# SPINES OF TOPOLOGICAL MANIFOLDS

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In this paper we prove that a closed 2-connected topological manifold has a PL-spine, i. e. there is a locally tamely embedded complex such that a regular neighborhood of this complex is the manifold with a disc deleted (dimension is assumed to be at least 6). This "spine method" together with the relative edition of regular neighborhoods of complexes in topological manifolds [5] makes it easy to use general position arguments in topological manifolds. This will be used in a forthcoming paper to extend various embedding theorems to the topological category.

The methods we use are PL-approximation theorems due to Cernavskii, Connally, Miller, Rushing ... as quoted in [5, theorem 2] and blocktransersality for PL complexes and PL submanifolds as was first considered by C. Morlet [4] and later extended by D. Stone [6].

Definition 1. A spine of a topological manifold $M$ with $\partial M \neq \emptyset$ is a locally tamely embedded complex $K \subset M$ so that $K$ is a strong deformation retract of $M$ and $K \subset M$ is a simple homotopy equivalence. In case $\partial M = \emptyset$ by the spine of $M$ we mean a spine of $M$ with a disc deleted.

Theorem 2. *Let* $(M, \partial_- M, \partial_+ M)$ *be a triad of topological manifolds* $\dim(M) = m$, *and assume* $m \geq 6$ *and*

$$\pi_j(M, \partial_+ M) = 0 \qquad \text{for} \qquad j < m - r, \qquad r \leq m - 3.$$

*Further assume there is a* PL-*complex* $P$ *locally tamely embedded in the interior of* $\partial_+ M$, $\dim(P) = p$ *and* $m - p \geq 4$. *Then there is a complex* $K$ *of dimension* $\max(p+1, r, 2)$, *locally tamely embedded in* $M$ *such that*

$$K \cap \partial_+ M = P$$
$$K \cap \partial_- M = K'$$

$K'$ *a subcomplex of* $K$, $K'$ *has a regular neighborhood of the form* $K' \times I$ *in* $K$ *and*

$$\partial_- M \cup K \subset M$$

*is a strong deformation retract and a simple homotopy equivalence.*

Theorem 2 has an immediate corollary:

Corollary 3. *Let* $M$ *be a closed topological manifold,* $\dim(M) \geq 6$ *and* $\pi_j(M) = 0$ *for* $j \leq r$, $r > 2$. *Then* $M$ *has a spine of dimension* $m - r$.

*Proof.* Let $\overline{M} = M -$ (interior of a disc). Put $\partial_- M = \emptyset, \partial_+ M = S^{m-1}, P = \emptyset$, and apply Theorem 2. □

*Proof of Theorem 2.* First let us consider the case where $P = \emptyset$. Put $k = \max(r, 2)$. According to Kirby and Siebenmann [3] $M$ has a handlebody decomposition relative to $\partial_- M$ with no handles of dimension greater than $k$: Kirby and Siebenmann prove that $(M, \partial_- M)$ has a handle body decomposition, and one can then cancel handles to get a minimal handle decomposition. Because of problems with torsion one needs at least 1- and 2-handles.

We filter $M$ by the handle filtration

$$\partial_- M \times I = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

where $M_{i+1}$ is obtained from $M_i$ by adjoining a single handle, no handles of dimension greater than $m - 3$. The proof will be by downwards induction on the statement:

There is a locally tamely embedded complex

$$K_i \subset \overline{M - M_i} \qquad \dim(K_i) \leq k$$

such that

$$K_i' = K_i \cap \partial_+ M_i$$

is contained in the interior of $\partial_+ M_i$, $K_i'$ has a neighborhood in $K_i$ of the form $K_i' \times [0, 1]$ and $M_i \cup K_i$ is a simple strong deformation retract of $M_i$.

