SMOOTHING H-SPACES II

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0. Introduction

In [8] we showed that certain H-spaces obtained by homotopy mixing are homotopy equivalent to smooth, parallellizable manifolds. Unfortunately (as was added in proof) we needed the restriction on the fundamental group π, that $D(\mathbf{Z}\pi) = 0$. It is the purpose of this sequel to [8] to remove this restriction, and also to generalize the main theorem considerably. I want to thank I. Hambleton for pointing out the error in [8].

1. Notation. Statement of results

Throughout the paper space will mean topological space of the homotopy type of a connected CW complex. For a set of primes l and a nilpotent space X, X_l denotes the localization at l in the sense of [5]. A space X is called quasifinite if $H_*(X) = \oplus H_i(X;\mathbf{Z})$ is a finitely generated abelian group. If $H_*(X)$ is a $\mathbf{Z}_l$-module X is called l-locally quasifinite if $H_*(X)$ is finitely generated as a $\mathbf{Z}_l$-module.

To state our main theorem we need a couple of definitions. Let S^i denote the i-sphere.

1.1. Definition. A nilpotent space X admits a special 1-torus if, up to homotopy, there is a diagram of orientable fibrations

$$\begin{array}{ccccc}
S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & X & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & A & \longrightarrow & A
\end{array}$$

such that

(a) A is quasifinite, B is stably reducible.
(b) Localized at 0 the diagram is homotopy equivalent to

$$\begin{array}{ccccc} S^1_0 & \longrightarrow & S^3_0 & \longrightarrow & S^2_0 \\ \downarrow & & \downarrow & & \downarrow \\ S^1_0 & \longrightarrow & A_0\times S^3_0 & \longrightarrow & A_0\times S^2_0 \\ \downarrow & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ * & \longrightarrow & A_0 & \longrightarrow & A_0 \end{array}$$

We remark that since X is nilpotent and the fibrations are orientable A and B are nilpotent so localization makes sense.

1.2. Example. *Given a bundle $S^3 \to X \to A$ with A quasifinite stably reducible and the bundle stably trivial then X admits a special 1-torus by dividing out the subgroup $S^1 \subset S^3$ (see* [8, lemma 3.6] *). All compact Lie groups other than $SO(3)^k \times T^l$ have subgroups isomorphic to S^3 as is seen by classification and all these Lie groups admit special 1-tori. This will be further discussed in section 5.*

We need a p-local version of definition 1.1. Let X be a nilpotent space, p a prime.

1.3. Definition. X admits a p-local special 1-torus if, up to homotopy, there is a diagram of orientable fibrations

$$\begin{array}{ccccc} S^1_p & \longrightarrow & S^3_p & \longrightarrow & S^2_p \\ \downarrow & & \downarrow & & \downarrow \\ S^1_p & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A & \longrightarrow & A \end{array}$$

such that

(a) A is p-locally quasifinite and B is p-locally stably reducible, i. e. there are integers n and i and a map $S^{n+1}_p \to \Sigma^i B$ inducing isomorphism in homology in dimensions $\geq n+i$.
(b) as in definition 1.1

It is clear that if X admits a special 1-torus then X_p admits a p-local special 1-torus.

We prove the following

1.4. Theorem. *Let X be a quasifinite H-space. Assume for every prime p that X_p is homotopy equivalent to a product $C(p) \times D(p)$ and $C(p)$ admits a p-local special 1-torus. Then X is homotopy equivalent to a smooth, stably parallellizable manifold.*

1.5. Remark. If $H_3(X) \supset \mathbf{Z}$ the condition of the theorem is trivially satisfied for all but finitely many primes. This is because for all but finitely many primes X_p is homotopy equivalent to a product of localized spheres which must include the 3-sphere. We also note that in view of example 1.2 (see section 5) this theorem is stronger than theorem 1.1 of [8].

2. Surgery

We use the special 1-tori for the following

2.1. Proposition. *Let X be a quasifinite, nilpotent Poincaré complex admitting a special 1-torus. Then X is homotopy equivalent to a stably parallellizable smooth manifold.*

Remark. This result only needs condition (a) of definition 1.1 Condition (b) is needed to ensure that the property of having a special 1-torus is a generic property.

