UNIVERSAL GEOMETRIC EXAMPLES FOR TRANSFER MAPS IN ALGEBRAIC K AND L-THEORY

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0. Introduction

In this paper we will be concerned with various geometrically defined transfer maps in algebraic K-theory. In [3], Ehrlich shows that given a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ with finitely dominated fiber and base, there is a homomorphism $p^* : \widetilde{K}_0(\mathbf{Z}\pi_1 B) \to \widetilde{K}_0(\mathbf{Z}\pi_1 E)$ which is related to the finiteness obstruction of E, $\sigma(E)$ by the formula: $\sigma(E) = p^*(\sigma(B)) + i_*(\sigma(F) \cdot \chi(B)$, where $\chi(B)$ denotes the Euler characteristic of B.

Similarly, given a PL bundle Anderson [2] defines a homomorphism $p^* : \widetilde{\mathrm{Wh}}(\mathbf{Z}\pi_1 B) \to \widetilde{\mathrm{Wh}}(\mathbf{Z}\pi_1 E)$ which, given a fiber homotopy equivalence

$$\begin{array}{ccc} F' & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{h} & B \end{array}$$

relates the torsion of the maps by the analogous formula: $\tau(g) = p^*(\tau(h)) + i_*(\tau(f)) \cdot \chi(B)$.

A final example to have in mind is the transfer like homomorphism in Wall's L-groups coming from bundles: Given a topological bundle $F \to E \to B$ with F a topological manifold, there is a transfer homomorphism $L_i(\mathbf{Z}\pi_1 B) \to L_{i+\dim M}(\mathbf{Z}\pi_1 E)$ which relates surgery obstructions of base and total space in a bundle, see [11]. In case of S^1-bundles Wall shows [11, p. 123] that this transfer only depends on the fundamental groups in the bundle and the orientation, in particular it does not depend on π_2 of the base. His method is to construct a universal S^1-bundle with given fundamental group data.

In this paper we generalize this result to arbitrary fibrations (bundles) $F \to E \to B$. We construct universal fibrations with given fundamental group data (see Section 1 for precise definition) such that a given fibration is the pullback of this universal fibration via a map inducing isomorphisms of fundamental groups. In course of the construction we determine which fundamental group sequences are possible for a fibration with fiber F, in terms of

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$G_1(F)$, a subgroup of $\pi_1(F)$ (see [4]). We apply the construction to show the following theorem.

Theorem. *The geometrically defined* $\widetilde{K}_0$*,* Wh *and L-group transfer maps of a fibration, respectively* PL*-bundle, respectively topological bundle* $F \to E \to B$*, only depend on* F *and the exact sequence* $\pi_1 F \to \pi_1 E \to \pi_1 B$*, and the orientation, that is* $\pi_1(E) \to \pi_0(H(F,*))$ *(*$H(F,*)$*= basepoint preserving homotopy equivalences) in the* $\widetilde{K}_0$*-case,* $\pi_1(E) \to \pi_0(\mathrm{PL}(F,*))$ *in the* Wh*-case and* $\pi_1(E) \to \pi_0(\mathrm{Homeo}(F,*))$ *in the L-group case.*

Remark. The $\widetilde{K}_0$-case was proved by Ehrlich in [3] for fibrations with $\pi_1(B) \to \pi_0(H(F))$ trivial, and conjectures in general ($H(F)$ = unbased homotopy equivalences of F).

The results facilitate computation of the transfer maps in a number of interesting cases. We exemplify this by giving an algebraic description of the $\widetilde{K}_0$ and Wh-transfer of S^1-fibrations, whenever the extension $\mathbf{Z}/n \to \pi_1 E \to \pi_1 B$ determined by the fibration is the reduction mod n of some integral extension $\mathbf{Z} \to \rho \to \pi_1 B$. In this algebraic description we use results of [8] and [9].

