PROJECTIVE SURGERY THEORY

ERIK KJÆR PEDERSEN AND ANDREW RANICKI

0. Introduction

A simple (resp. finite) n-dimensional Poincaré complex X ($n \geq 5$) is simple homotopy (resp. homotopy) equivalent to a compact n-dimensional CAT (= DIFF, PL or TOP) manifold if and only if the Spivak normal fibration ν_X admits a CAT reduction for which the corresponding normal map $(f, b) : M \to X$ from a compact CAT manifold M has Wall surgery obstruction $\sigma^s_*(f, b) = 0 \in L^s_n(\pi_1(X))$ (resp. $\sigma^h_*(f, b) = 0 \in L^h_n(\pi_1(X))$). The surgery obstruction groups $L^s_*(\pi)$ (resp. $L^h_*(\pi)$) of a group π are defined algebraically as Witt groups of quadratic structures on finitely based (resp. f. g. free) $\mathbb{Z}[\pi]$-modules, and geometrically as bordism groups of normal maps to simple (resp. finite) Poincaré complexes X with fundamental group $\pi_1(X) = \pi$.

The object of this paper is to extend the above theory to finitely dominated Poincaré complexes, that is Poincaré complexes in the sense of Wall [18], and to the Witt group $L^p_*(\pi)$ of quadratic structures on f. g. projective $\mathbb{Z}[\pi]$-modules introduced by Novikov [8], the groups denoted by $U_*(\mathbb{Z}[\pi])$ in Ranicki [12].

A normal map $(f, b) : M \to X$ from a compact n-dimensional manifold M to a finitely dominated Poincaré complex X has a normal bordism invariant, the "projective surgery obstruction"

$$\sigma^p_*(f, b) \in L^p_n(\pi_1(X)),$$

such that

$$\sigma^h_*((f, b) \times 1 : M \times S^1 \to X \times S^1) = (0, \sigma^p_*(f, b)) \in L^h_{n+1}(\pi_1(X \times S^1)) = L^h_{n+1}(\pi_1(X)) \oplus L^p_n(\pi_1(X)).$$

Thus a finitely dominated n-dimensional Poincaré complex X ($n \geq 4$) has $X \times S^1$ homotopy equivalent to a compact $(n+1)$-dimensional CAT manifold if and only if ν_X admits a CAT reduction for which the corresponding normal map $(f, b) : M \to X$ has projective surgery obstruction $\sigma^p_*(f, b) = 0 \in L^p_n(\pi_1(X))$. The point we are making here is that the Browder-Novikov transversality construction of normal maps from CAT reductions of ν_X applies equally well to finitely dominated Poincaré complexes X.

Partially supported by the Danish Natural Science Research Council.
Partially supported by the NSF Grant MCS 79-02017.

Given a space K let $L_n^{1,p}(K)$ be the bordism group of normal maps from compact n-dimensional manifolds to finitely dominated Poincaré complexes equipped with a reference map to K, defined exactly as the geometric L-groups $L_n^1(K)$ of §9 of Wall [19]. Our main result (Theorem 2.1) identifies $L_n^{1,p}(K) = L_n^p(\pi_1(K))$, by analogy with the identification $L_n'(K) = L_n(\pi_1(K))$ of §9 of Wall [19] (where $L \equiv L^s$).

The projective L-groups $L_n^p(\pi)$ have been previously interpreted geometrically by Maumary [6, 7] and Taylor [17], using normal maps from paracompact manifolds to "open" Poincaré complexes (which are quite distinct from finitely dominated Poincaré complexes). The various interpretations are discussed and compared in §7.

1. Algebraic L-Groups

Given a group π and a group morphism $w : \pi \to \{\pm 1\}$ let $L_n^s(\pi)$ (resp. $L_n^h(\pi)$, $L_n^p(\pi)$) be the algebraic L-groups defined for n (mod 4) using quadratic structure on bases (resp. f. g. free, f. g. projective) $\mathbb{Z}[\pi]$-modules, with respect to the involution

$$^- : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]; \quad \sum_{g\in\pi} n_g g \mapsto \sum_{g\in\pi} w(g) n_g g^{-1} \qquad (n_g \in \mathbb{Z}).$$

The L^s-groups are the original surgery obstruction groups of Wall [19]; the L^h-groups are due to Shaneson [14]; the L^p-groups are due to Novikov [8]. We recall the definitions of the various L-groups, as reformulated in Ranicki [12].

Let A be any ring with involution $^- : A \to A; a \mapsto \bar{a}$ (for example, $A = \mathbb{Z}[\pi]$). Given a f. g. projective A-module M let M^* be the dual f. g. projective A-module

$$M^* = \mathrm{Hom}_A(M, A), A \times M^* \to M^*; (a, f) \mapsto (x \mapsto f(x)\bar{a}),$$

and use the natural A-module isomorphism

$$M \to M^{**}; \qquad x \mapsto (f \mapsto \overline{f(x)})$$

to identify $M^{**} = M$. Given also a f. g. projective A-module N there is defined a duality isomorphism

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(N^*, M^*); \qquad f \mapsto (f^* : g \mapsto (x \mapsto g(f(x))).$$

In particular, for $N = M^*$, $\epsilon = \pm 1 \in A$ there is defined an ϵ-duality involution

$$T_\epsilon : \mathrm{Hom}_A(M, M^*) \to \mathrm{Hom}_A(M, M^*); \phi \to (\epsilon\phi^* : x \mapsto (y \mapsto \overline{\epsilon\phi(y)(x)})).$$

A (non-singular) ϵ-quadratic form over $A(M, \Psi)$ is a pair consisting of a f. g. projective A-module M and an element $\Psi \in Q_\epsilon(M)$ of the abelian group

$$Q_\epsilon(M) = \mathrm{Coker}(1 - T_\epsilon : \mathrm{Hom}_A(M, M^*) \to \mathrm{Hom}_A(M, M^*))$$

such that $(1 + T_\epsilon)\Psi \in \mathrm{Hom}_A(M, M^*)$ is an isomorphism. An isomorphism of forms

$$f : (M, \Psi) \to (M', \Psi')$$

is an A-module isomorphism $f \in \mathrm{Hom}_A(M, M')$ such that

$$f^*\Psi' f = \Psi \in Q_\epsilon(M).$$

The simple isomorphism classes of such forms (M, Ψ) with M based for which $(1 + T_\epsilon)\Psi \in \mathrm{Hom}_A(M, M^*)$ is a simple isomorphism are in a natural one-one correspondence with the isomorphism classes of triples $(M, \lambda : M \times M \to A, \quad \mu : M \to A/\{a - \epsilon\bar{a} | a \in A\})$ as in §5 of Wall [19], with $(M, \Psi) \mapsto (M, \lambda(x, y) = (1 + T_\epsilon)\Psi(x)(y), \quad \mu(x) = \Psi(x)(x))$.

A lagrangian of an ϵ-quadratic form over A (M, Ψ) is a direct summand L of M such that the inclusion $j \in \mathrm{Hom}_A(L, M)$ fits into an exact sequence

$$0 \to L \xrightarrow{j} M \xrightarrow{j^*(1+T_\epsilon)\Psi} L^* \to 0$$

and

$$j^*\Psi j = 0 \in Q_\epsilon(L).$$

In particular, for any f. g. projective A-module L there is defined the hyperbolic ϵ-quadratic form

$$H_\epsilon(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_\epsilon(L \oplus L^*))$$

with lagrangian L. An ϵ-quadratic form (M, Ψ) admits a lagrangian L if and only if it is isomorphic to the hyperbolic form $H_\epsilon(L)$.

