Semifree Topological Actions of Finite Groups on Spheres

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We recall that an action of a finite group G on a manifold M is *topological* if G acts via homeomorphisms and that it is *semifree* if for any $x \in X$ the isotropy subgroup $G_x = \{g \in G | gx = x\}$ is either trivial or G. The set of points of the latter type is called the *fixed point set* and is denoted by $\mathrm{Fix}(G)$.

Let G be a finite group and n and k be integers. The problem considered in this paper is the following: For which (G, n, k) does there exist a semifree topological action of G on S^{n+k} with $\mathrm{Fix}(G)$ a standard (i.e. unknotted) S^k? The emphasis in this question should be on the way in which G, n, and k are interrelated and not on viewing these parameters individually. For example, the following problems are of interest: Suppose G acts on S^{n+k} with $\mathrm{Fix}(G)$ a standard S^k. Does G act on S^{n+k-1} with $\mathrm{Fix}(G)$ a standard S^{k-1}? (The converse is obvious by suspending the action.) For fixed (G, n), what is the minimum value for k?

In the case when $k = -1$ [i.e. $\mathrm{Fix}(G) = \emptyset$ and G acts freely], this problem has an extensive literature. In particular, in [3] Cartan and Eilenberg show that if G acts freely on S^n for some n, then G has periodic cohomology of period d (necessarily even) and $n = rd - 1$ for some integer $r \geq 1$. Furthermore, in [14] Milnor shows that all subgroups of G of order $2p$ (p a prime) must be cyclic. In their work on the spherical space form problem [23, 11, 24, 10], Madsen, Thomas, and Wall proved the following converse to the above results.

Theorem (Madsen, Thomas, Wall). *Let G be a finite group such that every subgroup of G of order $2p$ (p a prime) is cyclic and suppose G has periodic cohomology of period d. Let $n = 2rd - 1$ where r is an integer ≥ 1. Then G acts freely on S^n.*

For many groups G satisfying the group theoretic hypothesis in this theorem, this result is best possible; i.e. G cannot act freely on S^{rd-1} where r is an odd integer. For some of these groups G, the problem of whether G acts on a sphere of the "period dimension" (i.e. S^{d-1}) or more generally S^{rd-1}, r an odd integer, is still unresolved due to difficulties in calculating Swan's finiteness obstructions [22] and surgery obstructions.

In the case when $k \geq 0$, the result of Cartan and Eilenberg [3] still applies and G must have periodic cohomology of even period d. Milnor's theorem [13], however, cannot be applied in this setting and it is unknown whether subgroups of G of order $2p$ must be cyclic. Indeed, settling this question is one of the interesting outstanding problems in this area.

If we concentrate on the groups of periodic cohomology which do satisfy the condition that subgroups of order $2p$ are cyclic, the main interest is in which values of n and k can occur and, in particular, in what happens when $n = rd$ for r an odd integer. In the case when $k = 0$, this problem has been considered in a slightly different form by Hambleton-Madsen [7] and by Milgram [12]. (One need only one point compactify their problem to get ours.)

The main thrust of this paper is to supply the topological arguments needed to reduce the existence problem stated above to a problem of calculating appropriate surgery obstructions. This result is stated precisely as Theorem A in Sect. 1. We also prove the following as a corollary to Theorem B (see Sect. 1 for a statement of that theorem):

Theorem. *Let $n \geq 4$. If the finite group G acts semifreely on S^{n+k} with $\mathrm{Fix}(G)$ a standard S^k, then G acts semifreely on S^{n+1} with $\mathrm{Fix}(G)$ a standard S^1.*

Thus, from the point of view of the existence of actions, it suffices to consider the cases when $k \leq 1$.

This paper is organized as follows: Our main results Theorems A and B are explained and stated in Sect. 1; Sect. 2 contains some results in lower L-theory needed to prove part of Theorem A; that part of the proof of Theorem A is in Sect. 3; Sects. 4–6 develop the results needed to prove Theorem B; while Sect. 7 contains that proof; Sect. 8 completes the proof of Theorem A, and Sect. 9 speculates on the classification problem.

1. Preliminary Observations and Statements of Theorems

For the sake of definiteness in this section and the rest of this paper, we shall consider

$$S^{n+k} = \{x \in R^{n+k+1} \mid |x| = 1\},$$

$$S^{n+k-1} = \{x = (x_1, \ldots, x_{n+k+1}) \in S^{n+k} \mid x_{k+1} = 0\}$$

and for $l = k - 1, k,$

$$S^l = \{x = (x_1, \ldots, x_{n+k-1}) \in S^{n+k} \mid x_i = 0 \text{ for } i \geq l+2\}.$$

In particular, $S^{n+k-1} \cap S^k = S^{k-1}$. We shall also consider the standard sphere pair (S^{n+k}, S^k) as being fixed and any action of G on S^{n+k} as arising from a homomorphism $\varrho: G \to \mathrm{Homeo}(S^{n+k}, S^k)$, the group of homeomorphisms of S^{n+k} that restrict to the identity on S^k and shall fix an identification of $S^{n+k} - S^k$ with $S^{n-1} \times R^{k+1}$.

If we now have a semifree action of G on S^{n+k} with $\mathrm{Fix}(G) = S^k$, then G acts freely on $S^{n+k} - S^k = S^{n-1} \times R^{k+1}$ and it follows from Cartan and Eilenberg [3] that G has periodic cohomology of period d and if $G \neq Z_2$, $n = rd$. (Whether G must satisfy the $2p$-condition of Milnor [13] is not known, however.)

The interrelationships among G, n, and k now become dependent on the period of G. In particular, if $d = 2$, G is finite cyclic and there are two possibilities: $G = Z_2$ or $G = Z_n$ ($n > 2$). In the first case there are linear semifree actions of G on S^{n+k} with Fix$(G) = S^k$ for any n and k. In the latter case, there are linear semifree actions of G on S^{n+k} with Fix$(G) = S^k$ for any even n (the only values of n possible) and k. Furthermore, in these two cases the linear actions exhaust the possible homotopy types of the orbit space $S^{n+k} - S^k/G$. In the rest of this paper, therefore, we shall assume that d (and thus n) is at least 4. We remark that in this case if Fix$(G) = S^k$ is locally flat, it is necessarily unknotted by [15].

Lemma 1.1. If $n \geq 3$, the orbit space $X = S^{n+k} - S^k/G$ is a polarized complex in the sense of [11].

Proof. We clearly have given a preferred homotopy equivalence $h : \tilde{X} \to S^{n-1}$ where $\tilde{X}$ is the universal cover of X and a preferred isomorphism $\pi_1(X, x_0) \to G$. Thus, it suffices to show X is finitely dominated. This, however, follows directly from [6, Lemma 3.5].

Since for $n = rd$, there is a bijection between homotopy classes of polarized complexes and generators $\kappa \in H^n(G; Z) \approx Z_{|G|}$ obtained by taking the first k-invariant [24], we shall let $X(\kappa)$ denote the polarized complex corresponding to κ. We recall that such polarized complexes are Poincaré duality complexes whose Spivak normal fibre space admits a topological (in fact, a smooth) normal invariant [11, Theorem 3.1]. We fix one such normal invariant $v(\kappa) : X(\kappa) \to \text{BTOP}$ and recall that any other normal invariant for $X(\kappa)$ differs from $v(\kappa)$ by an element $x \in [X(\kappa); G/\text{TOP}]$.

The following theorem is our first main result:

Theorem A. *Consider the following statements:*

i) *There exists an element* $x \in [X(\kappa); G/\text{TOP}]$ *whose surgery obstruction* $\theta(x) = 0$ *in* $L_{n-1}^{-k}(G)$.

ii) *There exists a semifree topological action of* G *on* S^{n+k} *with* Fix$(G) = S^k$ *and* $S^{n+k} - S^k/G$ *homotopy equivalent to* $X(\kappa)$.

If $n \geq 4$ *and* $n + k \geq 5$, *then* i) *implies* ii). *If* $n \geq 5$ *then* ii) *implies* i). *Thus* i) *and* ii) *are equivalent for* $n \geq 5$.

It seems quite likely that $n \geq 5$ in the second half of Theorem A can be replaced by $n \geq 4$. Thus, it is likely that i) and ii) are equivalent whenever $n \geq 4$.

In this theorem the functors $L_{n-1}^{-k}(\ )$ ($k \geq 0$) are defined inductively by setting $L_{n-1}^0(\pi) = L_{n-1}^p(\pi)$, the Wall group based on projective modules, and then setting

$$L_{n-1}^{-(k+1)}(\pi) = \text{coker}\{\sigma_* : L_n^{-k}(\pi) \to L_n^{-k}(\pi \times Z)\}$$

where $\sigma : \pi \to \pi \times Z$ is the obvious inclusion. These groups have been investigated by Ranicki [19]. In particular, he shows that there is a natural homomorphism $\varrho_k : L_{n-1}^{-k}(\pi) \to L_{n-1}^{-(k+1)}(\pi)$ that fits into a Rothenberg type exact sequence

$$\cdots \to H(Z_2; K_{-(k+1)}(\pi)) \to L_{n-1}^{-k}(\pi) \xrightarrow{\varrho} L_{n-1}^{-(k+1)}(\pi) \to H(Z_2; K_{-(k+1)}(\pi)) \to$$

Thus, although the surgery obstruction $\theta(x)$ of Theorem A originally lies in $L_{n-1}^p(G)$, it is only its image in $L_{n-1}^{-k}(G)$ that plays a role in Theorem A.

We remark that Theorem A actually contains more information than our problem, as originally stated, requires. In particular, it includes the homotopy type of $S^{n+k} - S^k/G$.