1. General notation

Let F be a fixed connected space of the homotopy type of a countable CW-complex. We will be concerned with fibrations with fibre F. Such fibrations have a universal classifying fibration $F \to EF \to BF$ see [1]. We shall also be concerned with fibrations with additional structure, e. g. topological bundles or PL-bundles, but always of a kind that have classifying spaces. Using the above notation we may readily think of EF and BF as classifying spaces of some restricted kind of fibrations as well. However in the case of fibrations if we denote the monoid of homotopy equivalences of F by $H(F)$ and basepoint preserving homotopy equivalences $H(F,*)$, we may reinterpret EF and BF as $BH(F,*)$ and $BH(F)$ respectively. The boundary map in homotopy $\pi_i BF \to \pi_{i-1} F$ then corresponds to the map induced by evaluation at basepoint $\pi_{i-1}(H(F)) \to \pi_{i-1}(F)$. In [4], Gottlieb defines and studies the subgroup $G_1(F) \subset \pi_1(F)$ given by $G_1(F) = \mathrm{Im}(\pi_2 BF \to \pi_1 F)$. Among other things it is shown in [4] that $G_1(F)$ is contained in $C\pi_1 F$, the center of $\pi_1 F$. By the fundamental group data of a fibration $F \to Y \to X$ we will understand the exact sequence $0 \to A \to \pi_1 F \to \pi_1 Y \to \pi_1 X$ together with the homomorphism induced by classification $\pi_1 Y \to \pi_1 EF$ which since $\pi_1 EF = \pi_0(H(F,*))$ records the way $\pi_1 Y$ acts on the fiber and hence also the action of $\pi_1 X$ on the fiber. In the exact sequence above A denotes $\mathrm{Im}(\pi_2 X \to \pi_1 F)$.

Two fibrations $F \to Y_i \to X_i$, $i = 1, 2$, are said to have isomorphic fundamental group data if there are isomorphisms $\pi_1(Y_1) \cong \pi_1(Y_2)$ and $\pi_1(X_1) \cong \pi_1(X_2)$ making the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(Y_1) & \longrightarrow & \pi_1(X) & \longrightarrow & 1 \\ & & \| & & \| & & \cong & & \cong & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(Y_2) & \longrightarrow & \pi_1(X_2) & \longrightarrow & 1 \end{array}$$

and

$$\begin{array}{ccc} \pi_1(Y_1) & & \\ & \searrow & \\ \cong & & \pi_1(EF) \\ & \nearrow & \\ \pi_1(Y_2) & & \end{array}$$

commutative. The main aim of this paper is to construct universal fibrations with given fundamental group data. However, the methods are general obstruction theory, so if we think of $F \to EF \to BF$ as e. g. the universal topological bundle with fibre F we also obtain universal bundles with given fundamental group data, so even though, throughout the paper, we will use the language of fibrations, in the applications we shall need the results for fibrations, as well as PL and topological bundles.

Throughout the paper we will use $K(\pi, n)$ to denote an Eilenberg–MacLane space of type (π, n). The letter B in front of a space will denote the classifying space of fibrations, thus $BK(\pi, 1)$ will be the classifying space of fibrations with fiber $K(\pi, 1)$. If π is a group we use $\mathrm{Aut}(\pi)$ to denote the group of automorphisms of π and $\mathrm{Out}(\pi)$ the group of outer automorphisms.

2. The obstruction to group extensions reinterpreted

Given discrete groups G and π and a homomorphism $\pi \to \mathrm{Out}(G)$ there is an obstruction in $H^3(K(\pi, 1); CG)$ (local coefficients) to the existence of an exact sequence $1 \to G \to \rho \to \pi \to 1$ inducing the given action of π on G. If this obstruction vanishes such extensions are classified up to isomorphism by $H^2(K(\pi, 1); CG)$[5]. On the other hand the above exact sequence does imply there is a fibration $K(G, 1) \to K(\rho, 1) \to K(\pi, 1)$. We want to relate the extension obstruction to the problem of constructing such a fibration. This involves a study of fibrations with fibre $K(G, 1)$. Such a study is done in [7] in the case G is abelian. However, using the same kind of semisimplicial methods it is clear that we may take $BK(G, 1)$ to be the realization of the classifying space of the semisimplicial group of automorphisms of $K(G, 1)$ where $K(G, 1)$ is the standard semisimplicial model (essentially the bar construction). Similarly $EK(G, 1)$ is obtained from basepoint preserving automorphisms of $K(G, 1)$.

