

# TOPOLOGICAL CONCORDANCES

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In this paper we will prove that concordance implies isotopy for codimension  $\geq 3$ , locally tame embeddings of locally triangulable spaces in topological manifolds, with the restriction that if the ambient manifold is 4-dimensional, we have to assume it simply-connected.

This theorem was first proved in the PL and Diff categories by J. F. P. Hudson [9]. Another method of proof has been developed by C. Rourke [17]. Our approach is to generalize the approach of C. Morlet [15] to the topological category. The theorem has also been proved by M. A. Armstrong [1] in case of a locally tame embedding of a complex in a PL-manifold, using PL-approximation [13] and Hudson's theorem [9]. We have also generalized Morlet's results on higher homotopy groups of concordance spaces, specifically the lemma of disjunction, to the topological case [3].

I would like to thank my thesis advisor professor R. Lashof for his help and encouragement in the preparation of this paper. Also I want to thank professor R. D. Edwards, who pointed out numerous improvements, using his stronger, relative version of the Miller-Connelly taming theorem [5]. In an earlier version of this paper we proved a slightly weaker theorem using only the non-relative PL-approximation theorem [4].

The results of this paper were announced in [16].

We use standard notation.  $D^n$  or  $B^n$  denotes the standard  $n$ -dimensional disc,  $S^n$  the  $n$ -sphere and  $\mathbf{R}^n$  Euclidean space. By an embedding  $M \subset V$  of topological manifolds we mean a locally flat embedding, transversal to the boundary. If  $K$  is a locally triangulable space in the topological manifold  $V$ , the embedding is said to be locally tame or locally PL-able if for every point in  $K$  there is a neighborhood  $U$  in  $V$  such that  $K \cap U$  and  $U$  have PL-structures such that  $K \cap U \subset U$  is PL.

We prove the following theorem:

**Theorem 8.** *Suppose  $V^n$  is a topological manifold with boundary (not necessarily compact) and  $(X, Y)$  is a compact locally triangulable pair of spaces, and suppose*

$$\phi : (X, Y) \times I \rightarrow (V, \partial V) \times I$$

*is a locally tame concordance (that is,  $\phi^{-1}(V \times I) = X \times i$  for  $j = 0, 1$ , and*

$$\phi^{-1}(\partial V \times I) = Y \times I,$$

*and the embedding  $\phi$  is locally PL). Suppose  $f|Y \times I$  is a product concordance and  $\dim(X - Y) \leq n - 3$  and if  $n = 4$ , that  $\pi_1(V) = 0$ . Then there is an ambient isotopy  $H_t : V \times I \rightarrow V \times I$ ,*

$t \in I$ , fixed on  $V \times 0 \cup \partial V \times I$  and having compact support, such that

$$H_1\phi = g \times 1_I$$

where  $g$  denotes  $\phi|X \times 0$ .

For the moment we will restrict ourselves to the case  $n \geq 5$ , and then we will discuss low dimensional difficulties later.

**Proposition 1.** *Suppose  $V^n$  is an  $n$ -dimensional topological manifold,  $n \geq 5$ , and let*

$$\phi : (D^p, \partial D^p) \times (I, 0, 1) \rightarrow (V, \partial V) \times (I, 0, 1)$$

be a concordance such that  $\phi|\partial D^p \times I = \phi|\partial D^p \times 0 \times 1_I$ . Denote  $\phi|D^p \times 0$  by  $g$  and assume  $g(D^p) \subseteq V \times 0$  has a PL neighborhood  $U$ . Then there exists an ambient isotopy

$$F_t : V \times I \rightarrow V \times I \quad 0 \leq t \leq 1$$

with  $F_t|\partial V \times I$  a product isotopy, and there exists a concordance  $H : V \times I \rightarrow V \times I$ , with  $H|\partial V \times I \cup V \times 0 = \text{identity}$ , such that the following images coincide:

$$H(g(D^p) \times I) = g(D^p) \times I$$

and

$$H(F_1\phi(D^p \times 1)) = F_1g(D^p) \times I.$$

(Note: it is not asserted that  $F_t|V \times 0 = \text{identity}$ , or that  $H$  is isotopic to the identity.)

The main work of this paper will be to prove Proposition 1. We shall need PL approximation theorems. We quote the Miller-Connelly-Edwards taming theorem [5]:

**Theorem 2.** *Let  $M^m$  be a PL manifold,  $\phi : K \rightarrow M$  a locally tame embedding of a PL complex  $K$  of codimension  $\geq 3$  such that  $\phi^{-1}(\phi(K) \cap \partial M) = L$  is a subcomplex of  $K$  and  $\phi|L$  is PL. Then there is an ambient  $\varepsilon$ -isotopy  $H_t$  of  $M$  fixing  $\partial M$  such that  $H_1 \circ \phi$  is a PL embedding.*

To use Theorem 2 we need to have local PL structures.

**Lemma 3.** *Let  $\phi : D^p \times I \rightarrow V^n \times I$  be a concordance as in Proposition 1. Then there is an extension*

$$\Phi : D^p \times \mathbf{R}^{n-p} \times I \rightarrow V^n \times I,$$

where  $\Phi$  is an embedding which is a product over the boundary.