Our next main result is the following theorem:

Theorem B. *Let G act semifreely on S^{n+k} with $\mathrm{Fix}(G) = S^k$. Let $n + k \geq 6$. Suppose that either $k \geq 2$ or that $k \leq 1$ and that an obstruction $O \in \tilde{K}_{-k}(G)$ vanishes, then there is a G-invariant subspace M of S^{n+k} meeting S^k in S^{k-1} and an ambient isotropy $h_t : S^{n+k} \to S^{n+k}$ ($0 \leq t \leq 1$) such that $h_0 = 1$, $h_1(M) = S^{n+k-1}$, and $h_t|S^k = 1$ for all t. In particular, G acts semifreely on S^{n+k-1} with $\mathrm{Fix}(G) = S^{k-1}$.*

If $k = 0$, O is essentially a Siebenmann end invariant and has already been investigated by Edmonds [6]. If $k = 1$, O is the Quinn invariant of a parametrized end which is described more carefully in Sect. 6.

We recall that two actions ϱ_0 and ϱ_1 of G on the manifold M are *isotopic* if there exists a level preserving action ϱ of G on $M \times I$ such that $\varrho|M \times \{i\} = \varrho_i$ ($i = 0, 1$).

A simple Alexander isotopy argument proves the following proposition:

Proposition. *Let $\varrho : G \to \mathrm{Homeo}(S^{n+k}, S^k)$ be a semifree action of G on S^{n+k} with $\mathrm{Fix}(G) = S^k$ and suppose S^{n+k-1} is G-invariant. Let $\varrho_0 : G \to \mathrm{Homeo}(S^{n+k-1}, S^{k-1})$ be the restriction of ϱ. Then ϱ is isotopic to the suspension of ϱ_0.*

If we combine this proposition with Theorem B we see that every semifree action of G on S^{n+k} with $\mathrm{Fix}(G) = S^k$ is isotopic to a suspension provided that $n + k \geq 6$ and $k \geq 2$ and that if $k \leq 1$, there is an obstruction to "desuspension up to isotopy" lying in $\tilde{K}_{-k}(G)$ ($k = 0, 1$).

2. Some Results in Lower L-Theory

Let X (respectively Y) be a finite (respectively, finitely dominated) Poincaré complex of dimension $n - 1$ whose Spivak normal fibre space admits a reduction to BTOP. [For example take Y to be the Swan complex $X(\kappa)$ in Sect. 1.] The following proposition, whose proof comes later in this section, is the starting point of the proof of the sufficiency part of Theorem A:

Proposition 2.1. i) *Let $n + k \geq 5$. If there exists $x \in [X; G/\mathrm{TOP}]$ whose surgery obstruction $\theta(x) = 0$ in $L^{-k}_{n-1}(\pi_1(X))$, then there is a homotopy structure $f : M \to X \times T^{k+1}$ whose normal invariant is $\pi^* x$ where $\pi : X \times T^{k+1} \to X$ is the projection.*

ii) *Part i) holds with X replaced by Y.*

The proof of this proposition depends on the following result concerning the lower L-groups:

Proposition 2.2. *For every $k \geq 1$ there exists a monomorphism $\chi_k : L^{-k}_{n-1}(\pi) \to L^{-k+1}_n(\pi \times Z)$ defining a natural transformation of functors. Furthermore,*

i) *If* $k = 1$, *the following diagram commutes*

$$
\begin{array}{ccc}
L_{n-1}^{p}(\pi) & \xrightarrow{\;\varrho_0\;} & L_{n-1}^{-1}(\pi) \\
\downarrow {\scriptstyle \times S^1} & & \downarrow {\scriptstyle \chi_1} \\
L_{n}^{h}(\pi \times Z) & \xrightarrow{\;\varrho\;} & L_{n}^{p}(\pi \times Z)
\end{array}
$$

where ϱ is the usual homomorphism and $\times S^1$ is obtained by taking the Cartesian product of an L^p surgery problem with S^1.

ii) *If* $k > 1$, *then* $\chi_{k+1}\varrho_k = \varrho_{k+1}\chi_k$.

Proof. The homomorphisms χ_k are defined inductively as follows: To define χ_1 consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_{n}^{p}(\pi) & \xrightarrow{\;\sigma_*\;} & L_{n}^{p}(\pi \times Z') & \longrightarrow & L_{n-1}^{-1}(\pi) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \times S_1} & & \downarrow {\scriptstyle \times S_1} & & \big\downarrow {\scriptstyle \chi_1} & & \\
 & & L_{n+1}^{h}(\pi \times Z) & \xrightarrow{\;(\sigma \times 1)_*\;} & L_{n+1}^{h}(\pi \times Z' \times Z) & & \big\downarrow & & \\
 & & \downarrow {\scriptstyle 1} & & \downarrow {\scriptstyle (1 \times \tau)_*} & & \big\downarrow & & \\
0 & \longrightarrow & L_{n+1}^{h}(\pi \times Z) & \xrightarrow{\;\bar{\sigma}_*\;} & L_{n+1}^{h}(\pi \times Z \times Z') & \longrightarrow & L_{n}^{p}(\pi \times Z) & \longrightarrow & 0
\end{array}
$$

in which Z' is infinite cyclic, $\sigma : \pi \to \pi \times Z'$ and $\bar{\sigma} : \pi \times Z \to \pi \times Z \times Z'$ are the inclusions as the "first" factor, and $\tau : Z' \times Z \to Z \times Z'$ switches the two factors. The top row is exact by the definition of L^{-1}; while the bottom row is exact by [20] or [19]. Since the left half of the diagram commutes, there is an induced homomorphism χ_1 making the whole diagram commute. Clearly χ_1 defines a natural transformation of functors.

Once χ_i is defined for $i \leq k$, χ_{k+1} is obtained by replacing h, p, -1, and $\times S^1$ in the above diagram, respectively by $k-1$, k, $k+1$, and χ_k. It is then clear once again that χ_{k+1} is a natural transformation of functors.

To show χ_k is monomorphic for all $k \geq 1$, it suffices to observe that for all $k \geq 0$, there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & L_{n-1}^{-(k+1)}(\pi) & \xrightarrow{\;\varepsilon\;} & L_{n}^{-k}(\pi \times Z') & \xrightarrow{\;\eta_*\;} & L_{n}^{-k}(\pi) & \to & 0 \\
 & & \big\downarrow {\scriptstyle \chi_{k+1}} & & \downarrow {\scriptstyle \chi_k} & & \downarrow {\scriptstyle \chi_k} & & \\
 & & \big\downarrow & & L_{n+1}^{-k+1}(\pi \times Z' \times Z) & \xrightarrow{\;(\eta \times 1)_*\;} & L_{n+1}^{-k+1}(\pi \times Z) & & \\
 & & \big\downarrow & & \downarrow {\scriptstyle (1 \times \tau)_*} & & \downarrow {\scriptstyle 1} & & \\
0 & \to & L_{n}^{-k}(\pi \times Z) & \xrightarrow{\;\varepsilon\;} & L_{n+1}^{-k+1}(\pi \times Z \times Z') & \xrightarrow{\;\bar{\eta}_*\;} & L_{n+1}^{-k+1}(\pi \times Z) & \to & 0
\end{array}
$$

where $\eta : \pi \times Z' \to \pi$ and $\bar{\eta} : \pi \times Z \times Z' \to \pi \times Z$ are projections on the "first" factor, $\varepsilon = 1 - \sigma_*\eta_*$, and $\bar{\varepsilon} = 1 - \bar{\sigma}_*\bar{\eta}_*$. [Thus ε is the splitting of the split short exact sequence

$$
0 \to L_{n}^{-k}(\pi) \xrightarrow[\;\sigma_*\;]{\;\eta_*\;} L_{n}^{-k}(\pi \times Z') \to L_{n-1}^{-(k+1)}(\pi) \to 0
$$

complementary to η_*.] In the case when $k=0$ and one replaces $-k$, $-k+1$, and χ_k in this diagram by p, h, and $\times S^1$ respectively, the vertical maps $(1 \times \tau)_* \chi_k$ and χ_k are split monomorphic by [20] or [19]. It follows that χ_1 is monomorphic. The general case now follows easily.

In order to prove i), we recall first that there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_n^h(\pi) & \xrightarrow{\ \sigma_*\ } & L_n^h(\pi \times Z') & \longrightarrow & L_{n-1}^p(\pi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times S^1} & & \downarrow{\scriptstyle \times S^1} & & \downarrow & & \\
 & & L_{n+1}^s(\pi \times Z) & \xrightarrow{(\sigma \times 1)_*} & L_{n+1}^s(\pi \times Z' \times Z) & & & & \downarrow{\scriptstyle \times S^1} \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle (1 \times \tau)_*} & & & & \\
0 & \longrightarrow & L_{n+1}^s(\pi \times Z) & \xrightarrow{\ \bar\sigma_*\ } & L_{n+1}^s(\pi \times Z \times Z') & \longrightarrow & L_{n-1}^h(\pi) & \longrightarrow & 0
\end{array}
$$

with exact rows by [20] and [16]. If one now maps this diagram into the diagram used above to define χ_1 via the usual maps of $L_*^a(\ ) \to L_*^b(\ )$ where (a, b) is (s, h), (h, p) or $(p, -1)$, the resulting (three dimensional) diagram commutes. The commutative square on the right hand end of this diagram yields i).

The proof of ii) is similar to that of i) and is omitted.

Proof of 2.1. The proof of i) is by induction on k. The case $k=0$ is proved in [16] essentially by the following argument: Observe that there is a commutative diagram

$$
\begin{array}{ccc}
[X ; G/\text{TOP}] \longrightarrow L_{n-1}^h(\pi_1(X)) \longrightarrow L_{n-1}^p(\pi_1(X)) \\
\downarrow{\scriptstyle \pi^*} \qquad\qquad \downarrow{\scriptstyle \times S^1} \qquad\qquad \downarrow{\scriptstyle \times S^1} \\
[X \times S^1 ; G/\text{TOP}] \longrightarrow L_n^s(\pi_1(X) \times Z) \longrightarrow L_n^h(\pi_1(X) \times Z)
\end{array}
$$

in which π^* is induced by the projection $\pi : X \times S^1 \to X$. Thus if $\theta(x)=0$ in $L_{n-1}^p(\pi_1(X))$, $\theta(\pi^* x)=0$ in $L_n^h(\pi_1(X) \times Z)$ and the result follows from the surgery exact sequence.