Proof of proposition 2.1. In the diagram of orientable fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A & \longrightarrow & A \end{array}$$

A is nilpotent and quasifinite, hence by [7] A is finitely dominated. It follows that X and B are finitely dominated [6]. Also A and B are Poincaré Duality spaces since X is [3]. It follows from [10] that X has 0 finiteness obstruction. Considering (B, X) a Poincaré Duality pair, we may use the stable reduction of B and a standard transversality procedure to produce a surgery problem

$$(M, \partial M) \xrightarrow{\phi} (B, X) \qquad \hat{\phi} : \nu_M \to \varepsilon$$

where ε is the trivial bundle. Let $\sigma(B)$ be the finiteness obstruction of B. Consider the exact sequence

$$\ldots \to H^{n+1}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}\pi)) \to L_n^h(\pi) \xrightarrow{\delta} L_n^p \to$$

where $\pi = \pi_1(B) = \pi_1(X)$. The class of $\sigma(B)$, $\{\sigma(B)\}$, is an element of $H^{n+1}(\mathbf{Z}_2, \widetilde{K}_0(\mathbf{Z}\pi))$. It follows from [9], that the surgery obstruction of $\partial M \to X$ is $\delta\{\sigma(B)\}$. However, since A is a P. D, space of dimension $n-3$ we have $\sigma(A) = (-1)^{n-3}\sigma(A)^*$ and by [10], $\sigma(B) = 2\sigma(A) = \sigma(A) + (-1)^{n+1}\sigma(A)^*$ and hence $\{\sigma(B)\} = 0$ in $H^{n+1}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}\pi))$ and we are done.

3. Reducibility of H-spaces.

Browder and Spanier have shown that a finite H-space is stably reducible [2]. This is one of the steps in the attempt to prove X is a manifold, since it implies that the Spivak normal fibre space is trivial. We need to generalize the results of Browder and Spanier to a p-local situation. This is mostly straightforward. We shall nevertheless indicate the line of argument in this section. The aim of this section is to prove:

3.1. THEOREM. *Let D be a p-locally quasi finite H-space. Then D is p-locally stably reducible.*

We need a p-local edition of S-duality.

3.2. PROPOSITION. *Let X be a simply connected p-locally quasifinite space. Then X admits a p-local* CW*-structure, i. e. X is homotopy equivalent to a space Y with a filtration $* = K_0 \subset K_1 \subset \ldots \subset K_n = Y$ such that K_i is the mapping cone of some map $f_i : S_p^{n(i)} \to K_{i-1}$, $n(i)$ a nondecreasing function of i.*

Proof. By the Hurewicz theorem we may find a finite wedge of local spheres and a map $f \bigvee S_p^k \to X$ such that $H_*(f)$ is onto in dimensions $\leq k$, $k \leq 2$. Using the relative Hurewicz theorem we inductively attach local cells to make $H_*(f)$ an isomorphism in higher dimensions. Since X is p-locally quasifinite and finitely generated $\mathbf{Z}_p$-module have free resolutions of length one, we eventually obtain a homotopy equivalence.

Let X and Y be p-locally quasifinite spaces. A p-local S-duality map is a map $X \wedge Y \to S_p^n$ so that slant product

$$f^*(i)/- : \widetilde{H}_*(X) \to \widetilde{H}^{n-*}(Y)$$

is an isomorphism. Here i is the generator of $H^n(S_p^n)$.

Given a p-locally finite space X we note that the suspension ΣX is a simply connected p-locally quasifinite space and thus admits a p-local CW structure by Proposition 3.2. We may now go through exercises F1-7 page 463 in Spanier [11] to prove existence and stable uniqueness of a p-local S-dual with the usual functorial properties. We need the concept to complete the

Proof of Theorem 3.1. $H^*(D;\mathbf{Q})$ and $H^*(D;\mathbf{Z}/p\mathbf{Z})$ are Hopf algebras and we may argue as in the finite case [1] that D satisfies Poincaré Duality with $\mathbf{Z}_p$ coefficients. We now only need to produce a map $D \to S_p^n$ inducing isomorphisms in dimensions $\geq n$. Then we may use Hopf algebra arguments (as in the finite case [2]) to prove that the composite $D^+ \wedge D^+ = (D \times D)^+ \to D \to S_p^n$ is a p-local S-duality map, so D^+ is selfdual and the dual of $D^+ \to S_p^n$ will be a stable reduction.

Localized at 0 D is a product of odd dimensional spheres so if we let l be the set of primes different from p and form the homotopy pullback

$$\begin{array}{ccc} Y & \longrightarrow & \prod S_l^{2n_i+1} \\ \downarrow & & \downarrow \\ D & \longrightarrow & \prod S_0^{2n_i+1} \end{array}$$

then Y is quasifinite, nilpotent and satisfies Poincaré Duality at all primes hence [5] and [7] is a finitely dominated Poincaré Duality space. By Wall [12] Y has the homotopy type of $K \cup e^n$ where K is $n-1$-dimensional and we may thus produce a map $Y \to K \cup e^n \to S^n$ by collapsing K to a point. Localizing at p we obtain $D \cong K_p \to S_p^n$ with the required property and we are done

4. Proof of main Theorem.

The proof will consist of two lemmas.