*Proof.* We need to deal specially with the case  $p = 0$ . For  $p = 0$  choose an embedded path in  $V \times 0$ ,  $\alpha$ , from  $\phi(D^0 \times 0)$  to a point  $q$  such that  $\phi(D^0 \times I) \cap q \times I = \emptyset$  and another embedded path  $\beta$  in  $V \times I$  from  $q \times 1$  to  $\phi(D^0 \times 1)$  and such that the loop  $\alpha \cup q \times I \cup \beta \cup \phi$  is trivial in  $V \times I$ . Doing general position patch by patch, this loop may be assumed to bound an embedded, locally flat disc in  $V \times I$ , and since  $q \times I$  does have a PL neighborhood, and we push  $\phi(D^0 \times I)$  into this neighborhood by an ambient isotopy, we are done with this case.  $\phi(D^p \times 0)$  is by assumption contained in a PL neighborhood  $U$  and by Theorem 2 we may

change the PL structure of  $U$  such that  $\phi : D^p \times 0 \rightarrow U$  is a PL-embedding. A regular neighborhood of  $\phi(D^p \times 0)$ ,  $N$ , is a regular neighborhood of a point hence a disc, and by Zeeman's unknotting theorem [21] we then get that  $\phi(D^p \times 0)$  is a standard subdisc hence an extension  $\phi : D^p \times 0 \times \mathbf{R}^{n-p} \rightarrow V$ . If we extend  $\phi : D^p \times I \rightarrow V \times I$  to a collar of  $V \times I$  by the product embedding we get

$$\bar{\phi} : D^p \times [0, 2) \rightarrow V \times [0, 2)$$

and it is now easy to find an ambient isotopy  $F_t$  of  $V \times [0, 2)$  fixing  $V \times 0 \cup \partial V \times [0, 2)$  such that  $F_1(\phi(D^p \times [0, 1])) \subseteq \Phi(D^p \times 0 \times \mathbf{R}^{n-p}) \times [0, 2)$  so

$$F_1^{-1}(\Phi(D^p \times 0 \times \mathbf{R}^{n-p}) \times [0, 2)) = W$$

is a PL neighborhood of  $\phi(D^p \times [0, 1])$ . Let  $S$  be a neighborhood of  $\phi(D^p \times 1)$  in  $V$  and  $\varepsilon \in \mathbf{R}$ , both chosen so small that  $S \times ]1 - \varepsilon, 1 + \varepsilon[ \subset W$ .  $S \times ]1 - \varepsilon, 1 + \varepsilon[$  is now an open subset of a PL manifold, hence inherits a PL structure [20] which is the product structure over the boundary, hence by the Kirby-Siebenmann product structure theorem [10] we may change the PL structure of  $W$  fixing it on the boundary to make  $S \times \{1\}$  a PL submanifold. We therefore get a PL neighborhood of  $\phi(D^p \times [0, 1])$  in  $V \times I$  extending the neighborhood of  $\phi(D^p \times 0 \cup \partial D^p \times [0, 1])$  in  $V \times 0 \cup \partial V \times I$  we already had. By Theorem 2 we may change the PL structure, fixing it on the intersection with  $V \times 0 \cup \partial V \times I$  to obtain that  $\phi(D^p \times I)$  is a subcomplex.

The proof is completed by taking a regular neighborhood of  $\phi(D^p \times I)$  extending the given regular neighborhood of  $\phi(D^p \times 0 \cup \partial D^p \times I)$  and using Zeeman's unknotting theorem twice [21]

□

We will now start constructing the isotopy of Proposition 1.

**Lemma 4.** *Assumptions as in Proposition 1. Then there is an extension*

$$\Phi : D^p \times \mathbf{R}^{n-p} \rightarrow V$$

of  $g : D^p \rightarrow V$ , and an ambient isotopy  $G_t$  of  $V \times I$  with compact support such that

- a)  $G_t|_{\partial V \times I}$  is a product isotopy.
- b)  $G_1(\phi(\partial D^p \times I)) \cap g(D^p) \times I = \emptyset$ .
- c)  $G_1 \circ \phi|(G \circ \phi)^{-1}(\Phi(D^p \times \mathbf{R}^{n-p}) \times I)$  is PL and intersects  $g(D^p) \times I$  simplicially transversally in the sense of [19].

*Proof.* By Lemma 3 we may find an extension  $\Phi : D^p \times \mathbf{R}^{n-p} \rightarrow V$  of  $g$  and it is easy to find an isotopy that moves  $\phi(\partial D^p \times I)$  off  $g(\partial D^p \times I)$  satisfying a). In the following we will then fix  $\partial V \times I$  so that we only need c). By Theorem 2 we may tame a subcomplex of  $\phi^{-1}(\Phi(D^p \times \mathbf{R}^{n-p}) \times I)$  by an  $\varepsilon$ -isotopy with compact support fixing  $\Phi(\partial D^p \times \mathbf{R}^{n-p})$  and by shrinking the fibres of  $\Phi$  we may then assume that  $\phi|\phi^{-1}(\Phi(D^p \times \mathbf{R}^{n-p}) \times I)$  is PL. Now since b) is satisfied we may find an ambient isotopy of  $\Phi(D^p \times \mathbf{R}^{n-p}) \times I$  with compact

support, fixing  $\Phi(\partial D^p \times \mathbf{R}^{n-p}) \times I$  that moves  $\phi(D^p \times I)$  to intersect  $g(D^p) \times I$  simplicially transversally, and hence so that the intersection is a PL manifold, using [19].  $\square$

The next step is to find isotopies so that we eventually move  $\phi$  so that the intersection  $X = \phi(D^p \times I) \cap g(D^p) \times I$  becomes a product. For a locally tame submanifold  $A$  of  $V \times I$  we use the notation:  $A^i = A \cap V \times i$ ,  $dA^i = A \cap \partial V \times i$  ( $i = 0, 1$ ) and  $dA = A \cap \partial V \times I$ . We say

$$\tau(X) \geq \alpha$$

if  $X$  is obtained from  $X^0 \times I$  by adding handles of index  $j$ , with  $\alpha \leq j \leq \dim X - \alpha$ , and the handles not touching  $d(X^0) \times I$ . Notice that  $\tau(X) > \frac{1}{2} \dim X$  implies that  $X$  is a product. By abuse of notation we will denote the result of Lemma 4,  $G_1 \circ \phi$  by  $\phi$ . (Whereas  $g$  always refers to the embedding given in Prop. 1.)

