

THE TOPOLOGICAL CATEGORY

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In this appendix we will extend a number of the results in [2] to the topological category. First we consider Morlet's lemma of disjunction: We succeed to the following extent:

THEOREM. *Let V^n be a topological manifold, $n \geq 5$, and if $n = 5$, assume ∂V stable. Let $g : (D^p, \partial D^p) \rightarrow (V, \partial V)$ and $h : (D^q, \partial D^q) \rightarrow (V, \partial V)$ be disjoint embeddings (locally flat) of discs with $n - p \geq 3$ and $n - q \geq 3$. Then*

$$\pi_i(C(D^p, V; g), C(D^p, V - h(D^q); g)) = 0$$

for $i \leq 2n - p - q - 5$.

We also generalize the addendum.

ADDENDUM. *If V^n is simply connected, then the result holds for $i \leq 2n - p - q - 4$.*

Proof of Theorem. We note that with the added assumption on V^n , everything in the PL (Diff) proof holds true up to [2, Proposition A'], replacing Hudson's theorem by [6], so the main problem is transversality. For the solution of the transversality problem we note that [2, Proposition A'] is only needed for the case where $W_i = D^{p_i}$, so replacing Hudson's theorem by [6], the proof goes directly through (also using embedding results due to J. Lees) once the following transversality considerations have been made:

1. TRANSVERSALITY IN PL AND TOPOLOGICAL MANIFOLDS.

We consider the problem of moving submanifolds to transversally intersecting submanifolds in PL and topological categories. When there are only two submanifolds, this presents no problem, using PL blocktransversality, but when there are more than two submanifolds one encounters particular problems in the topological category, namely whether or not blocktransversality is preserved under change of PL-structure.

The main motivation for these considerations is the following situation which arises in "Morlet's lemma of disjunction". Given embeddings

$$\phi_i : (D^{p_i}, \partial D^{p_i}) \times (I, 0, 1) \rightarrow (V, \partial V) \times (I, 0, 1),$$

denote $\phi_i|_{D^{p_i} \times 0}$ by g_i and assume

$$\phi_i|_{\partial D^{p_i} \times I} = (g_i|_{\partial D^{p_i}}) \times 1_I$$

and $n - p_i \geq 3$. Find small isotopies h_i^t satisfying that the restriction to $\partial V \times I$ is a product isotopy, and such that ϕ_i are moved to “transversal” embeddings.

We present a solution here to the topological problem which might also have some interest in the PL case.

First we consider the situation in the PL category: If U is a PL manifold and M_i are PL submanifolds, they are said to be locally transversally intersecting if every point in U has a neighborhood PL homeomorphic to $\overline{R^n}$ by a homeomorphism that sends $M_i \cap U$ to subvectorspaces in general position, i. e. , sends $M_{i_1}^{n_1} \cap M_{i_2}^{n_2} \cap \dots \cap M_{i_s}^{n_s}$ to a subspace of R^n of dimension $n_1 + n_2 + \dots + n_s - (s - 1)n$. In the case of a manifold with boundary, we replace R^n by H^n , halfspace, in the above definition and consider ∂U to be one of the manifolds.

We note that this definition of transversality is a local one and different from the concept of blocktransversality by an example due to Hudson [4] (see also [10]) Let $M_i, i \in J, J$ finite, be locally transversally intersecting PL submanifolds in a PL manifold P , and denote $\bigcap_{s \in J'} M_s$ by $M_{J'}$ and $M_{\{\emptyset\}} = P$. Whenever $J'' \subset J'$ we have $M_{J'} \subset M_{J''}$ and a neighborhood of $M_{J'}$ in $M_{J''}$ has the structure of a blockbundle $\nu_{J', J''}$ the normal blockbundle, see [7, 8, 9]. The blockbundle is described by giving a cellstructure to $M_{J'}$, and to each cell assign a block over the cell, the procedure of [7, 8, 9] being: triangulate so that $M_{J'} \subset M_{J''}$ is the inclusion of a full subcomplex, give $M_{J'}$ the cell structure of dual cells, and take as block over the dual cell $C_\sigma^{M_{J'}}$ of σ in $M_{J'}$, the dual cell $C_\sigma^{M_{J''}}$ of σ in $M_{J''}$. We use the notation $\nu_{M'}$ for $\nu_{J', \{\emptyset\}}$, the normal blockbundle of $M_{J'}$ in P .