4.1. Lemma. *If X is a quasifinite H-space and $X_p = C(p) \times D(p)$ where $C(p)$ admits a p-local special 1-torus, then X_p does.*

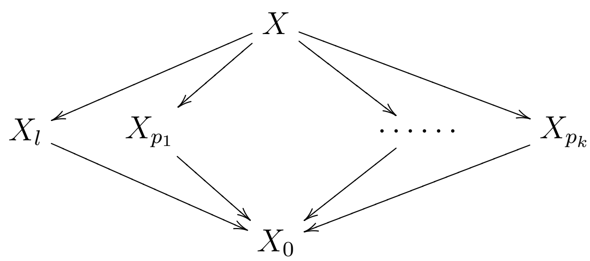
Proof. Crossing the special 1-torus diagram

$$\begin{array}{ccc} C(p) & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array}$$

with $D(p)$ reduces the lemma to showing $B \times D(p)$ is p-locally stably reducible. Now $D(p)$ is a retract of a p-locally quasifinite H-space and is thus itself a p-locally quasifinite H-space and thus p-locally stably reducible by theorem 3.1

4.2. Lemma. *If X is a quasifinite H-space such that each X_p admits a p-local special 1-torus, then X admits a special 1-torus.*

Proof. At all but finitely many primes X is a product of spheres, so we may consider X a homotopy pullback



where X_l is a product of odd dimensional spheres and X_{p_i} also admit special 1-tori. Mixing the special 1-tori in the obvious way we obtain $\overline{X}$ which admits a special 1-torus and such that $\overline{X}_l \cong X_l$ and $\overline{X}_{p_i} \cong X_{p_i}$; in other words $\overline{X}$ is in the genus of X. We now argue as in [8, proposition 3.2] to show that admitting a special 1-torus is a generic property for an H-space. The key step is the result of Zabrodsky that one obtains the whole genus of an H-space by mixings defined by diagonal matrices and the observation in [8] that one of these diagonal entries may be assumed to be 1.

5. Examples.

In this section we show that compact Lie groups other than $SO(3)^k \times T^l$ do admit special 1-tori. This implies that our theorem 1.4 is indeed stronger than theorem 1.1 of [8].

5.1. Proposition. *Let G be a compact connected Lie group which is not isomorphic to $SO(3)^k \times T^l$. Then G has a subgroup isomorphic to S^3.*

Proof. We use classification of compact Lie groups. Any compact connected Lie group is a quotient of $H \times T^l$ by a discrete central subgroup A. Here H is a simply connected compact Lie group. Furthermore H is a product of groups in a list, see [4, p. 346]. If we can find an S^3 subgroup of H that intersects A trivially we are done. There are two cases. First assume $H = (S^3)^k$. Then A can not contain the center of $(S^3)^k$ since if it does G will be isomorphic to $SO(3)^k \times T^l$. This being the case it is easy to find a subgroup isomorphic to S^3 not intersecting A. If H is not a product of S^3's it has a simple factor different from S^3 and we will be done if we can find a subgroup isomorphic to S^3 in this factor, intersecting the center trivially. We do this by checking the list. We have $S^3 = SU(2) \subset SU(n) \quad (n \leq 3)$ intersecting the center trivially since the central element of $SU(n)$ are the diagonal matrices with the same n'th root of unity as entry. Similarly $S^3 = SP(1) \subset SP(n) \quad (n \geq 2)$ and $S^3 = SU(2) \subset SO(4) \subset SO(n), \quad (n \geq 5)$ do not contain $-I$ which is the only central element $\neq I$. Furthermore $E_8 \supset E_7 \supset E_6 \supset SU(6) \supset S^3$ and since the center of E_6 has order 3, S^3 must intersect it trivially and E_6 must intersect the center of E_7 (cyclic of order 2) trivially. Finally $F_4 \supset SP(3)$ and $G_2 \supset SU(2)$ and these groups have trivial center. We are done.

5.2. Remark. It would be nice to have a conceptual proof of Proposition 5.1. Working in the Lie algebra it is not hard to find a subgroup isomorphic to $SO(3)$ or S^3 but it is crucial for us to be in the latter case.

5.3. Proposition. *Let G be a compact Lie group with S^3 as a subgroup $G \supset S^3$ Then*

$$\begin{array}{ccccc}
S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & G & \longrightarrow & G/S^1 \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & G/S^3 & \longrightarrow & G/S^3
\end{array}$$

is a special 1-torus in G.

Proof. Lemma 3.4 of [8] shows that G/S^1 is stably parallellizable. It follows from [8, lemma 3.3] that $H^3(G; \mathbf{Q}) \to H^3(S^3; \mathbf{Q})$ is onto. Let $G_0 \to K(\mathbf{Q}, 3) = S^3_0$ represent an element in $H^3(G; \mathbf{Q})$ hitting the generator of $H^3(S^3; \mathbf{Q})$ then one sees by a spectral sequence argument that $G_0 \to (G/S^3)_0 \times S^3_0$ is a homology equivalence hence a homotopy equivalence and we are done.

Final Remarks. In case $D(\mathbf{Z}\pi) = 0$ we could replace the concept special 1-torus by the concept 1-torus (see [8]). Since admitting a 1-torus is a weaker condition than admitting a special 1-torus, it is not entirely a loss to have both concepts.

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