**Lemma 5.** *With notation as above let  $B = \phi(D^p \times I) \cup g(D^p) \times I$ . Then if  $\tau(X) \geq \alpha - 1$  we have*

- 1)  $\pi_i(V \times I - B, V \times 0 - B^0) = 0$  for  $i \leq \alpha + 2$ .
- 2)  $\pi_i(\phi(D^p \times I) - X, \phi(D^p \times 0) - X^0) = 0$  for  $i \leq \alpha + 1$ .
- 3)  $\pi_i(g(D^p) \times I - X, g(D^p) - X^0) = 0$  for  $i \leq \alpha + 1$ .

*Proof.* We remind the reader that  $p \leq n - 3$ . First we prove 1. By general position (codimension 3) we have

$$\pi_1(V \times I - B) \cong \pi_1(V \times I)$$

and

$$\pi_1(V \times 0 - B^0) \cong \pi_1(V).$$

Let  $\tilde{V}$  be the universal cover of  $V$ , and let  $\tilde{B}$ ,  $\tilde{B}_0$  etc. be the preimages of  $B$ ,  $B_0$ , etc. in  $\tilde{V} \times I$ .  $\tilde{V} - \tilde{B}$  and  $\tilde{V} \times 0 - \tilde{B}^0$  are then simply connected by the above remarks, so it suffices to show that  $H_k(\tilde{V} \times I - \tilde{B}, \tilde{V} \times 0 - \tilde{B}^0) = 0$  for  $k \leq \alpha + 3$ . By Poincaré-Lefschetz duality we have

$$H_k(\tilde{V} \times I - \tilde{B}, \tilde{V} \times 0 - \tilde{B}^0) \cong H_c^{n+1-k}(\tilde{V} \times I - \tilde{B}, \tilde{V} \times 1 - \tilde{B}^1)$$

where the  $c$  denotes cohomology with compact supports. We now have

$$H_c^{n+1-k}(\tilde{V} \times I - \tilde{B}, \tilde{V} \times 1 - \tilde{B}^1) \cong \lim\{H_c^{n+1-k}(\tilde{V} \times I - \tilde{N}, \tilde{V} \times 1 - \tilde{N}^1)\}$$

where  $N$  is a closed neighborhood of  $B$  in  $V \times I$ . Applying excision to each term, it follows that this is isomorphic to

$$\lim\{H_c^{n+1-k}(\tilde{V} \times I, \tilde{V} \times 1 \cup \tilde{N})\}$$

which is simply  $H_c^{n+1-k}(\tilde{V} \times I, \tilde{V} \times 1 \cup \tilde{B})$ .

The exact sequence for the triple  $(\tilde{V} \times I, \tilde{V} \times 1 \cup \tilde{B}, \tilde{V} \times 1)$  shows that

$$H_c^{n+1-k}(\tilde{V} \times I, \tilde{V} \times 1 \cup \tilde{B}) \cong H_c^{n-k}(\tilde{V} \times 1 \cup \tilde{B}, \tilde{V} \times 1)$$

which in turn by excision is isomorphic to

$$H_c^{n-k}(\tilde{B}, \tilde{B}^1).$$

The Mayer-Vietoris sequence applied to the pairs  $(\phi_0(D^p \times I), \phi_0(D^p \times 1))$  and  $(\phi_1(D^p \times I), \phi_1(D^p \times 1))$  shows that

$$H_c^{n-k}(\tilde{B}, \tilde{B}^1) \cong H_c^{n-k-1}(\tilde{X}, \tilde{X}^1).$$

But since  $X$  is obtained from  $X^1$  by adjoining handles of dimension at most  $\dim X - \alpha + 1$  we have  $H_c^{n-k-1}(\tilde{X}, \tilde{X}^1)$  is 0 for  $n - k - 1 > \dim X - \alpha + 1$  i. e. for  $k < (n - \dim X) + \alpha - 2$ , and then since  $n - \dim X \geq 5$  we have that  $H_c^{n-k-1}(\tilde{X}, \tilde{X}^1) = 0$  for  $k \leq \alpha + 2$ . This completes the proof of 1). The proof of 2) and 3) is completely analogous replacing  $V$  by  $D^p$  and  $B$  by  $\phi^{-1}(X)$  respectively  $(g \times 1)^{-1}(X)$ .  $\square$

We are now ready for the main geometric construction. In the following lemma the picture is the following: We assume that  $\phi$  has been moved by the isotopy given in Lemma 4, and the result we again denote be  $\phi$ . We consider  $g(D^p) \times I \subset V \times I$  ( $g$  is the original  $g$ ) with its extension  $\Phi(D^p \times \mathbf{R}^{n-p}) \times I$  defining a PL structure in a neighborhood of  $g(D^p) \times I$ .

**Lemma 6.** *Picture as above. Assume that  $\tau(X) \geq \alpha - 1$  ( $\alpha \geq 1$ ). Then there is an ambient isotopy of  $H_t$  of  $V \times I$  fixing  $\partial V \times I$  such that  $H_1 \circ \phi | (H_1 \circ \phi)^{-1}(\Phi(D^p \times \mathbf{R}^{n-p}) \times I)$  is PL and intersects  $g(D^p \times I)$  simplicially transversally. Moreover*

$$\tau(\text{resulting } X) \geq \alpha.$$

(Here of course resulting  $X = H_1 \circ \phi(D^p \times I) \cap g(D^p \times I)$ .)

*Proof.* As above set  $B = g(D^p) \times I \cup \phi(D^p \times I)$ .