Definition 1. Notation as above: the normal blockbundle $\nu_{J', J''}$ of $M_{J'}$ in $M_{J''}$, $J'' \subset J'$ are said to be compatible if for every $J''' \subset J'' \subset J'$ and every cell C in $M_{J'}$, we have

- (1) $\nu_{J', J''}(C)$ is a cell in $M_{J''}$
- (2) $(\nu_{J'', J'''}(\nu_{J', J''}(C)), C) = (\nu_{J', J'''}(C), C)$

The property of being compatible can clearly be preserved under subdivision: In other words, if you subdivide the cell structure of $M_{J'}$, you can subdivide all $M_{J''}$ in such a way that condition 1. and 2. are preserved.

Lemma 2. *If M_i are locally transversally intersecting PL submanifolds of P then $M_{J'}$ have compatible normal blockbundles.*

Proof. Triangulate so that every inclusion $M_{J'} \subset M_{J''}$ is an inclusion of a full subcomplex, and use the standard dual cell construction. \square

We note that if N is blocktransversal to $M_{J'}$ with respect to the compatible normal block bundles, then it follows that N is block transversal to $M_{J''}$ in a neighborhood of $M_{J'}$, $J'' \subset J'$ since the block are the same. Hence it follows that

Lemma 3. *Let P be a PL manifold, M_1, M_2, \dots, M_s locally transversally intersecting submanifolds, and N another submanifold, everything assumed to be PL. Then N can be moved*

by an ambient ϵ -isotopy which is the identity outside an ϵ -neighborhood of N , to be simultaneously transverse to all $M_{J'}$ with respect to a compatible system of normal blockbundles. If $\partial N \subset \partial P$ is already blocktransverse to $\{\nu_{J',J''}\}$ then the isotopy can be chosen to fix ∂P .

Proof. Use [7, 8, 9] to move N so N is blocktransverse to $M_{J'}$, then relative to a neighborhood of M_J move N so N is blocktransverse to all $M_{J-\{i\}}, i \in J$ etc. In case we assume blocktransversality in the boundary we may do this relative to the boundary. If we want ϵ -isotopy we can obtain that by using a very fine triangulation to define the blockbundles. \square

We now consider variations in PL-structure.

Lemma 4. *Let U be a topological manifold, M_i be PL manifolds locally flatly embedded in U . Suppose U has a PL structure U' in which M_i are PL embedded submanifolds intersecting locally transversally. Then if U is given another PL structure U'' such that M_i are still PL embedded. Then they are still locally transversally intersecting if $\dim(U) \geq 6$ (≥ 5 if $\partial U \neq \emptyset$), and*

$$\dim(U) - \dim(M_i) \geq 3 \quad \text{for all } i$$

Proof. Let p be a point in U , and let V be an open neighborhood in U such that V , with PL structure from U' is PL homeomorphic to R^n by a homeomorphism h sending M_i to subvector spaces of H^n in general position. Denote the union of these subvectorspaces by K . Then K is a complex of codimension 3 in R^n . If we consider V with PL structure induced from U'' then $h : V'' \rightarrow H^n$ is not PL, but restricted to $V'' \cap M_i$ it is, and by uniqueness of PL structure of H^n there is a small isotopy h^t of h such that h^1 is PL. Let Z be a regular neighborhood of p in $\bigcup M_i \cap V$, then $h|Z$ and $h^1|Z$ are close PL embeddings of a finite codimension 3 complex into H^n so by [3] there is an ambient PL isotopy of H^n moving $h|Z$ to $h^1|Z$ i. e. there is a PL homeomorphism of a neighborhood W of p in U sending $\bigcup M_i \cap W$ to K i. e., to subvector spaces in general position. \square

Definition 5. Let V be a topological manifold and M_i be PL manifolds locally flatly embedded in V . Then M_i are said to be tamely transversally intersecting in V if every point p in V has a neighborhood U with a PL structure so that $U \cap M_i \subset U$ is a PL embedding. ($U \cap M_i$ has PL structure induced from M_i .) Furthermore $U \cap M_i$ are required to be locally transversally intersecting in the PL sense.