A (non-singular) ϵ-quadratic formation over A $(M, \Psi; F, G)$ is an ϵ-quadratic form over A (M, Ψ) together with an ordered pair of lagrangians (F, G). An isomorphism of formations

$$f : (M, \Psi; F, G) \to (M', \Psi'; F', G')$$

is an isomorphism of forms $f : (M, \Psi) \to (M', \Psi')$ such that $f(F) = F', f(G) = G'$. If $(M, \Psi; F, G)$ is an ϵ-quadratic formation an A-module isomorphism $F \to G$ (if any) extends to an automorphism $\alpha : (M, \Psi) \to (M, \Psi)$ such that $\alpha(F) = G$, by a generalization of Witt's theorem. Conversely, if $\alpha : (M, \Psi) \to (M, \Psi)$ is an automorphism of an ϵ-quadratic form (M, Ψ), and L is a lagrangian of (M, Ψ), then $(M, \Psi; L, \alpha(L))$ is an ϵ-quadratic formation. In particular, if (M, Ψ) is an ϵ-quadratic form with a f. g. free lagrangian L any base of L extends to a base of M, with a simple isomorphism $H_\epsilon(L) \to (M, \Psi)$. Thus if $(M, \Psi; F, G)$ is an ϵ-quadratic formation such that F and G are based A-modules of the same rank r, and such that the resulting two bases of rank $2r$ for M differ by a simple automorphism, then $(M, \Psi; F, G)$ is simple isomorphic to $(H_\epsilon(A^r); A^r, \alpha(A^r))$ for some simple automorphism $\alpha : H_\epsilon(A^r) \to H_\epsilon(A^r)$. The matrix of α is an element of the special unitary group $SU_r(A)$ considered in §6 of Wall [19], and conversely any element $\alpha \in SU_r(A)$ determines an ϵ-quadratic formation $(H_\epsilon(A^r); A^r, \alpha(A^r))$. Note, however, that for a formation $(M, \Psi; F, G)$ with projective lagrangians F, G there may be no A-module isomorphism $F \to G$, and hence no automorphism $\alpha : (M, \Psi) \to (M, \Psi)$ such that $\alpha(F) = G$.

Define $L^p_{2i}(\pi)$ $(i(\mathrm{mod}\ 2))$to be the abelian group with one generator for each isomorphism class of $(-)^i$-quadratic forms over $\mathbb{Z}[\pi]$, subject to the relations

$$(M,\Psi)=0 \text{ if } (M,\Psi) \text{ admits a lagrangian,}$$

with addition and inverses by

$$(M,\Psi)+(M',\Psi')=(M\oplus M',\Psi\oplus\Psi'), -(M,\Psi)=(M,-\Psi).$$

Define $L^p_{2i+1}(\pi)$ $(i(\mathrm{mod}\ 2))$ to be the abelian group with one generator for each isomorphism class of $(-)^i$-quadratic formations over $\mathbb{Z}[\pi]$, subject to the relations

$$\begin{aligned}&(M,\Psi;F,G)=0 \text{ if } M=F\oplus G\\&(M,\Psi;F,G)+(M,\Psi;G,H)=(M,\Psi;F,H),\end{aligned}$$

with addition and inverses by

$$\begin{aligned}(M,\Psi;F,G)+(M',\Psi';F',G')&=(M\oplus M',\Psi\oplus\Psi';F\oplus F',G\oplus G'),\\-(M,\Psi;F,G)&=(M,-\Psi;F,G).\end{aligned}$$

The groups $L^h_n(\pi)$ (resp. $L^s_n(\pi)$) $(n\ (\mathrm{mod}\ 4))$are defined in the same way as $L^p_n(\pi)$, using f. g. free $\mathbb{Z}[\pi]$-modules (resp. based $\mathbb{Z}[\pi]$-modules and simple isomorphisms). Here, an isomorphism of based $\mathbb{Z}[\pi]$-modules is simple if it has 0 torsion in the Whitehead group $\mathrm{Wh}(\pi)=\widetilde{K}_1(\mathbb{Z}[\pi])/\{\pi\}$.

The L^s-groups are related to the L^h-groups by the Rothenberg exact sequence

$$\ldots\to\hat{H}^{n+1}(\mathbb{Z}_2;\mathrm{Wh}(\pi))\to L^s_n(\pi)\to L^h_n(\pi)\to\hat{H}^n(\mathbb{Z}_2;\mathrm{Wh}(\pi))\to\ldots$$

obtained by Shaneson [14]. The reduced Tate cohomology groups $\hat{H}^*(\mathbb{Z}_2;G)$ of a $\mathbb{Z}_2$-module G are defined by

$$\hat{H}^i(\mathbb{Z}_2;G)=\{g\in G|Tg=(-)^ig\}/\{h+(-)^iTh|h\in G\}\qquad(i(\mathrm{mod}\ 2));$$

the Whitehead group $\mathrm{Wh}(\pi)$ is regarded as a $\mathbb{Z}_2$-module by the duality involution

$$*:\mathrm{Wh}(\pi)\to\mathrm{Wh}(\pi);\tau(\alpha:P\to P)\mapsto\tau(\alpha^*:P^*\to P^*).$$

The L^h-groups are related to the L^p-groups by the exact sequence

$$\cdots\to\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi]))\to L^h_n(\pi)\to L^p_n(\pi)\to\hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi]))\to\cdots$$

obtained by Ranicki [12]. The reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi])$ is regarded as a $\mathbb{Z}_2$-module by the duality involution

$$*:\widetilde{K}_0(\mathbb{Z}[\pi])\to\widetilde{K}_0(\mathbb{Z}[\pi]);[P]\mapsto[P^*].$$

2. Geometric L-groups

Given a CW complex K and a map $\pi_1(K) \to \{\pm 1\}$ let $L_n^{1,s}(K)$ (resp. $L_n^{1,h}(K)$) be the geometrically defined L-groups of §9 of Wall [19], involving normal maps from compact manifolds to simple (resp. finite) Poincaré complexes. (Here, we are adopting the terminology regarding simple and finite Poincaré complexes suggested in the footnote on p. 23 of Wall [19]). Let $L_n^{1,p}(K)$ be the L-groups defined in exactly the same way, but using normal maps from compact manifolds to finitely dominated Poincaré complexes. For the sake of completeness we spell out the definition of the appropriate "objects".

An "object" consists of the following: (1) a finitely dominated Poincaré pair (Y, X) and a bundle ν over Y, compact manifold N with boundary M, $\dim N = m$; (2) a map $\phi : (N, M) \to (Y, X)$ of pairs of degree 1, including a homotopy equivalence $M \to X$; (3) a stable framing F of $\tau_N \oplus \phi^*\nu$; and finally (4) a map $\omega : Y \to K$ such that w_Y factorizes as $\pi_1(Y) \xrightarrow{\omega_*} \pi_1(K) \to \{\pm 1\}$.

Similarly for relations.

For each of the superscripts $q = s, h, p$ there are defined "surgery obstruction" functions

$$\sigma_*^q : L_n^{1,q}(K) \to L_n^q(\pi_1(K))$$

working as in §1, §5 and §6 of Wall [19], the case $q = s$. Given a normal map $(f, b) : M \to X$ from a compact n-dimensional manifold M to a finitely dominated Poincaré complex X it is possible to perform surgery below the middle dimension as in Theorem 1.2 of Wall [19]. For $n = 2i$ there is obtained a non-singular $(-)^i$ quadratic form $(K_i(M), \lambda, \mu)$ as in Theorem 5.2 of Wall [19], except that now $K_i(M)$ is a f. g. projective module. For $n = 2i+1$ there is obtained a non-singular $(-)^i$ quadratic formation $(H_{(-)^i}(K_{i+1}(U, \partial U), K_{i+1}(U, \partial U), K_{i+1}(M_0, \partial U))$ as in §6 of Wall [19], except that now $K_{i+1}(M_0, \partial U)$ is a f. g. projective module, and there may not exist an automorphism α : of $H_{(-)^i}(K_{i+1}(U, \partial U))$ sending $K_{i+1}(U, \partial U)$ to $K_{i+1}(M_0, \partial U)$. Alternatively, the functions $\sigma_*^q (q = s, h, p)$ may be defined using the chain complex method of Ranicki [13].

