{A}$ in an ambient Abelian category will result in a different family of short exact sequences (see [12]). In particular, we cannot talk about $K_1(\mathcal{C}_i(\mathcal{A}))$ unless we know which sequences in $\mathcal{C}_i$ are "exact". It is not clear what the notion of "exact" should be, unless either (a) all exact sequences in $\mathcal{A}$ split (we insist the same is true of $\mathcal{C}_i$), or (b) $\mathcal{A}$ is embedded in an Abelian category $\widetilde{\mathcal{A}}$ closed under countably infinite direct sum (for then $\mathcal{C}_i$ is embeddable in $\widetilde{\mathcal{A}}$. In either case, it follows from (1.4.2) and Theorem 1.1 of [6] that

$$K_n(\mathcal{C}_i(\mathcal{A})) = K_n\mathcal{C}_i(\widehat{\mathcal{A}}) = K_n(\widehat{\mathcal{C}}_i(\mathcal{A})), \qquad n \geq 1.$$

Note that our proofs of theorem A and B only to situation (a).

1.5. **Example.** Let p_- be the idempotent natural transformation in $\mathcal{C}_1(\mathcal{A})$ given by

$$(p_-)_A : A \to A, \qquad p_-(j,k) = \begin{cases} 1 & \text{if } j = k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given an object A of $\mathcal{A}$, let A_- denote the image of p_- on the constant object $A(j) = A$ of $\mathcal{C}_1(\mathcal{A})$. Thus $A_-(j) = 0$ if $j > 0$ and $A_-(j) = A$ if $j \leq 0$. The map t is an endomorphism of the constant object $A \cong TA$; write s for the restriction of p_-t to A_-. Then $1 - s : A_- \to A_-$ is both a monomorphism and an epimorphism in $\mathcal{C}_1(\mathcal{A})$, but not an isomorphism. This is because the "inverse" $\sum s^n$ is not bounded. In particular, $\mathcal{C}_1(\mathcal{A})$ can never be an Abelian category, even if $\mathcal{A}$ is.

We conclude this section with the following result, which provides motivation for our Theorem B. It is also a consequence of Theorem B. Since we will not use this result, we merely sketch the proof.

1.6. **Proposition.** *If all short exact sequences in $\mathcal{A}$ split, then $K_1(\mathcal{C}_{i+1}(\mathcal{A})) = K_0(\widehat{\mathcal{C}}_i(\mathcal{A}))$. In particular, $K_1\mathcal{C}_1(\mathcal{A}) = K_0(\widehat{\mathcal{A}})$.*

Sketch of proof. This is proven in section 1 of [11], modulo terminology.

First of all we can assume that $\mathcal{A}$ is idempotent complete and that $i = 0$ by (1.4.2) and (1.2.2). The map from $K_0(\mathcal{A})$ to $K_1\mathcal{C}_1(\mathcal{A})$ sends the object A of $\mathcal{A}$ to the shift automorphism of the constant object $A(j) = A$ of $\mathcal{C}_1(\mathcal{A})$. The map $\phi : K_1(\mathcal{C}_1) \to K_0(\mathcal{A})$ is defined by sending the class of $\alpha \in \mathrm{Aut}(A)$ to the difference (for $d \gg 0$) in $K_0(\mathcal{A})$:

$$\phi(\alpha) = [(\alpha p_- \alpha^{-1})(\bigoplus_{j=-2d}^{2d} A(j))] - [p_-(\bigoplus_{j=-2d}^{2d} A(j))].$$

If α has filtration degree less than d, one shows as in [11, 1.11] that this map ϕ is well-defined and independent of d. Clearly the composition is the identity on $K_0(\mathcal{A})$. The proof of [11, (1.20)] applies to show that ϕ is monic, which proves the proposition. □

1.6.1. **Example.** Again, let $\mathcal{F}$ be the category of finitely generated free R-modules. Then for $i \geq 1$ we have $K_0\mathcal{C}_i(R) = 0$ but $K_0\widehat{\mathcal{C}_i}(R) = K_1\mathcal{C}_{i+1}(R) = K_{-i}(R)$.

