# TOPOLOGICAL $H_0 \times H_1$-ACTIONS ON SPHERES AND LINKING NUMBERS

ERIK KJÆR PEDERSEN

Following tom Dieck and Löffler [4] we consider the following situation:

A: *Let $G = H_0 \times H_1$ be a product of two finite groups acting orientably on the standard sphere $X = S^{n(0)+n(1)+1}$ with the following properties:*
   i) *The isotropy subgroups are 1, $H_0$ and $H_1$.*
   ii) *The fixed point set $X^{H_i}$ is a locally flatly embedded manifold homeomorphic to an $n(i)$-dimensional sphere.*

*We denote the linking number $X^{H_0}$ with $X^{H_1}$ by $k$. Note that it follows from* [5] *that $(X - X^{H_0} - X^{H_1})/G$ is finitely dominated. We denote the finiteness obstruction by $\sigma$*

Obviously $H_0$ and $H_1$ at least have to be periodic groups for situation A to have any chance to arise. That however is not our concern here. In [4] it is shown that for $H_i$ odd cyclic groups , the only obstruction to realize situation A smoothly is the finiteness obstruction $\sigma$, which of course must be 0 in the smooth of PL case. This finiteness obstruction in turn, is identified with the Swan homomorphism applied to $k$, see [4]

In the topological case however, there is no a priori reason that $\sigma$ should be 0. Also one does not have the same immediate identification of the finiteness obstruction with the Swan homomorphism applied to the linking number. The purpose of this note is to discuss these questions.

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With assumptions as in A we prove the following:

**Theorem 1.** *If $K_{-1}Z[H_0] = K_{-1}Z[H_1] = 0$ then the finiteness obstruction $\sigma$ lies in $j_{0*}\widetilde{K}_0Z[H_0] \oplus j_{1*}\widetilde{K}_0Z[H_1]$, where $j_i$ is the natural inclusion of $H_i$ in $G$*

**Theorem 2.** *If $H_0$ and $H_1$ are nilpotent groups, then $\sigma \in D(Z[G])$.*

**Theorem 3.** *If $\sigma = 0$, then $\sigma$ may be identified with the Swan homomorphism applied to the linking number.*

*Remark.* We do not think the assumption in Theorem 2 that $H_0$ and $H_1$ are nilpotent is necessary. This is the subject of further work.

**Main application.** If $H_0 = H_1 = C_p$, a cyclic group of odd prime order, then $\sigma = 0$ and hence by theorem 3 must be the Swan homomorphism applied to $k$, and that must be 0. This however is exactly the condition that ensures the existence of smooth actions [4], so one may realize no more linking numbers topologically than smoothly in this situation.

*Proof.* Let $p_i$ $i = 0, 1$ denote the projection of $G = H_0 \times H_1$ on $H_i$. Then $p_i$ sends $D(C_p \times C_p)$ to $D(C_p)$ which is 0. Since $K_{-1}(C_p) = 0$ [2], by theorem 2, $\sigma \in D(C_p \times C_p)$, hence $p_{i*}(\sigma) = 0$. By Theorem 1 we know that $\sigma$ can be written in the form $\sigma = j_{0*}(\sigma_0) + j_{1*}(\sigma_1)$.

Applying $p_{i*}$ to this equation it follows that $\sigma_0 = 0$ and $\sigma_1 = 0$ hence $\sigma = 0$. We may thus apply theorem 3 to conclude that the Swan homomorphism of $k$ must vanish. □

We now turn to the proofs of theorems 1,2 and 3:

*proof of theorem 1 and 3.* In the terminology of Quinn [8, 9], the action considered is ANR, that is, the fixed point sets of the various subgroups are ANR's. Hence the mapping cylinder obstruction theory of Quinn applies. We may thus try to construct equivariant mapping cylinder neighborhoods of the fixed sets $X^{H_0}$ and $X^{H_1}$. Notice that if we succeed, then $(X - X^{H_0} - X^{H_1})/G$ has collared ends, so is homotopy equivalent to a compact manifold with boundary, which of course is homotopy equivalent to a finite complex.