It is proved in Corollary 9.4.1 of Wall [19] that for a CW complex K with a finite 2-skeleton the functions

$$\sigma_*^q : L_n^{1,q} \to L_n^q(\pi_1(K))$$

are isomorphisms for $n \geq 5$, $q = s$. The proof applies equally well for $q = h$, so that the functions σ_*^h are also isomorphisms for $n \geq 5$. The stable 4-dimensional surgery technique of Cappell and Shaneson [1] applies to prove that σ_*^q $(q = s, h)$ is an isomorphism for $n = 4$.

Theorem 2.1. *If K has a finite 2-skeleton the function*

$$\sigma_*^p : L_n^{1,p}(K) \to L_n^p(\pi_1(K))$$

is an isomorphism for $n \geq 5$, and a monomorphism for $n = 4$.

The proof of Theorem 2.1 must necessarily differ from the cases $q = s, h$, since there is no analogue of the $\pi - \pi$ theorem of §4 of Wall [19] for normal maps from compact manifolds with boundary to finitely dominated Poincaré pairs. Our proof is by a 5-lemma argument applied to the morphisms of exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & L^{1,p,h}_{n+1}(K) & \longrightarrow & L^{1,h}_{n}(K) & \longrightarrow & L^{1,p}_{n}(K) & \longrightarrow & L^{1,p,h}_{n} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \sigma^{p,h}_*} & & \downarrow{\scriptstyle \sigma^{h}_*} & & \downarrow{\scriptstyle \sigma^{p}_*} & & \downarrow{\scriptstyle \sigma^{p,h}_*} & \\
\cdots \longrightarrow & \hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])) & \longrightarrow & L^h_n(\pi_1(K)) & \longrightarrow & L^p_n(\pi_1(K)) & \longrightarrow & \hat{H}^{n}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])) & \longrightarrow \cdots
\end{array}$$

The relative cobordism groups$L^{1,p,h}_{n+1}(K)$ are defined to be the evident equivalence classes of "objects" consisting of: (1) a finitely dominated Poincaré triad $(Z;Y,Y_+)$ with $Y\cap Y_+=X$, and a bundle μ over Z, such that (Y,X) is a finite Poincaré pair; (2) a compact manifold triad $(P;N,N_+)$ with $N\cap N_+=M$, $\dim P=n+1$; (3) a map $\Psi:(P;N,N_+)\to(Z;Y,Y_+)$ of degree 1, which restricts to a degree 1 map $\phi:(N,M)\to(Y,X)$, and to a homotopy equivalence $\phi_+:(N_+,M)\to(Y_+,X)$; (4) a stable framing G of $\tau_P\oplus\Psi^*\mu$ which restricts to a stable framing F of $\tau_N\oplus\phi^*\nu$, where $\nu=\mu|Y$; and finally (5) a map $\Omega:Z\to K$ such that w_Z factorizes as $\pi_1(Z)\xrightarrow{\Omega_*}\pi_1(K)\to\{\pm1\}$, and w_Y factorizes as $\pi_1(Y)\to\pi_1(K)\to\{\pm1\}$.

The map $\sigma^{p,h}_*:L^{1,p,h}_{n+1}(K)\to\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]))$ sends such an object to the image of the Wall finiteness obstruction $[Z]\in\widetilde{K}_0(\mathbb{Z}[\pi_1(Z)])$.

Lemma 2.2. *Let K be a space with finitely presented fundamental group $\pi_1(K)$, which is equipped with a map $\pi_1(K)\to\{\pm1\}$, and let $n\geq5$. For each f. g. projective $\mathbb{Z}[\pi_1(K)]$-module Q there exists a normal map $\Psi:(P,N)\to(Z,Y)$ from a compact n-dimensional manifold with boundary (P,N) to a finitely dominated Poincaré (Z,Y), which is equipped with a map $Z\to K$ such that*
the maps $\pi_1(Y)\to\pi_1(Z)$, $\pi_1(Z)\to\pi_1(K)$ are isomorphisms,
w_Z factorizes as $\pi_1(Z)\to\pi_1(K)\to\{\pm1\}$,
$[Z]=[Q]\in\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]),[Y]=[Q]+(-)^{n-1}[Q^*]\in\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$.

We defer the proof of Lemma 2.2 to §3. First, let us deduce Theorem 2.1 from Lemma 2.2. Every element $[Q]\in\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]))$ is represented by a f. g. projective $\mathbb{Z}[\pi_1(K)]$-module Q such that $[Q^*]=(-)^{n+1}[Q]\in\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$. The $(n+1)$-dimensional normal map $\Psi:(P,N)\to(Z,Y)$ given by Lemma 2.2 has $[Y]=[Q]+(-)^n[Q^*]=0\in\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$, so that Y is a finite n-dimensional Poincaré complex, and we have an element of $L^{1,p,h}_{n+1}(K)$ (with $N_+=Y_+=\emptyset$) whose image under $\sigma^{p,h}_*$ is $[Q]$, showing that $\sigma^{p,h}_*:L^{1,p,h}_{n+1}(K)\to\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]))$ $(n\geq4)$ is onto. In order to verify that $\sigma^{p,h}_*$ is one-one consider an object $\Psi:(P;N,N_+)\to(Z;Y,Y_+)$ as above, representing an element of $L^{1,p,h}_{n+1}(K)$ such that $\sigma^{p,h}_*(\Psi)=[Z]=0\in\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]))$ $(n\geq3)$. Let Q be a f. g. projective $\mathbb{Z}[\pi_1(K)]$-module such that

$$[Z]+[Q]+(-)^{n+1}[Q^*]=0\in\widetilde{K}_0(\mathbb{Z}[\pi_1(K)]),$$

and let $\overline{\Psi} : (\overline{P}, \overline{N}) \to (\overline{Z}, \overline{Y})$ be the $(n+2)$-dimensional normal map given by Lemma 2.2 with

$$[\overline{Z}] = [Q], [\overline{Y}] = [Q] + (-)^{n+1}[Q^*] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)]).$$

Let $\overline{\phi} = \overline{\Psi}| : \overline{N} \to \overline{Y}$, and form the connected sum

$$\Psi \# \overline{\phi} : (P \# \overline{N}; N, N_+) \to (Z \# \overline{Y}; Y, Y_+).$$

Now $\overline{\phi}$ bounds a normal map, so that Ψ and $\Psi \# \overline{\phi}$ represent the same element of $L^{1,p,h}_{n+1}(K)$. Now $[Z \# \overline{Y}] = [Z] + [\overline{Y}] = [Z] + [Q] + (-)^{n+1}[Q^*] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$, and by doing simultaneous 1-surgeries on $P \# \overline{N}$ and $Z \# \overline{Y}$ to ensure π_1-isomorphisms to K we can apply the $\pi - \pi$ theorem of §4 of Wall [19] to deduce that $\Psi \# \overline{\phi}$ represents 0. Hence ϕ represents 0, showing that $\sigma^{p,h}_* : L^{1,p,h}_{n+1}(K) \to \hat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(K)]))$ $(n \geq 3)$ is one-one.

