

CONTROLLED ALGEBRA AND THE NOVIKOV CONJECTURES FOR K- AND L-THEORY

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ABSTRACT. The aim of this paper is to split the assembly map in K- and L-theory for a class of groups of finite cohomological dimension, containing the word hyperbolic groups

0. INTRODUCTION

In this paper we combine the methods of [5] with the continuously controlled algebra of [1] and the L-theory of additive categories with involution [23] to split assembly maps in K- and L-theory. Specifically we prove the following theorems

Let Γ be a group with finite classifying space $B\Gamma$. Assume $E\Gamma$ admits a compactification X (meaning X compact, and $E\Gamma$ is an open dense subset) satisfying the following conditions, (denoting $X - E\Gamma$ by Y).

- (i) The Γ -action extends to X .
- (ii) X is metrizable.
- (iii) X is contractible.
- (iv) Compact subsets of $E\Gamma$ become small near Y i. e. for every point $y \in Y$, for every compact subset $K \subset E\Gamma$ and for every neighborhood U of y in X , there exists a neighborhood V of y in X so that if $g \in \Gamma$ and $gK \cap V \neq \emptyset$ then $gK \subset U$.

Conditions of this type were first utilized by Farrell and Hsiang in [12]. Let R be a ring, and let $K^{-\infty}(R)$ denote the Gersten-Wagoner (non-connective) K-theory spectrum of R . Then we have

Theorem A. *The spectrum $B\Gamma_+ \wedge K^{-\infty}(R)$ is a split summand of $K^{-\infty}(R\Gamma)$.*

Let R be a ring with involution, satisfying that $K_{-i}(R) = 0$ for sufficiently large i , and let $\mathbb{L}^{-\infty}(R)$ be the periodic L-theory spectrum of R with homotopy groups the Wall surgery obstruction groups [23]. Then we have

Theorem B. *The spectrum $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R)$ is a split summand in $\mathbb{L}^{-\infty}(R\Gamma)$.*

If $\tilde{K}_i(R) = 0$ for all $i \leq 1$, say for $R = \mathbb{Z}$ we may replace $\mathbb{L}^{-\infty}$ by \mathbb{L}^s . In this case we get the corollary

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Corollary C. *The spectrum $B\Gamma_+ \wedge \mathbb{L}^s(\mathbb{Z})$ is a split summand in $\mathbb{L}^s(\mathbb{Z}\Gamma)$.*

Novikov conjectured the homotopy invariance of higher signatures, and proved it for free abelian groups. It is well known that the Novikov conjecture for a group Γ is equivalent to the rational split injectivity of the assembly map. In section 3 and 5, we identify the above splitting maps with the usual assembly map, so Corollary C verifies the Novikov conjecture for the class of groups considered here (actually a stronger integral version of the Novikov conjecture), see [23, Prop. 24.5] for a discussion of this.

The condition that X be metrizable is not actually necessary. It is sufficient that X is compact Hausdorff. This is proved using a generalized Čech theory in a sequel to this paper [7], where we also give conditions to ensure splitting, depending intrinsically on the group. We choose to present a proof in the metrizable case since the proof is fairly easy, and it does cover a large class of groups, in particular by [2] the above conditions are satisfied for word hyperbolic groups.

Jointly with W. Vogell and J. Roe respectively, we have extended the methods presented here to work in A -theory and in topological K -theory as well [9, 8]

A theorem similar to Theorem B has been announced by S. Ferry and S. Weinberger. Specifically they replace the condition that the compactification is equivariant, by the condition that $E\Gamma$ is compactified by Y , a Z -set. The definition of a Z -set is that there is a homotopy h_t of the identity such that $h_t(Y) \subset E\Gamma$ for $t > 0$.

Assembly maps and related problems have been studied for a long time by many mathematicians under various assumptions on the group, and with various conclusions such as rational splitting, integral splitting, integral isomorphism, and for various functors such as K - L - A - or C^* -theory. see e. g. [26], [4], [12], [13], [19], [17], [21], [28], [3], [10], [5] and [14].

1. CONTINUOUSLY CONTROLLED ALGEBRA

In this section we recapitulate and extend results from [1].

Let E be a topological space, R a ring. We denote the free R -module generated by $E \times \mathbb{N}$ by $R[E]^\infty$. Here \mathbb{N} denotes the natural numbers. Notice that a set map $f : E \rightarrow F$ induces a map $R[E]^\infty \rightarrow R[F]^\infty$, in particular if $x \in E$, we have $R[x]^\infty$ a submodule of $R[E]^\infty$.

Definition 1.1. The category $\mathcal{B}(E; R)$ of finitely generated free R -modules parameterized by E has objects A , submodules of $R[E]^\infty$ such that denoting $A \cap R[x]^\infty$ by A_x

- (i) $A = \bigoplus A_x$.
- (ii) A_x is a finitely generated free R -module.
- (iii) $\{x | A_x \neq 0\}$ is locally finite in E .

morphisms are all R -module morphisms.

Given a morphism $\phi : A \rightarrow B$ and two points $x, y \in E$ we denote the component of ϕ from A_x to B_y by ϕ_y^x . Clearly ϕ is determined by, and determines $\{\phi_y^x\}$. We call ϕ_y^x the *components* of ϕ .

Notice we have an isomorphism $R[x]^\infty \times R[x]^\infty \cong R[x]^\infty$ sending $((x, i), 0)$ to $(x, 2i)$ and $(0, (x, i))$ to $(x, 2i - 1)$.

Definition 1.2. Direct sum \oplus in the category $\mathcal{B}(E; R)$ is defined by

$$(A \oplus B)_x = \begin{cases} A_x \subset R[x]^\infty & \text{if } B_x = 0 \\ B_x \subset R[x]^\infty & \text{if } A_x = 0 \\ A_x \oplus B_x \subset R[x]^\infty \oplus R[x]^\infty \cong R[x]^\infty & \text{otherwise} \end{cases}$$

The special case made when A_x or B_x is zero, ensures the convenient formulae $A \oplus 0 = A = 0 \oplus A$. With this direct sum we clearly have

Proposition 1.3. $\mathcal{B}(E; R)$ is a small additive category.

Remark 1.4. The reader may find it artificial to require the objects of $\mathcal{B}(E; R)$ to be submodules of $R[E]^\infty$, specially in view of the extra trouble in defining direct sum as above. The justification is that $\mathcal{B}(E; R)$ is a *small* category with convenient equivariant properties as exemplified by

Proposition 1.5. Assume the group Γ acts freely, properly discontinuously on the space E . The induced action on $R[E]^\infty$ gives an action on $\mathcal{B}(E; R)$. We then have an equivalence of categories

$$\mathcal{B}(E; R)^\Gamma \cong \mathcal{B}(E/\Gamma; R[\Gamma])$$

Proof. The Γ -action on E induces an $R[\Gamma]$ -module structure on $R[E]^\infty$. A set theoretic section $s : E/\Gamma \rightarrow E$ induces an $R[\Gamma]$ -module morphism

$$R\Gamma[E/\Gamma]^\infty \rightarrow R[E]^\infty$$

which is an isomorphism with inverse Φ given by

$$\Phi(\epsilon)([x], n) = \sum_g \epsilon(gs([x]), n)g.$$

An object A in $\mathcal{B}(E; R)$ fixed under the Γ -action is a submodule $A \subset R[E]^\infty \cong R\Gamma[E/\Gamma]^\infty$ which is setwise fixed under the Γ -action, hence an $R\Gamma$ -submodule. We have

$$A_{[x]} = A \cap R\Gamma[[x]]^\infty = \bigoplus_{g \in \Gamma} A_{g \cdot x}$$

hence $A_{[x]}$ is a free $R\Gamma$ -module of rank the R -rank of A_x , and the local finiteness of $\{x \in E | A_x \neq 0\}$ and proper discontinuity of the Γ -action implies that $\{[x] \in E/\Gamma | A_{[x]} \neq 0\}$ is locally finite in E/Γ . Obviously A is generated by the $\{A_{[x]}\}$ as $R\Gamma$ -modules. Morphisms of $\mathcal{B}(E; R)$, fixed under the Γ -action satisfy

$$\phi_{gx}^{gy} = g\phi_x^y g^{-1},$$

so they are $R\Gamma$ -module morphisms, so we have completed the proof. Notice the equivalence does depend on the set theoretic splitting $E/\Gamma \rightarrow E$. \square

We shall consider categories with various degrees of control on the morphisms. Examples of this is bounded control as in [20] and continuous control as in [1], but also something that might be considered a mixture of the two.

In the following definitions, let X be a topological space, Y a subspace, T an open subset of X and $p : T \rightarrow K$ a map to a topological space K , which is continuous at points of $Y \cap T$. Denote $X - Y$ by E .

Definition 1.6. Let U be a subset of X and A an object in $\mathcal{B}(E)$. We define $A|U$ by $(A|U)_x = A_x$ if $x \in U - Y$ and $(A|U)_x = 0$ if $x \in X - U - Y$.

Definition 1.7. A morphism $\phi : A \rightarrow B$ in the category $\mathcal{B}(E; R)$ is said to be *continuously controlled* at a point $y \in Y$, if for every neighborhood U of y in X , there is a neighborhood V of y in X , so that

$$\phi(A|V) \subset B|U \quad \text{and} \quad \phi(A|X - U) \subset B|X - V$$

in other words, if $a \in V - Y$ and $b \in X - U - Y$ implies $\phi_b^a = 0$ and $\phi_a^b = 0$.

This is the standard control definition in [1].

Lemma 1.8. *If X is compact Hausdorff, and $\phi : A \rightarrow B$ satisfies half the control condition at all points of Y , i. e. for every $y \in Y$ and every neighborhood U there is a neighborhood V so that $\phi(A|V) \subset B|U$, then ϕ is continuously controlled.*

Proof. Let y and U be given. Find $V \subset \bar{V} \subset U$ so that $\phi(A|V) \subset B|U$. For $z \in Y - U$ we may find a neighborhood W_z so that $\phi(A|W_z) \subset B|X - \bar{V}$. Since $X - U - \cup W_z$ is compact we have $A|X - U - \cup W_z$ is a finitely generated R -module, so we can find a compact set $K \subset X - Y$ so that $\phi(A|X - U - \cup W_z) \subset B|K$, and we may replace V by $V - K$ \square

The next definition deals with control, but only in the direction of a certain map, reminiscent of the long thin handles in [13]

Definition 1.9. A morphism $\phi : A \rightarrow B$ in the category $\mathcal{B}(E; R)$ is said to be controlled in the p -direction or p -controlled at a point $y \in Y \cap T$, if for every neighborhood U of $p(y)$ in K , there is a neighborhood V of y in X so that

$$\phi(A|V) \subset B|p^{-1}(U) \quad \text{and} \quad \phi(A|X - p^{-1}(U)) \subset B|X - V$$

in other words if $a \in V - Y$ and $b \in X - p^{-1}(U) - Y$ then $\phi_b^a = 0$ and $\phi_a^b = 0$.

Definition 1.10. The category $\mathcal{B}(X, Y; R)$ has the same objects as $\mathcal{B}(E; R)$, $E = X - Y$, but morphisms are required to be continuously controlled at all points of Y .

These are the categories defined in [1]. The reader should think of $E = X - Y$ as an open dense subset of X .

Definition 1.11. The category $\mathcal{B}(X, Y, p; R)$ has the same objects as $\mathcal{B}(E; R)$, $E = X - Y$ but morphisms have to be continuously controlled at all points of $Y - T$, and p -controlled at all points of $T \cap Y$.

Example 1.12. The main examples we shall consider in this paper are the categories

$$\mathcal{B}(CX, CY \cup X, p_X; R)$$

and

$$\mathcal{B}(\Sigma X, \Sigma Y, p_X; R),$$

where X is a compactification of $E\Gamma$, $Y = X - E\Gamma$ and p_X is the projection $X \times (0, 1) \rightarrow X$.

Finally we need to introduce germs. Using notation as above let W be a (typically open) subset of Y .

Definition 1.13. The category $\mathcal{B}(X, Y, p; R)^W$ has the same objects as $\mathcal{B}(X, Y, p; R)$, but morphisms are identified if they agree in a neighborhood of W . Specifically $\phi, \psi : A \rightarrow B$ are identified if there is a neighborhood U of W in X , so that $\psi_b^a = \phi_b^a$ when $a \in U - Y$ or $b \in U - Y$. Similarly the category $\mathcal{B}(X, Y; R)^W$ has the same objects as $\mathcal{B}(X, Y; R)$, but morphisms are identified if they agree in a neighborhood of W .