It is easy to start the induction, we let $K_{s-1}$ be the core of the last handle. Then clearly $M_{s-1} \cup K_{s-1}$ is a simple strong deformation of $M = M_s$, so assume the statement for $i + 1$. Now

$$M_{i+1} = M_i \cup_{S^{j-1} \times D^{m-j}} D^j \times D^{m-j}$$

for some $j \leq k$. Let

$$\overline{E} = D^j \times D^{m-j} \cap \partial M_{i+1} = D^j \times S^{m-j-1}$$

take an outside collar $\partial \overline{E} \times [0, 2]$ of $\partial \overline{E}$ in $\partial_+ M_{i+1}$ and let

$$E_1 = \overline{E} \cup \partial \overline{E} \times [0, 1], \qquad E_2 = \overline{E} \cup \overline{E} \times [0, 2].$$

$\overline{E}$ has a PL structure being a codimension 0 submanifold of the boundary of $D^j \times D^{m-j}$ and we can extend this PL structure to $E_1$ and $E_2$ using the collar. $K_{i+1}'$ is of codimension more than 3 in $\partial_+ M_{i+1}$, so by [1], see e. g. [5, Theorem 2], since $\dim \partial_+ M_{i+1} \geq 5$, there is an ambient $\varepsilon$-isotopy of $E_2$ fixing $\partial E_2$ that moves $K_{i+1}'$ to be PL embedded in $E_2$, except in a neighborhood of $\partial E_2$ which can be assumed small. So we may assume, since this can be taken to be the restriction of an ambient isotopy of $M$, that $K_{i+1}' \cap E_1 \subset E_1$ is PL. Using [4] we can isotop $K_{i+1}'$ further by a small ambient isotopy so that $K_{i+1}'$ intersects $\partial \overline{E} = S^{j-1} \times S^{m-j-1}$ blocktransversally. Assume this done and denote

$$Z = K_{i+1}' \cap \partial \overline{E}$$

Since the normal blockbundle of $\partial\overline{E}$ in $E_1$ is a trivial one dimensional bundle we obtain that $\partial\overline{E}$ has a neighborhood in $E$ of the form $\partial \to E \times (-1,1)$ and

$$\partial\overline{E} \times (-1,1) \cap K'_{i+1} = Z \times (-1,1)$$

since it is the restriction of the trivial blockbundle to $Z$, by blocktransversality. $Z$ is a PL subcomplex of $S^{j-1} \times S^{m-j-1}$, which is the boundary of $S^{j-1} \times D^{m-j}$ of dimension $m-1-r$, so of codimension at least 3. By [2, Theorem 5.2] there is a subcomplex $Z'$ of $S^{j-1} \times D^{m-j}$ of dimension $\min(\dim(Z)+1, j)$ so that

$$Z' \cap S^{j-1} \times S^{m-j-1} = Z$$

and $S^{j-1} \times D^{m-j}$ simplicially collapses to $Z'$ ($S^{j-1} \times D^{m-j}$ is the mapping cylinder of the projection $S^{j-1} \times S^{m-j-1} \to S^{j-1}$, so take $Z'$ to be the mapping cylinder of the restriction to $Z$). Using [2, lemma 2.20] this implies that

$$S^{j-1} \times D^{m-j} \times I$$

simplicially collapses to

$$S^{j-1} \times D^{m-j} \times 0 \cup Z' \times I \cup S^{j-1} \times D^{m-j} \times I$$

so taking $S^{j-1} \times D^{m-j} \times I$ to be a collar of $S^{j-1} \times D^{m-j}$ in $D^j \times D^{m-j}$ we see that if we define $D$ to be

$$D = \overline{D^j \times D^{m-j} - S^{j-1} \times D^{m-j} \times I}$$

there is a simple strong deformation retract of $M_{i+1} \cup K_{i+1}$ to $M_i \cup Z' \times I \cup K_{i+1} \cup D$. However $D$ is a disc, and $Z' \times 1 \cup K_{i+1} \cap \partial D$ is of codimension bigger than 3 in $\partial D$, so we may as before assume it is PL-embedded and $D$ now simplicially collapses to the cone of $Z' \times 1 \cup K_{i+1} \cap \partial D$ thus finishing the induction step. It is clear by construction that $K'_i$ has a product neighborhood in $K_i$.

In case $P \neq \emptyset$ the proof is the same except we have to go through the motions of the induction step in the initial step of the induction too. □

## References

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