It is immediate from Theorem 2.1 that if $\phi : (N, M) \to (Y, X)$ is a normal map of pairs to a finitely dominated n-dimensional Poincaré pair (Y, X) with X finite and $\pi_1(X) \cong \pi_1(Y)$ then the finite surgery obstruction $\sigma^h_*(\phi| : M \to X) \in L^h_{n-1}(\pi_1(X))$ is the image of the projective class $[Y] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ under the canonical map $\hat{H}^n(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])) \to L^h_{n-1}(\pi_1(X))$. See Pedersen [9] for an application of this observation.

3. Realizing Projective Classes

Let K, Q, n be as in the statement of Lemma 2.2. In view of the Browder-Novikov transversality construction of normal maps in order to prove Lemma 2.2 it suffices to exhibit an n-dimensional Poincaré pair (Z, Y) with a CAT ($=$ TOP, PL or DIFF) reduction of the Spivak normal fibration ν_Z, and with a map $Z \to K$ such that

$$\pi_1(Y) \cong \pi_1(Z) \cong_1 (K), \qquad w_Z : \pi_1(Z) \to \pi_1(K) \to \{\pm 1\},$$

$$[Z] = i[Q], [Y] = [Q] + (-)^{n-1}[Q^*] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)]).$$

We now proceed to do just this.

As $\pi_1(K)$ is finitely presented and $n \geq 5$ there exists a compact n-dimensional CAT-manifold with boundary $(X, \partial X)$ equipped with a map $X \to K$ such that

$$\pi_1(\partial X) \cong \pi_1(X) \cong \pi_1(K), \qquad w_X : \pi_1(X) \to \pi_1(K) \to \{\pm 1\}.$$

By the realization theorem of Siebenmann [15] (proved by the method of infinite repetition) there exists an open n-dimensional CAT manifold U with boundary $\partial U = \partial X$ and a tame end ϵ such that

$$\pi_1(\partial U) \cong \pi_1(U) \cong \pi_1(\epsilon), \qquad [U] = [\epsilon] = [Q] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)]).$$

Let $(Z, Y) = (\overline{O}, \partial U \cup -V)$, with $(\overline{U}; \partial U, V)$ as in Lemma 3.1;

Lemma 3.1. *Let $(U, \partial U)$ be an open n-dimensional* CAT *manifold U with compact boundary ∂U and a tame end ϵ such that $\pi_1(\partial U) \cong \pi_1(U) \cong \pi_1(\epsilon)$, $n \geq 5$. Then $(U, \partial U)$ is homotopy equivalent rel ∂U to $(\overline{U}, \partial U)$, for some finitely dominated n-dimensional Poincaré cobordism $(\overline{U}, \partial U, V)$ with a* CAT *reduction of $\nu_{\overline{U}}$, $\pi_1(V) \cong \pi_1(\overline{U})$ and*

$$[\overline{U}] = [U], [V] = [U] + (-)^{n-1}[U]^* \in \widetilde{K}_0(\mathbb{Z}[\pi_1(U)]).$$

Proof. As $U \times S^1$ has 0 finiteness obstruction by the main result of Siebenmann [15] there exists a compact $(n+1)$-dimensional cobordism $(\overline{U \times S^1}, \partial U \times S^1, \overline{V})$ such that $\overline{U \times S^1} - \overline{V} = U \times S^1$. Let $p : \overline{U} \to \overline{U \times S^1}$ be the infinite cyclic covering of $\overline{U \times S^1}$ obtained from the universal cover $\mathbb{R} \to S^1$ by pullback along the composite $\overline{U \times S^1} \xrightarrow{\simeq} U \times S^1 \xrightarrow{\text{projection}} S^1$, and let $V = p^{-1}(\overline{V}) \subset \overline{U}$. We shall produce a homotopy equivalence

$$(\overline{U \times S^1}; \partial U \times S^1, \overline{V}) \to (\overline{U}, \partial U \times \mathbb{R}, V) \times S^1,$$

from which it follows that $(\overline{U}; \partial U, V)$ is a finitely dominated n-dimensional Poincaré cobordism with a CAT reduction of $\nu_{\overline{U}}$. In the first instance we produce a homotopy equivalence $\overline{V} \to V \times S^1$.

Let $U_1 \supset U_2 \supset \cdots$ be a system of neighbourhoods of infinity in U consisting of manifolds with boundary, and let $[-, U_i]$ denote the functor associating to a CW, complex C the set $[C, U_i]$ of free homotopy classes of maps $C \to U_i$. Consider the functor $\lim_i[-, U_i] : C \to \lim_i[C, U_i]$. Since any two system of neighborhoods are cofinal this functor is independent of the system of neighborhoods.

Now $U_1 \times S^1 \supset U_2 \times S^1 \supset \cdots$ is a system of neighborhoods of infinity in $U \times S^1$. Let $\overline{V}_1 \supset \overline{V}_2 \supset \cdots$ be a system of neighborhoods of infinity in $U \times S^1 = \overline{U \times S^1} - \overline{V}$ consisting of collars of the boundary component $\overline{V}$ in $\overline{U} \times S^1$, so that

$$\lim_i [C, U_i \times S^1] = \lim_i [C, \overline{V}_i] = [C, \overline{V}].$$

Let $V_i = p^{-1}(\overline{V}_i) \subset \overline{U \times S^1}$. Passing to the covers we have

$$\lim_i [C, U_i \times \mathbb{R}] = \lim_i [C, V_i] = [C, V],$$

so that

$$[C, V \times S^1] = \lim_i [C, U_i \times S^1] = [C, \overline{V}].$$

Thus both the spaces $\overline{V}$ and $V \times S^1$ represent the functor $\lim_i[-, U_i]$, and there is determined a unique homotopy class of homotopy equivalences $\overline{V} \to V \times S^1$.

Next we give a geometric construction for a homotopy equivalence $(\overline{U \times S^1}; \partial U \times S^1, \overline{V}) \to (\overline{U}, \partial U \times \mathbb{R}, V) \times S^1$ which restricts to one of the specified homotopy equivalences $\overline{V} \to V \times S^1$. The map into S^1 is easily obtained, so we need only construct $\overline{U \times S^1} \to \overline{U}$. Choose a collar $\overline{V} \times [0, 1]$ of $\overline{V}$ in $\overline{U \times S^1}$, with $\overline{V} = \overline{V} \times \{0\} \subset \overline{U \times S^1}$. Let U_i be a neighborhood of infinity in U such that $U_i \times S^1 \subset \overline{V} \times (0, 1]$. Let $\eta > 0$ be so small that $\overline{V} \times (0, \eta] \subset U_i \times S^1$. Identify $\overline{U} - V$ with $U \times \mathbb{R}$. The map $\overline{U \times S^1} \to \overline{U}$ is defined to be the restriction of

$U \times S^1 \to U \times \mathbb{R} \subset \overline{U}; (u,s) \mapsto (u,0)$ on $\overline{U \times S^1} - \overline{V} \times [0,\eta]$, and on $\overline{V} \times [0,\eta]$ to be the composite

$$\overline{V} \times \{t\} \xrightarrow{1} V \times \{\eta\} \subset U_i \times S^1 \xrightarrow{1\times 0} U_i \times \mathbb{R} \subset V \times (0,1) \xrightarrow{\text{projection}} V \times \{t\} \subset \overline{U}(0 \le t \le \eta).$$

□

4. The Splitting theorem

The L^s-groups are related to the L^h-groups by the splitting theorem

$$L^s_n(\pi \times \mathbb{Z}) = L^s_n(\pi) \oplus L^h_{n-1}(\pi) \quad (n(\text{mod } 4))$$

obtained geometrically by Shaneson [14]. Following Novikov [8] this was obtained algebraically by Ranicki [12], along with the corresponding splitting theorem relating the L^h-groups to the L^p-groups

$$L^h_n(\pi \times \mathbb{Z}) = L^h_n(\pi) \oplus L^p_{n-1}(\pi) \quad (n(\text{mod } 4)).$$

The constructions of §2 and §3 can be used to also give a geometric proof of the latter theorem (for finitely presented π) i. e. to prove

$$L^{1,h}_n(K \times S^1) = L^{1,h}_n(K) \oplus L^{1,p}_{n-1}(K) \quad (n \ge 6).$$

We contend ourselves with a geometric description of the maps involved.