Proposition 2.1. *In the fibration $K(G, 1) \to EK(G, 1) \to BK(G, 1)$ we have $EK(G, 1) = B\,\mathrm{Aut}\, G$, $BK(G, 1)$ has only two nonzero homotopy groups,* $\mathrm{Out}\, G$ *and* CG.

Proof. We use the model for $K(G,1)$ with $K(G,1)_n = G \times G \times \cdots G$, n factors, and

$$\begin{aligned} \partial_0(x_1,\dots,x_n) &= (x_2,\dots,x_n), \\ \partial_i(x_1,\dots,x_n) &= (x_1,\dots,x_i x_{i+1},\dots,x_n), \\ \partial_n(x_1,\dots,x_n) &= (x_1,\dots,x_{n-1}, \end{aligned}$$

and degeneracies inserting 1's at appropriate places. Using [7] one sees that an n-simplex in the mapping space $\operatorname{Hom}(K(G,1),K(G,1))$ is a collection of maps $f_{ij} : G \to G$, $0 \le i \le j \le n$, where f_{ii} is a group homomorphism and f_{ij} is left f_{ii} and right f_{jj} equivariant, i. e. $f_{ij}(a \cdot b) = f_{ii}(a) \cdot f_{ij}(b) = f_{ij}(a) \cdot f_{jj}(b)$. We get the automorphisms by requiring all f_{ij} to be $1-1$ and onto. We have the relation of $f_{jj}(a) = f_{ij}(1)^{-1} \cdot f_{ii}(a) \cdot f_{ij}(1)$ from which it is seen that π_0 is automorphisms modulo inner automorphisms. It also shows that π_1 is the center of G since $f_{00} = f_{11} = 1_G$ implies that $f_{01}(1) \in CG$ and this determines π_1. The fixed point preserving automorphisms are the ones satisfying $f_{ij}(1) = 1$ for all $i \le j$ which shows that $f_{ii} = f_{ij} = f_{jj}$, so the fixed point preserving automorphisms are exactly $\operatorname{Aut} G$. The results now follow. □

Corollary 2.2. *There is up to homotopy a fibration $K(CG,2) \to BK(G,1) \to K(\operatorname{Out} G,1)$. Given a homomorphism $\pi \to \operatorname{Out} G$ we may identify the above mentioned obstruction to construct an exact sequence, with the obstruction to solve the lifting problem*

$$\begin{array}{ccc} & & BK(G,1) \\ & \nearrow & \downarrow \\ K(\pi,1) & \longrightarrow & K(\operatorname{Out} G,1) \end{array}$$

and if this obstruction vanishes, the set of liftings classified by $H^2(K(\pi,1);CG)$ corresponds to the classification of extensions.

Proof. Using the models for $HK(\pi,1)$ of the above proof, the considerations in [5, p. 124] translate directly to the lifting problem. □

3. Fibrations with given fundamental group data

Given a fibration $F \to Y \to X$ one immediately sees that $\operatorname{Im}(\pi_2 X \to \pi_1 F) = A$ is a $\pi_1 X$ invariant subgroup contained in $G_1 F$. If on the other hand we have a group π, a homomorphism $\pi \to \operatorname{Out}(\pi_1 F)$, and a subgroup A of $\pi_1 F$ invariant under this action, we may ask if there is a fibration realizing this data. The first condition that has to be fulfilled is that the map $\pi \to \operatorname{Out}(\pi_1 F)$ factors $\pi \to \pi_0(H(F)) \to \operatorname{Out} \pi_1 F$. If this is the case we choose such a factorization and fix it. We will now replace BF which has $\pi_1 BF = \pi_0(H(F))$ by some cover of BF with fundamental group $\operatorname{Im}(\pi \to \pi_0(H(F)))$. We pull back the universal fibration to this cover and get a universal fibration with some restriction on the way π_1 of the base is allowed to act on the fibre. We will still denote this by $F \to EF \to BF$.

To study the question of realizing fibrations with given properties of the fundamental groups we need to construct a space $BF(A)$ by the following procedure: Consider the canonical homomorphism $\pi_2 BF \to G_1 F/A$. The kernel is $\pi_1 BF$ invariant so we may attach 3-cells to kill this kernel. We proceed to attach higher dimensional cells to kill higher homotopy groups.