Following [1] we shall study the functoriality of $\mathcal{B}(X, Y; R)^W$, but first let us recall categorical terminology: Two functors between categories $F, G : \mathcal{A} \rightarrow \mathcal{B}$ are naturally equivalent, if there is a natural transformation from F to G which is an isomorphism for each object in \mathcal{A} . The categories \mathcal{A} and \mathcal{B} are equivalent if there are functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ and natural equivalences from FG to $1_{\mathcal{B}}$ and from GF to $1_{\mathcal{A}}$. Two additive categories are isomorphic when they are equivalent by functors which give a 1-1 correspondence of objects. We use the symbol $=$ in case an isomorphism is the identity on objects. A functor between additive categories is called *lax* if it commutes with direct sum up to natural equivalence, sends 0 to 0 and induces a homomorphism of Hom-sets.

Definition 1.14. A *set* map $f : (X, Y) \rightarrow (X', Y')$ satisfying the following conditions is said to be *eventually continuous* at Y

- (i) If K is a compact subset of $X' - Y'$, then $f^{-1}(K)$ has compact closure in $X - Y$.
- (ii) $f(X - Y) \subset X' - Y'$.
- (iii) f is continuous at points of Y .

Definition 1.15. The support at infinity $\text{supp}_{\infty}(A)$ of an object A , in $\mathcal{B}(X, Y; R)$ is the set of limit points of $\{x | A_x \neq 0\}$.

Clearly the support at infinity is a subset of Y .

There is a slight problem getting induced morphisms from eventually continuous maps. We need to consider two cases, as to whether f is monic on $X - Y$ or not. Let (X, Y) and (X', Y') be compact Hausdorff pairs.

Proposition 1.16. *An eventually continuous map $f : (X, Y) \rightarrow (X', Y')$ which is monic on $X - Y$, induces a functor*

$$\mathcal{B}(X, Y; R) \rightarrow \mathcal{B}(X', Y'; R)$$

sending $A \subset R[X - Y]^\infty$ to $A \subset R[X - Y]^\infty \xrightarrow{f_\sharp} R[X' - Y']^\infty$, and the map induced by the identity on morphisms. When f is not assumed to be 1-1, we get a functor by sending $C = \{C_x\}$ to $f_\sharp C$ with

$$(f_\sharp C)_{x'} = \bigoplus_{f(x)=x'} C_x$$

and choosing an embedding of $(f_\sharp C)_{x'} \subset R[x']^\infty$. The functor is induced by the identity on morphisms. In this case f_\sharp is only well-defined up to natural equivalence. If f sends $Y - W$ to $Y' - W'$ we get a functor

$$\mathcal{B}(X, Y; R)^W \rightarrow \mathcal{B}(X', Y'; R)^{W'}$$

Proof. The case when f is 1-1 is clear, so consider the case when f is not necessarily 1-1. Condition (i) ensures the direct sum is finite. We need to show that if ϕ is a continuously controlled map in $\mathcal{B}(X, Y; R)$, then $f_\sharp \phi$ is continuously controlled. Let $y' \in Y'$ and a neighborhood U' be given. Assume by contradiction that we can find a sequence $x'_i, y'_i \in X' - Y'$, so that $x'_i \notin U'$, $y'_i \rightarrow y'$ and $f_\sharp \phi$ has a nontrivial component between x'_i and y'_i . We then have $x_i, y_i \in X - Y$ so that $f(x_i) = x'_i$, $f(y_i) = y'_i$ and ϕ has a nontrivial component between x_i and y_i . Since X is compact Hausdorff $\{y_i\}$ has a convergent subsequence $\{y_j\}$ with limit point y . The local finiteness condition on objects ensures that y must lie in Y . By continuity $f(y) = y'$, and we may find a neighborhood V of y , so that $f(V) \subset V'$. Clearly $x_j \notin V$ and we have a contradiction. In the germ case note that two morphisms are identified if and only if the difference factors through an object A with $\text{supp}_\infty(A) \subset Y - W$. But then $\text{supp}_\infty(f_\sharp(A)) \subset Y' - W'$ because if $f_\sharp(A)_{y'_i} \neq 0$ and $y'_i \rightarrow y'$, then we can find y_i such that $A_{y_i} \neq 0$ and $f(y_i) = y'_i$. The set $\{y_i\}$ must have a limit point y . This point belongs to $\text{supp}_\infty(A) \subset Y - W$. By continuity $y' = f(y)$ so $y' \in Y' - W'$. \square

Remark 1.17. When composing eventually continuous maps f and g that are 1-1 in the interior, we do get $f_\sharp g_\sharp = (fg)_\sharp$. When they are not 1-1 in the interior we only get $f_\sharp g_\sharp$ is naturally equivalent to $(fg)_\sharp$.

Proposition 1.18. *If f_1 and f_2 are eventually continuous maps $(X, Y) \rightarrow (X', Y')$ and $f_1|_Y = f_2|_Y$, then $f_{1\sharp}$ and $f_{2\sharp}$ are naturally equivalent functors.*

Proof. The natural equivalence is induced by the identity. \square

Remark 1.19. Given any set map f we shall use the symbol f_\sharp as given in Proposition 1.16. In any given case of course one needs to make sure that this does indeed define a functor.

Remark 1.20. In case of a locally compact Hausdorff pair (X, Y) , $Y \subset X$ a proper inclusion, the obvious identification

$$\mathcal{B}(X, Y; R)^Y = \mathcal{B}(X_+, Y_+; R)^Y$$

ensures that an eventually continuous map $f : (X, Y) \rightarrow (X', Y')$ induces a functor

$$f_\sharp : \mathcal{B}(X, Y; R)^Y \rightarrow \mathcal{B}(X', Y'; R)^{Y'}$$

if $f : (X_+, Y_+) \rightarrow (X'_+, Y'_+)$ is an eventually continuous map, i. e. if f satisfies an appropriate properness condition.

Let C be a (typically closed) subset of Y .

Definition 1.21. The full subcategory of $\mathcal{B}(X, Y; R)$ on objects with support at infinity contained in C , is denoted by $\mathcal{B}(X, Y; R)_C$. Similarly we have $\mathcal{B}(X, Y, p; R)_C$, $\mathcal{B}(X, Y; R)_C^W$ and $\mathcal{B}(X, Y, p; R)_C^W$ as full subcategories of $\mathcal{B}(X, Y, p; R)$, $\mathcal{B}(X, Y; R)^W$ and $\mathcal{B}(X, Y, p; R)^W$ respectively.

Let Y be a closed subset of X , C a closed subset of Y , S a closed subset of C (e. g. $S = C$). Then $Y - C$ may be thought of as an open subset of Y/S . In case $S = C$ then Y/S is just the one-point compactification of $Y - C$.

Lemma 1.22. *We have*

$$\mathcal{B}(X, Y; R)^{Y-C} = \mathcal{B}(X/S, Y/S; R)^{Y/S-C/S}.$$

In particular we have

$$\mathcal{B}(X, Y; R)^{Y-C} = \mathcal{B}(X/C, (Y/C - C/C)_+; R)^{Y/C-C/C}.$$

Proof. The identity induces a homeomorphism

$$E = X - Y = X/S - Y/S.$$

Hence the objects on both sides are the objects of $\mathcal{B}(E; R)$. Any morphism has a representative which is 0 except in a neighborhood of $Y - C$, and for such a representative the control conditions are the same, since $Y - C$ is thought of as an open subset of Y as well as of Y/S \square

Next we reduce the dependency of X

Theorem 1.23. *Let X be a compact metrizable space, Y a closed subset, so that $X - Y$ is dense in X , CY denotes the cone on Y . We then have eventually continuous maps $f : (X, Y) \rightarrow (CY, Y)$ and $g : (CY, Y) \rightarrow (X, Y)$ so that $f|_Y = g|_Y = 1_Y$,*

Proof. We define the eventually continuous maps

$$f : (X, Y) \rightarrow (CY, Y) \text{ and } g : (CY, Y) \rightarrow (X, Y)$$

so that $f|_Y = g|_Y = 1_Y$. Choose a metric on X so that $\sup_{x \in X} (d(x, Y)) = 1$. We orient the cone so that the cone point is $Y \times 0$. Given $x \in X$ there exists $y \in Y$ so that $d(x, y) = d(x, Y)$. Choose one such y and define $f(x) = (y, 1 - d(x, Y))$. If $x \in Y$, obviously $f(x) = (x, 1)$, and if x is close to Y $f(x)$ will be close to $(x, 1)$. Given $(y, t) \in CY$. We can find x so that $d(x, Y) = \sup(\{d(x, Y) | d(x, y) \leq 1 - t\})$, using the open denseness condition and compactness. Choose one such x and define $g(y, t) = x$. If $t = 1$ clearly $g(y, t) = y$, and if t is close to 1, $g(y, t)$ must be close to y . \square

Corollary 1.24. *There is a natural equivalence*

$$f_{\#} : \mathcal{B}(X, Y; R) \cong \mathcal{B}(CY, Y; R),$$

natural in the sense that if h is an eventually continuous map $h : (X, Y) \rightarrow (X', Y')$, then the diagram

$$\begin{array}{ccc} \mathcal{B}(X, Y; R) & \longrightarrow & \mathcal{B}(CY, Y; R) \\ \downarrow & & \downarrow \\ \mathcal{B}(X', Y'; R) & \longrightarrow & \mathcal{B}(CY', Y'; R) \end{array}$$

commutes up to natural equivalence.

Proof. Let $f : (X, Y) \rightarrow (CY, Y)$ and $g : (CY, Y) \rightarrow (X, Y)$ be chosen as in Theorem 1.23. Then by Proposition 1.18, $f_{\#}$ will be an equivalence of categories with inverse $g_{\#}$ since f and g are the identity on Y . If $f' : (X', Y') \rightarrow (CY', Y')$ is chosen as in Theorem 1.23 we have $Ch \cdot f$ and $f' \cdot h$ agree when restricted to Y , so once again it follows from Proposition 1.18 that $Ch_{\#} \cdot f_{\#}$ and $f'_{\#} \cdot h_{\#}$ are naturally equivalent. \square

If W is an open subset of Y , the above technique also yields

Theorem 1.25. $\mathcal{B}(X, Y; R)^W \cong \mathcal{B}(CY, Y; R)^W$.

Following Karoubi [16] we have the following definition

Definition 1.26. An additive category is *flasque* if it admits an endofunctor Σ^{∞} and a natural equivalence $1 \oplus \Sigma^{\infty} \cong \Sigma^{\infty}$

Let \mathcal{A} be a small additive category. We let $K(\mathcal{A})$ be a functorial assignment of \mathcal{A} to a spectrum whose homotopy groups are the K -groups of the symmetric monoidal category obtained from \mathcal{A} by restricting to isomorphisms. It is well known (from the additivity theorem) that the K -groups of a flasque additive category are trivial. Following [20], we define $\mathcal{C}_i(\mathcal{A})$ to be the category of \mathcal{A} -objects parameterized by \mathbb{Z}^i and bounded morphisms. The inclusion $\mathbb{Z}^i = \mathbb{Z}^i \times 0 \subset \mathbb{Z}^{i+1}$ induces a map

$$K(\mathcal{C}_i(\mathcal{A})) \rightarrow K(\mathcal{C}_{i+1}(\mathcal{A}))$$

which is naturally homotopy trivial in two ways since it factors through the categories parameterized by $\mathbb{Z}^i \times \mathbb{Z}_+$ and $\mathbb{Z}^i \times \mathbb{Z}_-$ respectively, which have obvious natural flasque structures by Eilenberg swindle see e. g. [20]. This gives a functorial map

$$\Sigma K(\mathcal{C}_i(\mathcal{A})) \rightarrow K(\mathcal{C}_{i+1}(\mathcal{A}))$$

or

$$K(\mathcal{C}_i(\mathcal{A})) \rightarrow \Omega K(\mathcal{C}_{i+1}(\mathcal{A})).$$

We denote the homotopy colimit of

$$K(\mathcal{A}) \rightarrow \Omega K(\mathcal{C}_1(\mathcal{A})) \rightarrow \dots \rightarrow \Omega^i(K(\mathcal{C}_i(\mathcal{A}))) \rightarrow \dots$$

by $K^{-\infty}(\mathcal{A})$. It follows from [20], see [5] for a more detailed explanation, that $K^{-\infty}(\mathcal{A})$ is the nonconnective K -theory spectrum associated with the symmetric monoidal category obtained from \mathcal{A} by restricting to isomorphisms. Clearly $K^{-\infty}$ is a functor from the category of small additive categories and lax functors to the category of spectra.