The maps $L^{1,h}_n(K \times S^1) \rightleftarrows L^{1,h}_n(K)$ are the functorial splitting maps[1] induced by

$$K \times S^1 \rightleftarrows K,$$

and

$$L^{1,p}_{n-1} \to L^{1,h}_n(K \times S^1); (\phi : (N,M) \to (Y,X) \mapsto (\phi \times 1 : (N,M) \times S^1 \to (Y,X) \times S^1).$$

It remains to define $L^{1,h}_n(K \times S^1) \to L^{1,p}_{n-1}(K)$.

By the realization theorems of §5 and §6 of Wall [19] every element of $L^{1,h}_n(K \times S^1)$ is represented by a normal map of compact n-dimensional manifold triads

$$\phi : (W; M \times S^1, N) \to (M \times S^1 \times I; M \times S^1 \times 0, M \times S^1 \times 1)$$

for some compact $(n-2)$-dimensional manifold M equipped with a map $M \to K$, such that $\phi| = id : M \times S^1 \to M \times S^1 \times 0$ and $\phi| : N \to M \times S^1 \times 1$ is a homotopy equivalence. Making ϕ transverse regular at $M \times pt \times I (pt \times S^1)$, so that $\phi^{-1}(M \times pt \times I; M \times pt \times 0, M \times pt \times 1) =$

[1] *(Added in 1997) This statement is not correct. The map $L^{1,h}_n(K \times S^1) \to L^{1,h}_n(K)$ is functorial, but it is not the map induced by $K \times S^1 \to K$. The error is corrected in the second author's paper 'Algebraic and geometric splittings of the K- and L-groups of polynomial extensions' (Transformation Groups, Proceedings 1985 Poznań Topology Conference, Springer Lecture Notes 1217, 321–363 (1986)).*

$(W_0; M, N_0) \subset (W; M\times S^1, N)$ is a codimension 1 manifold triad, there is defined an $(n-1)$-dimensional normal map

$$\phi_0 = \phi| : (W_0; M, N_0) \to (M \times I; M \times 0, M \times 1).$$

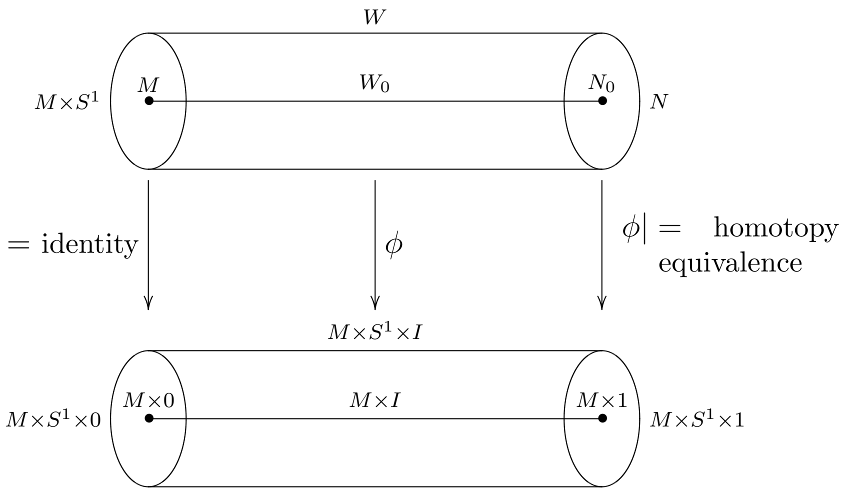


Fig. 1

Let $\phi_1 : (N_1; N_0, zN_0) \to (M \times [1,2]; M \times 1, M \times 2)$ be the normal map of $(n-1)$-dimensional manifold triads obtained from the homotopy equivalence $\phi| : N \to M \times S^1$ by cutting along $\phi_0| : N_0 \to M$, with $N^1 = N - N_0$ obtained from N by cutting along N_0, and zN_0 denoting a copy of N_0. Let U be the open $(n-1)$-manifold with compact boundary $\partial U = M$ and tame end ϵ defined by

$$U = W_0 \cup_{N_0} N_1 \cup_{zN_0} zN_1 \cup_{z^2N_0} z^2N_1 \cup \cdots ,$$

and let $\Psi : (U, \partial U) \to (M \times [0,\infty), M \times 0)$ be the proper degree 1 map defined by

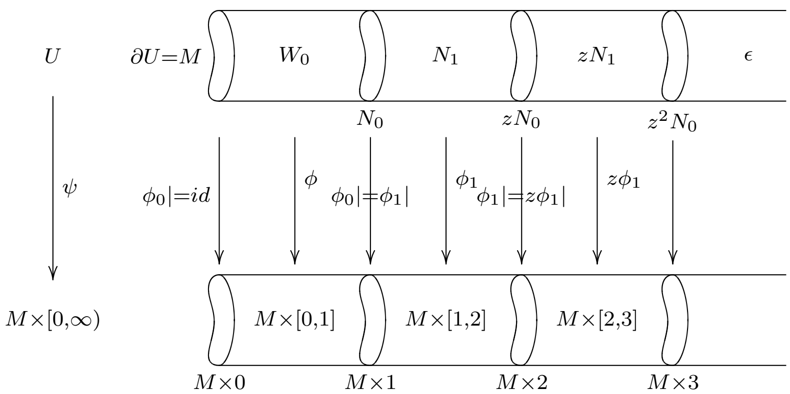


Fig. 2

The construction of Lemma 3.1 now gives a degree 1 map of finitely dominated Poincaré triads

$$\overline{\Psi} : (\overline{U}; \partial U, V) \to (M \times I; M \times 0, M \times 1)$$

such that $\overline{\Psi}| = \text{identity} : \partial U = M \to M \times 0$ and $\overline{\Psi} : V \to M \times 1$ is a homotopy equivalence. Moreover, $\overline{\Psi}$ is covered by a map of CAT reductions of the Spivak normal fibrations. The Browder-Novikov transversality construction now gives a normal map from a compact CAT manifold triad

$$\theta : (P; Q, R) \to (\overline{U}; \partial U, V)$$

such that $\theta| = \text{identity} : Q \to \partial U$ and $\theta| : R \to V$ is a homotopy equivalence. Define

$$\begin{aligned} L^{1,h}_n(K \times S^1) \to L^{1,p}_{n-1}(K); (\phi : (W; M \times S^1, N) \to (M \times S^1 \times I; M \times S^1 \times 0, M \times S^1 \times 1)) \\ \mapsto (\overline{\Psi}\theta : (P; Q, R) \to (M \times I; M \times 0, M \times 1)) - (\theta : (P; Q, R) \to (\overline{U}; \partial U, V)). \end{aligned}$$

5. Proper Manifold Theory

As noted in the Introduction a finitely dominated n-dimensional Poincaré complex X is such that $X \times S^1$ has the homotopy type of a compact CAT manifold ($n \geq 4$) if and only if the Spivak normal fibration ν_X admits a CAT reduction for which the corresponding normal map $(f, b) : M \to X$ has projective surgery obstruction $\sigma^p_*(f, b) = 0 \in L^p_n(\pi_1(X))$. Equivalently, the total projective surgery obstruction of X (as defined in Ranicki [11]) is $s(X) = 0 \in \mathcal{S}^p_n(X))$. We shall now characterize such Poincaré complexes in terms of homotopy type of certain open $(n+1)$-dimensional CAT manifolds.

