REGULAR NEIGHBORHOODS IN TOPOLOGICAL MANIFOLDS

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Regular neighborhoods have proved to be a very useful tool in the theory of PL manifolds. In this paper we want to make a very easy construction of regular neighborhoods in the topological category. F. E. A. Johnson [6] has constructs regular neighborhoods in the topological category, but only in the case of nonintersection with the boundary. R. D. Edwards [2] has announced a very general construction of regular neighborhoods; see also [3]. The present construction has the advantage of allowing a "relative" version , (Theorem 13), in the sense that if L is a complex, K is a subcomplex, and L is locally tamely embedded in a topological manifold V, then one may find a regular neighborhood of K in V, intersecting L in a regular neighborhood of K in L, in the usual PL sense. This is used in [10] to prove embedding theorems for topological manifolds. In [11] we have a proof that the opposite procedure is possible; namely a spine of a topological manifold.

We should emphasize that the regular neighborhoods we obtain are mapping cylinder neighborhoods; i. e. if $K \subset N$ where N is a regular neighborhood of K, then there is a map $\pi : \partial N \to K$ such that N is homeomorphic to the mapping cylinder of π (Theorem 15).

Let K be a compact topological space with a given simple homotopy structure; i. e. of the homotopy type of a finite CW-complex, with the homotopy equivalence specified up to torsion.

Definition 1. A *regular neighborhood* N of K in V is a locally flat, compact submanifold of V, of codimension 0, which is a topological neighborhood of K such that the inclusion $K \subset N$ is a simple homotopy equivalence, and K is a strong deformation retract of N. We also require that $\partial N \subset N - K$ induces an isomorphism on the fundamental groups for every component.

Definition 2. A regular neighborhood N of $K \subset V$ is said to *meet the boundary transversally* if $N \cap \partial V$ is a regular neighborhood of L in ∂V and $\eta(N) = \overline{\partial N - N \cap \partial V}$ meets ∂V transversally.

Remark 3. If a regular neighborhood meets the boundary regularly, it then follows from van Kampen's theorem that $\eta(N) \to N - K$ induces an isomorphism on the fundamental group.

Definition 4. $K \subset V$ is said to have *arbitrarily small* regular neighborhoods of for every neighborhood U of K there is a regular neighborhood N of K in V such that $N \subset U$.

Definition 5. Two regular neighborhoods of $K \subset V$, N and $\widetilde{N}$, are said to be *equivalent* if N is homeomorphic to $\widetilde{N}$ by a homeomorphism which is the identity on a neighborhood of K. If N and $\widetilde{N}$ meet the boundary regularly, the homeomorphism is required to restrict to a homeomorphism of $N \cap \partial V$ to $\widetilde{N} \cap \partial V$.

We now want to change a regular neighborhood into one that meets the boundary regularly.

PROPOSITION 6. *Let N be a regular neighborhood of K in V, and assume $L = K \cap \partial V$ has a regular neighborhood $\overline{N}$ in ∂V such that $\overline{N} \subset \operatorname{int}(N \cap \partial V)$. Then K has a regular neighborhood which meets the boundary regularly in $\overline{N}$.*

Proof. Push N off ∂V outside $\overline{N}$ using a collar of ∂V in V in V and of $\overline{N}$ in ∂V outside $\overline{N}$. □

We now make some observations essentially due to F. E. A. Johnson (see [6]).

PROPOSITION 7. *Let N and N' be regular neighborhoods of K such that $N \subset \operatorname{int}(N')$. If $K \cap \partial V = \emptyset$ and $\dim(V) \geq 6$, then $\overline{N' - N}$ is homeomorphic to $\partial N \times I$. If $K \cap \partial V \neq \emptyset$, N and N' meet the boundary regularly, and $\dim(V) \geq 7$, then $\overline{N' - N}$ is homeomorphic to $\eta(N) \times I$.*

Proof. The topological s-cobordism theorem applies, since van Kampen's theorem applied to

$$\begin{array}{ccccc} & & N - K & & \\ & \nearrow & & \nwarrow & \\ \overline{N' - N} & & & & N' - K \\ & \nwarrow & & \nearrow & \\ & & \partial N' & & \end{array}$$

and the factoring $\partial N \subset N' - N \subset N - K$ proved that $\partial N \subset N' - N$ and $\partial N' \subset N' - N$ both induce isomorphism on the fundamental group. Further, $K \subset N'$ is a simple homotopy equivalence which factors $K \subset N \subset N'$, where $K \subset N$ and $K \subset N'$ are both simple homotopy equivalences. Hence $\partial N \subset N' - N$ is a simple homotopy equivalence. □