Lemma 3.1. *Up to canonical homotopy equivalence $BF(A)$ is universal with respect to the properties*

(a) $\pi_i(BF(A)) = 0$ *for* $i \geq 3$.
(b) $\pi_2(BF(A)) = G_1 F/A$.
(c) $\pi_1 BF(A)) = \pi_1 BF$.
(d) *There is a map $BF \to BF(A)$ inducing the canonical maps on π_1 and π_2.*

Proof. Clearly $BF(A)$ satisfies conditions (a) through (d). Let X be some space satisfying the conditions; then we have

$$\begin{array}{ccc} BF & \longrightarrow & BF(A) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

and using elementary obstruction theory there is a unique homotopy class of maps we can fill in to make the diagram homotopy commutative. This map must induce isomorphism in homotopy so it is a homotopy equivalence. We now construct a diagram

$$\begin{array}{ccccc} F & \longrightarrow & K(\pi_1 F/A, 1) & \longrightarrow & K(\pi_1 F/A.1) \\ \downarrow & & \downarrow & & \downarrow \\ EF & \longrightarrow & K(\pi_1 EF, 1) & \longrightarrow & EK(\pi_1 F/A, 1) \\ \downarrow & & \downarrow & & \downarrow \\ BF & \longrightarrow & BF(A) & \longrightarrow & BK(\pi_1 F/A, 1) \end{array} \qquad (*)$$

Here all vertical sequences are fibrations up to homotopy. The map $EF \to K(\pi_1 EF, 1)$ classifies the universal cover of EF. The map $K(\pi_1 EF, 1) \to BF(A)$ is seen to exist and to be unique up to homotopy to make the diagram commutative. That the homotopy fibre of $K(\pi_1 EF, 1) \to BF(A)$ is a $K(\pi_1 F/A, 1)$ follows from the exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & G_1 F/A & \longrightarrow & \pi_1 F/A & \longrightarrow & \pi_1 EF & \longrightarrow & \pi_1 BF & \longrightarrow & 1 \\ & & \| & & & & & & \| & & \\ & & \pi_2 BF(A) & & & & & & \pi_1 BF(A) & & \end{array}$$

Finally $BF(A) \to BK(\pi_1 F/A, 1)$ is the classifying map. We notice that the homotopy exact sequences may be thought of as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1F/A & \longrightarrow & \pi_1 F/A & \longrightarrow & \pi_0(H(F,*)) & \longrightarrow & \pi_0 H(F) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & C\pi_1 F/A & \longrightarrow & \pi_1 F/A & \longrightarrow & \mathrm{Aut}\,\pi_1 F/A & \longrightarrow & \mathrm{Out}(\pi_1 F/A) & \longrightarrow & 0 \end{array}$$

Here we identify $\pi_0(H(F))$ with $\pi_1 BF$ and $\pi_0(H(F,*))$ with $\pi_1 EF$ and remember the remarks in the beginning of this section that imply we do not have an induced homomorphism $\pi_0(H(F,*)) \to \mathrm{Aut}(\pi_1 F/A)$. □

From the diagram $(*)$ we may construct a new diagram

$$\begin{array}{ccc} K(G_1F/A, 2) & \longrightarrow & K(C\pi/A, 2) \\ \big\downarrow & & \big\downarrow \\ BF(A) & \longrightarrow & BK(\pi_1 F/A, 1) \\ \big\downarrow & & \big\downarrow \\ K(\pi_1(BF), 1) & \longrightarrow & K(\mathrm{Out}(\pi_1 F/A), 1) \end{array} \qquad (**)$$

Proposition 3.2. *Given a group π, a homomorphism $\pi \to \pi_1 BF$ and a subgroup $A \subset G_1(F)$ invariant under the induced action of π, there is an obstruction $p \in H^3(K(\pi,1); G_1F/A)$ (local coefficients) to the existence of a fibration $F \to Y \to X$ with $\pi_1 X \cong \pi$, $\mathrm{Im}(\pi_2 X \to \pi_1 F) = A$ and the action of $\pi_1 X$ on F given by $\pi_1 X \cong \pi \to \pi_1 BF$. The image of this obstruction in $H^3(K(\pi,1); C\pi_1 F/A)$ is the obstruction of [5] to realize an exact sequence $\pi_1 F/A \to \rho \to \pi$ with π acting on $\pi_1 F/A$ through the given $\pi \to \pi_1 BF \to \mathrm{Out}(\pi_1 F/A)$.*