Note. Example (1.6.1) follows from [11], not from (1.6).

2. The passage to topology

In this section we recall various results on the passage from the categories $\mathcal{A}$, $\mathcal{C}_i$ etc. to infinite loop spaces and spectra. We also recall Thomason's simplified double mapping cylinder from section 5 of [13]. We urge the reader to consult [13] for more details.

A symmetric monoidal category $\underline{\underline{S}}$ is a category together with a functor $\oplus : \underline{\underline{S}} \times \underline{\underline{S}} \to \underline{\underline{S}}$ and natural isomorphisms

$$\begin{aligned} \alpha &: (A \oplus B) \oplus C \cong A \oplus (B \oplus C) \\ \gamma &: A \oplus B \cong B \oplus A. \end{aligned}$$

These natural isomorphism are subject to coherence conditions that certain diagrams commute. We refer he reader to [10] for a more detailed definition, contending ourselves with:

2.1. **Example.** If $\mathcal{A}$ is an additive category then $\mathcal{A}$ is a symmetric monoidal category under $\oplus$ = direct sum. The subcategory $\underline{\underline{A}}$ of the isomorphisms in $\mathcal{A}$ is also symmetric monoidal under $\oplus$ = direct sum. It follows that $\mathcal{C}_i(\mathcal{A})$ and its category $\underline{\underline{C}}_i(\mathcal{A})$ of isomorphisms are also symmetric monoidal.

There is a functor Spt from the category of small symmetric monoidal categories to the category of connective Ω-spectra (i. e. sequences of spaces X_n with X_n being $(n-1)$-connected and with $X_n = \Omega X_{n+1}$). This functor satisfies

(a) A functor $\underline{\underline{A}} \to \underline{\underline{B}}$ preserving $\oplus$ up to coherent natural transformation, a "lax" functor, induces a map $\mathrm{Spt}(\underline{\underline{A}}) \to \mathrm{Spt}(\underline{\underline{B}})$ of infinite loop spectra.

(b) The zeroth space $\mathrm{Spt}_0(\underline{\underline{A}})$ is the "group completion" of $B\underline{\underline{A}}$, the classifying space of the category $\underline{\underline{A}}$.

The construction of Spt is basically due to May and Segal, and Spt is unique up to homotopy equivalence. See [1]. One description of Spt may be found in the Appendix of [13].

2.2. **Lemma.** *Suppose that* $\underline{\underline{A}} \to \underline{\underline{B}}$ *is a lax functor of small symmetric monoidal categories, and that* $B\underline{\underline{A}} \to B\underline{\underline{B}}$ *is a homotopy equivalence of topological spaces. Then* $\mathrm{Spt}_0(\underline{\underline{A}}) \to \mathrm{Spt}_0(\underline{\underline{B}})$ *is a homotopy equivalence.*

Proof. See (2.3) of [13]. □

2.3. **Lemma.** *Suppose that* $\underline{\underline{A}}$ *is a full, cofinal subcategory of the small symmetric monoidal category* $\underline{\underline{B}}$*. Then the connected components of* $\mathrm{Spt}_0(\underline{\underline{A}})$ *and* $\mathrm{Spt}_0(\underline{\underline{B}})$ *are homotopy equivalent.*

Proof. This is well-known. The point is that

$$\begin{aligned} H_*[\mathrm{Spt}_0(\underline{\underline{A}})_0] &= \operatorname*{Colim}_{A\in\underline{\underline{A}}} H_*B\,\mathrm{Aut}(A) \\ &= \operatorname*{Colim}_{B\in\underline{\underline{B}}} H_*B\,\mathrm{Aut}(B) \\ &= H_*[\mathrm{Spt}_0(\underline{\underline{B}})_0]. \end{aligned}$$