PROPOSITION 8. *If $\dim(V) \geq 6$ and $K \subset \operatorname{int}(V)$ has arbitrarily small regular neighborhoods then any two are equivalent. If $\dim(V) \geq 7$, $K \cap \partial V \neq \emptyset$, and K has arbitrarily small neighborhoods meeting the boundary regularly, then any two such neighborhoods are equivalent.*

Proof. Let N_1 and N_2 be two regular neighborhoods. By assumption, there is a regular neighborhood $N \subset \operatorname{int}(N_1 \cap N_2)$. By Proposition 7, $\overline{N_1 - N}$ and $\overline{N_2 - N}$ are both homeomorphic to $\partial N \times I$ (resp., $\eta(N) \times I$). Hence N_1 is homeomorphic to N_2 by a homeomorphism that is the identity on N. □

PROPOSITION 9. *Let $K \subset V$ have arbitrarily small neighborhoods meeting the boundary regularly, and let N be a regular neighborhood meeting the boundary regularly. Then if $K \cap \partial V = \emptyset$ and $\dim(V) \geq 6$, $N - K$ is homeomorphic to $\partial N \times [0, \infty)$. If $K \cap \partial V \neq \emptyset$ and $\dim(V) \geq 7$, then $N - K$ is homeomorphic to $\eta(N) \times [0, \infty)$.*

Proof. By assumption we can find a decreasing sequence of regular neighborhoods $N \supset N_1 \supset N_2 \supset \cdots \supset N_i \supset \cdots \supset K$, each contained in the interior of the next, so that $K = \bigcap_i N_i$. A homeomorphism $N - K$ to $\eta(N) \times [0, \infty)$ is defined inductively sending $\overline{N_i - N_{i+1}}$ homeomorphically to $\eta(N) \times [i, i+1]$ using Proposition 7. □

We now finally consider the existence of regular neighborhoods. The main tool here is the existence of local PL structures, which follows essentially from [7], [9] and PL approximation theorems. The following theorem is due to R. T. Miller, R. Connelly, and R. D. Edwards; we quote from [4]

THEOREM 10. *Let V be a* PL *manifold and K a finite complex locally tamely embedded in V, such that $K \cap \partial V = L$ is a subcomplex of K,* PL*-embedded in ∂V. Further, assume $K - L$ is of codimension greater than or equal to 3 in V. Then there is an ambient ε-isotopy h^t of V, with compact support, fixing ∂V, such that the composition $K \subset V \xrightarrow{h_1} V$ is* PL.

LEMMA 11. *For $n \geq 5$, let $D^p \subset V^n$ be a locally flat embedding, meeting the boundary transversally, such that $\partial V \cap D^p = \partial D^p$. If $n = 5$ assume that ∂V is stable. Then D^p has a neighborhood U with a* PL *structure.*

Proof. By [5], if we let $\widetilde{V} = V \cap \partial V \times [0, 1)$, then D^p has a PL neighborhood $\widetilde{U}$ in $\widetilde{V}$. By Brown's collaring theorem [1], $\widetilde{U} \cap \partial V$ has a neighborhood $\widetilde{\widetilde{U}}$ in $\widetilde{U}$ such that $(\widetilde{\widetilde{U}}, \widetilde{U} \cap \partial V)$ is homeomorphic to $(\widetilde{U} \cap \partial V \times \mathbf{R}, \widetilde{U} \cap \partial V \times 0)$. By [7] we can now change the PL structure of $\widetilde{U}$ so that it is a product structure on $\widetilde{U}$ and hence induces a PL structure on $U = V \cap \widetilde{U}$, a neighborhood of $\phi(D^p)$ in V. To do this for $n = 5$, we need ∂V to be stable. □

Remark 12. Although we do not strictly need it in this paper, it follows from Theorem 10 and Lemma 11 that under the assumptions of Lemma 11, $D^p \subset V$ extends to an embedding

$$(D^p \times \mathbf{R}^{n-p}, \partial D^p \times \mathbf{R}^{n-p}) \subseteq (V, \partial V).$$

This follows for $n - p = 1$ and 2 by [1] an [8] respectively. For $n - p \geq 3$, first tame D^p and then either use block bundle theory to see that the normal block bundle is trivial, hence as described above; or use [12] to see that the "topological normal bundle" is trivial.