Proof. The homomorphism $\pi \to \pi_1 BF$ induces a map $K(\pi,1) \to K(\pi_1 BF, 1)$. We define $p \in H^3(K(\pi,1); G_1F/A)$ to be the obstruction to lift this map to $BF(A)$ in the diagram $(**)$. The final remarks on the image of p in $H^3(K(\pi,1); \pi_1 F/A)$ now follow from the diagram $(**)$, naturality of obstructions and Section 2. If this obstruction vanishes we choose a lift l and construct a pullback diagram defining $BF(A,l)$:

$$\begin{array}{ccc} BF(A,l) & \longrightarrow & K(\pi,1) \\ \big\downarrow & & \big\downarrow \\ BF & \longrightarrow & BF(A) \end{array}$$

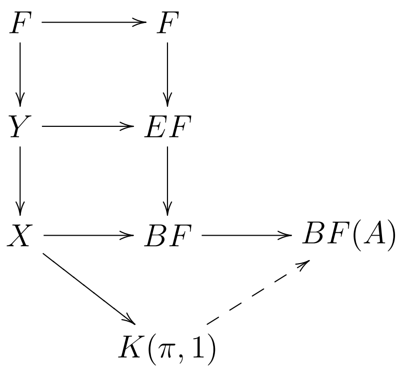
The pullback of the universal fibration $F \to EF \to BF$ to $BF(A,l)$ we denote $F \to EF(A,l) \to BF(A,l)$ and we claim this fibration satisfies the assumptions. To see this consider the homotopy fibre H of $K(\pi,1) \to BF(A,l)$. We have the exact sequence $0 \to$

$G_1F/A \to \pi_1 H \to \pi \to \pi_1 BF \to 1$ to compute the homotopy groups of H, so H is a $K(\pi_1 H, 1)$. This implies the homotopy fibre of $BF(A,l) \to BF$ is a $K(\pi_1 H, 1)$, and if we compare exact sequences

$$\begin{array}{ccccccccccccc}
0 & \to & \pi_2(BF(A,l)) & \to & \pi_2 BF & \to & \pi_1 H & \to & \pi_1 BF(A,l) & \to & \pi_1 BF & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow & & \| & & \\
0 & \to & 0 & \to & G_1F/A & \to & \pi_1 H & \to & \pi & \to & \pi_1 BF & \to & 1
\end{array}$$

we see, using some version of the five lemma, that $\pi_1(BF(A,l)) \cong \pi$ by an isomorphism making $\pi_1(BF(A,l)) \to \pi_1 BF$ the given map $\pi \to \pi_1 BF$, and further that the sequence $0 \to \pi_2(BF(A,l)) \to \pi_2 BF \to G_1F/A \to 0$ is exact. This implies that $\mathrm{Im}(\pi_2(BF(A,l)) \to \pi_1 F) = A$ and we have constructed a fibration with the sought properties.

On the other hand if such a fibration $F \to Y \to X$ does exist, we must prove that the obstruction vanishes. We have the following diagram



We may think of $K(\pi, 1)$ as X with 3-cells attached to kill $\pi_2(X)$ etc. and the map $X \to K(\pi, 1)$ as inclusion. By assumption, the image of $\pi_2(X)$ in $\pi_2(BF)$ is the kernel $\pi_2 BF \to G_1F/A$ so the map $\pi_2 X \to \pi_2(BF(A)) = G_1F/A$ is trivial so there is no obstruction to extend the map to the 3-skeleton. Since higher homotopy groups of $BF(A)$ vanish we meet no further obstructions. We also notice that the homotopy class of the extension $K(\pi, 1) \to BF(A)$ is uniques (using obstruction theory once again) and that it is indeed a lift of the given map $K(\pi, 1) \to K(\pi_1 BF, 1)$. □