A *proper n-dimensional* CAT *manifold* (CAT = TOP, PL or DIFF) consists of: (i) an open $(n+1)$-dimensional CAT manifold M; (ii) a free $\mathbb{Z}$-action $\mathbb{Z} \times M \to M$ such that the quotient $M/\mathbb{Z}$ is compact; (iii) a homotopy retraction $r : M/\mathbb{Z} \to M$ of the projection $M \to M/\mathbb{Z}$.

Then $r \times c : M/\mathbb{Z} \to M \times S^1$ is a homotopy equivalence, where $c : M/\mathbb{Z} \to B\mathbb{Z} = S^1$ is the classifying map of the free $\mathbb{Z}$-action, and M is a finitely dominated n-dimensional Poincaré complex.

Theorem 5.1. *A finitely dominated n-dimensional Poincaré complex X is such that $X \times S^1$ has the homotopy type of a compact* CAT *manifold if and only if X has the homotopy type of a proper n-dimensional* CAT *manifold.*

Proof. If $X \times S^1$ is homotopy equivalent to a compact $(n+1)$-dimensional CAT manifold N then the infinite cyclic cover $\widetilde{N}$ is a proper n-dimensional CAT manifold homotopy equivalent to X. The converse is obvious. □

6. Projective Poincaré Surgery

Given a space K and a group morphism $\pi_1(K) \to \{\pm 1\}$ let $\Omega_n^{\rm CAT}(K)$ (CAT = DIFF, PL or TOP) be the bordism group of maps $M \to K$ from compact n-dimensional CAT manifolds M for which the orientation map factors as $w_M : \pi_1(M) \to \pi_1(K) \to \{\pm 1\}$.

We shall say that an n-dimensional Poincaré complex is of type q for $q = s, h, p$ if it is simple, finite, finitely dominated respectively. Similarly for Poincaré pairs. Define the bordism group $\Omega_n^{q\,\rm CAT}(K)$ of maps $X \to K$ from n-dimensional Poincaré complexes X of type q with a CAT reduction of the Spivak normal fibration ν_X, such that $w_X : \pi_1(X) \to \pi_1(K) \to \{\pm 1\}$.

The result of Levitt [5]

$$\Omega_n^{q\,\rm CAT}(pt.) = \Omega_n^{\rm CAT}(pt.) \oplus L_n(\{1\}) \quad (n \geq 5, q = s, h)$$

admits the following generalization.

Theorem 6.1. *If K is a* CW *complex with a finitely presented $\pi_1(K)$ then*

$$\Omega_n^{q\,\rm CAT}(K) = \Omega_n^{\rm CAT}(K) \oplus L_n^q(\pi_1(K)) \quad (n \geq 5, q = s, h, p).$$

Proof. Define a map

$$\Omega_n^{q\,\rm CAT}(K) \to \Omega_n^{\rm CAT}(K) \oplus L_n^q(\pi_1(K)); (X \to K) \mapsto (M \xrightarrow{f} X \to K, \sigma_*^p((f,b) : M \to X)),$$

with $(f,b) : M \to X$ the normal map from a compact n-dimensional CAT manifold M obtained from the given CAT reduction of ν_X by the Browder-Novikov transversality construction. Define an inverse map

$$\Omega_n^{\rm CAT}(K) \oplus L_n^q(\pi_1(K)) \to \Omega_n^{q\,\rm CAT}(K); (L \to K, \sigma_*^p(f,b) : (N,M) \to (Y,X))) \to (L \cup W \to K)$$

to be the forgetful map on the first summand, and the following map on the second summand. By the isomorphisms $\sigma_*^q : L_n^{1,q}(K) = L_n^q(\pi_1(K)) \quad (n \geq 5)$ of §9 of Wall [19] $(q = s, h)$ and Theorem 2.1 above $(q = p)$ every element of $L_n^q(\pi_1(K))$ can be expressed as the surgery obstruction $\sigma_*^q(f,b)$ of a normal map $(f,b) : (N,M) \to (Y,X)$ to an n-dimensional Poincaré pair of type q (Y,X), which is equipped with a map $Y \to K$, such that the restriction $e = f| : M \to X$ is a simple homotopy equivalence for $q = s$, and a homotopy equivalence for $q = h, p$. Let $W = Y \cup_e -N$ be the n-dimensional Poincaré complex of type q obtained from N and Y by reversing the orientation of N and gluing by e. The Spivak normal fibration ν_W has a CAT reduction, such that the corresponding normal map from a compact CAT manifold is given by

$$(g,c) = (f,b) \cup_e id. : N \cup_{\partial N} -N \to W = Y \cup_e -N,$$

with surgery obstruction

$$\sigma_*^q(g,c) = \sigma_*^q(f,b) \in L_n^q(\pi_1(K)).$$

The required map is defined by

$$L_n^q(\pi_1(K)) \to \Omega_n^{q\,\rm CAT}(K); \sigma_*^q(f,b) \mapsto (W \to K).$$

□

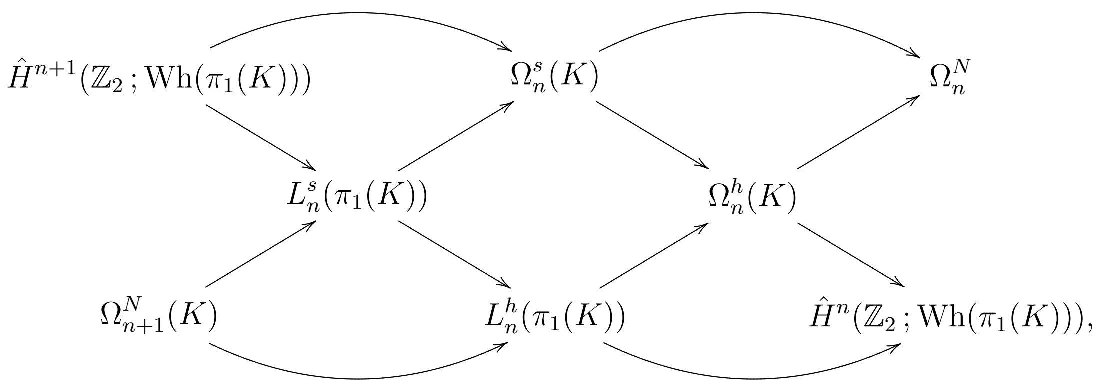
Let $\Omega^q_n(K)$ denote the bordism group of maps $X \to K$ from n-dimensional Poincaré complexes of type q for $q = s, h, p$.