We now finally consider the existence of regular neighborhoods.

THEOREM 13. *Let V^n be a topological manifold and L a locally tamely embedded* PL *complex of codimension greater than or equal to* 3 *such that $\partial L = L \cap \partial V$ is a sub complex of L of codimension greater than or equal to* 3 *in ∂V. Let K be a subcomplex of L. Denote $\partial L \cap K$ by ∂K. Then if $n \geq 7$, or if $n \geq 6$ and ∂K is empty, K has a regular neighborhood* *meeting the boundary regularly, so that the intersection with L is a regular neighborhood of K in L.*

Proof. First let us consider the case where ∂K is empty. Triangulate L so that K is a full subcomplex. The 0-skeleton of K is the disjoint union $K^{(0)} = \bigcup D_i^0$ of a finite number of 0-discs. We extend $D_i^0 \subset V$ to disjoint embeddings

$$D_i^0 \times \mathbf{R}^n \subset V,$$

and consider $L \cap D_i^0 \times \mathbf{R}^n$. By Theorem 10, we can change the PL structure of $D_i^0 \times \mathbf{R}^n$ so that a neighborhood of D_i^0 in L is PL-embedded in $D_i^0 \times \mathbf{R}^n$. Therefore, after shrinking $D_0^i \times \mathbf{R}^n$, we may assume that $L \cap D_i^0 \times \mathbf{R}^n$ is PL-embedded in $D_i^0 \times \mathbf{R}^n$. Triangulate $D_i^0 \times \mathbf{R}^n$ such that $L \cap D_i^0 \times \mathbf{R}^n$ is a full subcomplex and let N_i^0 be a derived neighborhood of D_i^0. Define

$$V_1 = \overline{V - \bigcup N_i^0}; \quad \partial_1 V_1 = \partial V_1 \cap \bigcup N_i^0; \quad \text{and} \quad \partial_2 V_1 = \partial V_1 \cap \partial V.$$

Clearly, $\partial_1 V_1 \cap \partial_2 V_1 = \emptyset$. Consider the higher skeleta $K^{(j)}$ of K. Note that $K^{(j)} \cap V_1$ is $K^{(j)}$ with a regular neighborhood of $K^{(0)}$ removed. Therefore, $K^{(1)} \cap V_1$ is a disjoint union of 1-discs meeting $\partial_1 V_1$ transversally; $K^{(1)} \cap V_1 = \bigcup D_i^1$. We use Lemma 11, or rather Remark 12, to extend $D_i^1 \subset V_1$ to disjoint embeddings $D_i^1 \times \mathbf{R}^{n-1} \subset V_1$, and we change PL structure and shrink so that $L \cap D_i^1 \times \mathbf{R}^{n-1} \subset D_i^1 \times \mathbf{R}^{n-1}$ is a PL embedding. We then triangulate so that

$$K \cap D_i^1 \times \mathbf{R}^{n-1} \subset L \cap D_i^1 \times \mathbf{R}^{n-1} \subset D_i^1 \times \mathbf{R}^{n-1}$$

are inclusions of full subcomplexes, and take a derived neighborhood N_i^1 of D_i^1. We put

$$V_2 = \overline{V_1 - \bigcup N_i^1}; \quad \partial_1 V_2 = \partial V_2 \cap \left(\bigcup N_i^0 \cup \bigcup N_i^1\right); \quad \text{and} \quad \partial_2 V_2 = \partial V_2 \cap \partial V.$$

Again, $\partial_1 V_2 \cap \partial_2 V_2 = \emptyset$. Now $K^{(2)} \cap V_2$ is a disjoint union of 2-discs meeting the boundary regularly, since at every point in the boundary they meet the boundary transversally in some PL structure.

In the inductive step, we have

$$V_j = V - \bigcup_{s<j} (N_i^s); \quad \partial_1 V_j = \partial V_j \cap \bigcup_{s<j} (N_i^s); \quad \text{and} \quad \partial_2 V_j = V_j \cap \partial V;$$

and $L \cap V_j$ is L with a regular neighborhood of $K^{(j-1)}$ removed, just as $K^{(s)} \cap V_j$ is $K^{(s)}$ with a regular neighborhood of $K^{(j-1)}$ removed. Thus $K^{(j)} \cap V_j$ is a disjoint union of j-discs meeting the interior of $\partial_1 V_j$ regularly. The inductive step is now completely analogous to the first step. Let $N = \bigcup_{j=1}^{\dim K} \left(\bigcup N_i^j\right)$. We claim N is a regular neighborhood of K in V, and N intersects L in a regular neighborhood of K in L. The latter is clear by construction.