Remark. We note that the question of tame transversality is independent of the PL structure on U in codimension 3 by Lemma 4 above.

We now return to our motivating situation: given

$$\phi_i : (D^{p_i}, \partial D^{p_i}) \times (I, 0, 1) \rightarrow (V^n, \partial V) \times (I, 0, 1)$$

$n \geq 6$, $n - p_i \geq 3$, denote $\phi_i|D^{p_i} \times 0$ by g_i and assume

$$\phi_i|\partial D^{p_i} \times I = g_i|\partial D^{p_i} \times 1_I.$$

Then there are small isotopies h_i^t that are product isotopies when restricted to $\partial B \times I$ so that $h_i^1 \circ \phi_i$ are tamely transversally intersecting.

Proof. To facilitate notation whenever we have found isotopies as above such that $h_i^1 \circ \phi_i$ satisfies some condition we may as well assume this was true originally and thus denote $h_i^1 \circ \phi_i$ by ϕ_i . First we take an inside collar of ∂V in V , $\partial V \times [0, 1]$ and of ∂D^{p_i} in D^{p_i} , $\partial D^{p_i} \times [0, 1]$, and after a small ambient isotopy we can assume that ϕ_i agree with the collars near the boundary, see e. g. [12]. Also we may assume ϕ_i is a product isotopy when restricted to $V \times [0, \epsilon]$ and $V \times [1 - \epsilon, 1]$ for some sufficiently small ϵ . Having done this we assume inductively that ϕ_i are tamely transversally intersecting for $i < r$, and still agreeing with collars near the boundary. We let

$$\Phi : D^{p_i} \times I \times R^{n-p_i} \rightarrow V \times I$$

be an embedding extending ϕ_r , such that Φ is a product embedding when restricted to $\partial V \times [0, \epsilon] \times I$, $V \times [0, \epsilon]$ and $V \times [1 - \epsilon, 1]$, this being a trivial extension of Lemma 4 of [6]. Denote

$$\phi_i(D^{p_i} \times I) = L_i$$

and

$$U = \Phi(D^{p_r} \times I \times R^{n-p_r})$$

and let

$$B = L_1 \cup L_2 \cup \dots \cup L_{r-1}.$$

Let Z be a finite subcomplex of $B \cap U$ such that

$$Z \supset B \cap \Phi(D^{p_r} \times I \times B^{n-p_r})$$

where B^{n-p_r} is the unit ball of R^{n-p_r} . Then by [3] as quoted in [6] there is an ambient ϵ -isotopy of $U \cap V \times 0$ which is the identity outside a compact set and moves $Z \cap V \times 0$ to a PL embedding. Extending by the identity this is an ambient isotopy of $V \times 0$ and we use the product isotopy to obtain an isotopy of $V \times I$. Our considerations about collars now assure that $Z \cap V \times 1$ is PL embedded in a neighborhood of $\partial V \times 1$ and thus by [3] may be moved to a PL embedding by an isotopy as above which is the identity near the boundary. We extend to an isotopy of $V \times I$ using a product isotopy near $V \times 1$, the identity near $V \times 0$ and tapering the ambient isotopy off in between. By this we obtain that Z is PL embedded in a neighborhood of $\partial(V \times I)$ and we then finally obtain an isotopy of $V \times I$ relative to a neighborhood of $\partial(V \times I)$ so that Z is finally PL embedded in U . All these isotopies may be chosen so small that $B - Z$ stays outside $\Phi(D^{p_r} \times I \times B^{n-p_r})$. If we now shrink the fibre of U we can thus assume that $B \cap U \subset U$ is a PL embedding.