Recall the notion of an \mathcal{A} -filtered additive category \mathcal{U} [16]

Definition 1.27. Let \mathcal{A} be a full subcategory of an additive category \mathcal{U} . Denote objects of \mathcal{A} by the letters A through F and objects of \mathcal{U} by the letters U through W . We say that \mathcal{U} is \mathcal{A} -filtered, if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$, so that

- (i) For each U , the decomposition form a filtered poset under the partial order that $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$, whenever $U_\beta \subseteq U_\alpha$ and $E_\alpha \subseteq E_\beta$.
- (ii) Every map $A \rightarrow U$, factors $A \rightarrow E_\alpha \rightarrow E_\alpha \oplus U_\alpha = U$ for some α .
- (iii) Every map $U \rightarrow A$ factors $U = E_\alpha \oplus U_\alpha \rightarrow E_\alpha \rightarrow A$ for some α .
- (iv) For each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$ i. e. to $U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)$

Karoubi defines \mathcal{U}/\mathcal{A} to be the category with the same objects as \mathcal{U} , but with $\phi, \psi : U \rightarrow V$ identified if $\phi - \psi$ factors through \mathcal{A} . We have the following [20, Cor. 5.7]:

Theorem 1.28. $K^{-\infty}(\mathcal{A}) \rightarrow K^{-\infty}(\mathcal{U}) \rightarrow K^{-\infty}(\mathcal{U}/\mathcal{A})$ is a homotopy fibration.

Proof. This follows from [20, Cor. 5.7]. By taking the spectra that include the negative homotopy groups we avoid idempotent completions. \square

Let X be a compact Hausdorff space, Y a closed subspace, so that $X - Y$ is dense, C a closed subspace of Y , W an open subset of Y so that $C \subset W$. Put $\mathcal{U}_1 = \mathcal{B}(X, Y; R)$, $\mathcal{A}_1 = \mathcal{B}(X, Y; R)_C$, $\mathcal{U}_2 = \mathcal{B}(X, Y; R)^W$, $\mathcal{A}_2 = \mathcal{B}(X, Y; R)_C^W$. Then we have

Lemma 1.29. \mathcal{U}_i is \mathcal{A}_i -filtered and

$$\mathcal{U}_1/\mathcal{A}_1 = \mathcal{B}(X, Y; R)^{Y-C}$$

$$\mathcal{U}_2/\mathcal{A}_2 = \mathcal{B}(X, Y; R)^{W-C}$$

Proof. Direct from definitions. \square

Theorem 1.28 gives the following

Corollary 1.30. We have fibrations up to homotopy

$$K^{-\infty}(\mathcal{B}(X, Y; R)_C) \rightarrow K^{-\infty}(\mathcal{B}(X, Y; R)) \rightarrow K^{-\infty}(\mathcal{B}(X, Y; R)^{Y-C})$$

and

$$K^{-\infty}(\mathcal{B}(X, Y; R)_C^W) \rightarrow K^{-\infty}(\mathcal{B}(X, Y; R)^W) \rightarrow K^{-\infty}(\mathcal{B}(X, Y; R)^{W-C})$$

Lemma 1.31. *Let X be a compact metrizable space, Y a closed subspace so that $X - Y$ is dense in X , and $*$ a point of Y . The natural map*

$$\mathcal{B}(X, Y; R) \rightarrow \mathcal{B}(X, Y; R)^{Y-*}$$

induces an isomorphism on K -theory (i. e. a weak homotopy equivalence of K -theory spectra).

Proof. Let $\mathcal{A} = \mathcal{B}(X, Y; R)_*$. By Corollary 1.30 we shall be done if we can prove that $K^{-\infty}(\mathcal{A})$ is weakly contractible. Choose a metric on X and a sequence of points x_0, x_1, \dots , in $X - Y$ converging to $*$, and such that $d(x_i, Y)$ is strictly decreasing. This is possible since $X - Y$ is open dense in X . Let \mathcal{A}' be the full subcategory of $\mathcal{B}(X, Y; R)$ with $A_x = 0$ except for $x \in \{x_i\}$. Clearly \mathcal{A}' is a subcategory of \mathcal{A} , but actually \mathcal{A} and \mathcal{A}' are equivalent. One way is the inclusion. To get the other direction define a set map $f : X \rightarrow X$ by $f(x) = x_i$ if $d(x, Y) \geq d(x_i, Y)$ and $d(x, Y) < d(x_{i-1}, Y)$, when $x \in X - Y$, and $f(y) = y$ for $y \in Y$. This map is clearly continuous at $*$ (and discontinuous at all other points of Y), but since objects of \mathcal{A} are required to be 0 in a neighborhood of $Y - *$, we nevertheless get $f_{\#}A$, an object of \mathcal{A}' and an equivalence of categories. To finish off the proof of the lemma notice that \mathcal{A}' is flasque i. e. it admits an endofunctor $U^\infty : \mathcal{A}' \rightarrow \mathcal{A}'$ so that $1 \oplus U^\infty \cong U^\infty$ (an Eilenberg swindle), and hence $K^{-\infty}(\mathcal{A}')$ is contractible. Specifically U^∞ is given by

$$(U^\infty(A))_{x_i} = \bigoplus_{j < i} A_{x_j}$$

This completes the proof of Lemma 1.31. □

Lemma 1.32. *Let X be a compact metrizable space, Y a closed nowhere dense subset, and W an open subset of Y . Then*

$$\mathcal{B}(X, Y; R)^W = \mathcal{B}(X - (Y - W), W; R)^W$$

Proof. The functor forgets the control along $Y - W$. The categories have the same objects, namely the objects of $\mathcal{B}(X - Y; R)$. If a morphism in $\mathcal{B}(X, Y; R)^W$ becomes 0 in $\mathcal{B}(X - (Y - W), W; R)^W$, that means that the components of the morphism are 0 in a neighborhood of W , but that means the morphism is 0 in $\mathcal{B}(X, Y; R)^W$. This shows the functor is monic on Hom-sets. To see it is epic on Hom-sets, consider a morphism in $\mathcal{B}(X - (Y - W), W; R)^W$ represented by $\phi : A \rightarrow B$. We may assume A and B are 0 except in a neighborhood of W , so ϕ is automatically controlled at interior points of $Y - W$. Hence it is no loss of generality to assume that Y is the closure of W . We need to show that ϕ is equivalent to a morphism which is controlled at points of ∂W . For every point $z \in W$ we find a neighborhood $U_z \subset B(z, \frac{1}{2}d(z, \partial W))$ so that no non-zero component of ϕ reaches from U_z outside the ball $B(z, \frac{1}{2}d(z, \partial W))$. We let U be the union of all the U_z 's and replace ϕ by a map whose components are equal to ϕ 's when two points are in U and 0 otherwise. Rechoosing ϕ to be this new representative, we claim ϕ represents a morphism in $\mathcal{B}(X, Y; R)^W$ i. e. that it is controlled at points of ∂W . Let y be a point in ∂W and consider a δ -ball around y . We let $V = B(y, \delta/4)$. If $a \in V$ and ϕ_a^a or ϕ_a^b is different from 0, we must have $a \in U_z$ some

$z \in W$. We have

$$d(z, y) \leq d(z, a) + d(a, y) < d(z, \partial W)/2 + \delta/4 \leq d(z, y)/2 + \delta/4$$

so $d(z, y) < \delta/2$. Since $a \in U_z$ and ϕ_b^a or ϕ_a^b is different from 0, we have $b \in B(z, \frac{1}{2}d(z, \partial W))$ and

$$d(b, y) \leq d(b, z) + d(z, y) < d(z, \partial W)/2 + \delta/2 \leq d(z, y)/2 + \delta/2 < \delta.$$

Hence a non-zero component of ϕ does not reach outside the δ -ball. Thus ϕ represents a map in $\mathcal{B}(X, Y)^W$ and we have shown the functor is epic on Hom-sets. \square

Definition 1.33. A (reduced) Steenrod homology theory [15, 11] is a functor from the category of compact metrizable spaces and continuous maps, to graded abelian groups (with $h_*(pt) = 0$), satisfying the following axioms

- (i) h is homotopy invariant.
- (ii) Given any closed subset A of X there is a natural transformation

$$\partial : h_*(X/A) \rightarrow h_{*-1}(A)$$

fitting into a long exact sequence

$$\dots \rightarrow h_n(A) \rightarrow h_n(X) \rightarrow h_n(X/A) \rightarrow h_{n-1}(A) \rightarrow \dots$$

- (iii) Given a compact metric space, which is the countable union of metric spaces along a single common point (like Hawaiian earrings), $\bigvee X_i$, then the projection maps $p_i : \bigvee X_i \rightarrow X_i$ induce an isomorphism

$$h_*(\bigvee X_i) \rightarrow \prod h_*(X_i)$$

These axioms are sometimes called the Kaminker-Schochet axioms.

Given any generalized homology theory, there is a unique Steenrod homology extension. Uniqueness was proved by Milnor [18], existence by Kahn-Kaminker-Schochet and Edwards-Hastings [15, 11].

Definition 1.34. A functor k from compact metrizable spaces to spectra is called a *Steenrod functor* if it satisfies the following conditions

- (i) $k(CX)$ is contractible for any cone CX .
- (ii) If $A \subset X$ is closed, then

$$k(A) \rightarrow k(X) \rightarrow k(X/A)$$

is a fibration (up to natural weak homotopy equivalence).

- (iii) Given a compact metric space $\bigvee X_i$ which is the countable union along a single point of metric spaces X_i , then the projections induce a weak homotopy equivalence $k(\bigvee X_i) \simeq \prod k(X_i)$

We have the following

Proposition 1.35. *Let k be a Steenrod functor. Then $\pi_*(k(X))$ is the unique Steenrod homology theory associated with the spectrum $k(S^0)$. If X is a finite CW-complex, then $k(X)$ is weakly homotopy equivalent to $X \wedge k(S^0)$.*

Proof. The maps $X \rightarrow X \times I$ sending x to $(x, 0)$ or $(x, 1)$ fit into $X \rightarrow X \times I \rightarrow CX$ giving homotopy invariance. The connecting homomorphism of the fibration gives the long exact sequence, and axiom (iii) gives the wedge axiom, hence $X \rightarrow \pi_*(k(X))$ is a Steenrod homology theory. We need to show that when we restrict to finite CW-complexes we get the homology theory associated to the spectrum $k(S^0)$. This however is proved in [27] (see Theorem 3.1 for the full statement). In particular it is proved in [27] that for any finite CW complex X , there is a weak homotopy equivalence of spectra

$$X \wedge k(S^0) \cong k(X)$$

thus $\pi_*(k(X))$ restricts to the usual homology theory on the category of finite CW-complexes. \square

Our definition of a Steenrod functor might be called a reduced Steenrod functor, since it sends a point to a contractible spectrum. We may of course get an unreduced Steenrod functor by the usual device of adding an extra basepoint.

Theorem 1.36. *Let X be a compact metrizable space, then $K^{-\infty}(\mathcal{B}(CX, X; R))$ is a reduced Steenrod functor with value $\Sigma K^{-\infty}(R)$ on S^0 . In particular if X a finite CW-complex then $\Omega K^{-\infty}(\mathcal{B}(CX, X; R))$ is weakly homotopy equivalent to $X \wedge K^{-\infty}(R)$.*

Proof. We need to prove conditions (i), (ii) and (iii) for the functor sending X to $K^{-\infty}(\mathcal{B}(CX, X; R))$. (i) is proved in [1], and so is (ii), but not quite in this generality. Consider

$$K^{-\infty}(\mathcal{B}(CX, X; R)_A) \rightarrow K^{-\infty}(\mathcal{B}(CX, X; R)) \rightarrow K^{-\infty}(\mathcal{B}(CX, X)^{X-A})$$

This is a fibration by Corollary 1.30. We first want to show that the inclusion

$$\mathcal{B}(CA, A; R) \rightarrow \mathcal{B}(CX, X; R)_A$$

is an equivalence of categories. Define a set map f from CX to CX to be the identity on X , and on $CX - X$ a point x is sent some point a in CA such that $d(x, a)$ realizes $\inf\{d(x, a) | a \in CA, d(a, A) \geq d(x, X)\}$. If this set is empty, send x to the cone point. Clearly f is continuous at points of A , (and discontinuous at points of $X - A$, but since objects of $\mathcal{B}(CX, X; R)_A$ are required to be 0 in a neighborhood of $X - A$, we still get a functor

$$f_{\#} : \mathcal{B}(CX, X; R)_A \rightarrow \mathcal{B}(CA, A; R),$$

which is an inverse to the inclusion up to natural equivalence. It follows from Lemma 1.22, Lemma 1.31 and Corollary 1.24 that $K^{-\infty}(\mathcal{B}(CX, X; R)^{X-A})$ is naturally weakly homotopy

equivalent to $K^{-\infty}\mathcal{B}(C(X/A), X/A; R)$, so we have verified condition (ii). Finally consider $\mathcal{B}(C(\bigvee X_i), \bigvee X_i; R)$. Let $*$ denote the wedge point. We have

$$K^{-\infty}(\mathcal{B}(C(\bigvee X_i), \bigvee X_i; R)) \simeq K^{-\infty}(\mathcal{B}(C(\bigvee X_i), \bigvee X_i; R)^{\bigvee X_i-*})$$

by Lemma 1.31. The category of germs $\mathcal{B}(C(\bigvee X_i), \bigvee X_i; R)^{\bigvee X_i-*}$ is equivalent to the product of the categories $\mathcal{B}(CX_i, X_i; R)^{X_i-*}$ since only small neighborhoods of X_i-* matter, so there can be no interaction between the components at the germ. It is proved in [6] that $K^{-\infty}$ commutes with infinite products, so

$$K^{-\infty}(\prod (\mathcal{B}(CX_i, X_i; R)^{X_i-*})) \simeq \prod K^{-\infty}(\mathcal{B}(CX_i, X_i; R)^{X_i-*}).$$

Moreover

$$K^{-\infty}(\mathcal{B}(CX_i, X_i; R)^{X_i-*}) \simeq K^{-\infty}(\mathcal{B}(CX_i, X_i; R))$$

by Lemma 1.31, and we are done. The final remark follows from Proposition 1.35 \square

We define Steenrod homology of a pair by $h_n(X, A) = h_n(X/A)$. Combining Theorem 1.36 with Corollary 1.24 we have proved

Theorem 1.37. *Let X be a compact metrizable space, Y a closed subspace so that $X - Y$ is dense in X , W an open subset of Y . Then there is a natural isomorphism*

$$K_*^{-\infty}(\mathcal{B}(X, Y; R)^W) \cong h_{*-1}(Y, Y - W; K^{-\infty}R)$$

where $h_{*-1}(-; K^{-\infty}R)$ denotes the Steenrod homology theory associated to the algebraic K -theory spectrum of the ring R (non-connective version).