Let  $\mu_s : (D^{\alpha-1}, S^{\alpha-2}) \rightarrow C$  be the cores of the  $\alpha - 1$  handles mod  $X^0(X^1)$  in a handlebody decomposition of  $X$  with no handles of dimension less than  $\alpha - 1$  mod  $X^0(X^1)$ . Since everything is PL near  $g(D^p) \times I$  and  $g(D^p) \times I$  intersects  $\phi$  transversally, we can extend the embedding

$$\mu_s : (D^{\alpha-1}, S^{\alpha-2}) \rightarrow (X, X^0)$$

to

$$F_s : (D^{\alpha-1}, S^{\alpha-2}) \times I \rightarrow (\phi(D^p \times I), \phi(D^p \times 0))$$

and

$$G_s : (D^{\alpha-1}, S^{\alpha-2}) \times I \rightarrow (g(D^p) \times I, g(D^p) \times 0)$$

such that

$$\begin{aligned} F_s|_{D^{\alpha-1} \times 0} &= G_s|_{D^{\alpha-1} \times 0} = \mu_s, \\ F_s(D^{\alpha-1} \times I) \cap g(D^p) \times I &= \mu_s(D^{\alpha-1}) \end{aligned}$$

and

$$G_s(D^{\alpha-1} \times I) \cap \phi(D^p \times I) = \mu_s(D^{\alpha-1}).$$

$F_s|D^{\alpha-1} \times 1$  then represent an element of

$$\pi_{\alpha-1}(\phi(D^p \times I) - X, \phi(D^p \times 0) - X^0)$$

and  $G_s|D^{\alpha-1} \times 1$  represents an element of

$$\pi_{\alpha-1}(g(D^p) \times I - X, g(D^p) \times 0 - X^0).$$

These groups, however, are both zero by Lemma 5. Therefore since without loss  $2\alpha < p+1$ ,  $G_s$  and  $F_s$  extend to PL embeddings

$$f_s : (D^\alpha, S_+^{\alpha-1}, S_-^{\alpha-1}) \rightarrow (\phi(D^p \times I), X, \phi(D^p \times 0)),$$

$$g_s : (D^\alpha, S_+^{\alpha-1}, S_-^{\alpha-1}) \rightarrow (g(D^p) \times I, X, g(D^p) \times 0)$$

where  $S_+^{\alpha-1}$ ,  $S_-^{\alpha-1}$  are respectively the upper and lower hemisphere of  $\partial D^\alpha$ , and  $D^{\alpha-1} \times I$  is identified with a collar of  $S_+^{\alpha-1}$  in  $D^\alpha$ . The embeddings  $f_s$  and  $g_s$  are made so that

$$f_s(D^\alpha) \cap g(D^p) \times I = \mu_s(D^{\alpha-1}),$$

$$g_s(D^\alpha) \cap \phi(D^p \times I) = \mu_s(D^{\alpha-1}).$$

We now glue  $f_s$  and  $g_s$  together along  $S_+^{\alpha-1}$  to obtain

$$\kappa_s = (f_s \cup_{\mu_s} g_s) : D^\alpha \rightarrow B.$$

We now want to extend  $\kappa_s$  to

$$H_s : (D^\alpha, S^{\alpha-1}) \times I \rightarrow (V \times I, V \times 0)$$

that satisfies

$$H_s|D^\alpha \times 0 = \kappa_s,$$

$$H_s(D^\alpha \times I) \cap B = \kappa_s(D^\alpha)$$

and  $B \cup H_s(D^\alpha \times I)$  is locally tamely embedded in  $V \times I$ . This is done by first extending  $g_s$  to an embedding

$$(D^\alpha, S_+^{\alpha-1}, S_-^{\alpha-1}) \times I \rightarrow (V \times I, \phi(D^p \times I), V \times 0)$$

using the PL structure in a neighborhood of  $g(D^p) \times I$  and the transverse intersection of  $g(D^p) \times I$  and  $\phi(D^p \times I)$  and then use the PL neighborhood of  $\phi(D^p \times I)$  (Lemma 3) and the PL approximation theorem (Theorem 2) to extend this extension to the desired extension  $H_s$ .  $H_s|D^\alpha \times 1$  now represents an element in  $\pi_\alpha(V \times I - B, V \times 0 - B^0)$ . This group is 0 by Lemma 5, so we may get a map

$$h_s : (D^\alpha, S_+^{\alpha-1}, S_-^{\alpha-1}) \rightarrow (V \times I, V \times 0, B)$$

extending  $H_s$  (we identify  $D^\alpha \times I$  with a collar of  $S_-^\alpha$ ) and such that

$$h_s D^{\alpha+1} \cap B = \kappa_s(D^\alpha).$$

Since, without loss,  $2(\alpha+1) < n+1$  we can change  $h_s$  relative to  $H_s$  by an arbitrarily small homotopy to make it a locally tame embedding. This is an elementary chart by chart

argument using PL general position in each chart and Miller's theorem (Theorem 2) to go from chart to chart.

We are now ready to find the desired isotopy. Firstly we may find an ambient isotopy of  $V \times I$  fixing  $\partial V \times I$  and carrying  $g(D^p) \times I \cup h_s(D^{\alpha+1})$  into the given PL neighborhood of  $g(D^p) \times I$  fixing a smaller neighborhood of  $g(D^p) \times I$ . This can be done since  $h_s(D^{\alpha+1})$  is locally flat and hence may inductively be pushed into an arbitrarily small neighborhood of  $g(D^p) \times I$ . Having changed the picture by this isotopy we use Theorem 2 to find an isotopy that fixes a neighborhood of  $g(D^p) \times I$  and such that  $\Phi(D^p \times \mathbf{R}^{n-p}) \times I \cap B \cup h_s(D^{\alpha+1})$  is PL embedded in  $\Phi(D^p \times \mathbf{R}^{n-p}) \times I$ .