Suppose i) holds for $k-1$ where $k \geq 1$. It then follows from 2.2 that the diagram

$$
\begin{array}{ccc}
[X ; G/\text{TOP}] \xrightarrow{\ \theta\ } L_{n-1}^{-(k-1)}(\pi_1(X)) \longrightarrow L_{n-1}^{-k}(\pi_1(X)) \\
\downarrow{\scriptstyle \pi^*} \qquad\qquad\qquad \downarrow{\scriptstyle \bar\chi^k} \qquad\qquad\qquad \downarrow{\scriptstyle \chi^k} \\
[X \times T^k ; G/\text{TOP}] \longrightarrow L_{n-1+k}^h(\pi_1(X) \times Z^k) \longrightarrow L_{n-1+k}^p(\pi_1(X) \times Z^k)
\end{array}
$$

commutes where $\pi : X \times T^k \to X$ is projection, $\chi^k = \chi_0 \chi_1 \cdots \chi_k$, $\bar\chi^k = (\times S^1) \chi^{k-1}$, and η is the composite of the usual surgery obstruction map

$$\theta : [X ; G/\text{TOP}] \to L_{n-1}^h(\pi_1(X))$$

and the maps ϱ_i of L-theories. In particular, $\theta(\pi^* x)=0$ in $L_{n-1+k}^p(\pi_1(X \times T^k))$ and this case now follows from the case when $k=0$.

Part ii) is an easy consequence of part i).

3. The Proof that i) implies ii) in Theorem A

It follows immediately from 2.1 that there is a homotopy equivalence $f: M \to X(\kappa) \times T^{k+1}$, where M is a compact manifold whose normal invariant is $\pi^* x$. Let $\tilde{M}$ be the universal cover of M and notice that $\tilde{M}$ supports a free action of G (in fact, of $G \times Z^{k+1}$). The proof of the sufficiency part of Theorem A is completed by proving the following proposition.

Proposition 3.1. *There is a homeomorphism* $\tilde{d}: \tilde{M} \to S^{n-1} \times R^{k+1}$ *which induces a free action of* G *on* $S^{n-1} \times R^{k+1}$. *This induced action is bounded in the* R^{k+1} *factor; hence, it extends to a semifree topological action of* G *on* S^{n+k} *with* $\mathrm{Fix}(G) = S^k$.

We recall that if Y is a compact space, a function $f: Y \times R^{k+1} \to Y \times R^{k+1}$ is *bounded in the* R^{k+1} *factor* if there exists a number N such that for all $x \in Y \times R^{k+1}$, $|\pi(x) - \pi f(x)| < N$ where $\pi: Y \times R^{k+1} \to R^{k+1}$ is the projection.

Let $\mathrm{Homeo}_b(Y \times R^{k+1})$ be the group of homeomorphisms of $Y \times R^{k+1}$ that are bounded in R^{k+1} factor and $\mathrm{Homeo}(Y * S^k, S^k)$ be the group of homeomorphisms of the join $Y * S^k$ which are the identity on S^k. We recall that there is a homomorphism

$$\gamma: \mathrm{Homeo}_b(Y \times R^{k+1}) \to \mathrm{Homeo}(Y * S^k, S^k)$$

obtained by the following modification of an idea due to Kirby [9] (cf. also [1, 2]):
Let $h: Y \times R^{k+1} \to Y \times R^{k+1}$ be bounded in the R^{k+1} factor and consider the diagram

$$
\begin{array}{ccccccc}
Y \times R^{k+1} & \xrightarrow{1 \times \varrho} & Y \times D^{k+1} & \longrightarrow & Q & \xrightarrow{g} & Y * S^k \\
\downarrow h & & & & \downarrow h' & & \downarrow h'' \\
Y \times R^{k+1} & \xrightarrow{1 \times \varrho} & Y \times D^{k+1} & \longrightarrow & Q & \xrightarrow{g} & Y * S^k
\end{array}
$$

where $\varrho: R^{k+1} \to \mathrm{Int}\, D^{k+1}$ is a radial homeomorphism and Q is the quotient space of $Y \times D^{k+1}$ obtained by identifying (y, t) with (y', t) for all $y, y' \in Y$ and $t \in S^k$. It is easy to see that setting $h' = 1$ on $y \times S^k$ (for any $y \in Y$) yields a homeomorphism h' of Q extending h. It is also easy to see that there is a homeomorphism of pairs $g: (Q, y \times S^k) \to (Y_* S^k, S^k)$. Let $h'' = gh'g^{-1}$. Then $\gamma(h) = h''$ is the desired homeomorphism.

Proof of 3.1. The remarks above show that the last clause of 3.1 follows directly from the rest of it. Thus we need only find $\tilde{d}$ and show the induced action of G is bounded in the R^{k+1} direction.

To do this we consider first the diagram

$$
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{f}} \tilde{X}(\kappa) \times T^{k+1} \xrightarrow{h \times 1} S^{n-1} \times T^{k+1} \\
\downarrow p & \qquad \cdot \qquad \downarrow q \times 1 \\
M & \xrightarrow{f} X(\kappa) \times T^{k+1}
\end{array}
$$

in which $q: \tilde{X}(\kappa) \to X(\kappa)$ is the universal cover of $X(\kappa)$, p is a pull back, $\hat{f}$ covers f, and h is the preferred homotopy equivalence given by the polarization of $X(\kappa)$.

Then $(\hat{M}, \hat{f}) \in hS^h(\tilde{X}(\kappa) \times T^{k+1})$ and the normal invariant

$$\eta(\hat{M}, \hat{f}) = (q \times 1)^* \eta(M, f) = (q \times 1)^* \pi^*(x) = (\tilde{\pi})^* q^* x$$

where $\tilde{\pi} : \tilde{X}(\kappa) \times T^{k+1} \to \tilde{X}(\kappa)$ is the projection. It now follows that

$$(\hat{M}, (h \times 1)\hat{f}) \in hS^h(S^{n-1} \times T^{k+1})$$

and that

$$\eta(\hat{M}, (h \times 1)\hat{f}) = (\tilde{\pi})^*(h^{-1})^*(q^* x)$$

where h^{-1} is a homotopy inverse for h and $\tilde{\pi} : S^{n-1} \times T^{k+1} \to S^{n-1}$ is the projection. Since $n-1 = rd-1$ is odd, $\pi_{n-1}(G/\text{TOP})$ vanishes and so does $\eta(\hat{M}, (h \times 1)\hat{f})$. Hence

$$(\hat{M}, (h \times 1)\hat{f}) \in \text{Im}\{\partial : L^s_{n+k+1}(Z^{k+1}) \to hS^s(S^{n-1} \times T^{k+1})\}.$$

[Since $Wh(Z^{k+1}) = 0$, L^h and L^s theory coincide.] It now follows from [8, Theorem 10.2] if $n \geq 5$, or from [8, Remark 3, p. 44] if $n = 4$, that there is a homeomorphism $\hat{d} : \hat{M} \to S^{n-1} \times T^{k+1}$ which we may assume induces the identity on fundamental groups.

Let $\tilde{d} : \tilde{M} \to S^{n-1} \times R^{k+1}$ be a lift of $\hat{d}$ to universal covers. Then $\tilde{d}$ is a homeomorphism and the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{d}} & S^{n-1} \times R^{k+1} \\
p' \downarrow & & \downarrow 1 \times e \\
\hat{M} & \xrightarrow{\hat{d}} & S^{n-1} \times T^{k+1}
\end{array}
$$

where e is the exponential map. Since $\tilde{M}$ and $\hat{M}$ support free actions of G with respect to which p' is equivariant, there are free actions of G on $S^{n-1} \times R^{k+1}$ and $S^{n-1} \times T^{k+1}$ with respect to which $1 \times e$ is equivariant. Since the action of G on $\hat{M}$ induces the identity on the fundamental group, the same is true of the action of G on $S^{n-1} \times T^{k+1}$. It now follows that the lifted action of G on $S^{n-1} \times R^{k+1}$ is bounded in the R^{k+1} factor. This completes the proof of 3.1 and of the fact that if $n + k \geq 5$, then i) implies ii) in Theorem A.

4. Some Lemmas Concerning Quinn Ends

The proofs of the ii) implies i) part of Theorem A and of Theorem B are both based on the observation that the part of the orbit space S^{n+k}/G near the singular set S^k can be regarded as an end in the sense of Quinn, [17] and [18], and on an analysis of that end. In this section we shall establish some general lemmas about Quinn ends (henceforth called simply ends) that will be used in studying the end of S^{n+k}/G near S^k. We assume the reader is familiar with the terminology of [17, Sects. 1 and 2].

Let $e : M \to X$ be an end where M is a manifold with compact boundary. We say that e is *eventually collared* if for every neighborhood U of the end there is a

manifold neighborhood N of the end such that $N \subset U$, $\partial N \cap \partial M = \emptyset$ and such that there is a commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ e\ } & X \\
\scriptstyle h \downarrow & & \uparrow \scriptstyle e \\
\partial N \times [0, \infty) & \xrightarrow{\ p_1\ } & \partial N
\end{array}
$$

where $h : (N, \partial N) \to (\partial N \times [0, \infty), \partial N \times 0)$ is a homeomorphism of pairs and p_1 is projection on the first factor.

If $e_i : M_i \to X$ $(i = 1, 2)$ are ends, we shall say that e_1 and e_2 are *eventually locally equivalent* if either $M_1 \subset M_2$ or $M_2 \subset M_1$, and if for every neighborhood U of the end e_i $(i = 1, 2)$ and every neighborhood V of $x \in X$, there is a neighborhood U' of the end e_{3-i} and a neighborhood V' of x such that $U' \cap e_{3-i}^{-1}(V') \subset U \cap e_i^{-1}(V)$.