Lemma 6.2. *If K is a* CW *complex with finitely presented $\pi_1(K)$ the various Poincaré bordism groups $\Omega^q_n(K)$ $\quad(q = s, h, p)$ are related to each other by exact sequences*

$$\cdots \to \hat{H}^{n+1}(\mathbb{Z}_2; \mathrm{Wh}(\pi)) \to \Omega^s_n(K) \to \Omega^h_n(K) \to \hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(\pi)) \to \cdots$$
$$\cdots \to \hat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \to \Omega^h_n(K) \to \Omega^p_n(K) \to \hat{H}^n(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \to \cdots$$
$$(n \geq 5, \pi = \pi_1(K))$$

Proof. This is immediate from Lemma 2.2, and from its analogue realizing Whitehead torsion elements using normal maps from compact manifolds with boundary to finite Poincaré pairs. □

It follows from the Poincaré surgery theories of Levitt [5], Jones [4] and Quinn [10] that there is defined a braid of exact sequences



$(n \geq 5)$

with $\Omega^N_n(K) = H_n(K; \underline{MSG})$ the normal space bordism groups. We deduce from this and from Lemma 3.1:

Theorem 6.3. *If K is a* CW *complex with finitely presented $\pi_1(K)$ there is defined a braid of exact sequences*

$$\begin{array}{ccccccc}
\hat{H}^{n+1}(\mathbb{Z}_2\,;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])) & & \Omega^h_n(K) & & \Omega^N_n(K) & & \\
& L^h_n(\pi_1(K)) & & \Omega^p_n(K) & & & (n \geq 5) \\
\Omega^N_{n+1}(K) & & L^p_n(\pi_1(K)) & & \hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(K)])). & &
\end{array}$$

Furthermore, as a consequence of the splitting theorems for L-groups (see §4 above) we have

$$\begin{aligned}
\Omega^s_n(K \times S^1) &= \Omega^s_n(K) \oplus \Omega^h_{n-1}(K) \\
\Omega^h_n(K \times S^1) &= \Omega^h_n(K) \oplus \Omega^p_{n-1}(K).
\end{aligned} \qquad (n \geq 5)$$

7. Surgery on Open Manifolds

We shall now relate our projective surgery theory to the open surgery theory of Maumary [6, 7] and Taylor [17].

In his thesis Taylor [17] sets up a surgery theory along the lines of §9 of Wall [19], involving paracompact open manifolds and open Poincaré complexes (which are not in general Poincaré complexes in the sense of Wall [19]). We outline this theory:

Let K be a locally finite CW complex , and let $\mathrm{Wh}(K)$ denote the Whitehead group of K in the sense of Siebenmann [16]. Using the algebraic description of $\mathrm{Wh}(K)$ in terms of locally finite infinite matrices due to Farrell and Wagoner [3] there is defined a duality involution $*$ on $\mathrm{Wh}(K)$. An n-dimensional open Poincaré complex is a locally finite CW complex X together with a fundamental class $[X] \in H^{l.f.}_n(X)$ in the homology theory defined by locally finite chains – we refer to Taylor [17] for the details of the open Poincaré duality itself. At any rate, if X is a locally finite CW complex such that $X = \bigcup_{i=1}^{\infty} X_i$ for some subcomplexes X_1 $(i = 1, 2, \ldots)$ such that each $(X_i; X_i \cap X_{i-1}, X_i \cap X_{i+1})$ $(i \geq 1, X_0 = \emptyset)$ is a finite n-dimensional n-dimensional Poincaré triad (in the usual sense) then X is an n-dimensional open Poincaré complex. In particular, paracompact open manifolds have such decompositions and are open Poincaré complexes. It is shown in Taylor [17] that the open Poincaré duality of an n-dimensional open Poincaré complex X has a proper Whitehead torsion $\tau(X) \in \mathrm{Wh}(X)$ such that $\tau(X)^* = (-)^n\tau(X) \in \mathrm{Wh}(X)$, and such that $\tau(X) = 0$ if X is a paracompact open manifold. Following §9 of Wall [19] there are defined geometric L-groups $L^{q,\mathrm{open}}_n(K)$ for $q = s$ (resp. h) involving proper normal maps from paracompact open manifolds to open Poincaré complexes with zero (resp. arbitrary) proper Whitehead torsion, with proper reference maps to K. A proper analogue of the $\pi - \pi$ theorem of §4 of Wall [19] is obtained, proving that a proper normal map of pairs $\phi : (N, M) \to (Y, X)$ with $X \subset Y$ a proper 1-equivalence can be made a proper homotopy equivalence by open surgery. It then follows that a proper normal map $M \to X$ with an n-dimensional open Poincaré complex of type q can be made a proper q-homotopy equivalence by open surgery if and only if it represents 0 in $L_n^{q,\text{open}}(X)$ $(q = s, h, n \geq 6)$. Furthermore, if $X \to K$ is a proper 1-equivalence of locally finite CW complexes then the maps $L_n^{q,\text{open}}(X) \to L_n^{q,\text{open}}(K)$ are isomorphisms $(q = s, h)$.

The theory of Maumary [6, 7] is primarily concerned with the algebraic determination of the groups $L_n^{h,\text{open}}(K)$, as follows. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a sequence of neighborhoods of infinity in K, so that each K_i is cocompact and $\bigcap_{i=0}^{\infty} K_i = \emptyset$. Let $\Pi_n^q(K) = \prod_{i=1}^{\infty} L_n^q(\pi_1(K_i))$ $(q = p, h)$, and define maps

$$1 - s : \Pi_n^q(K) \to L_n^q(\pi_1(K)) \oplus \Pi_n^q(K); (a_1, a_2, \ldots) \mapsto (-j_*(a_1), a_1 - j_*(a_2), \ldots),$$

where j_* denotes the maps induced by the inclusions $j : K_i \subset K_{i-1}$ $(i \geq 1, K_0 = K)$. (The problem of base points is solved by choosing an appropriate tree.)

Theorem 7.1. *(Maumary) The groups $L_n^{h,open}(K)$ fit into an exact sequence*

$$\Pi_n^h(K) \xrightarrow{1-s} L_n^h(\pi_1(K)) \oplus \Pi_n^h(K) \to L_n^{h,open}(K) \to \Pi_{n-1}^p(K) \to L_{n-1}^p(\pi_1(K)) \oplus \Pi_{n-1}^p(K).$$

Taylor [17] obtains the following realizability theorem for open surgery obstruction.

Theorem 7.2. *(Taylor) Every element of $L_n^{h,open}(K)$ $(n \geq 6)$ is the open surgery obstruction of a proper normal map of n-dimensional open manifold triads $\Psi : (W; M, N) \to (M \times I; M \times 0, M \times 1)$ with $\Psi| = id. : M \to M \times 0$, $\Psi| : N \to M \times 1$ a proper homotopy equivalence.*

Remark. A similar result was also obtained for $L_n^{s,\text{open}}(K)$ $(n \geq 6)$.

We proceed with some specific computations of the $L_*^{p,\text{open}}$-groups, which will enable us to relate them to the geometric construction of the projective L-groups in §2.

Proposition 7.3. *If K is a finite* CW *complex*

$$L_n^{q,open}(K \times [0, \infty)) = 0 \quad (q = s, h, n \geq 6).$$

Proof. In the first instance note that $\text{Wh}(K \times [0, \infty)) = 0$ (Siebenmann [16], so that $L_n^{h,\text{open}}(K \times [0, \infty)) = L_n^{s,\text{open}}(K \times [0, \infty))$. The exact sequence of Theorem 7.1 implies that $L_n^{h,\text{open}}(K \times [0, \infty)) = 0$. Alternatively, this may be deduced from Theorem 7.2 by an inductive application of the usual $\pi - \pi$ theorem. □

Remark. Let $\mathbb{R}^n \to [-, \infty)$; $x \mapsto ||x||$ be the norm map. Then $K \times \mathbb{R}^n \to K \times [0, \infty)$ is a proper 1-equivalence for $m \geq 3$, so that $L_n^{q,\text{open}}(K \times \mathbb{R}^n) = 0$ $(q = s, h, m \geq 3, n \geq 6)$.