Now we are entirely reduced to the PL situation and we now slide  $\phi(D^p \times I)$  across  $h_s(D^{\alpha+1})$  to remove a regular neighborhood of  $\mu_s(D^{\alpha+1})$  in  $X$  as in the PL case, see [15] or [3]. Then  $\tau(\text{resulting } X) \geq \alpha$  since we have removed a regular neighborhood of  $\bigcup \mu_s(D^{\alpha-1})$  in  $X$  i. e. all  $\alpha - 1$  handles mod.  $X^0(X^1)$ , and at the end of the sliding we still have  $g(D^p) \times I$  simplicially transversal to  $\phi(D^p \times I)$ .  $\square$

*Proof of Proposition 1.* First we assume  $n \geq 6$ . By Lemma 4 and Lemma 6 we inductively obtain an isotopy  $F_t$  of  $V \times I$  so that  $F_t|_{\partial V \times I}$  is a product isotopy and  $F_1(\phi(D^p \times I)) \cap g(D^p) \times I = \text{resulting } X$  is a product,  $X^0 \times I$ , and what is left is to find the concordance  $H : V \times I \rightarrow V \times I$ . In the following we will use the notation  $B_0 = D(D^p) \times I$ ,  $B_1 = F_1(\phi(D^p \times I))$ ,  $B = B_0 \cup B_1$ ,  $X = B_0 \cap B_1$ ,

$$A_0 = \Phi(D^p \times B^{n-p} \times 0) \times I,$$

$$A_1 = F_1\Phi(D^p \times B^{n-p} \times I)$$

where  $\Phi$  is the extension of  $\phi$  given by Lemma 3. Since  $F_1 \circ \phi|_{F_1 \circ \phi}^{-1}(\text{int } A_0)$  is PL we can find a regular PL neighborhood  $N$  of  $X$  in  $A_0$  such that

- a)  $Z_i = N \cap B_i$  is a regular neighborhood of  $X$  in  $B_i$ ,  $i = 0, 1$ .
- b)  $N \subset \text{int}(A_0 \cap A_1)$ .
- c)  $N \cap \partial V \times I = (N^0 \cap \partial V \times 0) \times I \subset \partial V \times I$ .
- d)  $N$  is PL homeomorphic to  $N^0 \times I$  via a homeomorphism  $h$  that restricts to a homeomorphism  $(h|_{Z_i}) : Z_i \rightarrow Z_i^0 \times I$  and is the identity on  $N \cap \partial V \times I = (N^0 \cap \partial V \times 0) \times I$ .

To see this we triangulate a neighborhood  $U$  of  $X$  in  $A_0$  such that  $B \cap U$  and  $X$  and boundaries are triangulated as full subcomplexes. We then take  $N$  to be a derived neighborhood of  $X$ , so a) is fulfilled, and so is b) if we take  $N$  small enough. To obtain c) we just need to arrange that the triangulation on  $A_0 \cap \partial V \times I = (A_0^0 \cap \partial V \times 0) \times I$  is the product triangulation. To find  $h : N \rightarrow N^0 \times I$  we note that  $X$  is PL homeomorphic to  $X^0 \times I$  by a homeomorphism which is the identity on  $dX = X \cap \partial V \times I$ . We denote this homeomorphism by  $h$ .  $h$  can by Hudson's theorem [9] and uniqueness of regular neighborhoods, be extended to a product structure on  $Z_i$  agreeing with the identity on  $Z_i \cap \partial V \times I$ . Repeating this argument  $h$  can be extended to a product structure on  $N$  agreeing with the identity on  $(N \cap \partial V \times 0) \times I$ , so d) is proved.

We next consider

$$Y_i = \overline{B_i - Z_i}, \quad i = 0, 1.$$

$Y_i$  is contained in  $C_i = \overline{A_i - N}$ .

We claim:

e)  $Y_i$  is PL homeomorphic to  $Y_i^0 \times I$  via a homeomorphism which agrees with  $h$  on  $Y_i \cap Z_i$  and with the identity on  $Y_i \cap \partial V \times I$ .

f)  $C_0$  is PL homeomorphic to  $C_0^0 \times I$  via a homeomorphism agreeing with  $h$  on  $C_0 \cap N$  and with the identity on

$$C_0 \cap \partial V \times I = (C_0^0 \cap \partial V \times 0) \times I.$$

g)  $C_1$  is homeomorphic to  $C_1^0 \times I$  via a homeomorphism  $k$  that agrees with  $h$  on  $C_1 \cap N$  and with the identity on  $C_1 \cap \partial V \times I = (C_1^0 \cap \partial V \times 0) \times I$ .

h)  $C_1^0$  can be given a PL structure such that there is an ambient isotopy  $l^t$  of  $C_1^0 \times I$  fixing  $\partial C_1^0 \times I \cup C_1^0 \times 0$  and such that

$$Y_1 \rightarrow C_1 \xrightarrow{k} C_1^0 \times I \xrightarrow{\mu} C_1^0 \times I$$

is PL.

e) follows from the same argument that proved that  $Z_i$  is homeomorphic to  $Z_i^0 \times I$ . Since that homeomorphism was realized by a PL ambient isotopy, the complement,  $Y_i$ , will also be a product.

f) follows from the PL  $s$ -cobordism theorem.  $N$  is PL embedded in  $A_0$  so  $C_0$  is a PL manifold.  $(C_0, C_0^0, C_0^1)$  is an  $s$ -cobordism since  $\pi_1(C_0^1) \cong \pi_1(C_0) \cong \pi_1(A_0)$ , by general position, and the Mayer-Vietoris sequence applied to the universal covering space shows that  $H_*(\tilde{C}_0^i) \cong H_*(\tilde{C}_0)$ ,  $i = 0, 1$ . The torsion is zero, since torsion is additive over unions. The  $s$ -cobordism is already given a product structure over  $\partial C_0^0$ , by gluing the product structures on the boundary of  $A_0$  and  $N$  together. We now extend this product structure to  $C_0$  using the PL  $s$ -cobordism theorem. To prove g) we proceed as follows:  $C_1^0 = \overline{A_a^0 - N^0}$  and  $N^0$  is a codimension 0 submanifold of  $A_1^0$ , meeting the boundary transversally. Therefore if we let  $Q = \overline{\partial N^0 - \partial V \times 0}$   $Q$  is a codimension 1 submanifold of  $C_1^0$  as well as of  $N^0$ . Taking a collar neighborhood in each we obtain an open neighborhood  $Q \times \mathbf{R} \subset A_1^0$  of  $Q$  in  $A_1^0$ . The induced PL structure on  $Q \times \mathbf{R}$  can be moved to a product structure near  $Q \times 0$  i. e. one can change the PL structure of  $A_1^0$  so that  $Q$  inherits a structure as PL submanifold, see [11] or [12]. (Note this is the only point in this paper where the handle straightening does not have a starting point on the boundary, so we have to straighten 0-handles, thus we use the Annulus conjecture and hence surgery theory.) This then give a PL structure to  $C_1^0$  since  $\partial C_1^0 \cap (\partial V \times I \cup \partial N)$  is a product we may extend the PL structure to  $C_1^0 \cup \partial C_1^0 \cap (\partial V \times I \cup \partial N)$  and then further to  $C_1$  using triangulation theory since  $C_1^0$  is a deformation retract of  $C_1$ , and we have ensured that the structure on  $\partial C_1^0 \cap (\partial V \times I \cup \partial N)$  is the product structure.  $Y_1^0$  is locally tamely embedded in  $C_1^0$  so by Theorem 2, we can move the PL structure of  $C_1^0$  to make  $Y_1^0$  a PL-manifold on the boundary.  $Y_1$  is PL-embedded since it is a product