Lemma 4.1. *Let* $e_1, e_2 : M \to X$ *be eventually locally equivalent ends.*

i) *Let* P *denote any of the following properties: onto*, $0 - LC$, *or* $1 - LC$. *Then* e_1 *has property* P *if and only if* e_2 *has property* P.

ii) *If* X *is compact and* e_1 *is eventually collared, then* e_2 *is tame.*

Proof. Part i) is obvious. To prove part ii) we need the following sublemma:

Sublemma 4.2. *Let* $e_1, e_2 : M \to X$ *be eventually locally equivalent ends where* X *is a compact metric space with metric* ϱ. *Let* $U \subset M_1 \cap M_2$. *Then* U *is a neighborhood of the end* e_1 *if and only if* U *is a neighborhood of* e_2. *Furthermore, given* $\delta > 0$ *there is a neighborhood* U *of the end* e_1 *such that for all* $y \in U$, $\varrho(e_1(y), e_2(y)) < \delta$.

Proof. Let U be a neighborhood of e_i $(i = 1, 2)$. Then $e_i | : M_i - U \to X$ is proper. Hence, since X is compact, $(e_i|)^{-1}(X) = M_i - U$ is compact. But then $e_{3-i} | : M_{3-i} - U \to X$ is clearly proper and U is a neighborhood of the end e_{3-i}. The first part of the sublemma follows.

To prove the second part of the sublemma, let $x \in X$ and V_x be the ball of radius $\delta/2$ with center x. Then there exists a neighborhood U'_x of the end e_2 and a neighborhood $V'_x \subset V_x$ of x such that $U'_x \cap e_2^{-1}(V'_x) \subset e_1^{-1}(V_x)$. Since X is compact, there are finitely many of the V'_x, say $V'_{x_1}, \ldots, V'_{x_r}$, which cover X. Let $U = \bigcap\limits_{i=1}^{r} U_{x_i}$. Then U is a neighborhood of e_2 hence also of e_1. Let $y \in U$. Then for some j $(1 \leq j \leq r)$,

$$
y \in U \cap e_2^{-1}(V'_{x_j}) \subset U_{x_j} \cap e_2^{-1}(V'_{x_j}) \subset e_1^{-1}(V_{x_j}).
$$

Thus $e_2(y) \in V'_{x_j} \subset V_{x_j}$ and $e_1(y) \subset V_{x_j}$. Since the diameter of V_{x_j} is less than δ, $\varrho(e_1(y), e_2(y)) < \delta$ and the sublemma follows.

We return to the proof of part ii) of Lemma 4.1. Let U be a neighborhood of the end e_2 and $\varepsilon > 0$. (Since X is compact, we can replace the function $\varepsilon : X \to (0, \infty)$ by a constant.) By the sublemma there is a neighborhood U' of the end e_1 such that for all $y \in U'$, $\varrho(e_1(y), e_2(y)) < \varepsilon/3$. Without loss of generality, we may assume $U' \subset U$. Since e_1 is eventually collared, there is a manifold neighborhood N of the end such that $N \subset U'$ and a homeomorphism $h : N \to \partial N \times [0, \infty)$ such that $e_1 | N = (e_1 | \partial N)$ $\cdot p_1 h$ where $p_1 : \partial N \times [0, \infty) \to \partial N$ is projection on the first factor. Let

$g_t: \partial N \times [0, \infty) \to \partial N \times [0, \infty)$ $(0 \leq t \leq 1)$ be given by $g_t(x,s) = (x,(1-t)s)$, and $h_t = h^{-1} g_t h$ $(0 \leq t \leq 1)$. Then $h_t | \partial N = 1$ and we extend h_t over M by setting $h_t = 1$ on $M - N$. Clearly $h_0 = 1$, $h_1(M) \subset \overline{M-N}$, and $h_t(M-U) \subset \overline{M-N}$ for all t. Furthermore if $x \in M$, then the diameter of $\{e_2 h_t(x) | 0 \leq t \leq 1\}$ is 0 if $x \in M - N$; while if $x \in N$, then for any r, s satisfying $0 \leq r$, $s \leq 1$ we have

$$\varrho(e_2 h_r(x), e_2 h_s(x)) \leq \varrho(e_2 h_r(x), e_1 h_r(x)) + \varrho(e_1 h_r(x), e_1 h_s(x)) + \varrho(e_1 h_s(x), e_2 h_s(x)).$$

Since $x \in N$, $h_t(x) \in N \subset U'$ for all t. Hence the first and last terms are less than $\varepsilon/3$ by the choice of U'. The middle term is zero by the choice of h_t. Thus, $\mathrm{diam}\{e_2 h_t(x) | 0 \leq t \leq 1\} < t$ independent of x and e_2 is tame. This completes the proof of the lemma.

Lemma 4.3. *Let* $e_i: M_i \to X$ $(i = 1,2)$ *be ends and suppose there exists a covering map* $q: M_1 \to M_2$ *with finitely presented group of covering transformations such that* $e_1 = e_2 q$. *Then*

i) *The end* e_1 *is onto (respectively,* $0-LC$*) if and only if* e_2 *is onto (respectively,* $0-LC$*).*

ii) *If* e_1 *is* $1-LC$*, then* e_2 *has constant fundamental group the group of covering transformations of* $q: M_2 \to M_1$*.*

iii) *Suppose* X *is locally compact and locally simply connected,* e_i $(i = 1,2)$ *is onto and* $0-LC$*, and* e_1 *is* $1-LC$*. Then* e_2 *is tame if and only if* e_1 *is tame.*

Proof. Part i) is obvious; while ii) follows from the observation that since e_1 is $1-LC$, the group of covering transformations of M_1 over M_2 is the (constant) fundamental group of e_2. To prove iii) observe first that since e_1 is $1-LC$ it has constant fundamental group the trivial group. It follows from [17, Proposition 1.7] that e_i $(i = 1,2)$ is tame if and only if it is homologically tame. Since $e_1 = e_2 q$, the calculations necessary to verify the homological tameness of e_1 and e_2 are identical. Hence iii) follows.

5. The Ends of the Action near S^k

In this section we describe three ends related to the action of G on S^{n+k} (the non-equivariant end of S^{n+k} near S^k, the equivariant end of S^{n+k} near S^k, and the end of S^{n+k}/G near S^k) and derive their basic properties.

To describe the first end we first fix a homeomorphism $f: S^{n+k} \to S^{n-1} * S^k$ that is the identity on S^k and let $c_1: S^{n+k} \to D^k$ be the composite

$$S^{n+k} \xrightarrow{\ f\ } S^{n-1} * S^k \xrightarrow{\ c*1\ } x * S^k \to D^{k+1}$$

where $c: S^{n-1} \to x$ is the constant map and the last map sends the formal sum (in the sense of Milnor [13]) $(1-t)x \oplus ty$ $(0 \leq t \leq 1; y \in S^k)$ to $ty \in D^{k+1}$. Then $c_1 | S^k$ is the identity and $c_1^{-1}(S^k) = S^k$.

Let rD^{k+1} be the disk of radius r about the origin and rS^k be its boundary. Clearly $M_1 = S^{n+k} - c_1^{-1}(\mathrm{int}\, 1/3 D^{k+1} \cup S^k)$ is a submanifold of S^{n+k} homeomorphic to $S^k \times D^n - S^k \times 0$. Let $e_1: M_1 \to S^k$ be the composite

$$c_1 | : M_1 \to D^{k+1} - \mathrm{Int}(1/3 D^{k+1}) \xrightarrow{\ \varrho\ } S^k$$

where ϱ is the radial retraction $\varrho(x) = x/|x|$. We note that if $p : S^{n-1} * S^k - (S^{n-1} \cup S^k) \to S^k$ sends the formal sum $(1-t)z \oplus ty$ $(0 < t < 1;\ z \in S^{n-1};\ y \in S^k)$ to $y \in S^k$, then e_1 is the restriction of the composite

$$S^{n+k} - (f^{-1}(S^{n-1}) \cup S^k) \xrightarrow{f} S^{n-1} * S^k - (S^{n-1} \cup S^k) \xrightarrow{p} S^k \quad \text{to } M_1.$$

Thus, there is a homeomorphism h such that the following diagram commutes

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\ e_1\ } & S^k \\
h \downarrow & & \downarrow 1 \\
S^{n-1} \times S^k \times (0,1] & \xrightarrow{\ p_2\ } & S^k
\end{array}
$$

where p_2 is projection on the second factor (i.e. e_1 is topologically a projection). The following lemma is obvious:

Lemma 5.1. *The end e_1 is onto, $1 - LC$, and eventually collared.*

We now define $\bar{c}_2 : S^{n+k} \to D^{k+1}$ by $\bar{c}_2(x) = |G|^{-1} \sum c_1(gx)$ where $|G|$ is the order of G and the summation runs over $g \in G$. It is easy to see that $\bar{c}_2(x) \in D^{k+1}$, $\bar{c}_2|S^k$ is the identity and $\bar{c}_2^{-1}(S^k) = S^k$, and that $\bar{c}_2$ is G-equivariant with respect to the trivial action of G on D^{k+1}.

Let $\bar{c}_3 : S^{n+k}/G \to D^{k+1}$ be the map induced by $\bar{c}_2$. Thus $\bar{c}_2 = \bar{c}_3 q$ where $q : S^{n+k} \to S^{n+k}/G$ is the quotient map. Clearly $\bar{c}_3|S^k$ is the identity and $\bar{c}_3^{-1}(S^k) = S^k$. By altering $\bar{c}_3$ via a homotopy fixing $\bar{c}_3^{-1}(D^{k+1} - 2/3 D^{k+1})$, we may replace $\bar{c}_3$ by a new map c_3 which is transverseregular to $1/3 S^k$. Let

$$M_3 = S^{n+k}/G - (c_3^{-1}(\operatorname{Int} 1/3 D^{k+1}) \cup S^k)$$

and let $e_3 : M_3 \to S^k$ be the composite

$$c_3| : M_3 \to D^{k+1} - \operatorname{Int}(1/3 D^{k+1}) \xrightarrow{\ \varrho\ } S^k.$$

Finally let $M_2 = q^{-1}(M_3)$ and $e_2 = e_3 q$.