□

2.4. **Lemma.** (Quillen). *Let* $\underline{\underline{S}}$ *be a small symmetric monoidal category in which all morphisms are isomorphisms, and assume that all translation* $S\oplus : \underline{\underline{S}} \to \underline{\underline{S}}$ *are faithful. Then there is a category* $\underline{\underline{S}}\,\underline{\underline{S}}^{-1}$ *whose objects are pairs* (S_1, S_2) *of objects in* $\underline{\underline{S}}$*, such that* $B\underline{\underline{S}}\,\underline{\underline{S}}^{-1}$ *is homotopy equivalent to* $\mathrm{Spt}_0(\underline{\underline{S}})$*.*

Proof. See [5, p. 221] or p. 1657 of [13]. □

2.4.1. **Corollary.** If $\mathcal{A}$ is a small additive category, let $\underline{\underline{A}}$ denote the category of isomorphisms in $\mathcal{A}$. Then $B\underline{\underline{A}}\,\underline{\underline{A}}^{-1}$ is homotopy equivalent to $\mathrm{Spt}_0(\underline{\underline{A}})$.

2.4.2. **Example.** Let R be a ring for which $R^m \cong R^n$ implies that $m = n$, and let $\underline{\underline{F}}$ be the category of finitely generated free R-modules and isomorphisms. The basepoint component of $\underline{\underline{F}}^{-1}\underline{\underline{F}}$ has objects $R^m = (R^m, R^m)$ and

$$\hom(R^m, R^{m+n}) = Gl_{m+n} \times_{Gl_n(R)} Gl_{m+n}(R).$$

In particular, $\hom(0, R^m)$ is $Gl_m(R)$. The family of the $\hom(0, R^m)$ gives a map from $BGl(R)$ to the basepoint component $BGl^+(R)$ of $B\underline{\underline{F}}^{-1}\underline{\underline{F}}$

The main ingredient in the proof of Theorem B is the simplified mapping cylinder construction of R. W. Thomason, described in (5.1) of [13]. Let $\underline{\underline{A}}$ be a symmetric monoidal category with all morphisms isomorphisms and $u : \underline{\underline{A}} \to \underline{\underline{B}}$, $v : \underline{\underline{A}} \to \underline{\underline{C}}$ strong functors of symmetric monoidal categories (i. e. functors preserving direct sum up to natural isomorphism). Define $\underline{\underline{P}} = (\underline{\underline{P}}(\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, u, v)$ to be the category with objects triples (B, A, C) with A an object of $\underline{\underline{A}}$, B of $\underline{\underline{B}}$, and C of $\underline{\underline{C}}$. A morphism $(B, A, C) \to (B', A', C')$ is a 5-tuple $(\psi, \psi_1, \psi_2, U.V)$ where U, V are objects of $\mathcal{A}$, $\psi : A \cong U \oplus A' \oplus V$, $\psi_1 : B \oplus uU \to B'$ and $\psi_2 : C \oplus vV \to C'$. U and V may be varied up to isomorphism. Composition of $(\psi, \psi_1, \psi_2, U, V) : (B, A, C) \to (B', A', C')$ with $(\overline{\psi}, \overline{\psi}_1, \overline{\psi}_2, \overline{U}.\overline{V}) : (B', A', C') \to (B'', A'', C'')$ is given by

$$A \cong U \oplus A' \oplus V \cong (U \oplus \overline{U}) \oplus A'' \oplus (\overline{V} \oplus V)$$

$$B \oplus u(U \oplus \overline{U}) \cong (B \oplus uU) \oplus u\overline{U}) \to B' \oplus u\overline{U} \to B''$$

$$v(\overline{V} \oplus V) \oplus C \cong v\overline{V} \oplus vV \oplus C \to v\overline{V} \oplus C' \to C''$$

and direct sum in $\underline{\underline{P}}$ is induced by direct sum in $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{C}}$. We then have