embedding, so we can use Theorem 2 to find the desired isotopy  $l^t$  fixing  $C_1^0 \cup \partial C_1^0 \times I$ . We may now proceed as in g) to get h).

Using PL regular neighborhood theory and Lemma 3 we may find regular neighborhoods  $N_i$  of  $Y_i$  in  $C_i$  and homeomorphisms

$$h_i : N_i \rightarrow N_i^0 \times I$$

extending the homeomorphisms of  $Y_i$  to  $Y_i^0 \times I$  given by e), and so that  $h_i$  agrees with the identity on  $N_i \cap \partial V \times I$ , and with  $h$  on  $N \cap N_i$ . Further we choose  $N_i$  so small that  $N_0 \cap N_1 = \emptyset$ .  $N_0 \cup N \cup N_1$  is now a (topological) regular neighborhood of  $B$  in  $V \times I$ , so we can argue as before that  $D = V \times I - N_0 \cup N \cup N_1$  is a topological  $s$ -cobordism with a given product structure on the boundary.

Therefore the homeomorphism

$$h_0 \cup h \cup h_1 : N_0 \cup N \cup N_1 \rightarrow (N_0^0 \cup N^0 \cup N_1^0) \times I$$

extends by [18] to a homeomorphism.

$$H : V \times I \rightarrow V \times I$$

which is the identity on  $\partial V \times I \cup V \times 0$ , and since it is an extension of  $h_0 \cup h \cup h_1$  it satisfies  $H(B) = B^0 \times I$ .

For  $n = 5$  we note that  $X$  is at most 0-dimensional, so if  $\tau(X) > 0$   $X$  is empty. We can therefore find extensions  $\Phi_0$  and  $\Phi_1$  that are disjoint, so there are disjoint regular neighborhoods  $N_0$  and  $N_1$ , and we may proceed as above. □

We may now prove:

**Theorem 7.** *Suppose  $V^n$  is an  $n$ -dimensional topological manifold,  $n \geq 5$ , and let*

$$\phi : (D^p, \partial D^p) \times (I, 0, 1) \rightarrow (V, \partial V) \times (I, 0, 1)$$

*be a concordance such that  $\phi|_{\partial D^p \times I} = (\phi|_{\partial D^p \times 0}) \times 1_I$ . Denote  $\phi|_{D^p \times 0}$  by  $g$  and assume  $g(D^p) \subseteq V \times 0$  has a PL neighborhood  $U$ . Then there is a ambient isotopy with compact support*

$$K_t : V \times I \rightarrow V \times I \quad 0 \leq t \leq 1$$

*with  $K_t|_{\partial V \times I \cup V \times 0}$  the identity and*

$$K_1 \circ \phi = g \times 1_I$$

*Proof.* Since we have the same assumptions as in Proposition 1, the proof of that provides us with an ambient isotopy  $F_t$  and a concordance  $H$ . We denote  $F_t|_{V \times 0}$  by  $F_t^0$ , and as a first approximation we define

$$\begin{aligned} L_t &= F_{2t} & \text{for } 0 \leq t \leq \frac{1}{2}, \\ L_t &= H^{-1} \circ (F_{2-2t}^0 \times 1_I) \circ (F_1^0 \times 1_I)^{-1} \circ H \circ F_1 & \text{for } \frac{1}{2} \leq t \leq 1 \end{aligned}$$

then  $L_t$  is another solution to Proposition 1, so we do have

$$H(L_1(\phi(D^p \times I))) = L_1(g(D^p)) \times I$$

and

$$H(g(D^p \times I)) = g(D^p) \times I$$

but we now further have

$$L_1(g(D^p)) \times I = L_1^0(g(D^p)) \times I = g(D^p) \times I$$

and hence

$$L_1(\phi(D^p \times I)) = g(D^p) \times I$$

as images. However,  $L_t|V \times 0 \cup \partial V \times I$  is not the identity. We note, however, that  $L_t|V \times 0 \cup \partial V \times I$  is  $F_{2t}$ , for  $t \in [0, \frac{1}{2}]$  and  $F_{2t-1}$  for  $t \in [\frac{1}{2}, 1]$ . We first modify  $L_t$  to get  $T_t$  with  $T_0 = L_0$ ,  $T_1 = L_1$  and  $T_t|_{\partial V \times I}$  the identity as follows:

Let  $\partial V \times [0, 2] \subset V$  be a collar of the boundary [2]. and denote

$$V(t) = V - \partial V \times [0, t] \subset V$$

and

$$k^t : V \rightarrow V(t)$$

the homeomorphism defined to be the identity for  $x \notin \partial V \times [0, 2]$  and

$$k^t(y, s) = (y, t + \frac{1}{2}(2-t)s) \quad \text{for } (y, s) \in \partial V \times [0, 2].$$