Lemma 5.2. *The ends e_1 and e_2 are eventually locally equivalent. Hence e_2 is onto, $1 - LC$, and tame.*

Proof. The first sentence follows immediately from the observation that if U is a neighborhood of the end e_i $(i = 1, 2)$ and $V \subset S^k$ is a neighborhood of $x \in S^k$, then $U \cap e_i^{-1}(V) \cup V$ is a neighborhood of x in S^{n+k} and that such neighborhoods form a cofinal family of neighborhoods of x in S^{n+k}. The last part of the lemma is immediate from 4.1 and 5.1.

Lemma 5.3. *The end e_3 is onto, $0 - LC$, has constant fundamental group G, and is tame.*

Proof. This is immediate from 4.3.

In the sequel, our interest is not so much in the whole of the ends e_i $(i = 1, 2, 3)$ as in a part of them which we now describe.

6. The Ends of the Action near R^k

The proofs of the necessity part of Theorem A and of Theorem B actually use some ends obtained by restricting the ends of Sect. 5 to a certain copy of $R^k \subset S^k$. This section describes these ends, their main properties, and the construction based on them that will be needed later in this paper.

We regard D^k (respectively, S^k) as $\{x \in R^k | |x| \leq 1\}$ (respectively, $\{x \in R^{k+1} | |x| = 1\}$) and define $a : D^k \to S^k$ by $a(x) = ((1 - |x|^2)^{1/2}, x)$. Let D^k_+ be the image of a, $N_i = e_i^{-1}(\mathrm{Int}\,D^k_+)$, and

$$f_i = \varrho^{-1} a^{-1} e_i : N_i \to R^k \quad (i = 1, 2, 3)$$

where $\varrho : R^k \to D^k$ is a radial embedding onto $\mathrm{Int}\,D^k$. We shall call f_i the part of e_i over R^k $(i = 1, 2, 3)$.

It is clear that there exists a homeomorphism h such that the following diagram commutes

$$
\begin{array}{ccc}
N_1 & \xrightarrow{\ f_1\ } & R^k \\
\scriptstyle h \downarrow & & \downarrow \scriptstyle 1 \\
S^{n-1} \times R^k \times (0, 1] & \xrightarrow{\ p_2\ } & R^k
\end{array}
$$

where p_2 is projection on the second factor. Thus f_1 is onto, $1 - LC$, is eventually collared, and has a completion $\bar{f}_1 : \bar{N}_1 \to R^k$. Furthermore, h extends to a homeomorphism $\bar{h} : \bar{N}_1 \to S^{n-1} \times R^k \times [0, 1]$ such that $f_1 = p_2 \bar{h}$.

Since the LC properties, the possession of a constant fundamental group and homological tameness are obviously inherited by the part of an end over an open set, we see that f_i $(i = 2, 3)$ is onto, $0 - LC$, has constant fundamental group, and is homologically tame. Thus, f_i $(i = 2, 3)$ is tame by [17, Proposition 1.7]. In fact, the following stronger result holds:

Proposition 6.1. i) If $k \geq 2$, then the end $f_3 : N_3 \to R^k$ has a completion $\bar{f}_3 : \bar{N}_3 \to R^k$.

ii) If $k = 0, 1$ and $n + k \geq 6$, then the end $f_3 : N_3 \to R_k$ has a completion $\bar{f}_3 : \bar{N}_3 \to R^k$ if and only if an obstruction $O \in \tilde{K}_{-k}(G)$ vanishes.

iii) If $k = 0, 1$ and $n + k \geq 6$, then the end $N_3 \times S^1 \xrightarrow{\ p\ } N_3 \xrightarrow{\ f_3\ } R^k$ has a completion where p is projection on the first factor.

It is the end obstruction O of part ii) of 6.1 that is the obstruction of Theorem B in the case when $k \leq 1$.

Proof. Since f_3 is a tame end with constant fundamental group G, it follows from [18] that there is a sequence obstructions $O_j(f_3) \in \check{H}_j^{lf}(R^k; K_{-j}(G))$ whose vanishing is necessary and sufficient for the existence of a completion of f_3. Since $\check{H}_j^{lf}(R^k; K_{-j}(G))$ vanishes if $j \neq k$ and equals $K_{-k}(G)$ if $j = k$, there is only one possible non-zero obstruction among the $O_j(f_3)$; namely $O_k(f_3) \in K_{-k}(G)$. In particular ii) follows.

The rest of the proof now divides into two cases. If $k \geq 2$, then $K_{-k}(G) = 0$ by a result of Carter [4] and part i) follows immediately. If $k \leq 1$, $K_{-k}(G)$ is non-zero in general; hence, $O_k(f_3)$ may not vanish. However Proposition 1.8 of [18] shows that

the end obstruction of $N_3 \times S^1 \to N_3 \to R^k$ is essentially obtained from $O_k(f_3)$ by a multiplication by $\chi(S^1)$. Hence, this end obstruction vanishes and part iii) of the proposition follows.

Corollary 6.2. *If the end* f_3 *has a completion* $\bar{f}_3 : \bar{N}_3 \to R^k$, *then there is a principal G-bundle* $\bar{q} : \bar{N}_2 \to \bar{N}_3$ *such that* $\bar{f}_2 = \bar{f}_3 \bar{q}$ *is a completion of* f_2.

Proof. This is obvious.

Let $\bar{f} : \bar{N} \to R^k$ be a completion of the end $f : N \to R^k$, suppose that $\partial \bar{N}$ has two components ∂N and $\partial_0 \bar{N}$, and fix a collar $c : \partial_0 \bar{N} \times I \to \bar{N}$ such that $c(x, 0) = x$. A *tapering* is a function $t : R^k \to (0, 1]$ such that $\lim\limits_{x \to \infty} t(x) = 0$. A *tapered embedding* $T : \partial_0 \bar{N} \times (0, 1] \to N$ is one of the form $T(x, s) = c(x, st\bar{f}(x))$ where t is a tapering. In this case T is said to be *associated with* t. The shaded region in Fig. 1 below shows the image of a tapered embedding in $\partial_0 \bar{N} \times (0, 1]$. If t_1 and t_2 are taperings with
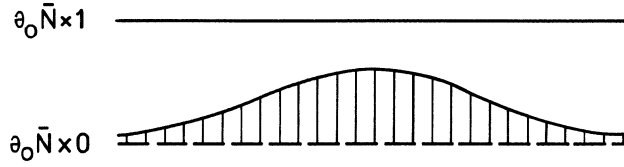

Fig. 1

$t_2(x) < t_1(x)$ for all $x \in R^k$ and $T_1, T_2 : \partial_0 \bar{N} \times (0, 1] \to N$ are the associated tapered embeddings, then

$$T_2(\partial_0 \bar{N} \times (0, 1]) \subset \mathrm{Int}\, T_1(\partial_0 \bar{N} \times (0, 1])$$

and the closure W of the region between these two images has the *canonical product structure* $h : \partial_0 \bar{N} \times [0, 1] \to W$ given by $h(x, s) = c(x, (1 - s)t_2 \bar{f}(x) + st_1 \bar{f}(x))$.

In order to describe the construction underlying the proof of Theorem B we assume that either $k \geq 2$ or that $k \leq 1$ and $O \in \tilde{K}'_{-k}(G)$ vanishes. In either case $f_3 : N_3 \to R^k$ has a completion. We now fix completions $\bar{f}_i : \bar{N}_i \to R^k$ of f_i ($i = 1, 2, 3$) such that $\bar{f}_2 = \bar{f}_3 \bar{q}$ as in 6.2. We also fix collars $c_i : \partial_0 \bar{N}_i \times I \to \bar{N}_i$ with $c_i(x, 0) = x$ ($i = 1, 2, 3$) such that $\bar{h} c_1 = (\bar{h} | \partial_0 \bar{N}_1) \times 1$, where $\bar{h} : \bar{N}_1 \to S^{n-1} \times R^k \times [0, 1]$ is the homeomorphism above, and such that $c_3(\bar{q} \times 1) = \bar{q} c_2$. Then corresponding to any tapered embedding $T_3 : \partial_0 \bar{N}_3 \times (0, 1] \to N_3$ there is a unique tapered embedding $T_2 : \partial_0 \bar{N}_2 \times (0, 1] \to N_2$ satisfying $T_3(\bar{q} \times 1) = \bar{q} T_2$ which is said to be *associated with* T_3.

Lemma 6.3. *Let* $T_1 : \partial_0 \bar{N}_1 \times (0, 1] \to N_1 \subset S^{n+k}$ *be a tapered embedding. Then*

i) *There exists a tapered embedding* $T_3 : \partial_0 \bar{N}_3 \times (0, 1] \to N_3$ *whose associated tapered embedding* $T_2 : \partial_0 \bar{N}_2 \times (0, 1] \to N_2 \subset S^{n+k}$ *has*

$$T_2(\partial_0 \bar{N}_2 \times (0, 1]) \subset \mathrm{Int}(T_1(\partial_0 \bar{N}_1 \times (0, 1]))$$

ii) *For any tapered embeddings* T_2, T_3 *as in* i), *there is a tapered embedding* $T_1' : \partial_0 \bar{N}_1 \times (0, 1] \to N_1 \subset S^{n+k}$ *such that*

$$T_1'(\partial_0 \bar{N}_1 \times (0, 1]) \subset \mathrm{Int}\, T_2(\partial_0 \bar{N}_2 \times (0, 1]).$$

Proof. We observe that if T_3 is a tapered embedding with associated embedding T_2, then $T_2(\partial_0 \bar{N}_2 \times (0, 1]) \cup \mathrm{Int} D^k_+$ is a neighborhood of D^k_+ in S^{n+k} and that such neighborhoods are cofinal in the collection of all neighborhoods of D^k_+. A similar observation holds for $T_1(\partial_0 \bar{N}_1 \cup (0, 1]) \cup \mathrm{Int} D^k_+$. The lemma is then an immediate consequence of these observations.