2.5**. Theorem.** R. W. Thomason [13, (5.2)]. *Up to homotopy the diagram*

$$\begin{array}{ccc} \mathrm{Spt}_0\, \underline{\underline{A}} & \longrightarrow & \mathrm{Spt}_0\, \underline{\underline{B}} \\ \downarrow & & \downarrow \\ \mathrm{Spt}_0\, \underline{\underline{C}} & \longrightarrow & \mathrm{Spt}\, \underline{\underline{P}}_0 \end{array}$$

is a pullback diagram

3. The proof of Theorem A and B

In this section we prove Theorems A and B. We make the standing assumption that $\mathcal{A}$ is a small filtered additive category and that $\underline{\underline{A}}$ is the (symmetric monoidal) category of isomorphisms of $\mathcal{A}$. Similarly we write $\underline{\underline{C}}_i$, $\underline{\underline{C}}_+$, and $\underline{\underline{C}}_-$ for the categories of isomorphisms of $\mathcal{C}_i(\mathcal{A})$, $\mathcal{C}_+(\mathcal{A})$ and $\mathcal{C}_-(\mathcal{A})$. The idea is to show that the diagram

$$\begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_+ \\ \downarrow & & \downarrow \\ \underline{\underline{C}}_- & \longrightarrow & \underline{\underline{C}}_1 \end{array}$$

induces a pullback diagram of spectra, and use the following result:

3.1. **Proposition.** $\mathrm{Spt}_0(\underline{\underline{C}}_+)$ *and* $\mathrm{Spt}_0(\underline{\underline{C}}_-)$ *are contractible.*

Proof. By symmetry it is enough to consider $\underline{\underline{C}}_+$. Recall from the discussion before (1.3) that there is a shift functor $T : \underline{\underline{C}}_+ \to \underline{\underline{C}}_+$ and a natural transformation t from A to TA. The category $\underline{\underline{C}}_+$ has an endofunctor $\sum_{n=0}^{\infty} T^n$ with

$$(\sum_{n=0}^{\infty} T^n)A(j) = \bigoplus_{n=0}^{j} A(j-n).$$

(Recall that $A(j) = 0$ for $j < 0$.) We can define $\sum_{n=1}^{\infty} T^n$ similarly. The natural isomorphism t induces a natural isomorphism t from $\sum_{n=0}^{\infty} T^n A$ to $\sum_{n=1}^{\infty} T^n A$. But as endofunctors of $\underline{\underline{C}}_+$ we have $1 \oplus \sum_{n=1}^{\infty} T^n \cong \sum_{n=0}^{\infty} T^n$. Hence as self maps of the H-space $B\underline{\underline{C}}_+$ we have

$$1 \sim (\sum_{n=0}^{\infty} T^n) - (\sum_{n=1}^{\infty} T^n) \overset{t}{\sim} 0.$$

This shows that B is contractible. But then $\mathrm{Spt}(\underline{\underline{C}}_+)$ is contractible by Lemma (2.2). □

Proof that Theorem B implies Theorem A. Write $\widehat{B}_i$ for $\mathrm{Spt}_0(\underline{\underline{\widehat{C}}}_i)$. Since we have $\pi_0(\widehat{B}_i) = K_{-i}(R)$ by (1.6.1) and since translations are faithful in $\underline{\underline{\widehat{C}}}_i$, it follows that $\widehat{B}_i$ is homotopy equivalent to $B_i \times K_{-i}(R)$. Since $\Omega B_i = \Omega \widehat{B}_i$, the result is now immediate. □

We now begin the proof of theorem B by making a series of reductions. Since

$$\pi_0(B_i) = \pi_0 \,\mathrm{Spt}_0(\underline{\underline{A}}_i) = K_0(\underline{\underline{A}}_i),$$

connectedness of the B_i for $i \neq 0$ follows from (1.3.1). Noq $\underline{\underline{C}}_i$ is full and cofinal in $\underline{\underline{C}}_i$ by (1.4.2), so by (2.3) the connected space $B_i = \mathrm{Spt}(\underline{\underline{C}}_i)$. By construction (or by (2.4.1)), $\widehat{B}_0 = \mathrm{Spt}_0(\underline{\underline{\widehat{A}}})$ is the group completion of $B\underline{\underline{\widehat{A}}}$. Thus the proof of Theorem B is reduced to showing that $\Omega \widehat{B}_{i+1} = \widehat{B}_i$ for $i \geq 0$.