We define  $T_t$  by

$$T_t(x) = \begin{cases} k^{1-|1-t|} \times 1_I \circ L_t \circ (k^{1-|1-t|})^{-1} \times 1_I & \text{for } x \in V(1 - |1-t|) \\ (L_s^0(y), r) & \text{for } x = (y, s, r) \in \partial V \times [0, 1 - |1-t|] \times I. \end{cases}$$

In a similar fashion we modify  $T_t$  to get  $K_t$  with  $K_0 = L_0$ ,  $K_1 = L_1$  but now  $K_t$  also fixes  $V \times 0$ . Specifically we let  $\rho_s : [0, 1] \rightarrow [s, 1]$  be defined by  $\rho_s(t) = s + (1-s)t$  and define

$$K_t(x) = \begin{cases} (1_v \times \rho_{\frac{1}{2}-|\frac{1}{2}-t|} \circ T_t \circ 1_v \times \rho_{\frac{1}{2}-|\frac{1}{2}-t|}^{-1})(x) & \text{for } x \in V \times [\frac{1}{2} - |\frac{1}{2} - t|, 1] \\ (T_s^0(y), s) & \text{for } x = (y, s) \in V \times (0, \frac{1}{2} - |\frac{1}{2} - t|). \end{cases}$$

One easily checks that  $K_t|_{\partial V \times I \cup V \times 0}$  is the identity (since  $T_0^0 = 1_V$ ) and we thus have found an isotopy with the required properties, except we only have  $K_1\phi(D^p \times I) = g(D^p) \times I$  as sets, but this may now be straightened by an Alexander isotopy. We finally note that since  $F_t$  by construction has compact support so does  $K_t$ .  $\square$

We are now ready to state and prove our main theorem,

**Theorem 8.** *Suppose  $V^n$  is a paracompact topological manifold and  $(X, Y)$  is a compact locally triangulable pair of spaces, and suppose*

$$\phi : (X, Y) \times (I, 0, 1) \rightarrow (V, \partial V) \times I$$

*is locally tame concordance (that is  $\phi^{-1}(\partial V \times I) = Y \times I$  and  $\phi$  is locally PL with varying structures both in  $X$  and  $V$ ). Suppose  $f|Y \times I$  is a product concordance and  $\dim(X - Y) \leq n - 3$  and if  $n = 4$ , that  $\pi_1(V) = 0$ . Then there is an ambient isotopy  $H_t : V \times I \rightarrow V \times I$ ,  $t \in I$ , fixed on  $V \times 0 \cup \partial V \times I$  and having compact support, such that if we denote  $g = \phi|X \times 0$*

$$H_1 \phi = g \times 1_I.$$

We first consider the case where  $\dim(V) \geq 5$ .

The proof will be in a number of steps. First we straighten  $\phi$  near  $Y$ .

*Step 1.* There is an ambient isotopy  $H_t$  fixing  $V \times 0 \cup \partial V \times I$  such that  $H_1 \circ \phi$  is a product embedding in a neighborhood of  $Y$ .

*Proof.* Write  $Y = C_1 \cup \dots \cup C_n$  where each  $C_i$  is a closed subset of  $Y$  having an open neighborhood of  $U_i$  in  $V$  (we consider  $g$  an inclusion) such that both  $U_i$  and  $U_i \cap X$  have PL structures. By the taming theorem we may change the PL-structure of  $U_i$  such that  $(X, Y) \cap U_i$  is a subpolyhedron of  $U_i$ . We inductively suppose that  $f$  has been straightened

over some neighborhood  $V_{j-1}$  of  $\bigcup_{i=1}^{j-1} C_i$  in  $X$ . Let  $E$  be a compact PL neighborhood of  $C_j$  in

$\partial U_j \cap Y$  (where  $\partial U_j = U_j \cap \partial V$ ), and let  $F$  be a compact PL neighborhood of  $\left(\bigcup_{i=1}^{j-1} C_i\right) \cap E$

in  $\partial V_j \cap Y$ , both chosen so small that  $F \subset V_{j-1}$ . Let  $N$  be a PL regular neighborhood (which meets the boundary regularly) of  $F$  in  $U_j$  such that  $N \cap X \subset V_{j-1}$ . Let  $V_0 = V - \text{int}_V N$  and  $X_0 = X - \text{int}_V N$  and  $Y_0 = X_0 \cap \partial V_0$  where  $\text{int}_V$  denotes the interior of  $N$  with respect to  $V$ .

In a neighborhood of  $E_0 \times I$  we now have PL structures on both  $(X_0, Y_0)$  and on a neighborhood of  $H_0 \times I$  in  $V \times I$  so we may apply the taming theorem (Theorem 2) to change the situation to a PL situation and then proceed as in the PL-case, using the Haefliger trick (see [17, Lemma 7]) to straighten the concordance of  $X_0 \times I$  in  $V_0 \times I$  near  $E_0 \times I$  keeping fixed  $V_0 \times 0 \cup \partial V_0 \times I$ . We extend this straightening isotopy over  $N \times I$  via the identity and note that by making support of the straightening isotopy small enough we have not

disturbed a neighborhood of  $\bigcup_{i=1}^{j-1} C_i$ , thus by induction, since the induction step also proves

step 1 by putting  $C_0 = \emptyset$  we have finished step 1 of the proof of Theorem 8.

*Step 2.* Assumptions as in Theorem 8 and further that  $\phi$  is a product concordance over a neighborhood of some closed subset  $C$  of  $X$  with  $C \supset Y$ . Suppose  $D$  is a simplex in some triangulated open subset of  $X$  such that  $D \cap C = \partial D$ . Then there is an ambient isotopy

$F_t$ , fixed on  $V \times 0 \cup \partial V \times I$ , having compact support, such that  $F_1 f$  is a product over a neighborhood of  $C \cup D$  in  $X$ .