We seek to exhibit isomorphisms for a finite CW complex K

$$L^{1,p}_n(K) \to L^{n,\mathrm{open}}_{n+1}(K \times \mathbb{R}),$$

thus proving $L^{n,\mathrm{open}}_{n+1}(K\times\mathbb{R}) = L^p_n(\pi_1(K))$ (which can also be obtained directly from theorem 7.1). An element of $L^{1,p}_n(K)$ is an equivalence class of n-dimensional normal maps $\phi : (N, M) \to (Y, X)$ to finitely dominated Poincaré pairs (Y, X) equipped with a reference map to K, such that $\phi| : M \to X$ is a homotopy equivalence. We cross the normal map with S^1 and choose a finite CW complex Z with $X \times S^1$ as a subcomplex and $(Z, X \times S^1)$ homotopy equivalent to $(Y \times S^1, X \times S^1)$. The map $Z \to K \times S^1$ is proper since both spaces are compact. The pullback of $K \times \mathbb{R} \to K \times S^1$ will thus produce an $(n + 1)$-dimensional open surgery problem

$$(N \times \mathbb{R}, M \times \mathbb{R}) \to (\overline{Z}, X \times \mathbb{R}) \to K \times \mathbb{R}$$

defining an element of $L^{h,\mathrm{open}}_{n+1}(K\times\mathbb{R})$. Noting that $\mathrm{Wh}(K\times\mathbb{R}) = \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$ (Siebenmann [16]) it is easy to see that the torsion of the open Poincaré duality pair $(\overline{Z}, X \times \mathbb{R})$ is the Wall finiteness obstruction $[Y] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$ (assuming $\pi_1(Y) \cong \pi_1(K)$).

Theorem 7.4. *If K is a finite* CW *complex the map*

$$L^{1,p}_n(K) = L^p_n(\pi_1(K)) \to L^{h,open}_{n+1}(K \times \mathbb{R})$$

is an isomorphism for $n \geq 7$.

Proof. Taylor [17] obtains an analogue of the Rothenberg exact sequence

$$\cdots \to \hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(T)) \to L^{s,\mathrm{open}}_n(T) \to L^{h,\mathrm{open}}_n(T) \to \hat{H}^{n+1}(\mathbb{Z}_2; \mathrm{Wh}(T)) \to \cdots$$

for any locally finite CW complex T. The map $\hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(T)) \to L^{s,\mathrm{open}}_n(T)$ is given by constructing a proper homotopy equivalence of open manifolds with prescribed proper Whitehead torsion and regarding it as an open surgery problem of type s. The map $L^{h,\mathrm{open}}_n(T) \to \hat{H}^{n-1}(\mathbb{Z}_2; \mathrm{Wh}(T))$ is given by sending $\phi : (W; M, N) \to (M \times I, M \times 0, M \times 1)$ (as in Theorem 7.2) to the torsion of the proper homotopy equivalence $\phi| : N \to M \times 1$. In the case of interest to us $T = K \times \mathbb{R}$, so $\mathrm{Wh}(K \times \mathbb{R}) \cong \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$ (as a $\mathbb{Z}_2$-module) and we have a morphism of exact sequences

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \hat{H}^n(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])) & \longrightarrow & L^h_{n-1}(\pi_1(K)) & \longrightarrow & L^p_{n-1}(\pi_1(K)) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(K \times \mathbb{R})) & \longrightarrow & L^{s,\mathrm{open}}_n(K \times \mathbb{R}) & \longrightarrow & L^{h,\mathrm{open}}_n(K \times \mathbb{R}) & \longrightarrow \cdots ,
\end{array}$$

the map not yet defined being

$$L^h_{n-1}(\pi_1(K)) \to L^{s,\mathrm{open}}_n(K\times\mathbb{R}); (\phi : (N, M) \to (Y, X)) \mapsto (\phi\times 1 : (N, M)\times\mathbb{R} \to (Y, X)\times\mathbb{R}).$$

In view of the 5-lemma it now suffices to prove that the latter maps are isomorphisms. They are monomorphisms by Siebenmann [15]. To see that they are also epimorphisms represent an element of $L_n^{s,\text{open}}(K \times \mathbb{R})$ by an open surgery problem $W \xrightarrow{\Psi} M \times I \times \mathbb{R}$ with M a compact $(n-2)$-dimensional manifold. $\partial W = M \times 0 \times \mathbb{R} \cup \partial_1 W$, $\partial_1 W \to M \times 1 \times \mathbb{R}$ a simple proper homotopy equivalence. Using Siebenmann [15] again, we can make Ψ transverse to $M \times I \times 0$ obtaining a homotopy equivalence on $\partial(M \times I)$. Crossing with $\mathbb{R}$ we obtain a surgery problem with the same open surgery obstruction as $W \xrightarrow{\Psi} M \times I \times \mathbb{R}$, as is seen by a double application of Proposition 7.3. □

Remark. Using Maumary's work the isomorphism of Theorem 7.4 can be extended to $n = 6$.

References

1. S. E. Cappell and J. L. Shaneson, *On four dimensional surgery and applications*, Comment. Math. Helv. **46** (1971), 500–528.
2. F. T. Farrell, L. R. Taylor, and J. B. Wagoner, *The Whitehead theorem in the proper category*, Compositio Math. **27** (1973), 1–23.
3. F. T. Farrell and J. B. Wagoner, *Infinite matrices in algebraic K-theory and topology*, Comment. Math. Helv. **47** (1972), 475–501.
4. L. Jones, *Patch spaces: a geometric representation for Poincaré spaces*, Ann. of Math. (2) **97** (1973), 306–343.
5. N. Levitt, *Poincaré duality cobordism*, Ann. of Math. (2) **96** (1972), 211–244.
6. S. Maumary, *Proper surgery groups for non-compact manifolds of finite dimension*, Notes, 1972.
7. ———, *Proper surgery groups and Wall-Novikov groups*, Algebraic K-theory (Battelle Seattle, 1972), Lecture Notes in Mathematics, vol. 343, Springer, Berlin, 1973, pp. 526–539.
8. S. P. Novikov, *Algebraic construction and properties of hermitian analogs of k-theory over rings with involution from the viewpoint of the hamiltonian formalism. Applications to differential topology and the theory of characteristic classes I, II*, Izv. Akad. Nauk SSSR, Ser. Mat. **34** (1970), 253–288, 475–500 (Russian).
9. E. K. Pedersen, *Smoothing H-spaces II*, Math. Scand. **46** (1980), 216–222.
10. F. Quinn, *Surgery on Poincaré spaces and normal spaces*, Bull. Amer. Math. Soc. (N.S.) **78** (1972), 262–267.
11. A. A. Ranicki, *The total surgery obstruction*, Algebraic Topology, (Aarhus, 1978), Lecture Notes in Mathematics, vol. 763, Springer, Berlin, 1979, pp. 275–316.
12. ———, *The algebraic theory of surgery, I. Foundations*, Proc. London Math. Soc. (3) **40** (1980), 87–192.
13. ———, *The algebraic theory of surgery II. Applications to topology*, Proc. London Math. Soc. (3) **40** (1980), 193–283.
14. J. L. Shaneson, *Walls's surgery on obstruction groups for* $\mathbb{Z} \times G$, Ann. of Math. (2) **90** (1969), 296–334.
15. L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than 5*, Ph.D. thesis, Princeton University, 1965.
16. ———, *Infinite simple homotopy types*, Indag. Math. **32** (1970), 479–495.
17. L. Taylor, *Surgery on paracompact manifolds*, Ph.D. thesis, UC at Berkeley, 1972.
18. C. T. C. Wall, *Poincaré complexes*, Ann. of Math. (2) **86** (1970), 213–245.
19. ———, *Surgery on Compact Manifolds*, Academic Press, New York, 1970.

Odense Universitet, Campusvej 55, 5230, Odense M, Denmark

Princeton University, Princeton, NJ 08540, U.S.A.