Next , observe that $\widehat{\mathcal{C}}_{i+1}(\underline{\underline{A}}) = \widehat{\mathcal{C}}_1 \widehat{\mathcal{C}}_i(\underline{\underline{A}})$, so that $\widehat{B}_{i+1} = \mathrm{Spt}_0(\underline{\underline{\widehat{C}}}_1(\widehat{\mathcal{C}}_i(\mathcal{A}))$ and $\widehat{B}_i = \mathrm{Spt}_o(\widehat{\mathcal{C}}_i(\mathcal{A}))$.

Since we can replace $\mathcal{A}$ by $\widehat{\mathcal{C}}_i(\mathcal{A})$, it is enough to prove that $\Omega \widehat{B}_{i+1} = \widehat{B}_0 = \operatorname{Spt}(\underline{\underline{\widehat{A}}})$. There is also no loss in generality in assuming that $\underline{\underline{A}}$ is idempotent complete, since

$$\Omega \widehat{B}_1 = \Omega \operatorname{Spt}_0(\widehat{\mathcal{C}}_1(\mathcal{A})) = \Omega \operatorname{Spt}_0(\underline{\underline{\widehat{C}}}_1(\widehat{\mathcal{A}}))$$

by (2.3). In fact by (2.3) we also have

$$\Omega \operatorname{Spt}_0(\underline{\underline{\widehat{C}}}_1) = \Omega \operatorname{Spt}_0(\underline{\underline{C}}_1).$$

Therefore, Theorem B will follow from:

3.2. Theorem. *Let $\mathcal{A}$ be a small, filtered additive category which is idempotent complete. Then $\Omega \operatorname{Spt}(\underline{\underline{C}}_1)$ is homotopy equivalent to $\operatorname{Spt}_0(\underline{\underline{A}})$.*

3.3. Lemma. *Let $\mathcal{A}$ be a small filtered additive category. Recall that $\underline{\underline{C}}_+$ and $\underline{\underline{C}}_-$ are subcategories of $\underline{\underline{C}}_+$ whose intersection is $\underline{\underline{A}}$. Let $\underline{\underline{P}}$ be the simplified double mapping cylinder construction applied to $\underline{\underline{A}} \to \underline{\underline{C}}_-$ and $\underline{\underline{A}} \to \underline{\underline{C}}_+$. Then $\Omega \operatorname{Spt}_0(\underline{\underline{P}})$ is homotopy equivalent to $\operatorname{Spt}_0(\underline{\underline{A}})$.*

Proof. This is immediate from Thomason's Theorem (2.5), since by (3.1) the spaces $\operatorname{Spt}_0(\underline{\underline{C}}_+)$ and $\operatorname{Spt}_0(\underline{\underline{C}}_-)$ are contractible. □

By the universal mapping property of $\underline{\underline{P}}$ (see p. 1648 of [13]), there is a strong symmetric monoidal functor $\Sigma : \underline{\underline{P}} \to \underline{\underline{C}}_1$. This functor is defined on objects by

$$\Sigma(A^-, A, A^+) = A^- \oplus A \oplus A^+$$

where A^-, A, A^+ are objects of $\underline{\underline{C}}_+$, $\underline{\underline{A}}$ and $\underline{\underline{C}}_-$, respectively. A morphism $(\psi^-, \psi, \psi^+, U^-, U^+)$ is $\underline{\underline{P}}$ from (A^-, A, A^+) to (B^-, B, B^+) is sent by Σ to the composite

$$A^- \oplus A \oplus A^+ \xrightarrow{1\oplus\psi\oplus 1} A^- \oplus U^- \oplus A \oplus U^+ \oplus A^+ \xrightarrow{\psi^-\oplus 1\oplus\psi^+} B^- \oplus B \oplus B^+.$$