*Proof.* We can find a PL neighborhood  $U$  of  $\text{int } D$  in  $\text{int } V$  since  $D$  is locally flatly embedded in  $V$ . By taming  $X \cap U$  and possibly shrinking  $U$  we may assume that  $X \cap U$  is a subpolyhedron of  $U$ . Let  $V$  be a neighborhood of  $C$  in  $X$  over which  $\phi$  is straight. Let  $D_0 \subset \text{int } D$  be a slightly smaller PL copy of  $D$  such that  $D - V \subset \text{int } D$  and let  $N$  be a PL regular neighborhood of  $\partial D_0$  in  $U$ , such that  $N$  intersects  $X$  and  $D_0$  in regular neighborhoods and  $N \cap X \subset V$ .

Define  $B = D_0 - \text{int } N$  which is a ball, and  $V_0 = V - (C \cup \text{int } N)$ . Since  $B$  does have a PL neighborhood in  $V_0$ , we may straighten  $\phi|B \times I$  in  $V_0$ , by an ambient isotopy with compact support and then apply the taming theorem and the Haefliger trick [17, Lemma 7] to straighten  $\phi$  over a neighborhood of  $B$  in  $X - \text{int } N$ . We may now extend this isotopy to all of  $V$  by the identity, since it fixes the boundary and has compact support, so we have now straightened  $\phi$  over a neighborhood of  $C \cup D$  in  $X$ .

*Step 3.* Assume that  $\phi$  is straight over a neighborhood of  $C$ ,  $C_0$  some closed subset of  $X$  containing  $Y$ . This we may assume by step 1. Write  $\overline{X - C_0}$  as a union of closed subsets  $C_1, \dots, C_n$  such that each  $C_i$  has a neighborhood  $U_i$  in the interior of  $V$  with a PL structure in which  $U_i \cap X$  is a subpolyhedron (we consider  $g$  an inclusion and use the taming theorem). By induction we may assume that  $f$  has been straightened over a neighborhood of  $V_{j-1}$  of

$\bigcup_{i=0}^{j-1} C_i$  in  $X$ . Let  $E$  be a compact PL neighborhood of  $C_j$  in  $U_j \cap X$  and let  $F$  be a compact PL neighborhood of  $(\bigcup_{i=0}^{j-1} C_i) \cap E$  in  $U_j \cap X$ , both chosen so small that  $F \subset V_{j-1}$ . Define  $C = (\bigcup_{i=0}^{j-1} C_i) \cup F$  and  $G = E - F$ , so that  $C \cup G = \bigcup_{i=0}^j C_i$ . Now  $C \cap G$  is a subpolyhedron of  $G$  and  $\phi$  is straight over a neighborhood of  $C$  so step 2 can be applied repeatedly to the simplices  $\{D\}$  of  $G - C$ , in order of increasing dimension, for some triangulation of  $(G, G \cap C)$  to ultimately straighten  $\phi$  on a neighborhood of  $C \cup G = \bigcup_{i=0}^j C_i$ . This completes the proof for  $n \geq 5$ . We now consider the remaining cases. For  $n < 3$  there is nothing to prove.

For  $n = 3$  we may use the theorem of Homma-Gluck that homotopy (and hence concordance) implies isotopy for 1-complexes in 4-manifolds see [7] and [6]. For  $n = 4$  and  $V$  1-connected we use the following trick due to Morlet [14]

First consider

$$\phi : D^0 \times I \rightarrow V^4 \times I$$

as in the proof of Lemma 4 we extend  $\phi$  to an embedding

$$G : D^1 \times I \rightarrow V^4 \times I$$

so that  $G|_{\partial D^1 \times I}$  is the disjoint union of  $\phi$  and a trivial concordance. It is easy to find a PL neighborhood of  $G(D^1 \times I)$  (as in Lemma 4) and having that it is easy to isotope  $\phi$  to a product embedding, not fixing  $V \times 0$ , but that isotopy may be changed to fix  $V \times 0$  as in

proof of Proposition 1. Finally consider

$$\phi : D^1 \times I \rightarrow V^4 \times I$$

$\phi|_{D^1 \times 1}$  is homotopic hence by [6] isotopic to  $\phi|_{D^1 \times 0}$ . So we may change  $\phi$  by an isotopy to make  $\phi|_{D^1 \times 1} = \phi|_{D^1 \times 0}$ . We may further, since the isotopy given by [6] is in the homotopy class of the given homotopy relative endpoints, assume that  $\phi$  is homotopic to  $g \times 1_I$  relative to  $\phi|_{\partial(D^1 \times I)} = g \times 1_I|_{\partial(D^1 \times I)}$ . This homotopy may be realized as an embedding

$$\Phi : D^1 \times I \times I \rightarrow V^4 \times I \times I$$

since we may first general position  $\Phi$  patch by patch and finally delete the 0-dimensional double points using the standard Whitney trick and the fact that  $V$  is 1-connected. We thus have  $\phi$  concordant to  $g \times 1_I$  and hence isotopic relative to  $V \times 0 \cup \partial V \times I$ , to  $g \times 1_I$  by the 5 dimensional case.

*Remark.* One could get rid of the 1-connected assumption if one could prove that it is possible to isotop  $\phi$  so that the following map is an embedding

$$\phi : D^1 \times I \rightarrow V \times I \rightarrow V.$$

If this composition is in general position this may be done using sunny collapsing ala the PL proof due to Hudson. The difficulty seems to be to isotop  $\phi$  so that the composition is in general position since at least the naive approach requires fibrewise PL approximation which is only known in codimension  $\geq 4$ . Of course it may be interesting to notice that what goes wrong in the present line of proof is only the final application of the  $s$ -cobordism theorem since it is easy to isotop  $\phi : D^p \times V \times I$  off  $(\phi|_{D^p \times 0}) \times 1_I$ , by a patch by patch argument.

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