By a standard codimension 3 argument, $\partial N \subset N - K$ induces an isomorphism on the fundamental group. The inclusion $K \subset N$ factors

$$K \subset K \cup \left(\bigcup N_i^0\right) \subset K \cup \left(\bigcup N_i^0 \cup \bigcup N_i^1\right) \subset \cdots \subset N.$$

Since N_i^j was obtained as a PL-regular neighborhood, $K \cup \bigcup_{s\leq j}(\bigcup N_i^s)$ can be strongly deformed into $K \cup \bigcup_{s\leq j-1}(\bigcup N_i^s)$ by a sequence of elementary simplicial collapses, so it follows by induction that K is a strong deformation retract of N and $K \subset N$ is a simple homotopy equivalence. This uses the result of Edwards [2] that the simple homotopy type of a topological manifold is given by the handlebody structure.

In case $\partial K \neq \emptyset$, we proceed as above except at boundary points. We first tame K in the boundary, and then relative to the boundary. we triangulate L such that the inclusions $\partial K \subset K \subset L$ and $\partial K \subset \partial L$ are inclusions of full subcomplexes. In the inductive step of the proof, we have constructed V_j, $\partial_1 V_j$, and $\partial_2 V_j$, where

$$\partial_1 V_j = \partial V_j \cap \left(\bigcup_{s<j}\left(\bigcup N_i^s\right)\right), \quad \partial_2 V_j = \partial V_j \cap \partial V,$$

and $\partial_2 V_j \cap \partial L$ is L with a regular neighborhood of $\partial K^{(j-1)}$ deleted, while $V_j \cap L$ is L with a regular neighborhood of $K^{(j-1)}$ deleted. Further, $\partial_1 V_j$ has a collar in V_j and $\partial V_j \cap L$ has a PL collar in L such that in a neighborhood of $\partial_1 V_j$, the inclusion $V_j \cap L \subset V_j$ is a product inclusion $\partial_1 V_j \cap L \times [0,1) \subset \partial_1 \times [0,1) \subset V_j$. As before, $K^{(j)} \cap V_j$ is a disjoint union of j-discs, but now some of these are contained in $\partial_2 V_j$, meeting $\partial(\partial_2 V_j) = \partial_1 V_j \cap \partial_2 V_j$ regularly. Thus extend $D_i^j \subset \partial_2 V_j$ to $D_i^j \times \mathbf{R}^{n-j-1} \subset \partial_2 V_j$, and extend to $D_i^j \times \mathbf{R}^{n-j-1} \times [0,1) \subset V_j$, using a collar of $\partial_2 V_j$ in V_j. The collar of $\partial_1 V_j \cap L$ in L gives a collar of ∂D_i^j in D_i^j, which induces a collar of $\partial D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ in $D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$. It is easy to see that the extension can be made so that this collar agrees with the given collar of $\partial_1 V_j$. We now change the PL structure of $D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ by an isotopy to make $L \cap D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ be PL embedded in a neighborhood of D_i^j. We do this by first finding an isotopy of $\partial D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ moving a neighborhood of ∂D_i^j in $L \cap D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ to a PL embedding. We extend this isotopy to a neighborhood of $\partial D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ as a product isotopy, using the given collar, and further to $D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$. After shrinking the fibres, we may then assume that

$$L \cap D_i^j \times \mathbf{R}^{n-j-1} \times [0,1) \subset D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$$

is a PL embedding in a neighborhood of $\partial D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$, so we can change PL structure relative to a neighborhood and shrink fibre to be able to assume that we have $L \cap D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$ PL embedded and the PL structure near

$$\partial D_i^j \times \mathbf{R}^{n-j-1} \times [0,1)$$

is the product structure given by the collar of $\partial_1 V$. We triangulate so that all relevant inclusions are inclusions of full subcomplexes, and let N_i^j be a second derived neighborhood of D_i^j. We can still assume that the triangulations near $\partial_1 V_j$ is the product triangulation, so that N_i^j is a product given by the collar bear ∂V_j, and we then proceed as before. In the region between a second derived neighborhood and a first derived neighborhood of D_i^j

everything looks like a product, and this product fits together with the collar of $\partial_1 V_j$ to give a collar of $\partial_1 V_{j+1}$. as desired. □

THEOREM 14. *Let V^n be a topological manifold, $n \geq 5$. If $n = 5$ assume also that V is a stable manifold. Let K be a locally flatly embedded topological handlebody of codimension greater than or equal to* 3. *Then K has a regular neighborhood in V.*

Proof. We proceed totally analogously to the above construction, doing it handle by handle. □

Remark. It is usual in regular neighborhood theory to require the existence of a map $\pi : \partial N \to K$ (N a regular neighborhood of K) such that N is the mapping cylinder of π. In this direction R. D. Edwards pointed out to me that we may prove the following, using a trick due to M. M. Cohen.