Remark 1.38. Notice that the proof of Theorem 1.36 identifies the map

$$K^{-\infty}(\mathcal{B}(CX, X; R)) \rightarrow K^{-\infty}(\mathcal{B}(CX, X; R)^{X-A})$$

with the Steenrod functor of $K^{-\infty}R$ applied to the collapse map $X \rightarrow X/A$.

Definition 1.39. Let E be a locally compact space such that E_+ , the one-point compactification is metrizable (e. g. E a finite dimensional CW -complex), and \mathbb{S} a spectrum. We define

$$h_*^{l.f.}(E; \mathbb{S}) = h_*^{st}(E_+; \mathbb{S})$$

where h^{st} is the unique reduced Steenrod homology theory associated to the homology theory with spectrum \mathbb{S} .

Theorem 1.40. *Let E be a locally compact space with metrizable one point compactification. Consider $E = E \times 1 \subset E \times (0, 1]$. Then*

$$K_*^{-\infty}(\mathcal{B}(E \times (0, 1], E; R)^E) \cong h_{*-1}^{l.f.}(E, K^{-\infty}R)$$

Proof. By Lemma 1.32

$$\mathcal{B}((E \times (0, 1])_+, E_+; R)^E \cong \mathcal{B}(E \times (0, 1], E; R)^E$$

and by Lemma 1.31 $\mathcal{B}((E \times (0, 1])_+, E_+; R)$ and $\mathcal{B}((E \times (0, 1])_+, E_+; R)^E$ have the same K -theory. Finally Theorem 1.37 shows that the K -theory of $\mathcal{B}((E \times (0, 1])_+, E_+; R)$ is the reduced Steenrod homology of E_+ , with a shift in dimension. \square

2. SPLITTING THE K-THEORY ASSEMBLY MAP

Let Γ be a group. We shall consider a finite free Γ -CW complex E with a compactification X (meaning X compact, and E is an open dense subset). This compactification is supposed to satisfy conditions (i) through (iv) from the introduction.

If E is an $E\Gamma$ with a compactification satisfying these conditions, (so in particular $B\Gamma$ is a finite complex) the aim of this section is to show that the K-theory assembly map splits.

Theorem 2.1. $B\Gamma_+ \wedge K^{-\infty}(R)$ is a split factor of $K^{-\infty}(R\Gamma)$.

Some of the lemmas needed do not require all the conditions. The proof follows the strategy of [5], but replacing $h^{l.f.}$ as well as bounded K-theory by continuously controlled K-theory, and the natural transformation by an induced map. In view of Theorem 1.40, what we are doing is of course using a continuously controlled model for $h^{l.f.}$. Given a spectrum A with Γ -action, recall the definition of the homotopy fixed set

$$A^{h\Gamma} = \text{Map}_{\Gamma}(E\Gamma_+, A_+).$$

The collapse map $E\Gamma_+ \rightarrow S^0$ induces a map from the fixed set $A^{\Gamma} = \text{Map}_{\Gamma}(S^0, A_+)$ to the homotopy fixed set.

Remark 2.2. There is an important special case where the map from the fixed set to the homotopy fixed set is a weak homotopy equivalence. When $A = \prod_{\Gamma} B$, and the Γ -action permutes the factors we have

$$\begin{aligned} A^{h\Gamma} &= (\text{Map}(\Gamma_+, B))^{h\Gamma} = \text{Map}_{\Gamma}(E\Gamma_+, \text{Map}(\Gamma_+, B)) = \text{Map}_{\Gamma}((E\Gamma \times \Gamma)_+, B) = \\ &= \text{Map}_{\Gamma}(\Gamma_+, \text{Map}(E\Gamma_+, B)) = \text{Map}(E\Gamma_+, B) \simeq B = A^{\Gamma} \end{aligned}$$

Let CX be the cone of X , with $X \times 0$ as the cone point, and identify $X \times 1$ with X . We denote $X - E$ by Y and the projection map $X \times (0, 1) \rightarrow X$ by p_X , and consider the Γ -equivariant functor induced by collapsing X , from CX to ΣX :

$$\mathcal{B}(CX, CY \cup X, p_X; R) \rightarrow \mathcal{B}(\Sigma X, \Sigma Y, p_X; R)$$

Notice that the modules on both sides are parameterized by $E \times (0, 1)$, but the control conditions are quite different. Denoting $\Omega K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R))$ by S and

$\Omega K^{-\infty}(\mathcal{B}(\Sigma X, \Sigma Y, p_X; R))$ by T . We study the diagram

$$\begin{array}{ccc} S^\Gamma & \longrightarrow & T^\Gamma \\ \downarrow & & \downarrow \\ S^{h\Gamma} & \longrightarrow & T^{h\Gamma} \end{array}$$

and prove the following statements

- (i) $S^\Gamma \simeq (E/\Gamma)_+ \wedge K^{-\infty}(R)$.
- (ii) $T^\Gamma \simeq K^{-\infty}(R\Gamma)$.
- (iii) $S^\Gamma \simeq S^{h\Gamma}$.
- (iv) $S^{h\Gamma} \simeq T^{h\Gamma}$.

It is only (iv) that requires X to be contractible.

The proof will be a sequence of lemmas. First consider T^Γ .

Lemma 2.3. $\Omega K^{-\infty}(\mathcal{B}(\Sigma X, \Sigma Y, p_X; R))^\Gamma \simeq K^{-\infty}(R\Gamma)$.

Proof. Taking fixed sets and applying $K^{-\infty}$ clearly commutes, so we need to study the category $\mathcal{B}(\Sigma X, \Sigma Y, p_X; R)^\Gamma$. We shall show that

$$\mathcal{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma) \cong \mathcal{B}(I, 0 \cup 1; R\Gamma).$$

A priori we require control at 0 and 1 and along $Y \times (0, 1)$, but only in the p_X -direction. Control along $Y \times (0, 1)$ in the p_X -direction is automatic by equivariance because of condition (iv). To see this let $\phi : A \rightarrow B$ be an equivariant morphism, $(y, t) \in Y \times (0, 1)$ a point and U a neighborhood of y in X . We need to show that we can find a neighborhood V of (y, t) in CX so that

$$\phi(A|V) \subset B|(U \times (0, 1)).$$

Choose $0 < \epsilon_1 < t < \epsilon_2 < 1$ and a compact fundamental domain $K \subset E$ (so $E = \cup_\Gamma gK$). By compactness there are only finitely many R -module generators in $K \times [\epsilon_1, \epsilon_2]$, so we may find $L \subset E$ compact and $0 < \delta_1 < t < \delta_2 < 1$ so that $K \subset L$ and

$$\phi(A|K \times [\epsilon_1, \epsilon_2]) \subset B|L \times [\delta_1, \delta_2].$$

By condition (iv) we may find a neighborhood W of y in X so that if $gL \cap W \neq \emptyset$ then $gL \subset U$. We claim $V = W \times (\epsilon_1, \epsilon_2)$ will do: For any point $a = (p, s) \in V$, p lies in some translate gK of K hence $gL \cap W \neq \emptyset$, but that means $gL \subset U$ and by equivariance

$$\phi(A_a) \subset \phi(A|gK \times [\epsilon_1, \epsilon_2]) \subset B|gL \times [\delta_1, \delta_2] \subset B|U \times (0, 1)$$

and the proof that control is automatic along $Y \times (0, 1)$ is finished by Lemma 1.8. Arguing as in Proposition 1.5 we thus get

$$\mathcal{B}(\Sigma X, \Sigma Y, p_X; R)^\Gamma \cong \mathcal{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma)$$

Since E/Γ is compact Corollary 1.24 shows that

$$\mathcal{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma) \cong \mathcal{B}(I, 0 \cup 1; R\Gamma)$$

and by Theorem 1.36 $\Omega K^{-\infty}(\mathcal{B}(I, 0 \cup 1; R\Gamma))$ is weakly homotopy equivalent to $K^{-\infty}(R\Gamma)$. \square

We now go on to study S^Γ . Thinking of $E = E \times 1 \subset E \times (0, 1]$ we get

Lemma 2.4. $\mathcal{B}(CX, CY \cup X, p_X; R)^E = \mathcal{B}(E \times (0, 1], E; R)^E$

Proof. We have $\mathcal{B}(CX, CY \cup X, p_X; R)^E = \mathcal{B}(CX, CY \cup X; R)^E$ and $\mathcal{B}(CX, CY \cup X; R)^E = \mathcal{B}(CX, X; R)^E$ because the categories have the same objects as in $\mathcal{B}(E \times (0, 1]; R)$ and morphisms subquotients of the morphisms in $\mathcal{B}(E \times (0, 1]; R)$. The equalities follow because any morphism has a representative which is 0 except for a neighborhood of E , so we may assume it is 0 in a neighborhood of $CY - Y$, and control conditions along $CY - Y$ are thus automatically satisfied. Finally $\mathcal{B}(CX, X; R)^E = \mathcal{B}(E \times (0, 1], E; R)^E$ is a direct consequence of Lemma 1.32 \square

Lemma 2.5. *We have weak homotopy equivalences*

$$K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R)) \simeq K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R)^E)$$

and

$$K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R))^\Gamma \simeq K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R)^E)^\Gamma$$

Proof. Let \mathcal{A} denote $\mathcal{B}(CX, CY \cup X, p_X; R)_{CY}$. Clearly the quotient category

$$(\mathcal{B}(CX, CY \cup X, p_X; R)/\mathcal{A} = \mathcal{B}(CX, CY \cup X, p_X; R)^E,$$

but \mathcal{A} has a flasque structure, by an Eilenberg swindle shifting modules to the left as follows: Choose a continuous function $a : X \rightarrow [1, \infty)$ satisfying $a(m) > 1$ when $m \in E$ and $a(Y) = 1$. Define $U^\infty : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(U^\infty(A))_{(m,t)} = \bigoplus A_{m,a(m)^nt}$$

where we let $A_{(m,s)} = 0$ when $s \geq 1$. This sum is clearly finite, and

$$\{(m,t) | (U^\infty(A))_{(m,t)} \neq 0\}$$

is locally finite because A is 0 in a neighborhood of E . To get the statement on fixed sets, since $K^{-\infty}$ commutes with taking fixed sets, we need to consider the fixed category. Defining $\mathcal{A} = (\mathcal{B}(CX, CY \cup X, p_X; R)_{CY})^\Gamma$, the argument goes through as above, noting that an $R\Gamma$ -module which is 0 in a neighborhood of E must be 0 in a neighborhood of X . \square

Remark 2.6. This is the point where we need control along the map p_X . If we required continuous control, then equivariant maps could not have a non-zero component between (p, s) and (q, t) when $s \neq t$ because such a component would be translated by the group to points close to $Y \times (0, 1)$ contradicting control. Hence \mathcal{A} would not be flasque, because the flasque structure requires a natural transformation with a non-zero component between (p, s) and (q, t) with $s \neq t$.