We consider $(X - X^{H_0} - X^{H_1})/G$ an open manifold with two ends. One end is parameterized by $X^{H_0}$ and has $H_0$ as locally constant fundamental group, the other end by $X^{H_1}$ with $H_1$ as locally constant fundamental group. Since the argument is symmetric, consider the end parameterized by $X^{H_0}$.

To build an equivariant mapping cylinder neighborhood, we encounter obstructions in $H_i^{lf}(X^{H_0}/H_1; K_{-i}Z[H_0])$. When $i > 1$ the coefficients are 0 by Carter [3] and when $i = 1$ by assumption. We are left with the obstruction in $H_0^{lf}(X^{H_0}/H_1; \widetilde{K}_0 Z[H_0])$. If we however replace $X^{H_0}/H_1$ by $X^{H_0}/H_1 - *$ this group vanishes too. Therefore, if we consider the end restricted to $X^{H_0}/H_1 - *$ we encounter no obstruction to build an equivariant mapping cylinder neighborhood. In a neighborhood of $*$, the point we took out, we taper down the mapping cylinder neighborhood to the point (see [1] for details of this construction). It is now clear that $(X - X^{H_0} - X^{H_1})/G$ is homotopy equivalent to a manifold with boundary, and rather than an end parameterized by $X^{H_0}$, we have an end parameterized by a point and fundamental group $H_0$. Treating the other end similarly we find $(X - X^{H_0} - X^{H_1})/G$ homotopy equivalent to an open manifold with two pieces of boundary, and two tame ends, one with fundamental group $H_0$, the other with fundamental group $H_1$. By Siebenmann's sum formulae, the finiteness obstruction is the sum of the finiteness obstructions of the two ends, thus finishing the proof of theorem 1. To prove theorem 3 note that $j_{0*}\widetilde{K}_0 Z[H_0]$ and $j_{1*}\widetilde{K}_0 Z[H_1]$ intersect trivially in $\widetilde{K}_0(Z[H_0 \times H_1])$, so the vanishing of the finiteness obstruction implies that both end obstruction vanish. Thus we may indeed complete the program of building equivariant mapping cylinder neighborhoods of the two ends. Thus $(X - X^{H_0} - X^{H_1})$ is equivariantly homotopy equivalent to a compact manifold $W$ with two pieces of boundary, $\delta_+ W$ and $\delta_- W$, on which $G$ acts freely. The homology $H_*(W, \delta_- W)$ is by excision isomorphic to $H_*(X - X^{H_1}, X^{H_0})$ which is 0 in all dimensions except $* = n$, where it is $Z/kZ$, $k$ the linking number of $X^{H_0}$ and $X^{H_1}$. Since $k$ is relatively prime to $p$, the finiteness obstruction is equal to the Swan homomorphism applied to $k$. This ends the proof of theorem 3.

*proof of theorem 2.* It suffices to prove that $(X - X^{H_0} - X^{H_1})/G$ is a nilpotent space, since the finiteness obstruction then must lie in $N(G)$ (see [6] for definition) which by Mislin and Varadarajan [7], see also [6] is included in $D(Z[G])$.

By assumption $G$ is nilpotent, so we must check that $G$ acts nilpotently on the homology of $X - X^{H_0} - X^{H_1}$, or equivalently that it acts trivially on cohomology. The cohomology however is very simple. Algebra generators are detected by the inclusions $X - X^{H_0} - X^{H_1} \subseteq X - X^{H_0}$ and $X - X^{H_0} - X^{H_1} \subseteq X - X^{H_1}$. But in $X - X^{H_0}$ we have the invariant subspace $X^{H_1}$ on which both $H_0$ and $H_1$ act homologically trivial, so $G$ acts trivially on a multiple of the generator and thus on the generator itself. The other generator is treated similarly. This implies that $(X - X^{H_0} - X^{H_1})/G$ is nilpotent and we are done. □

## References


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Sonderforschungsbereich Geometrie und Analysis Matematisches Institut der Georg August Universit at Bunsenstrasse 3-5 and Matematisk Institut, Odense Universitet, Odense Danmark.