It now follows from Lemma 4 that $L_i \cap U$ are locally transverse in The PL sense for $i < r$, and we can by Lemma 3 move L_r by an ambient ϵ -isotopy as above so that L_1, L_2, \dots, L_r are transversally intersecting in the PL sense. This isotopy, as before, can be done in stages assuring productisotopy when restricted to $\partial V \times [0, \epsilon] \times I$, etc. \square

2. STRAIGHTENING CONCORDANCES

The proof of the addendum takes a little more doing, the problem being the starting point. We need to know that a codimension 2 concordance of a disc can be straightened if the complement is simply connected. To do this all we need to know is that concordance of a simply connected topological manifold can be straightened. So we proceed to prove

Theorem 6. *Let V^n be a simply connected topological manifold, $n \geq 5$, if $n = 5$ assume V is a handlebody. Let $h : V \times (I, 0, 1) \rightarrow V \times (I, 0, 1)$ be a homeomorphism which restricts to the identity on $\partial V \times I \cup V \times 0$. Then there is an isotopy ϕ_t of $V \times I$ fixing $\partial V \times I \cup V \times 0$ such that $\phi^1 \circ h = 1_{V \times I}$.*

Proof. By [5], V is a handlebody, so let $V^{(k)}$ be a handlebody filtration relative to \emptyset . First we deform h to a homeomorphism which is the identity on $\overline{V - V^{(2)}}$. This is done by inductive straightening of the dual handles. First, straighten the core of the handle using [6] and then a neighborhood, using, e. g. the h -cobordism theorem. This procedure breaks down when we come to codimension 2 handles, i. e. the 2-skeleton. The 2-skeleton however is smoothable so by [1] we can finish off the straightening. \square

This makes all the results on embeddings, automorphisms, and concordance spaces in [2, Chapter 3] hold as in the PL case, with the added assumption that the ambient manifold be of dimension at least 6 and the embedded manifold be a handlebody. The only place one needs to put in information is replacing $(\widetilde{\text{PL}}_n, \text{PL})$ by $(\widetilde{\text{Top}}, \text{Top})$ using [11].

One should specifically mention the following result:

Theorem 7. *Let V^n , $n \geq 5$, be a topological manifold; if $n = 5$, assume V is a handlebody. Then*

$$\pi_0(C(V)) = \text{Wh}_2(\pi_1(V)) \oplus \text{Wh}_1(\pi_2(V); \pi_1(V)).$$

Proof. As in the PL case, prove that $\pi_0(C(V)) = \pi_0(C(V^{(3)}))$, where $V^{(3)}$ is the 3-skeleton of some handlebody decomposition. Then apply [1] to conclude that it is the same as in the Diff case so one may refer to Hatcher-Wagoner. \square

3. HOMOTOPING A MAP TO A BUNDLE MAP

Finally, we want to extend the application in [2] of homotoping a map to a bundle map to the topological case. We succeed to the following extent.

Consider the problem: Given a map $f : V^v \rightarrow M^m$ of closed topological manifolds, when is f homotopic to a bundle map? The methods of [2, Chapter 5] along with topological transversality [5] and the topological Lemma of Disjunction give the same results as in the PL case, assuming M is triangulable. It is the purpose of this note, which is not in its most general form, to reduce the question to that case. Let D be the total space of the normal

disc bundle ν of M . Then we have a diagram

$$\begin{array}{ccc} V & \xleftarrow{p'} & E \\ f \downarrow & & \downarrow f' \\ M & \xleftarrow{i_C} & D \end{array}$$

where p is the normal bundle projection, i is the zero section, f' is the pullback of f . We note that by symmetry p' is the pullback of p over f , so E is the total space of a bundle p' over V with zero section i' . Hence E is a manifold and $f' : (E, \partial E) \rightarrow (D, \partial D)$ is a map of manifolds. Also D is triangulable, being a codimension 0 submanifold of euclidean space, and the following theorem thus completes our goal.

Theorem 8. *Notation as above. Assume V is simply connected $v - m \neq 3$, $m \neq 4$, $v \geq 5$. Then f' is homotopic to a bundle map if and only if f is.*

Proof. The ‘if’ part is trivial and does not need any assumptions. Assume f is homotopic to a bundle map, let F be the homotopy

$$\begin{array}{ccc} F^*(D) & \xrightarrow{F'} & D \\ \tilde{p} \downarrow & & \downarrow p \\ V \times I & \xrightarrow{F} & M \end{array}$$

Then since $F^*(D)$ is a bundle over $V \times I$ it is a product bundle

$$\begin{array}{ccc} E \times I = F^*(D) & \xrightarrow{F'} & D \\ \searrow p' \times 1_I & & \downarrow p \\ & & V \times I \xrightarrow{F} M \end{array}$$

so F' is a homotopy from f' to $F'|E \times 1$, but

$$\begin{array}{ccc} E \times 1 & \xrightarrow{F'|E \times 1} & D \\ p|E \times 1 \downarrow & & \downarrow p \\ V & \xrightarrow{F|V \times 1} & M \end{array}$$

can also be considered a pullback over p , so if $F|V \times 1$ is a bundle map then $F'|E \times 1$ is also.