Lemma 2.7. $(\mathcal{B}(CX, CY \cup X, p_X; R)^E)^\Gamma = (\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma$

Proof. Immediate from Lemma 2.4. □

Lemma 2.8. *Assume Γ acts freely properly discontinuously on a space E . Then*

$$(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \cong \mathcal{B}(E/\Gamma \times (0, 1], E/\Gamma; R)^{E/\Gamma}$$

Proof. The map

$$p_\# : R[E]^\infty \rightarrow R[E/\Gamma]^\infty$$

induced by projection induces an isomorphism

$$R[E]^\infty \otimes_{R\Gamma} R \rightarrow R[E/\Gamma]^\infty.$$

An $R\Gamma$ -module A parameterized by $E \times (0, 1)$ is sent to the R -module $A \otimes_{R\Gamma} R$. Going backwards, an R -module B parameterized by E/Γ is sent to $p_\#^{-1}(B)$. Since we are taking germs at E and components of a morphism have to become small near E , there is only one choice when lifting a morphism. □

To finish off showing $S^\Gamma = (E/\Gamma)_+ \wedge K^{-\infty}(R)$ we need the following theorem

Theorem 2.9. *Let E be a finite, free Γ -CW-complex. Then*

$$\Omega K^{-\infty}(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \simeq (E/\Gamma)_+ \wedge K^{-\infty}R$$

Proof. By Lemma 2.8

$$(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \cong \mathcal{B}(E/\Gamma \times (0, 1], E/\Gamma; R)^{E/\Gamma}.$$

Lemma 1.32 shows that

$$\mathcal{B}(E/\Gamma \times (0, 1], E/\Gamma; R)^{E/\Gamma} = \mathcal{B}(C(E/\Gamma), E/\Gamma \cup *; R)^{E/\Gamma},$$

and Lemma 1.31 shows that

$$K^{-\infty}(\mathcal{B}(C(E/\Gamma), E/\Gamma \cup *; R)^{E/\Gamma}) \simeq K^{-\infty}(\mathcal{B}(C(E/\Gamma), E/\Gamma \cup *; R)).$$

Finally Theorem 1.36 shows that

$$\Omega K^{-\infty}(\mathcal{B}(C(E/\Gamma), E/\Gamma \cup *; R)) \simeq (E/\Gamma)_+ \wedge K^{-\infty}R.$$

□

Corollary 2.10. $S^\Gamma \cong (E/\Gamma)_+ \wedge K^{-\infty}(R)$.

Proof. Combine Lemma 2.5, Lemma 2.7 and Theorem 2.9. □

Theorem 2.11. *Let E be a finite free Γ -CW complex. Then*

$$K^{-\infty}(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \rightarrow K^{-\infty}(\mathcal{B}(E \times (0, 1], E; R)^E)^{h\Gamma}$$

is a weak homotopy equivalence.

Proof. The proof is by induction on the Γ -cells of E . To start the induction assume $E = \Gamma$. In this case the category $\mathcal{B}(E \times (0, 1], E; R)^E$ is equivalent to the product category $\prod_{g \in \Gamma} \mathcal{B}(g \times (0, 1], g; R)^g$. This follows since we take germs at E and demand control at points of E , so near E a component of a map can not reach from one element in $E = \Gamma$ to another.

$$(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma = \mathcal{B}((0, 1], 1; R)^1$$

by Lemma 2.8. The projection maps induce a map

$$K^{-\infty} \left(\prod_{g \in \Gamma} \mathcal{B}(g \times (0, 1], g; R)^g \right) \rightarrow \prod_{g \in \Gamma} K^{-\infty}(\mathcal{B}(g \times (0, 1], g; R)^g)$$

which is Γ -equivariant, in the action that permutes the factors, and a weak homotopy equivalence by [6]. The fixed set on the product is the diagonal, which may be identified with $K^{-\infty}(\mathcal{B}((0, 1], 1; R)^1)$. Under this identification the map of fixed sets

$$K^{-\infty}(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \rightarrow \left(\prod_{g \in \Gamma} K^{-\infty}(\mathcal{B}(g \times (0, 1], g; R)^g) \right)^\Gamma$$

is the identity. On homotopy fixed sets we get a weak homotopy equivalence since the map is equivariant, and unequivariantly it is a weak homotopy equivalence. Finally

$$\left(\prod_{\Gamma} K^{-\infty}(\mathcal{B}((0, 1], 1; R)^1) \right)^\Gamma \rightarrow \left(\prod_{\Gamma} K^{-\infty}(\mathcal{B}((0, 1], 1; R)^1)^{h\Gamma} \right)^\Gamma$$

is a weak homotopy equivalence, see Remark 2.2. Assume inductively that E is obtained from N by attaching one free G -cell, e^n . Consider

$$\mathcal{B}(N \times (0, 1], N; R)^N \xrightarrow{f} \mathcal{B}(E \times (0, 1], E; R)^E \xrightarrow{g} \mathcal{B}(E \times (0, 1], E; R)^{E-N}.$$

Let's denote $\Omega K^{-\infty}$ applied to this sequence by $A \rightarrow B \rightarrow C$. The sequence $A^\Gamma \rightarrow B^\Gamma \rightarrow C^\Gamma$ is

$$(N/\Gamma)_+ \wedge K^{-\infty}R \rightarrow (E/\Gamma)_+ \wedge K^{-\infty}R \rightarrow (E/\Gamma)/(N/\Gamma) \wedge K^{-\infty}R,$$

by Theorem 2.9 and Remark 1.38 hence a fibration of spectra. The composite functor gf factors through $\mathcal{B}(N \times (0, 1], N; R)^\emptyset$ which is equivariantly equivalent to the trivial category. Unequivariantly $A \rightarrow B \rightarrow C$ is a homotopy fibration (A Steenrod functor applied to $N_+ \rightarrow E_+ \rightarrow E_+/N_+$). Letting D denote the homotopy fibre of $B \rightarrow C$ we get an equivariant map from $A \rightarrow D$ which unequivariantly is a weak homotopy equivalence, hence a weak homotopy equivalence of homotopy fixed sets. Since homotopy fibre and homotopy fixed sets commute we have shown that

$$A^{h\Gamma} \rightarrow B^{h\Gamma} \rightarrow C^{h\Gamma}$$

is a fibration sequence. By our induction hypothesis we have $A^\Gamma \rightarrow A^{h\Gamma}$ is a weak homotopy equivalence, so to finish off the proof it suffices to show that $C^\Gamma \rightarrow C^{h\Gamma}$ is a weak homotopy

equivalence. This however is entirely similar to the start of the induction. Since $E - N = \Gamma \times e^n$ where e^n is an open n -cell and we are considering germ categories we have

$$\mathcal{B}(E \times (0, 1], E; R)^{E-N} \cong \prod \mathcal{B}(g \cdot e^n \times (0, 1], g \cdot e^n; R)^{g \cdot e^n}$$

and the Γ action permutes the factors, so this completes the induction step \square

Corollary 2.12. $S^\Gamma \cong S^{h\Gamma}$.

Proof. Consider the equivariant functor combining taking germs and forgetting control

$$\mathcal{B}(CX, CY \cup X, p_X; R) \rightarrow \mathcal{B}(CX, CY \cup X, p_X; R)^E \rightarrow \mathcal{B}(E \times (0, 1], E; R)^E.$$

We want to show that when we apply $K^{-\infty}$ and take fixed and homotopy fixed sets respectively, we get a weak homotopy equivalence. On fixed sets this follows from Lemma 2.5 and Lemma 2.4. On homotopy fixed sets it follows since the functor is equivariant, and unequivariantly it induces a weak homotopy equivalence by Lemma 2.5. The proof is now finished by Theorem 2.11. \square

So far we have not used the assumption that X is contractible. That however is needed to show that $S^{h\Gamma} \rightarrow T^{h\Gamma}$ is a weak homotopy equivalence. The proof of Theorem 2.1 will be completed once we have

Theorem 2.13.

$$K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R))^{h\Gamma} \rightarrow K^{-\infty}(\mathcal{B}(\Sigma X, \Sigma Y, p_X; R))^{h\Gamma}$$

is a weak homotopy equivalence.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{B}(CX, CY \cup X; R) & \xrightarrow{b} & \mathcal{B}(\Sigma X, \Sigma Y; R) \\ \downarrow a & & \downarrow c \\ \mathcal{B}(CX, CY \cup X, p_X; R) & \xrightarrow{d} & \mathcal{B}(\Sigma X, \Sigma Y, p_X; R) \end{array}$$

we need to show that d induces a weak homotopy equivalence when applying $K^{-\infty}$ and taking homotopy fixed sets. Since d is equivariant it suffices to show that d induces a weak homotopy equivalence unequivariantly, we show this by showing that a, b and c induce weak homotopy equivalences. We can conclude that a is a weak homotopy equivalence by the following three reasons. First

$$K^{-\infty}(\mathcal{B}(CX, CY \cup X; R)) \rightarrow K^{-\infty}(\mathcal{B}(CX, CY \cup X; R)^E)$$

is a Steenrod functor applied to collapsing the contractible space CY , see Remark 1.38 and Theorem 1.25. Secondly, we have a weak homotopy equivalence

$$K^{-\infty}\mathcal{B}(CX, CY \cup X, p_X; R) \rightarrow K^{-\infty}(\mathcal{B}(CX, CY \cup X, p_X; R)^E)$$

by Lemma 2.5. Thirdly, the categories

$$\mathcal{B}(CX, CY \cup X; R)^E \text{ and } \mathcal{B}(CX, CY \cup X, p_X; R)^E$$

are equal by Lemma 2.4. Similarly c induces a weak homotopy equivalence because $K^{-\infty}$ applied to $\mathcal{B}(\Sigma X, \Sigma Y; R)$ is a deloop of $K^{-\infty}(\mathcal{B}(X, Y; R))$ by Theorem 1.23 and Theorem 1.36, but so is $K^{-\infty}(\mathcal{B}(\Sigma X, \Sigma Y, p_X; R))$ by an application of Theorem 1.28. Finally b induces a weak homotopy equivalence because it is a Steenrod functor applied to collapsing the contractible subset X by Theorem 1.37. \square

This completes the proof of Theorem 2.1

3. IDENTIFYING THE ASSEMBLY MAP

In the previous section we have described a map

$$B\Gamma_+ \wedge K^{-\infty}(R) \rightarrow K^{-\infty}(R[\Gamma]).$$

We called it the assembly map. In this section we justify this by proving the map is the same as what is usually called the assembly map. This is based on results of Weiss Williams [27]. To describe the result we need to recall some definitions from [27]: Let F be a homotopy invariant functor from finite complexes to spectra, sending the empty set to a contractible space. The functor F is called excisive if it sends a homotopy pushout of spaces to a homotopy pushout of spectra (i. e. if $\pi_*(F(B))$ is a homology theory). Weiss-Williams prove

Theorem 3.1. *Let G be a homotopy functor from finite CW complexes to spectra, such that $G(\emptyset)$ is contractible. Then there is an excisive functor $G^{\%}$ and a natural transformation $G^{\%}(B) \rightarrow G(B)$, which is the identity on a point. If $F(B) \rightarrow G(B)$ is a natural transformation from an excisive functor F which is a homotopy equivalence on a point, then $F(B) \simeq G^{\%}(B)$ by a homotopy equivalence making $F(B) \simeq G^{\%}(B) \rightarrow G(B)$ the given natural transformation. Furthermore $F(B) \simeq G^{\%}(B) \simeq B_+ \wedge F(*)$.*

The natural transformation $F(B) \rightarrow G(B)$ is called an assembly map, and the theorem thus says that a homotopy invariant functor has an essentially unique assembly map.

To apply this theorem, we need to describe two functors F and G . Let

$$F(B) = \Omega K^{-\infty}(\mathcal{B}(B \times (0, 1], B; R)^B)$$

Then F on the category of finite complexes is an excisive homotopy invariant functor to the category of spectra. We define G as follows: Let B be a finite complex, E a universal covering space for B , and Γ the group of covering transformations (so Γ is isomorphic to the fundamental group of B). Give E a length metric induced from B and let q be the projection on the second factor $E \times (0, 1] \rightarrow (0, 1]$. Notice that if X is a compactification of E satisfying conditions (i) and (iv) of Theorem 2.1, then $(\mathcal{B}(E \times (0, 1], E, q; R)^E)^\Gamma$ and

$(\mathcal{B}(CX, CY \cup Y, p_X; R)^E)^\Gamma$ are the same subcategories of $\mathcal{B}(E \times (0, 1))$. Hence we may try to define G by

$$G(B) = \Omega K^{-\infty}((\mathcal{B}(E \times (0, 1], E, q; R)^E)^\Gamma)$$

The problem is that the assignment of E to B is not functorial without a basepoint. This however is solved by the following lemma:

Lemma 3.2. *Let $p : E \rightarrow B$ be a universal covering and $g : E \rightarrow E$ a map so that $pg = p$. Then the induced map on $\mathcal{B}(E \times (0, 1], E, q; R)^E$ fixes $(\mathcal{B}(E \times (0, 1], E, q; R)^E)^\Gamma$*

Proof. By covering space theory, g is multiplication by some element γ . Hence g is equivariant with respect to the Γ -action and the Γ -action conjugated by γ , but the fixed category with respect to the Γ -action and the conjugated Γ -action is the same. \square

We thus do have that G is a functor of B . Applying $\Omega K^{-\infty}$ to the equivalence

$$\mathcal{B}(B \times (0, 1], B; R)^B \cong (\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma$$

followed by the forget control map

$$(\mathcal{B}(E \times (0, 1], E; R)^E)^\Gamma \rightarrow (\mathcal{B}(E \times (0, 1], E, q; R)^E)^\Gamma$$

gives a natural transformation $F(B) \rightarrow G(B)$ which by the above mentioned results of Weiss–Williams is the assembly map, since it is the identity on a point.