Suppose now that we have tapered embeddings

$$T_{i,j} : \partial_0 \bar{N}_i \times (0, 1] \to N_i \subset S^{n+k} \quad (i, j = 1, 2, 3)$$

such that

    a) The associated taperings $t_{i,j}$ satisfy $t_{i,3}(x) < t_{i,2}(x) < t_{i,1}(x)$ $(x \in R^k)$ $(i = 1, 2, 3)$;

    b) $T_{2,j}$ is associated with $T_{3,j}$ $(j = 1, 2, 3)$;

    c) $T_{2,j}(\partial_0 \bar{N}_2 \times (0, 1]) \subset \mathrm{Int} T_{1,j}(\partial_0 \bar{N}_1 \times (0, 1])$ $(j = 1, 2, 3)$; and

    d) $T_{1,j}(\partial_0 \bar{N}_1 \times (0, 1]) \subset \mathrm{Int} T_{2,j-1}(\partial_0 \bar{N}_2 \times (0, 1])$.

We set

$$W = \mathrm{Cl}(\mathrm{Im}\, T_{1,2} - \mathrm{Im}\, T_{2,2}),$$

$$\partial_i W = T_{i,2}(\partial_0 \bar{N}_i \times 1) \quad (i = 1, 2),$$

$$V_1 = \mathrm{Cl}(\mathrm{Im}\, T_{1,2} - \mathrm{Im}\, T_{1,3}),$$

$$V_2 = \mathrm{Cl}(\mathrm{Im}\, T_{2,1} - \mathrm{Im}\, T_{2,2}),$$

$$U_1 = \mathrm{Cl}(\mathrm{Im}\, T_{1,2} - \mathrm{Im}\, T_{2,3}),$$

and

$$U_2 = \mathrm{Cl}(\mathrm{Im}\, T_{1,1} - \mathrm{Im}\, T_{2,2}),$$

where $\mathrm{Im}\, T_{i,j}$ is the image of $T_{i,j}$. In Fig. 2 below we draw these images inside N_1 and over R^k. In this figure the image of $T_{i,j}$ lies below the curve labelled $T_{i,j}$. The region W is shaded.
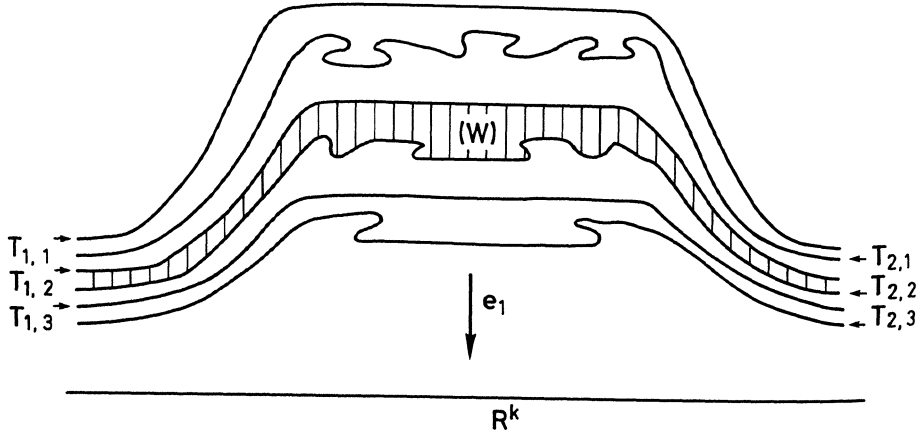

Fig. 2

In Fig. 3 we show the same regions as subsets of S^{n+k}. Here R^k is included in S^{n+k} as the northern hemisphere of $S^k \subset S^{n+k}$ so we have inverted the above picture. In particular, the image of $T_{i,j}$ lies above the curve labelled $T_{i,j}$. Notice that all the tapered regions "converge" to $\partial D^k_+ = S^{k-1} \subset S^k \subset S^{n+k}$.
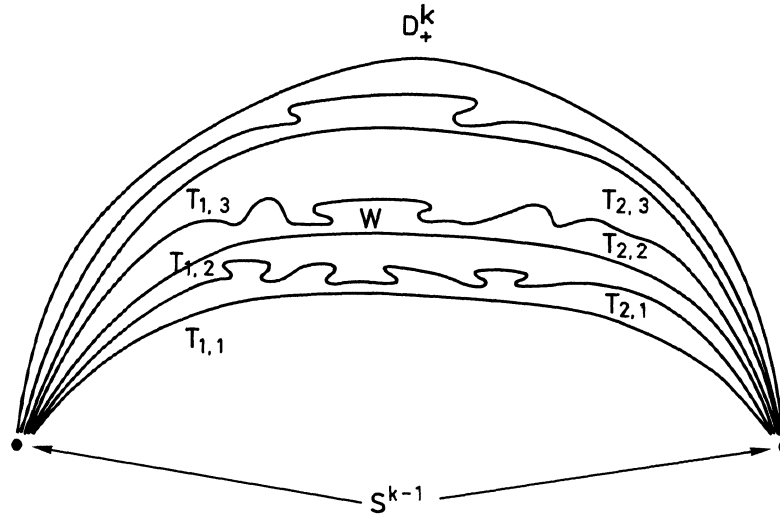
$$D^k_+$$

Fig. 3

We now define a deformation retraction R_i of W onto $\partial_i W$ to be the composite

$$W \times I \xrightarrow{\text{inc} \times 1} V_i \times I \xrightarrow{d_i} V_i \subset U_i \xrightarrow{r_i} W$$

where d_i pulls V_i into $\partial_i W$ relative to $\partial_i W$ via the canonical product structure on V_i and r_i pulls U_i into W relative to W via the canonical product structure on $\text{Cl}(\text{Im}\,T_{2,2} - \text{Im}\,T_{2,3})$ in the case of r_1 or on $\text{Cl}(\text{Im}\,T_{1,1} - \text{Im}\,T_{1,2})$ in the case of r_2. We have therefore established the following lemma:

Lemma 6.4. *There are deformation retractions* $R_i : W \times I \to W$ $(i = 1, 2)$ *of the region* W *onto* $\partial_i W$ $(i = 1, 2)$. *Thus* W *is an h-cobordism.*

7. The Proof of Theorem B

This section contains the proof of Theorem B. The basic idea of the proof is to make an appropriate choice for the embeddings $T_{i,j}$ $(i, j = 1, 2, 3)$ described at the end of Sect. 6. The key step in the proof is the following proposition whose proof is temporarily deferred:

Proposition 7.1. *If the end* $f_3 : N_3 \to R^k$ *has a completion, it is possible to choose the embeddings* $T_{i,j}$ $(i, j = 1, 2, 3)$ *of Sect. 6 such that there is a homeomorphism* h *of* $(S^{n-1} \times I) * S^{k-1}$ *onto the subspace* $W \cup S^{k-1}$ *of* S^{n+k} *such that* $h|S^{k-1} = 1$.

Assuming 7.1 the proof of Theorem B goes as follows: Let $O \in \tilde{K}_{-k}(G)$ $(k = 0, 1)$ be the end invariant of $f_3 : N_3 \to R^k$ as in 6.1. Then if, $k \leq 1$ and $O = 0$, or, if $k \geq 2$, then $f_3 : N_3 \to R^k$ has a completion and we choose embeddings $T_{i,j}$ $(i, j = 1, 2, 3)$ as in 7.1. Let $M = T_{2,2}(\partial_0 \bar{N}_2 \times 1) \cup S^{k-1}$. Since $T_{2,2}$ covers $T_{3,2}$, M is G-invariant. But also 7.1 provides a homeomorphism $h : (S^{n-1} \times I) * S^{k-1} \to W \cup S^{k-1}$ such that $h((S^{n-1} \times 1) * S^{k-1}) = \partial_2 W \cup S^{k-1} = M$. Thus M is an $(n+k-1)$ sphere. Furthermore, we can ambient isotope M onto $\partial_1 W \cup S^{k-1}$ fixing S^k by pushing M in S^{n+k} across the product structure on W coming from h. Since

$$\partial_1 W \cup S^{k-1} = T_{1,2}(S^{n-1} \times R^k) \cup S^{k-1}$$

can easily be ambient isotoped onto S^{n+k-1} fixing S^k, the proof of Theorem B is completed.

We turn now to the proof of 7.1. The first step is to show that considered as a subspace of N_1, W has a "small" product structure. The main ingredient in the proof of this fact is an extension of the thin h-Cobordism Theorem [17, Theorem 2.7]. We refer the reader to [17] for the definitions of the terms used in this extension:

Proposition 7.2. *Let* $\varepsilon > 0$ *and* $m \geq 6$ *be given. Then there is a* $\delta > 0$ *such that if the proper map* $e : (W^m, \partial_1 W^m) \to R^k$ *is* $(\delta, 1)$-*connected and is a* (δ, h) *cobordism then there is an* ε-*product structure* $h : (\partial_1 W \times I, \partial_1 W \times 0) \to (W, \partial_1 W)$ *with* $h|\partial_1 W \times 0$ *the identity.*

Proof. This follows from the arguments given in [17] to prove the thin h-Cobordism Theorem and from refinements of them due to Chapman [5].

Now let $\varepsilon > 0$ be given and let $\delta > 0$ be the constant given by 7.2 corresponding to ε and $n+k$. (Recall $n+k \geq 6$!)

Lemma 7.3. *It is possible to choose the embeddings* $T_{i,j}$ *($i,j = 1,2,3$) of Sect. 6 such that if* $(W, \partial_1 W)$ *is the pair described in Sect. 6 and*

$$e = f_1 | W : (W, \partial_1 W) \to R^k$$

then e *is* $(\delta, 1)$-*connected and is a* (δ, h)-*cobordism.*

Corollary 7.4. *Let* W *be considered as a subspace of* N_1. *Then there is a homeomorphism*

$$h_1 : (S^{n-1} \times R^k \times I, S^{n-1} \times R^k \times 0) \to (W, \partial_1 W)$$

such that $h_1 | S^{n-1} \times R^k \times 0 = T_{1,2}$ *and for every* $(x, y) \in S^{n-1} \times R^k$,

$$\mathrm{diam}\{eh_1(x, y, t)|0 \leq t \leq 1\} \leq \varepsilon.$$

Proof. The corollary follows immediately from 7.2 and 7.3.