Now assume f' is homotopic to a bundle map, and let F' be the homotopy $F' : E \times I \rightarrow D$. $F'|E \times 0$ is f' so is clearly transverse to $M \subset D$, and $F'|E \times 1$ is also transverse to $M \subset D$ by the following argument: Since $i \circ p$ is homotopic to $1_{D'}$, $F'|E \times 1$ being a bundle map can

be identified with the pullback of

$$\begin{array}{ccc} E \times 1 & \longrightarrow & i^*(E \times 1) \\ \downarrow F'|_{E \times 1} & & \downarrow \\ D & \xrightarrow{p} & M \end{array}$$

but p is a disc bundle projection, so considering the pullback diagram the other way around $E \times 1 \rightarrow i^*(E \times 1)$ is a disc bundle, and $F'|_{E \times 1}$ is a disc bundle map, and that is what it means to be transverse to $M \subset D$. Since $v - m$ is assumed to be $\neq 3$ and $m \neq 4$ F' is homotopic relative to $E \times 0 \cup E \times 1$ to a map F which is transverse regular to M . So let $W = F^{-1}(M)$. Then W is a cobordism from V to the total space of $i^*(F|_{E \times 1})$ which we denote V' and we have a diagram which is homotopy commutative

$$\begin{array}{ccccc} V & \xleftarrow{p'} & E & = & E & \supset & V' \\ \downarrow f & & \searrow f' & & \swarrow F|_{E \times 1} & & \downarrow \\ M & \xleftarrow{p} & D & \supset & M \end{array}$$

and $V' \rightarrow M$ is a bundle map, so all we need to prove is that there is a homeomorphism $h : V \rightarrow V'$ so that the composition $V \xrightarrow{h} V' \subset E \xrightarrow{p'} V$ is homotopic to the identity. To prove this we set up a surgery problem. We have

$$(W, V, V') \subset (E \times I, E \times 0, E \times 1) \xrightarrow{p' \times 1} (V \times I, V \times 0, V \times 1)$$

is a degree 1 map and the restriction to $\partial W = V \cup V'$ is a homotopy equivalence. Let the normal bundle of W in $E \times I$ be denoted ξ , then ξ is the pullback of ν over F by transversality, so the above map is covered by a bundle map

$$\begin{array}{ccc} \xi & \longrightarrow & (\xi|_{V \times 0}) \times I \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \times I \end{array}$$

We want the normal bundle of W over W rather than ξ , but this is obtained by adding the restriction of the normal bundle of $E \times I$ to both sides. We now consider the surgery problem relative to $V \cup V'$. If the surgery obstruction is nonzero, then since $\pi_1(V) = 0$ by Kervaire, Milnor plumbing theory, we may add a problem over a sphere with minus this obstruction, thus replacing W by a cobordism W' with trivial surgery obstruction. We may then complete surgery to obtain a cobordism W'' and a homotopy equivalence

$$g : (W'', V, V') \rightarrow (V \times I, V \times 0, V \times 1)$$

where $g|_V = 1_V$ and $g|_{V'}$ is the composition $V' \subset E \xrightarrow{p'} V$. It follows that W'' is an h -cobordism so there is a homeomorphism

$$g' : (V \times I, V \times 0, V \times 1) \rightarrow (W'', V, V')$$

which is the identity on $V \times 0$. We let $h = g'|_{V \times 1}$. Then $g \circ g'$ is a homotopy from 1_V to

$$V \xrightarrow{h} V' \subset E \xrightarrow{p'} V$$

□

We therefore have that all the results in [2, section 5] pertaining to the PL category also hold in the topological category for deforming $f : V \rightarrow M$ to a bundle projection, provided $\pi_1(V) = 0$.

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