Remark 3.3. In the above discussion $G(B)$ is homotopy equivalent to $K^{-\infty}(R[\Gamma])$. This identification is of course not independent of choice of basepoint.

Remark 3.4. The category $\mathcal{B}(E \times (0, 1], E, q; R)$ is easily seen to be flasque, whenever E is noncompact, say $E = E\Gamma$ with $B\Gamma$ finite. It follows that $\mathcal{B}(E \times (0, 1], E, q; R)^E$ has trivial K-theory since $\mathcal{B}(E \times (0, 1], E, q; R)_\emptyset$ is also flasque. Hence we obtain, that whenever we can find a Γ -equivariant subcategory $\mathcal{U} \subset \mathcal{B}(E \times (0, 1], E, q; R)^E$ containing $(\mathcal{B}(E \times (0, 1], E, q; R)^E)^\Gamma$ such that $\mathcal{B}(E \times (0, 1], E; R)^E \rightarrow \mathcal{B}(E \times (0, 1], E, q; R)^E$ factors through \mathcal{U} by a functor inducing isomorphism in K-theory, we get a splitting of the assembly map. In this paper we choose \mathcal{U} by continuous control in a compactification of $E\Gamma$. One may always try to let \mathcal{U} be the subcategory where morphisms also are required to be bounded. This remark recovers a result from [5], saying that if bounded K-theory with labels in $E\Gamma$ is isomorphic to $h^{l.f.}(E\Gamma; K^{-\infty}(R))$, we get a splitting of the assembly map.

4. L-THEORY OF ADDITIVE CATEGORIES

In this section we recall and expand various results due to Ranicki. Specifically we prove:

Theorem 4.1. *Let \mathcal{U} be an \mathcal{A} -filtered additive category with involution preserving \mathcal{A} , thus inducing an involution on \mathcal{A} . Let K be the inverse image of $K_0(\mathcal{U})$ under $K_0(\mathcal{A}^\wedge) \rightarrow K_0(\mathcal{U}^\wedge)$. Then up to homotopy there is a fibration of spectra*

$$\mathbb{L}^h(\mathcal{A}^\wedge K) \rightarrow \mathbb{L}^h(\mathcal{U}) \rightarrow \mathbb{L}^h(\mathcal{U}/\mathcal{A})$$

Theorem 4.2. *Let \mathcal{U} be an \mathcal{A} filtered additive category with involution preserving \mathcal{A} . Then up to homotopy there is a fibration of spectra*

$$\mathbb{L}^{-\infty}(\mathcal{A}) \rightarrow \mathbb{L}^{-\infty}(\mathcal{U}) \rightarrow \mathbb{L}^{-\infty}(\mathcal{U}/\mathcal{A})$$

To explain the terms in these theorems we recall Ranicki's L-theory of additive categories with involution: Let \mathcal{U} be an additive category. We shall consider chain complexes U_* in \mathcal{U} . If not otherwise stated, the chain complexes are always supposed to be finite i. e. $U_i = 0$ except for finitely many i , but we do not require $U_i = 0$ in $i < 0$.

An additive category with involution \mathcal{U} , is an additive category together with a contravariant functor $*$: $\mathcal{U} \rightarrow \mathcal{U}$, sending U to U^* , and a natural equivalence $** \cong 1$. We shall be working with small categories, and in our applications $**$ will be equal to the identity.

Given a chain complex U_* , the n -dual chain complex U^{n-*} is defined by $(U^{n-*})_r = U^{n-r} = (U_{n-r})^*$, and boundary $(-1)^r$ times $*$ applied to the boundary map. If U_* and V_* are two chain complexes, $\text{Hom}(U_*, V_*)$ is the chain complex of abelian groups which in degree r is $\bigoplus_{p+q=r} \text{Hom}(U_p, V_q)$.

Let U_* be a chain complex, then $\text{Hom}(U^*, U_*)$ has a \mathbb{Z}_2 action given by $T(f) = (-1)^{pq} f^*$, $f : U^p \rightarrow U_q$ and T the nontrivial element in \mathbb{Z}_2 . Notice an n -cycle in $\text{Hom}(U^*, U_*)$ is a chain map $\phi : U^{n-*} \rightarrow U_*$.

Let W be the standard $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z}

$$W : \dots \rightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

define the $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex

$$W\%U = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (\text{Hom}(U^*, U_*)),$$

An n -chain ψ is a collection $\{\psi_s \in \text{Hom}(U^*, U_*)_{n-s} \mid s \geq 0\}$, so with an n -cycle ψ comes a chain map $\psi_0 : U^{n-*} \rightarrow U_*$. Ranicki defines an n -dimensional quadratic complex in \mathcal{U} , (U_*, ψ) to consist of a chain complex U_* in \mathcal{U} and an n -cycle $\psi \in W\%U_*$. The quadratic complex is called *Poincaré* if the chain map $(1+T)\psi_0 : U^{n-*} \rightarrow U_*$ is a chain homotopy equivalence. Similarly Ranicki defines quadratic pairs, and quadratic Poincaré pairs, so that bordism and gluing operations are defined as in a manifold category. The n -dimensional quadratic L^h -group $L_n^h(\mathcal{U})$ is defined to be the cobordism group of n -dimensional quadratic Poincaré complexes. Notice that the $(n+4)$ -dual of the double suspension of a chain complex U_* , is the double suspension of the n -dual of U_* , and this together with the 2-periodicity of W gives a 1-1 correspondence between n -dimensional and $n+4$ -dimensional Poincaré complexes. This correspondence is called double skew suspension, and it works in all dimensions (including n negative).

Important for our purposes is that to a quadratic n -dimensional complex, a pair is assigned so that the pair is Poincaré specifically [24, Prop. 13.1]

Proposition 4.3. *There is a natural one-one correspondence between the homotopy equivalence classes of n -dimensional Poincaré pairs in \mathcal{U} and the homotopy equivalence classes of n -dimensional quadratic complexes in \mathcal{U} .*

This correspondence is given by sending the pair to the algebraic mapping cone. Roughly speaking, the algebraic mapping cone of the duality map of an n -dimensional quadratic complex is a null cobordant $(n - 1)$ -dimensional Poincaré complex.

Ranicki relates this categorical approach to the usual approach in the following theorems: Let R be a ring with involution

Theorem 4.4. *If \mathcal{U} is the category of f. g. free R -modules, and involution given by duality, then $L_n^h(\mathcal{U})$ coincides with the usual L -groups $L_n^h(R)$.*

Idempotent completion enters as related to L^p -groups.

Theorem 4.5. *Let \mathcal{U} be as above, If K is an involution invariant subgroup of $K_0(\mathcal{U}^\wedge)$, then $L_n^h(\mathcal{U}^{\wedge K}) = L_n^K(R)$*

The L -groups being defined as bordism groups, Ranicki [23] defines a (Kan)- Δ -set $\mathbb{L}_i^h(\mathcal{U})$ where the n -simplexes are n -ads of $n + i$ -dimensional Poincaré complexes. This Δ -set is naturally based by the 0-chain complex. A Δ -set model for the circle has one 0-simplex, the basepoint, and two 1-simplexes, the basepoint and one more simplex. The Δ -set model for the loop space is determined by what that one-simplex is sent to, so

$$\Omega \mathbb{L}_i^h(\mathcal{U}) = L_{i+1}^h(\mathcal{U})$$

in the world of Δ -sets. Upon realization of the Δ -set there is a map

$$|\Omega \mathbb{L}_i^h(\mathcal{U})| \rightarrow \Omega |\mathbb{L}_i^h(\mathcal{U})|$$

which is a homotopy equivalence. The skew double suspension

$$\mathbb{L}_i^h(\mathcal{U}) \rightarrow \mathbb{L}_{i+4}^h(\mathcal{U})$$

which is the double suspension of the chain complex, and the identity on the quadratic structure gives an isomorphism of Δ -sets. In Ranicki's original treatment of algebraic Poincaré complexes an n -dimensional Poincaré complex was assumed to be concentrated between dimensions 0 and n . By giving that up, and only assuming chain complexes to be finite [23] the 4-periodicity becomes exact, and negative dimensional Poincaré complexes make perfectly good sense. Realization of the Δ -sets $\mathbb{L}_i^h(\mathcal{U})$ now gives a four periodic spectrum. We denote this spectrum, with homotopy groups, the L -groups of \mathcal{U} functorially assigned to \mathcal{U} by $\mathbb{L}^h(\mathcal{U})$. We wish to establish the analogue of Theorem 1.28 for this spectrum.

Definition 4.6. Let \mathcal{A} be a full subcategory of \mathcal{U} . A chain complex U_* is \mathcal{A} -dominated if there is a chain complex C_* in \mathcal{A} and chain maps $r : C_* \rightarrow U_*$ and $i : U_* \rightarrow C_*$ such that ri is chain homotopic to the identity.

Recall the idempotent completion \mathcal{U}^\wedge of an additive category \mathcal{U} has objects (U, p) , $p : U \rightarrow U, p^2 = p$ and $\text{Hom}((U, p), (V, q)) \subset \text{Hom}(U, V)$, the subset for which $q\phi p = \phi$. If $K \subset K_0(\mathcal{U}^\wedge)$ is a subgroup, we denote the full subcategory of \mathcal{U}^\wedge with objects (A, p) so that the stable isomorphism class of (A, p) lies in K by $\mathcal{U}^{\wedge K}$.

We have an inclusion of \mathcal{U} in \mathcal{U}^\wedge as a full subcategory sending A to $(A, 1_A)$. Clearly this inclusion factors through $\mathcal{U}^{\wedge K_0(\mathcal{U})}$.

Proposition 4.7. *Let \mathcal{U} be an \mathcal{A} -filtered category. A chain complex U_* in \mathcal{U} is \mathcal{A} -dominated if and only if the induced \mathcal{U}/\mathcal{A} -chain complex is contractible.*

Proof. Assume U_* is \mathcal{A} -dominated by C_* . The induced \mathcal{U}/\mathcal{A} -complex of C_* is isomorphic to the 0-chain complex, hence U_* is \mathcal{U}/\mathcal{A} -homotopy equivalent to the 0-chain complex, so contractible. If U_* is \mathcal{A}/\mathcal{U} -contractible, we have maps $\bar{\Gamma}_i : U_i \rightarrow U_{i+1}$ in \mathcal{U}/\mathcal{A} so that $d_{i+1}\bar{\Gamma}_i + \bar{\Gamma}_{i-1}d_i = 1$ in \mathcal{U}/\mathcal{A} . Choose representatives Γ_i in \mathcal{U} for $\bar{\Gamma}_i$. Assume $U_i = 0$ for $i > n$. The map $1 - \Gamma_{n-1}d_n : U_n \rightarrow U_n$ factors through \mathcal{A} , hence (since \mathcal{U} is \mathcal{A} -filtered) we may choose $U_n = A_n \oplus V_n$ so that $1 - \Gamma_{n-1}d_n$ factors through A_n . Next decompose $U_{n-1} = A_{n-1} \oplus V_{n-1}$ so that $d_n|_{A_n}$ factors through A_{n-1} and $1 - d_n\Gamma_{n-1} - \Gamma_{n-2}d_{n-1} : U_{n-1} \rightarrow U_{n-1}$ factors through A_{n-1} . Continuing this process we get a chain complex A_* in \mathcal{A} which we claim dominates U_* . We define $r : A_* \rightarrow U_*$ to be the inclusion and $i : U_* \rightarrow A_*$ to be $1 - \Gamma d - d\Gamma$. It is easy to see that i is a chain map and $ri = 1 - \Gamma d - d\Gamma$ so Γ is a chain homotopy from ri to the identity. \square

This is an algebraic analogue of the connection established in [25] between a proper homotopy equivalence and a finitely dominated space.