3.4. Theorem. *Let $\mathcal{A}$ be idempotent complete, and let $\underline{\underline{P}}$ be the double mapping cylinder of Lemma (3.3). Then the functor $\Sigma : \underline{\underline{P}} \to \underline{\underline{C}}_1$ induces a homotopy equivalence between the classifying spaces $B\underline{\underline{P}}$ and $B\underline{\underline{C}}_1$.*

Note that Theorem (3.4) immediately implies Theorem (3.2) by (3.3) and (2.2). Thus we have reduced the proof of Theorem B to the proof of Theorem (3.4).

Proof. We will show that this functor satisfies the conditions of Quillen's Theorem A from [12]. Fix an object Y of $\underline{\underline{C}}_1$; we need to show that $Y \downarrow \Sigma$ is a contractible category. To do this, we use the bound d for $\mathcal{C}_1(\underline{\underline{A}})$ to filter $Y \downarrow \Sigma$ as the increasing union of sub-categories Fil_d, and show that each Fil_d has an initial object $*_d$. Therefore Fil_d is contractible; their union $Y \downarrow \Sigma$ must also be contractible by standard topology.

The category Fil_d is the full subcategory of all $\alpha : Y \to \Sigma(A^-, A, A^+)$ where both α and α^{-1} are bounded by d. Define Y_d, Y_d^- and Y_d^+ in $\underline{\underline{A}}$, $\underline{\underline{C}}_-$ and $\underline{\underline{C}}_+$ respectively by setting

$$Y_d = Y(-d) \oplus \ldots \oplus Y(d) \text{ in } \underline{\underline{A}}$$

$$Y_d^- = Y(j) \text{ if } j < -d, \text{ and } = 0 \text{ otherwise}$$
$$Y_d^+ = Y(j) \text{ if } j > -d, \text{ and } = 0 \text{ otherwise.}$$

The obvious isomorphism $\sigma : Y \cong Y_d^- \oplus Y_d \oplus Y_d^+$ in $\underline{\underline{C}}_1$ is bounded by d, and forms the object $*_d : Y \to \Sigma(Y_d^-, Y_d, Y_d^+)$ of Fil_d. We will show that $*_d$ is an initial object of Fil_d.

Given an object $\alpha : Y \to \sum(A^-, A, A^+)$, we have to show that there is a unique morphism

$$\eta = (\psi, \psi^-, \psi^+, e_-(Y_d), e_+(Y_d)) : (Y_d^-, Y_d, Y_d^+) \to (A^-, A, A^+)$$

in $\underline{\underline{P}}$ so that $\Sigma(\eta) = \alpha\sigma^{-1}$ in $\underline{\underline{C}}_1$. Let pr_-, pr, pr_+ be the projections of $\Sigma(A^-, A, A^+)$ onto A^-, A and A^+, respectively. Since α^{-1} is bounded by d, $\alpha^{-1}(A)$ is contained in Y_d, or rather in the image $\sigma^{-1}(Y_d)$ of Y_d in Y. Hence it makes sense to let e be $\sigma\alpha^{-1}(\mathrm{pr})\alpha\sigma^{-1}$ restricted to Y_d, and it is clear that e is an idempotent of Y_d. Similarly $\sigma\alpha^{-1}(A^-)$ is contained in $Y_d^- \oplus Y_d$, and $\alpha^{-1}(A^+)$ is contained in $Y_d \oplus Y_d^+$. Let e_- and e_+ be $\sigma\alpha^{-1}(\mathrm{pr}_-)\alpha\sigma^{-1}$ and $\sigma\alpha^{-1}(\mathrm{pr}_+)\alpha\sigma^{-1}$ restricted to Y_d. These maps are also idempotents of Y_d, and it is easy to see that $e_- + e + e_+ = 1$. Since $\mathcal{A}$ is idempotent complete, the composition