The proof of 7.3 requires two sublemmas:

Sublemma 7.5. *Let* $\eta > 0$ *be given. Then there exists a tapered embedding* $T : \partial_0 \bar{N}_1 \times (0, 1] \to N_1$ *such that for any* $x \in \mathrm{Im}\, T$, $|f_1(x) - f_2(x)| < \eta$ *where* f_1 *and* f_2 *are the ends of Sect. 6 and* $|\ |$ *denotes the metric in* R^k.

Proof. Since the ends f_1 and f_2 are eventually locally equivalent by 5.2 and the fact that $f_i = e_i|N_i$ ($i = 1, 2$), for each $z \in R^k$ we can find a neighborhood U_2 of the end e_1 and a neighborhood V_z of z in R^k such that for all $x \in U_z \cap f_1^{-1}(V_z) \subset N_1$, $|f_1(x) - f_2(x)| < \eta$. A standard argument shows that there is a countable set $\{z_l | l = 1, 2, \ldots\}$ such that only finitely many z_l lie in any closed disk and the corresponding V_{z_l}'s cover that disk. It is now straightforward to find a tapering function $t : R^k \to (0, \infty)$ with associated tapered embedding $T : \partial_0 \bar{N}_1 \times (0, 1] \to N_1$ such that for any integer $m > 0$, $T(f_1^{-1}(mD^k) \times [0, \infty)) \subset \cap U_{z_l}$ where the intersection runs over $\{z_l | z_l \in mD^k\}$. Clearly T is the desired tapered embedding.

Sublemma 7.6. *Let* $\eta > 0$ *be given. Then there exists a tapered embedding* $T : \partial_0 \bar{N}_3 \times (0, 1] \to N_3$ *such that for any* $x \in \partial_0 \bar{N}_3$, $\mathrm{diam}\{f_2 T(x, t)|0 < t \leq 1\} < \eta$.

Proof. This follows directly from the fact that f_3 has a completion $\bar{f}_3 : \bar{N}_3 \to R^k$ by a standard argument.

Proof of 7.3. We use 7.5 to choose the tapered $T_{1,1}$ such that for all x in $\operatorname{Im} T_{1,1}$, $|f_1(x) - f_2(x)| < \delta/6$. We then apply 6.3 and 7.6 to find a tapered embedding $T_{3,1} : \partial_0 \bar{N}_3 \times (0,1] \to N_3$ such that for all $x \in \partial_0 \bar{N}_3$, $\operatorname{diam}\{f_3 T_{3,1}(x,t)|0 < t \leq 1\}$ $< \delta/6$ and such that $\operatorname{Im}(T_{2,1}) \subset \operatorname{Int}(\operatorname{Im} T_{1,1})$ where $T_{2,1} : \partial_0 \bar{N}_2 \times (0,1] \to N_2$ is the tapered embedding associated with $T_{3,1}$. Notice that for all $x \in \partial_0 \bar{N}_2$, $\operatorname{diam}\{f_2 T_{2,1}(x,t)|0 < t \leq 1\} < \delta/6$ since $T_{2,1}$ is associated with $T_{3,1}$. Finally we apply 6.3 repeatedly to find the tapered embeddings $T_{i,j}$ ($i = 1,2,3$; $j = 2,3$) satisfying the conditions a) to d) given in Sect. 6.
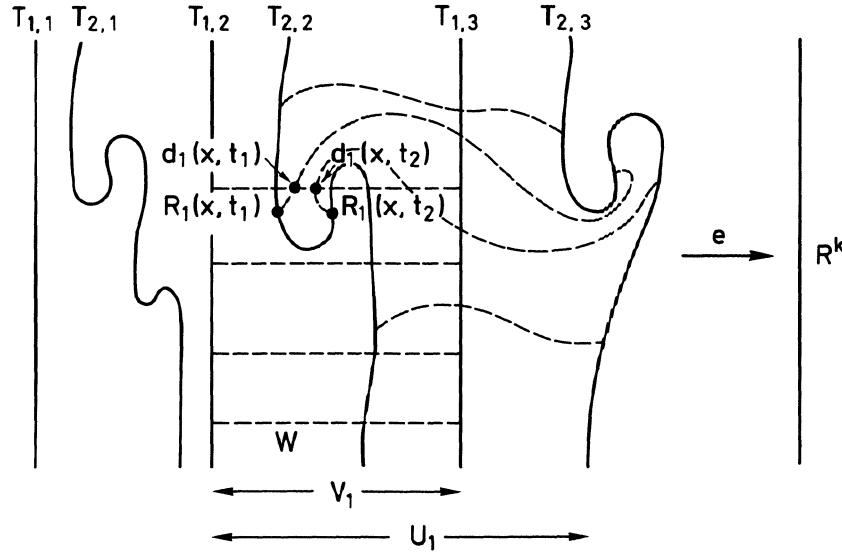

Fig. 4

We claim that for these choices of $T_{i,j}$ ($i,j = 1,2,3$), $e : (W, \partial_1 W) \to R^k$ has the properties required by 7.3. To see that e is a (δ, h)-cobordism, we show that the deformation retractions $R_i : W \times I \to \partial_i W$ ($i = 1,2$) of Sect. 6 have diameter $< \delta$. (The reader may find Fig. 4 above helps to understand the following estimates.) To estimate the diameter of R_1, let $x \in W$ and $0 \leq t_1, t_2 \leq 1$. Then

$$|eR_1(x,t_1) - eR_1(x,t_2)| \leq |eR_1(x,t_1) - ed_1(x,t_1)|$$
$$+ |ed_1(x,t_1) - ed_1(x,t_2)|$$
$$+ |ed_1(x,t_2) - eR_1(x,t_2)|.$$

Now $|ed_1(x,t_1) - ed_1(x,t_2)| = 0$ since $d_1(x,t_1)$ and $d_1(x,t_2)$ lie on the ray $T_{1,2}(x \times (0,1])$ and $e = f_1|$ is constant on this ray. Furthermore, if $l = 1,2$

$$|eR_1(x,t_l) - ed_1(x,t_l)| \leq |eR_1(x,t_l) - f_2 R_1(x,t_l)|$$
$$+ |f_2 R_1(x,t_l) - f_2 d_1(x,t_l)|$$
$$+ |f_2 d_2(x,t_l) - ed_1(x,t_l)|$$

where the first and last terms are $< \delta/6$ since $e = f_1|$ by the choice of $T_{1,1}$. On the other hand $R_1(x,t_l)$ and $d_1(x,t_l)$ lie on a ray $T_{2,1}(y \times (0,1])$ for some $y \in \partial_0 \bar{N}_2$.

Hence the middle term $<\delta/6$ by the choice of $T_{2,1}$. Thus for $l=1,2$, $|eR_1(x,t_l)-ed_1(x,t_l)|<\delta/2$. That R_1 has diameter $<\delta$ now follows easily.

A similar proof shows that R_2 has diameter $<\delta$.

Figure 4 is closely related to Fig. 2. Here the vertical straight lines are the images $T_{1,j}(\partial_0\bar{N}_1)$ $(j=1,2,3)$; while the "vertical" curves are the images $T_{2,j}(\partial_0\bar{N}_2)$ $(j=1,2,3)$. The horizontal broken lines (respectively, "horizontal" broken curves) are the canonical product structure on the region between $T_{1,2}$ and $T_{1,3}$ (respectively, between $T_{2,2}$ and $T_{2,3}$). We recall that the segments in the canonical product structures lie on rays of the form $T_{i,1}(z\times(0,1])$ for i either 1 or 2. The deformation R_1 pulls a point of W along a broken straight line until it moves outside W. At this point it is pushed back inside W by projecting along an appropriate broken curve.

We now show that $e:(W,\partial_1 W)\to R^k$ is $(\delta,1)$-connected. To do this, let (R,S) be a relative 2-complex and suppose given a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{s} & W \\
\cap & & \downarrow{e} \\
R & \xrightarrow{r} & R^k.
\end{array}
$$

We wish to construct $g:R\to W$ with $g|S=r$ and such that $|eg(x)-r(x)|<\delta$ for all $x\in R$. To do this we observe that the above diagram can be enlarged to the following diagram

$$
\begin{array}{ccccc}
S & \xrightarrow{s} & W & \subset & V_1 \xrightarrow{h} S^{n-1}\times R^k\times I \\
\cap & & \downarrow{e\,=\,f_1} & & \downarrow{f_1} \qquad\qquad \downarrow{p_2} \\
R & \xrightarrow{r} & R^k & = & R^k \xrightarrow{\quad 1 \quad} R^k
\end{array}
$$

where h is a homeomorphism and p_2 is projection on the second factor (cf. Sect. 6). Thus, we can regard hs as a triple of functions s_1',s_2',s_3' and r as an extension of s_2' to R. But since $n\geq4$, s_1' extends to $r_1':R\to S^{n-1}$; while s_3' obviously extends to $r_3':R\to I$. Then $g':R\to V_1$ given by $g'(x)=h^{-1}(r_1'(x),r(x),r_3'(x))$ satisfies $g'|S=s$ and $f_1g'=r$.

Let $\varrho:V_1\to W$ be the composite $V_1\subset U_1\xrightarrow{r_1} W$, where r_1 is as described at the end of Sect. 6, and set $g=\varrho g'$. Clearly $g|S=s$ and since the argument estimating the diameter of R_1 given above shows that $|f_1r_1(z)-f_1(z)|<\delta/2$ for all $z\in V_1$, $|eg(x)-r(x)|<\delta$ for all x. Thus g is the desired map.

This completes the proof of 7.3.