We need the following lemma from [22]

Lemma 4.8. *Let \mathcal{A} be a full subcategory of \mathcal{U} , U_* an \mathcal{A} -dominated chain complex in \mathcal{U} . Let K be the inverse image of $K_0(\mathcal{U})$ under the induced map $K_0(\mathcal{A}^\wedge) \rightarrow K_0(\mathcal{U}^\wedge)$, and let $\mathcal{U}^{\wedge K}$ be the full subcategory of \mathcal{U}^\wedge with objects $(A, p) \oplus U$, $[(A, p)] \in K$. Then the induced chain complex in $\mathcal{U}^{\wedge K}$, under the inclusion $\mathcal{U} \rightarrow \mathcal{U}^{\wedge K}$ is chain homotopy equivalent to a chain complex in $\mathcal{A}^{\wedge K}$,*

Proof. The explicit formulas in [22] show that U_* is homotopy equivalent to an infinite chain complex of the form

$$\dots A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} A \rightarrow A_{n-1} \rightarrow \dots \rightarrow$$

But in \mathcal{A}^\wedge this is homotopy equivalent to

$$0 \rightarrow (A, p) \rightarrow (A_{n-1}, 1) \rightarrow \dots \rightarrow$$

and $[(A, p)] \in K_0(\mathcal{A}^\wedge)$ must map to an element of $K_0(\mathcal{U})$. \square

We may now prove the main theorem of this section. Special cases of this theorem have been proved by Ranicki.

Proof of theorem 4.1. Let $\mathcal{U}^{\wedge K}$ be the full subcategory of \mathcal{U}^{\wedge} with objects $(A, p) \oplus U$, where (A, p) is an object of $\mathcal{A}^{\wedge K}$. Clearly $\mathcal{U}^{\wedge K}$ is $\mathcal{A}^{\wedge K}$ -filtered, and $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}$ is equivalent to \mathcal{U}/\mathcal{A} . By the bordism approach to L -theory we get a fibration

$$\mathbb{L}^h(\mathcal{A}^{\wedge K}) \rightarrow \mathbb{L}^h(\mathcal{U}^{\wedge K}) \rightarrow \mathbb{L}^h(\mathcal{A}^{\wedge K} \rightarrow \mathcal{U}^{\wedge K}).$$

We have $\mathbb{L}^h(\mathcal{U}) \rightarrow \mathbb{L}^h(\mathcal{U}^{\wedge K})$ is a weak homotopy equivalence since any chain complex in $\mathcal{U}^{\wedge K}$ is homotopy equivalent to a chain complex in \mathcal{U} . Given a Poincaré pair $((A_*, \psi_A) \rightarrow (U_*, \psi_U))$ with A_* in $\mathcal{A}^{\wedge K}$ and U_* in $\mathcal{U}^{\wedge K}$, the quadratic chain complex (U_*, ψ_U) in $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}$ is Poincaré, since chain complexes in $\mathcal{A}^{\wedge K}$ are equivalent to the 0-chain complex in $\mathcal{U}^{\wedge K}/\mathcal{A}^{\wedge K}$. Applying this to n -ads of Poincaré pairs induces a map $\mathbb{L}(\mathcal{A}^{\wedge K} \rightarrow \mathcal{U}^{\wedge K}) \rightarrow \mathbb{L}(\mathcal{U}/\mathcal{A})$. Given a chain complex \bar{U}_* in \mathcal{U}/\mathcal{A} . We want to find a chain complex U_* in \mathcal{U} such that the induced chain complex in \mathcal{U}/\mathcal{A} is isomorphic to \bar{U}_* . Choose representatives $\bar{d}_i : \bar{U}_i \rightarrow \bar{U}_{i-1}$ in \mathcal{U} for the boundary morphisms. This may not be a chain complex because $\bar{d}_{i-1}\bar{d}_i$ is not necessarily 0. It factors through \mathcal{A} however, so we define U_* inductively by $U_0 = \bar{U}_0$, $U_1 = \bar{U}_1$ and $d_1 = \bar{d}_1$. Write $\bar{U}_2 = A_2 \oplus U_2$ so that $\bar{d}_2 d_1$ factors $U_2 \oplus A_2 \rightarrow A_2 \rightarrow \bar{U}_0$, and let d_2 be $U_2 \rightarrow U_2 \oplus A_2 \xrightarrow{d_2} U_1$. Then $d_2 d_1 = 0$ and renaming \bar{d}_3 to be the composite $\bar{U}_3 \rightarrow \bar{U}_2 \rightarrow U_2$, we may decompose \bar{U}_3 similarly, and by a finite induction we get the chain complex U_* . Noting that the projection maps $\bar{U}_i \rightarrow U_i$ induce isomorphisms in \mathcal{U}/\mathcal{A} it is clear that the induced chain complex in \mathcal{U}/\mathcal{A} is isomorphic to \bar{U}_* . Similarly a quadratic Poincaré structure $\bar{\psi}$ on \bar{U}_* lifts to a quadratic structure on U_* , by choosing representatives, but not necessarily to a quadratic Poincaré structure. Since quadratic complexes are in one to one correspondence with Poincaré pairs (The correspondence being the algebraic mapping cone) there is a Poincaré pair $(A_* \rightarrow V_*)$ such that the algebraic mapping cone is homotopy equivalent to U_* . In \mathcal{U}/\mathcal{A} , U_* is Poincaré which is the same as saying that A_* is contractible by Proposition 4.7. That in turn by Lemma 4.8 means that A_* is homotopy equivalent to a chain complex in $\mathcal{A}^{\wedge K}$, so we have found a Poincaré pair in $\mathcal{A}^{\wedge K} \rightarrow \mathcal{U}^{\wedge K}$ mapping to \bar{U}_* , hence

$$\mathbb{L}(\mathcal{A}^{\wedge K} \rightarrow \mathcal{U}^{\wedge K}) \rightarrow \mathbb{L}(\mathcal{U}/\mathcal{A})$$

is an epimorphism on homotopy groups. A relative version of this argument shows it is a monomorphism. \square

So far we have not discussed torsion. Following Ranicki, given an additive category \mathcal{U} with involution, and a system of stable isomorphisms $\phi_{X,Y} : X \rightarrow Y$ so that any composite which happens to be an automorphism represents $0 \in K_1(\mathcal{U})$, and given a $*$ -invariant subgroup S of $K_1(\mathcal{U})$, one may define the groups $L_n^S(\mathcal{A})$ and the simple \mathbb{L} -spectra $\mathbb{L}_i^S(\mathcal{A})$. Slightly more general, if we only require that any composite which happens to be an automorphism lies in a $*$ -invariant subgroup H of $K_1(\mathcal{A})$, $\mathbb{L}_i^S(\mathcal{A})$ may be defined for any $*$ -invariant subgroup such that $H \subset S$. By methods entirely analogous to the above one obtains

Theorem 4.9. *Let \mathcal{U} be an \mathcal{A} -filtered additive category with involution. Assume \mathcal{A} and \mathcal{U} have compatible systems of stable isomorphisms. Then we have a quasifibration*

$$\mathbb{L}^s(\mathcal{A}) \rightarrow \mathbb{L}^s(\mathcal{U}) \rightarrow \mathbb{L}^K(\mathcal{U}/\mathcal{A})$$

where $K = \text{Im}(K_1(\mathcal{U}) \rightarrow K_1(\mathcal{U}/\mathcal{A}))$, and s refers to the trivial subgroup.

Remark 4.10. If E is non-compact, it is usually quite easy to find a system of stable isomorphisms for the various subcategories of $\mathcal{B}(E; R)$, by Eilenberg swindle on the *objects*.

In the above remark it is important to note that we only have Eilenberg swindle on the objects, not a functorial Eilenberg swindle, since this would imply that the L -theory of the category vanishes as seen in the next Lemma.

Definition 4.11. An additive category \mathcal{A} with involution is flasque if there is a functor $\Sigma^\infty : \mathcal{A} \rightarrow \mathcal{A}$ and natural equivalences $\Sigma^\infty * \cong * \Sigma^\infty$ and $1 \oplus \Sigma^\infty \cong \Sigma^\infty$.

Lemma 4.12. *If \mathcal{A} is flasque then $L_i(\mathcal{A}) = 0$.*

Proof. Let (A_*, ν) be an element of $L_i(\mathcal{A})$. Then

$$(\Sigma^\infty A_*, \Sigma^\infty \nu) \cong (\Sigma^\infty A_* \oplus A, \Sigma^\infty \nu \oplus \nu)$$

hence $(A_*, \nu) = 0$. □

Let $\mathcal{C}_i(-)$ be the functor from small additive categories to itself with objects parameterized by the \mathbb{R}^i , and bounded morphisms. We recover a theorem due to Ranicki.

Theorem 4.13.

$$L_{n+1}^s(\mathcal{C}_{i+1}(\mathcal{U})) \cong L_n^h(\mathcal{C}_i(\mathcal{U}))$$

and

$$L_{n+1}^h(\mathcal{C}_{i+1}(\mathcal{U})) \cong L_n^h(\mathcal{C}_i(\mathcal{U})^\wedge).$$

In particular

$$L_{n+1}^s(\mathcal{C}_1(\mathcal{U})) \cong L_n^h(\mathcal{U})$$

and

$$L_{n+1}^h(\mathcal{C}_1(\mathcal{U})) \cong L_n^h(\mathcal{U}^\wedge).$$

Proof. We prove the particular case, since the general case is entirely similar. Let \mathcal{A} be the full subcategory of $\mathcal{C}_1(\mathcal{U})$ with objects A such that there is an n with $A_i = 0$ for $i > n$ sufficiently large. Clearly \mathcal{A} is flasque so

$$\text{Im}(K_1(\mathcal{U}) \rightarrow K_1(\mathcal{C}_1(\mathcal{U}/\mathcal{A})) = K_1(\mathcal{C}_1(\mathcal{U}/\mathcal{A}))$$

and the quasifibration

$$\mathbb{L}^h(\mathcal{A}) \rightarrow \mathbb{L}^h(\mathcal{C}_1(\mathcal{U})) \rightarrow \mathbb{L}^h(\mathcal{C}_1(\mathcal{U}/\mathcal{A}))$$

shows that $L_n^h(\mathcal{C}_1(\mathcal{U})) \cong L_n^h(\mathcal{C}_1(\mathcal{U})/\mathcal{A})$. Let $\mathcal{C}_+(\mathcal{U})$ be the full subcategory of $\mathcal{C}_1(\mathcal{U})$ with objects A such that there is an n with $A_i = 0$ for $i < n$. The category $\mathcal{C}_+(\mathcal{U})$ is flasque, and

$\mathcal{C}_+(\mathcal{U})$ is \mathcal{B} -filtered, where \mathcal{B} is the full subcategory of $\mathcal{C}_1(\mathcal{U})$ with objects that are 0 both at large positive and negative values of the point in \mathbb{R} . We thus get $L_n^h(\mathcal{B}) \cong L^s(\mathcal{C}_+(\mathcal{U})/\mathcal{B})$ but \mathcal{B} is equivalent to \mathcal{U} and $\mathcal{C}_+(\mathcal{U})/\mathcal{B}$ is equivalent to $\mathcal{C}_1(\mathcal{U})/\mathcal{A}$ and we are done. \square

Following Ranicki we may define

Definition 4.14. Let \mathcal{A} be an additive category with involution

$$L_n^{-i}(\mathcal{A}) = L_{n+i+2}^s(\mathcal{C}_{i+2}(\mathcal{A}))$$

where we write s for $-i$ when $-i = 2$, h for $-i$ when $-i = 1$, and p for $-i$ when $-i = 0$.

The point of Ranicki's definition is that he shows it agrees with the usual definitions in case \mathcal{A} is the category of finitely generated R -modules, R a ring with involution.