$$Y_d \cong e_-(Y_d) \oplus e(Y_d) \oplus e_+(Y_d)$$

makes sense in $\mathcal{A}$. Define ψ to be the composite

$$Y_d \cong e_-(Y_d) \oplus e(Y_d) \oplus e_+(Y_d) \xrightarrow{1\oplus\alpha\oplus 1} e_-(Y_d) \oplus A \oplus e_+(Y_d)$$

Similarly, define maps

$$\psi^- : Y_d^- \oplus e_-(Y_d) \xrightarrow{\alpha\sigma^{-1}} A^- \text{ in } \underline{\underline{C}}_-$$

$$\psi^+ : e_+(Y_d) \oplus Y_d^+ \xrightarrow{\alpha\sigma^{-1}} A^+ \text{ in } \underline{\underline{C}}_+.$$

This completes the definition of the map $\eta : (Y_d^-, Y_d, Y_D^+) \to (A^-, A, A^+)$ in $\underline{\underline{P}}$. By definition of Σ we have $\Sigma(\eta) = \alpha\sigma^{-1}$. Because all maps in $\underline{\underline{A}}$, $\underline{\underline{C}}_-$ and $\underline{\underline{C}}_+$ are isomorphisms, it is an easy task to verify that η is the unique map with $\Sigma(\eta) = \alpha\sigma^{-1}$. It follows that $*_d$ is an initial object of Fil_d □

4. An overview

To place our construction in perspective, it is appropriate to review a little history. The definition of the functors $K_{-i}(R)$ was given by Bass [2] in 1966 during an attempt to formalize his decomposition of $K_1(R[t_1, t_1^{-1}, \dots t_n, t_n^{-1}])$. In 1967, Karoubi [8] gave another definition of $K_{-i}(R)$ by defining $K_{-i}(\mathcal{A})$ for any Abelian category. A third and fourth definition of $K_{-i}(R)$ were given independently by Karoubi Villamayor [9] using the ring $S(R)$ and by Wagoner [14] using the subring $\mu(R)$ of $S(R)$. Happily all these definitions were shown to agree by Karoubi's axiomatic treatment in [7].

In 1971, Gersten [4] constructed a nonconnective delooping of $K_0(R) \times BGl^+(R)$ using the fact that $\Omega BGl^+(S(R)) = K_0(R) \times BGl^+(R)$. Wagoner [15] then constructed the Ω-spectrum $K_0(\mu^i(R)) \times BGl^+(\mu^i(R))$ and showed that the inclusions $\mu(R) \to S(R)$ induced an equivalence of spectra. To our knowledge, nonconnective deloopings of K-theory of other additive categories besides $\mathcal{F}$ has not been studied until now.

The construction in [11] is very much in the spirit of the early definitions of $K_{-i}(R)$, but works for any additive category. Needless to say, an open question in our work is whether or not the $\Omega BQ\mathcal{C}_n(\underline{\underline{A}})^\wedge$ yield a nonconnective delooping of any (idempotent complete) additive category with exact sequences. A major difference between the categories $\mathcal{C}_i(\mathcal{A})$ and Karoubi's categories $S^i\mathcal{A}$ is that $S\mathcal{A}$ is defined as a quotient of the flasque category $C\mathcal{A}$ (see [7]) while $\mathcal{C}_1(\mathcal{A})$ may be viewed as an enlargement of the flasque category $\mathcal{C}_+(\mathcal{A})$. It would be interesting to see if the natural inclusion of $C\mathcal{A}$ in $\mathcal{C}_+(\mathcal{A})$ could be made to induce an isomorphism between K-groups.

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