Proof of Proposition 7.1. Choose the embeddings $T_{i,j}$ $(i,j=1,2,3)$ such that 7.3 and 7.4 apply and consider the diagram

$$(S^{n-1}\times I)*S^{k-1}\xleftarrow{q}(S^{n-1}\times I)\times D^k\xleftarrow{1\times\varrho}S^{n-1}\times I\times R^k\xrightarrow{h_1}W\xrightarrow{l}S^{n+k}$$

where $\varrho:R^k\to\mathrm{Int}\,D^k$ is a radial homeomorphism, q is the quotient map described in Sect. 3, h_1 is the homeomorphism of 7.4, and l is the inclusion. We now define $h:(S^{n-1}\times I)*S^{k-1}\to W\cup S^{k-1}$, where the latter space is considered as a subspace of S^{n+k}, by setting $h=lh_1(1\times\varrho)^{-1}q^{-1}$ on $(S^{n-1}\times I)*S^{k-1}-S^{k-1}$ and $h=1$ on S^{k-1}.

Since the product structure h_1 has diameter $< \varepsilon$ (i.e. is "bounded"), h is continuous. Since h is obviously bijective, it follows that h is a homeomorphism. This completes the proof of 7.1.

8. The Proof that ii) Implies i) in Theorem A

This section contains the proof of the result that if $n \geq 5$ then ii) implies i) in Theorem A. It is based on Theorem B and the following observations:

Lemma 8.1. *Let $n \geq 5$ and suppose there exists a semifree topological action of G on S^{n+1} with $\text{Fix}(G) = S^1$ and $S^{n+1} - S^1/G$ homotopy equivalent to $X(\kappa)$. Then $hS^h(X(\kappa) \times T^2) \neq \emptyset$.*

We remark that since $X(\kappa)$ is finitely dominated, $X(\kappa) \times S^1$ has the homotopy type of a finite CW complex; hence, $X(\kappa) \times T^2$ has a well defined simple homotopy type. It is really the homotopy structure set of this space which is non-empty in 8.1.

Lemma 8.2. *Let $n \geq 5$. If $hS^h(X(\kappa) \times T^2) \neq \emptyset$, then there exists a homotopy structure $f : M^{n+1} \to X(\kappa) \times T^2$ whose normal invariant lies in*

$$\text{Im}\{\pi^* : [X(\kappa); G/\text{TOP}] \to [X(\kappa) \times T^2; G/\text{TOP}]\}$$

where $\pi : X(\kappa) \times T^2 \to X(\kappa)$ is projection on the first factor.

Assuming 8.1 and 8.2 the proof that i) implies ii) in Theorem A if $n \geq 5$ goes as follows: If $k = -1$ [i.e. $\text{Fix}(G) = \emptyset$], the result is a standard fact in surgery theory; while if $k = 0$, the result follows from the arguments of [7] based on [16].

Suppose now that $k \geq 1$. It follows from Theorem B that there is a semifree action of G on S^{n+1} with $\text{Fix}(G) = S^1$. Since $n \geq 5$, by 8.1 and 8.2 there exists a homotopy structure $f : M^{n+1} \to X(\kappa) \times T^2$ whose normal invariant is in

$$\text{Im}\{\pi^* : [X(\kappa); G/\text{TOP}] \to [X(\kappa) \times T^2; G/\text{TOP}]\}.$$

But then $f \times 1 : M^{n+1} \times T^{k-1} \to X(\kappa) \times T^{k+1}$ is a homotopy structure y whose normal invariant $\eta(y)$ lies in

$$\text{Im}\{\pi^* : [X(\kappa); G/\text{TOP}] \to [X(\kappa) \times T^{k+1}; G/\text{TOP}]\}.$$

The results of Sect. 2, however, show that there is a commutative diagram

$$
\begin{array}{ccc}
[X(\kappa); G/\text{TOP}] & \xrightarrow{\quad\theta\quad} & L_{n-1}^{-k}(G) \\
\downarrow{\scriptstyle\pi^*} & & \downarrow{\scriptstyle\chi} \\
hS^h(X(\kappa) \times T^{k+1}) \xrightarrow{\eta} [X(\kappa) \times T^{k+1}; G/\text{TOP}] & \xrightarrow{\theta} & L_{n+k}^h(G \times Z^{k+1})
\end{array}
$$

where χ, the composite $(\times S^1)\chi_1, \ldots, \chi_k$, is monomorphic. Let $\eta(y) = \pi^*(x)$. A diagram chase now shows that $\theta(x) = 0$ in $L_{n-1}^{-k}(G)$. This completes the proof that ii) implies i) in Theorem A when $n \geq 5$ assuming 8.1 and 8.2.

Proof of 8.1. Consider the end $f_3 : N_3 \to R^1$ of Sect. 6. Since $n + 1 \geq 6$, by 6.1 the end $N_3 \times S^1 \to N_3 \xrightarrow{f_3} R^1$ has a completion $F : W \to R^1$. Thus $N_3 \times S^1 \subset W$ and $W - (N_3 \times S^1) = V$ is a component of ∂W. If we embed V in $\text{Int} W$, and thus in

$N_3 \times S^1$, by using a collar it is not hard to see that the composite

$$V \to N_3 \times S^1 \xrightarrow{\text{inc} \times 1} (S^{n+1} - S^1/G) \times S^1 \xrightarrow{h \times 1} X(\kappa) \times S^1 ,$$

where h is a homotopy equivalence, is a homotopy equivalence. We shall call this composite g.

We observe now that since V is the "ideal" boundary of the completion $F: W \to R^1$, $F|V: V \to R^1$ is a proper map. Thus V is a manifold with two ends in the "classical" Siebenmann [21] sense. Since these ends are obviously tame and $\dim V = n+1 \geq 6$ the manifold $V \times S^1$ has a Siebenmann completion N with two boundary components. Let M be one such boundary component and embed M in $V \times S^1$ by using a collar as above. The composite

$$M \to V \times S^1 \xrightarrow{g \times 1} (X(\kappa) \times S^1) \times S^1$$

is the desired homotopy structure on $X(\kappa) \times T^2$.

Proof of 8.2. Let $h: V \to X(\kappa) \times T^2$ be a homotopy structure with arbitrary normal invariant and consider the map of infinite cyclic covers $\hat{h}: \hat{V} \to X(\kappa) \times S^1 \times R^1$. Observe that $\hat{V}$ has two tame ends. Since $n+1 \geq 6$, we may again apply the trick of crossing with S^1 and embedding a boundary component W in $\hat{V} \times S^1$ via a collar to obtain the composite homotopy equivalence

$$W \to \hat{V} \times S^1 \xrightarrow{\hat{h} \times 1} X(\kappa) \times S^1 \times R^1 \times S^1 \to X(\kappa) \times T^2$$

which we denote by g. It follows from the construction that (W, g) is a homotopy structure $X(\kappa) \times T^2$ whose normal invariant lies in

$$\text{Im}\{\pi_1^*: [X(\kappa) \times S^1; G/\text{TOP}] \to [X(\kappa) \times T^2; G/\text{TOP}]\}$$

where $\pi_1: X(\kappa) \times S^1 \times S^1 \to X(\kappa) \times S^1$ is projection on the first two factors.

If we now apply the above argument to W and the infinite cyclic cover corresponding to the first S^1 factor, we obtain the desired homotopy structure (M, f) on $X(\kappa) \times T^2$ with normal invariant in $\text{Im}\,\pi^*$. This completes the proof of 8.2.

9. Remarks on Classification

Although given a semifree action of a group G on a sphere S^{n+k} with fixed point set S^k we produce a manifold structure on $S^{n+k} - S^k/G \times T^k$ and thus a new semifree action of G on S^{n+k} with fixed point set S^k, we have made no attempt of comparing the two actions. This would be necessary if one was to make any attempt to classify actions. In this section we have a few remarks on these matters.

Definition 9.1. Two semifree actions of G on S^{n+k} with fixed point set are concordant if there is a semifree action of G on $S^{n+k} \times I$ with fixed point set $S^k \times I$ restricting to the given actions on $S^{n+k} \times 0$ and $S^{n+k} \times 1$. If the action is level preserving, i.e. preserves the $t \in I$ coordinate, we say the actions are isotopic. The actions are homeomorphic if there is a G-invariant homeomorphism of S^{n+k} sending one action to the other.

If we have a semifree action on S^{n+k} with fixed point set S^k and we choose some cellular subset of S^k, C then $(S^{n+k}/C, S^k/C)$ will be homeomorphic to (S^{n+k}, S^k) and there will clearly be an induced action. It is an interesting question how this new action relates to the original action. It is fairly easy to see that this action is isotopic to the original action since $(S^{n+k} \times I/C \times 1, S^k \times I/C \times 1)$ is homeomorphic to $(S^{n+k} \times I, S^k \times I)$ in a level preserving way. Altogether isotopy classes of actions are somewhat easier to handle than homeomorphism classes as one may e.g. strengthen Theorem B as follows:

Theorem B'. *If G acts semifreely on S^{n+k} with fixed point set S^k, k' then the action is isotopic to a suspended action.*

Proof. By Theorem B there is an invariant S^{n+k-1} meeting S^k in S^{k-1}. On the two halves of S^{n+k} that S^{n+k-1} splits S^{n+k} we may perform an Alexander isotopy on each of the homeomorphisms given by the action of specific group elements.

The behaviour of topological semifree group actions near the fixed point set seems a mystery, the problem being that a free action on $S^n \times R^k$, bounded in the R^k-factor certainly may be completed to a semifree action on S^{n+k}, but in how many ways is unclear and seems to involve intricate point set topology. However we think the following is a reasonable conjecture:

Consider $X(\kappa)$ as a polarized complex, and consider $X(\kappa) \times T^*$ (in its preferred simple homotopy type). Then there is a $1 - 1$ correspondence between isotopy classes of semifree G-actions on S^{n+k} with k-invariant κ and the topological structure set $S[X(\kappa) \times T^*]/\sim$ where $\sim$ means two structures are equivalent if they become equal after a finite cover in the T^*-factor.

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