Lemma 4.15. *If a functor $f : \mathcal{A} \rightarrow \mathcal{B}$ of additive categories with involution induces isomorphism on K -theory (when restricted to isomorphisms) and on L -theory with one decoration, then it induces isomorphism on L -theory with any decoration.*

Proof. This follows from Ranicki-Rothenberg exact sequences. \square

Let \mathcal{A} be an additive category and consider the inclusion

$$\mathcal{C}_i(\mathcal{A}) \subset \mathcal{C}_{i+1}(\mathcal{A})$$

The map on Δ -sets

$$\mathbb{L}_j^h(\mathcal{C}_i(\mathcal{A})) \rightarrow \mathbb{L}_j^h(\mathcal{C}_{i+1}(\mathcal{A}))$$

is homotopic to the constant map in two ways: The cellular chain complex of $([0, \infty), 0)$ obtained by subdividing at integral lattice points

$$\begin{array}{ccccccccc} 0 & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 0 \\ \downarrow & \swarrow^{-1} & \downarrow & \swarrow^{-1} & \downarrow & \swarrow^{-1} & \downarrow & \swarrow^{-1} & \downarrow & & \downarrow \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & \cdots & \mathbb{Z} \end{array}$$

defines an element in symmetric L -theory

$$L^1(\mathcal{C}_0(\mathbb{Z}) \rightarrow \mathcal{C}_{[0, \infty)}(\mathbb{Z}))$$

and pairing with this element gives a map

$$\mathbb{L}_j^h(\mathcal{C}_i(\mathcal{A})) \rightarrow \mathbb{L}_{j+1}^h(\mathcal{C}_{i+1}(\mathcal{A}))$$

which is a Δ -set homotopy from the inclusion to the constant map. (The element in $L^1(\mathcal{C}_0(\mathbb{Z}) \rightarrow \mathcal{C}_{[0, \infty)}(\mathbb{Z}))$ is actually the generator under the isomorphism $L^1(\mathcal{C}_0(\mathbb{Z}) \rightarrow \mathcal{C}_{[0, \infty)}(\mathbb{Z})) \cong L^0(\mathbb{Z}) \cong \mathbb{Z}$). Similarly pairing with the chain complex of $[0, -\infty)$ gives a null homotopy, so combined we get a map

$$\Sigma \mathbb{L}^h(\mathcal{C}_i(\mathcal{A})) \rightarrow \mathbb{L}^h(\mathcal{C}_{i+1}(\mathcal{A}))$$

and hence

$$\mathbb{L}^h(\mathcal{C}_i(\mathcal{A})) \rightarrow \Omega\mathbb{L}^h(\mathcal{C}_{i+1}(\mathcal{A}))$$

and the composite

$$\mathbb{L}^h(\mathcal{C}_i(\mathcal{A})) \rightarrow \Omega\mathbb{L}^h(\mathcal{C}_{i+1}(\mathcal{A})) \rightarrow \Omega\mathbb{L}^h(\mathcal{C}_{i+1}(\mathcal{A})^\wedge)$$

is a weak homotopy equivalence, since on homotopy groups this is exactly the map described in Theorem 4.13.

Definition 4.16. The spectrum $\mathbb{L}^{-\infty}(\mathcal{A})$ is defined to be the homotopy colimit of

$$\mathbb{L}^h(\mathcal{A}) \rightarrow \Omega\mathbb{L}^h(\mathcal{C}_1(\mathcal{A})) \rightarrow \Omega^2\mathbb{L}^h(\mathcal{C}_2(\mathcal{A})) \rightarrow \dots$$

It is clear that $\mathbb{L}^{-\infty}$ is a functor from the category of small additive categories and lax involution preserving functors to the category of spectra.

Lemma 4.17. *Let \mathcal{B} be an additive category. If \mathcal{A} is a full subcategory, inducing isomorphism on K_0 , then*

$$\mathbb{L}(\mathcal{A}) \rightarrow \mathbb{L}(\mathcal{B})$$

is a weak homotopy equivalence (any decoration on the L -theory).

Proof. We need to show the map induces isomorphism on homotopy groups, but any chain complex in \mathcal{B} is homotopy equivalent to a chain complex in \mathcal{A} , so from this it follows that $L^h(\mathcal{A}) \rightarrow L^h(\mathcal{B})$ is an isomorphism, and since the inclusion induces isomorphism on K -theory Lemma 4.15 finishes off the proof. \square

Finally we are ready for

Proof of Theorem 4.2. We get this result using Theorem 4.1. For any additive category with involution \mathcal{C} we have $\mathbb{L}^h(\mathcal{C}) \simeq \mathbb{L}^h(\mathcal{C}^{\wedge 0})$ by Lemma 4.17 above. Let

$$K_i = \ker(K_0(\mathcal{C}_i(\mathcal{U})) \rightarrow K_0(\mathcal{C}_i(\mathcal{U}/\mathcal{A}))).$$

Using Theorem 4.1, we then have fibrations

$$\mathbb{L}^h(\mathcal{C}_i(\mathcal{A})^{\wedge K_i}) \rightarrow \mathbb{L}^h(\mathcal{C}_i(\mathcal{U})) \rightarrow \mathbb{L}^h(\mathcal{C}_i(\mathcal{U})/\mathcal{C}_i(\mathcal{A}))$$

Noting that $\mathcal{C}_i(\mathcal{U})/\mathcal{C}_i(\mathcal{A})$ is equivalent to $\mathcal{C}_i(\mathcal{U}/\mathcal{A})$ by an equivalence which is the identity on objects, the proof is completed by contemplating the diagram

$$\begin{array}{ccccccc} \mathbb{L}^h(\mathcal{A}^{\wedge K_0}) & & \Omega\mathbb{L}^h(\mathcal{C}_1(\mathcal{A})^{\wedge K_1}) & & \Omega^2\mathbb{L}^h(\mathcal{C}_2(\mathcal{A})^{\wedge K_2}) & & \nearrow \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ \Omega\mathbb{L}^h(\mathcal{C}_1(\mathcal{A})^{\wedge 0}) & \longrightarrow & \Omega^2\mathbb{L}^h(\mathcal{C}_2(\mathcal{A})^{\wedge 0}) & \longrightarrow & \Omega^3\mathbb{L}^h(\mathcal{C}_3(\mathcal{A})^{\wedge 0}) & \longrightarrow & \dots \end{array}$$

which shows that taking homotopy colimit of the upper row is the same as taking homotopy colimit of the lower row. \square

5. THE L-THEORY SPLITTING

In this section we apply the results of the last section to obtain a splitting of the L-theory assembly map. Throughout this section R will be a ring with involution satisfying that $K_{-i}(R) = 0$ for sufficiently large i . The proofs are extremely similar to the K-theory proofs, given the requisite techniques from the preceding section. First however we need to worry about the involution. Given an object of one of the categories $\mathcal{B}(X; R), \mathcal{B}(X, Y), \dots$, we denote the full subcategory with objects A such that A_x has a basis of the form $\{(x, i)\}$ by $\mathcal{B}_b(X; R), \mathcal{B}_b(X, Y; R), \dots$. All the results of section 1 hold for these categories since they are cofinal and have the same K_0 .

If R is a ring with involution, we get an involution on $\mathcal{B}_b(X; R)$ as follows: On objects $*$ is the identity. On a morphism $\phi : A \rightarrow B$, we define $\phi^* : B^* \rightarrow A^*$ by $(\phi^*)_y^x : B_x^* \rightarrow A_y^*$ is the map $B_x \rightarrow A_y$ with matrix, the conjugate transpose of $\phi_x^y : A_y \rightarrow B_x$. Here elements of the ring R are conjugated by the involution on the ring.

Remark 5.1. We are identifying A_x with its dual via the basis, and dual basis, and use the matrix description of the dual map.

We shall prove L-theory results similar to the K-theory results, replacing Theorem 1.28 by Theorem 4.1 and Theorem 4.2.

Notice equivalent additive categories (with equivalence preserving the involution) have the same L-theory.

As in K-theory we get

Lemma 5.2. *Let X be a compact metrizable space, Y a closed subspace, so that $X - Y$ is dense in X . Let C be a closed subset of Y , W an open subset of Y with $C \subset W$. We then have fibrations up to homotopy*

$$\mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)_C) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)^{Y-C})$$

and

$$\mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)_C^W) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)^W) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}(X, Y; R)^{W-C})$$

Proof. Direct from Lemma 1.29 and Theorem 4.2. □

Lemma 5.3. *Let X be a compact metrizable space, Y a closed subspace, so that $X - Y$ is dense in X . The natural map*

$$\mathcal{B}_b(X, Y; R) \rightarrow \mathcal{B}_b(X, Y; R)^{Y-*}$$

induces an isomorphism in L-theory with any decoration.

Proof. The flasque structure on $\mathcal{B}_b(X, Y; R)_*$ preserves the involution, so this follows from Lemma 4.12 and Lemma 5.2. □

Theorem 5.4. *Let X be a compact metrizable space, R a ring with involution. Then $\mathbb{L}_*^{-\infty}(\mathcal{B}_b(CX, X; R))$ is a Steenrod functor, associated to the 4-periodic $\mathbb{L}^{-\infty}(R)$ -spectrum.*

Proof. We need to verify the conditions of Definition 1.34. In [1] it is proved that the K-theory of $\mathcal{B}(CCX, CX; R)$ is 0 by showing one can write $\mathcal{B}(CCX, CX; R)$ as the union of flasque categories. If (C_*, ν) represents an element in $L_n^h(\mathcal{B}_b(CCX, CX; R))$, the (C_*, ν) will have to be in one of these flasque categories, hence it must represent zero, so $L^h(\mathcal{B}_b(CCX, CX; R)) = 0$, and hence so is $L^{-\infty}(\mathcal{B}_b(CCX, CX; R))$ by Lemma 4.15. Assume \mathcal{A}_i are additive categories with involution satisfying that there exists j independent of i , so that $K_{-j}(\mathcal{A}_i) = 0$. It is clear that $\mathbb{L}_n^h(\amalg \mathcal{A}_i) \simeq \amalg \mathbb{L}_n^h(\mathcal{A}_i)$ as Δ -sets since a quadratic Poincaré complex in a product category is just a product of quadratic Poincaré complexes and any quadratic Poincaré complex is represented by a length one or 2 chain complex. Since these Δ -sets satisfy the Kan condition, we get a weak homotopy equivalence of spectra $\mathbb{L}^h(\amalg \mathcal{A}_i) \simeq \amalg \mathbb{L}^h(\mathcal{A}_i)$ upon realization. Combining with the K-theory result and Rothenberg-Ranicki exact sequences we get

$$\mathbb{L}^{-\infty}(\amalg \mathcal{A}_i) \simeq \amalg \mathbb{L}^{-\infty}(\mathcal{A}_i)$$

It thus follows that

$$\mathbb{L}^{-\infty}(\mathcal{B}_b(C(\bigvee X_\alpha), \bigvee X_\alpha; R)) \simeq \amalg \mathbb{L}^{-\infty}(\mathcal{B}_b(CX_\alpha, X_\alpha; R))$$

since the K-theory proof only involved flasque subcategories and equivalences of categories. Finally using Theorem 4.2, we get a fibration

$$\mathbb{L}^{-\infty}(\mathcal{B}_b(CX, X; R)_A) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}_b(CX, X; R)) \rightarrow \mathbb{L}^{-\infty}(\mathcal{B}_b(CX, X; R)^{X-A})$$

but then the proof is finished as in Theorem 1.36, using Lemma 5.3. \square

We can now formulate and prove a splitting theorem in L -theory.

Theorem 5.5. *Let Γ be a group satisfying the conditions in the introduction, and let R be a ring with involution so that $K_{-i}(R) = 0$ for sufficiently large i . Then $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R)$ is a split factor of $\mathbb{L}^{-\infty}(R\Gamma)$.*

Proof. Having provided the requisite technical tools in Theorem 4.2, and Theorem 5.4, the proof proceeds formally as in the K-theory case: Apply $\Omega\mathbb{L}^{-\infty}$ to

$$\mathcal{B}(CX, CY \cup X, p_X; R) \rightarrow \mathcal{B}(\Sigma X, \Sigma Y, p_X; R)$$

and call the resulting spectra S and T . We now study the diagram

$$\begin{array}{ccc} S^\Gamma & \longrightarrow & T^\Gamma \\ \downarrow & & \downarrow \\ S^{h\Gamma} & \longrightarrow & T^{h\Gamma} \end{array}$$

and prove the statements

- (i) $S^\Gamma \simeq (E/\Gamma)_+ \wedge \mathbb{L}^{-\infty}(R)$.
- (ii) $T^\Gamma \simeq \mathbb{L}^{-\infty}(R\Gamma)$.
- (iii) $S^\Gamma \simeq S^{h\Gamma}$.

(iv) $S^{h\Gamma} \simeq T^{h\Gamma}$

(i) and (ii) follow from Theorem 4.2, (iii) is an inductive argument over the cells as in the K-theory case using Theorem 4.2, and the above mentioned fact that L -theory commutes with products. Finally (iv) follows because $\mathbb{L}^{-\infty}$ is a Steenrod functor, and we are collapsing a contractible subspace. \square

Finally we need to identify the splitting map with the usual assembly map.

Theorem 5.6. *The map of spectra $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R) \rightarrow \mathbb{L}^{-\infty}(R[\Gamma])$ is the usual assembly map.*

Proof. This follows formally arguing as in Section 3 \square

This is needed to see that our results imply the Novikov conjecture for the class of groups considered, see [23, Prop. 24.5]. It also has the useful consequence that a diagram of assembly maps

$$\begin{array}{ccc} B\Gamma_+ \wedge \mathbb{L}^q(R) & \longrightarrow & \mathbb{L}^q(R[\Gamma]) \\ \downarrow & & \downarrow \\ B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R) & \longrightarrow & \mathbb{L}^{-\infty}(R[\Gamma]) \end{array}$$

will be commutative. Here q denotes any decoration. In particular if the decoration is chosen so that $\mathbb{L}^q(R) \cong \mathbb{L}^{-\infty}(R)$ we obtain the splitting with other decorations than $-\infty$. This can also be proved more directly using Theorem 4.1.

Remark 5.7. The analogue of Remark 3.4 holds in L -theory as well.

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