THEOREM 15. *Let K be a complex or a closed topological handlebody locally flatly embedded in V^n, V a topological manifold and* $\dim V - \dim K \geq 3$, $n = \dim V \geq 6$. *Let N be a regular neighborhood of K in V. Then there is a map $\pi : \partial N \to K$ such that N is homeomorphic to the mapping cylinder Z_π of π, by a homeomorphism which is the identity on K.*

Proof. By uniqueness of regular neighborhoods, we may assume that N is obtained as in the construction in Theorems 13 and 14. Let us consider the case of Theorem 13, where K is a complex. Assume we have constructed a regular neighborhood N^k, of K^k, the k-skeleton of K, and map $\pi^k : \partial N^k \to K^k$ such that $N^k = Z_\pi k$. Further assume inductively that $N^k \cap K$ and the mapping cylinder of $\pi^k | \partial N^k \cap K$ are equal as sets. The procedure of Theorem 13 is now to attach handles $D^{k+1} \times D^{n-k-1}$ to N^k via a map $S^k \times D^{n-k-1} \subset N^k$ such that $D^{k+1} \times D^{n-k-1}$ is a regular neighborhood of a $(k+1)$-cell in $\overline{K^{k+1} - N^k}$, in some PL structure defined locally, intersecting $\overline{K^{k+1} - N^k}$ in a regular neighborhood of the $(k+1)$-cell. We want to find $\pi^{k+1} : \partial N^{k+1} \to K^{k+1}$. We may assume without loss of generality that N^{k+1} is obtained from N^k by attaching only one $(k+1)$- handle, since otherwise we may repeat the argument.

Given $f : X \to Y$, we orient the mapping cylinder Z_f so that $x \in X$ is identified with $(x, 0) \in Z_f$, $(x, 1) = f(x)$. Since the handle $D^{k+1} \times D^{n-k-1}$ was constructed in an entirely PL situation, there is a map $p : D^{k+1} \times S^{n-k-2} \to D^{k+1}$ such that if we identify the handle with the mapping cylinder Z_p, $K \cap D^{k+1} \times D^{n-k-1}$ is the mapping cylinder of $p | P \cap D^{k+1} \times S^{n-k-2}$. We denote the part of N^k which is the mapping cylinder of $\pi^k | S^k \times D^{n-k-1}$ by B and denote $B \cap N^k$ by $\eta(B)$. Since $\eta(B)$ is the mapping cylinder of $p | \partial\eta(B)$, a point in B can be denote by (x, s, t) where $x \in \partial\eta(B) = S^k \times S^{n-k-2}$, and $s, t \in [0, 1]$.

Let C be a smaller copy of the handle $D^{k+1} \times D^{n-k-1} = Z_p$ corresponding to s-coordinate in $[\frac{1}{2}, 1]$. We now define $\pi^{k+1} : \partial(N^k \cup C) \to K^{k+1}$ by $\pi^{k+1} = \pi^k$ when restricted to $\overline{\partial N^k - \eta B}$. Since $\pi^{k+1}(x, \frac{1}{2}) = p(x)$ for $(x, \frac{1}{2}) \in \overline{\partial C - B}$, we now need to define π^{k+1} on $\overline{\eta B - C \cap \partial N^k}$, which are the points of B with coordinates $(x, s, 0)$, $s \in [0, \frac{1}{2}]$. We may consider $[0,1] \times [0,1]$ as the mapping cylinder of a map $\chi : [0, \frac{1}{2}] \times 0 \to [0,1] \times 1 \cup 1 \times [0,1]$ and we now finish the inductive step by defining $\pi^{k+1}(x, s, 0) = (x, \chi(s, 0))$. It is easy to see that π^{k+1} has all the required properties, since the points in $B \cap K$ are exactly the points (x, s, t) with either $s = 1$ or $t = 1$ or $x \in \partial\eta B \cap K$. □

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