Remark 3.3. *If we have a fibration $F \to Y \to X$ with fundamental group data $0 \to A \to \pi_1(F) \to \pi_1(Y) \to \pi_1 X \to 1$ and $\pi_1 Y \to \pi_1 EF$, it follows easily from the proof of proposition 3.2 that the exact sequences*

$$\pi_1 F/A \to \rho \to \pi_1 X$$

we may obtain from fibrations, with $\pi_1 X \to \mathrm{Out}(\pi_1 F/A)$ induced by $\pi_1 Y \to \pi_1 EF$, are given by the action of $H^2(K(\pi_1 X, 1); G_1F/A)$ on the exact sequence $\pi_1 F/A \to \pi_1(Y) \to \pi_1 X$ through the standard action of $H^2(K(\pi_1 X, 1); (\pi_1 F/A))$, see [5].

4. The universal fibrations

In the proof of Proposition 3.2 we constructed spaces $BF(A,l)$ as the pullback of the diagram

$$\begin{array}{ccc} BF(A,l) & \longrightarrow & K(\pi,1) \\ \downarrow & & \downarrow{\scriptstyle l} \\ BF & \longrightarrow & BF(A) \end{array}$$

where l is a lift of a given map $K(\pi,1) \to K(\pi_1 BF,1)$. We also saw that a fibration $F \to Y \to X$ with $A = \mathrm{Im}(\pi_2 X \to \pi_1 F)$ determines a unique lift $l : K(\pi_1 X,1) \to BF(A)$, and we thus get a pullback diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ Y & \longrightarrow & EF(A,l) \\ \downarrow & & \downarrow \\ X & \longrightarrow & BF(A,l) \end{array}$$

It is easily seen that the horizontal maps induce isomorphisms of fundamental group data. To see that $F \to EF(A,l) \to BF(A,l)$ may be considered a universal fibration with the given fundamental group data, it thus suffices to show that two fibrations with isomorphic fundamental group data give rise to the same lift $l : K(\pi,1) \to BF(A)$. Let us assume we have two fibrations with isomorphic fundamental; group data $F \to Y_i \to X_i$, $i = 1,2$. Denote the common fundamental group of X_i by π. We obtain diagrams of fibrations

$$\begin{array}{ccc} F & \longrightarrow & K(\pi_1 F/A,1) \\ \downarrow & & \downarrow \\ Y_i & \longrightarrow & K(\pi_1 Y_i,1) \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & K(\pi,1) \end{array}$$

and a pullback diagram

$$\begin{array}{ccccc}
K(\pi_1 F/A,1) & = & K(\pi_1 F/A,1) & = & K(\pi_1 F/A,1) \\
\downarrow & & \downarrow & & \downarrow \\
K(\pi_1 Y_i,1) & \longrightarrow & K(\pi_1 EF,1) & \longrightarrow & EK(\pi_1 F/A,1) \\
\downarrow & & \downarrow & & \downarrow \\
K(\pi,1) & \xrightarrow{l_i} & BF(A) & \longrightarrow & BK(\pi_1 F/A,1)
\end{array}$$

The composites $K(\pi,1) \xrightarrow{l_i} BF(A) \to BK(\pi_1 F/A,1)$, $i = 1,2$ are homotopic since we have isomorphic fundamental group sequences. We may thus think of l_1 and l_2 as lifts of the same map into $BK(\pi_1 F/A,1)$ covering given maps into $K(\pi_1 BF,1)$. The difference between l_1 and l_2 is by obstruction theory given by an element of $H^1(K(\pi,1); C\pi_1 F/G_1 F)$. However, the maps $K(\pi_1 Y_i,1) \to K(\pi_1 EF,1)$ are by assumption homotopic and since they are determined by the fundamental groups they are homotopic covering the map into $EK(\pi_1 F/A,1)$. We thus obtain that the difference between l_1 and l_2 in $H^1(K(\pi,1); C\pi_1 F/G_1 F)$ maps to 0 in $H^1(K(\pi_1 Y_i,1); C\pi_1 F/G_1 F)$. This map however is a monomorphism since $\pi_1 Y_i \to \pi$ is onto so l_1 is homotopic to l_2 and we have proved the following:

Theorem 4.1. *Suppose there exists a fibration $X \to Y$ with fibre F whose projection induces the homomorphism $f : A \to B$ of fundamental groups and $\phi : A \to \pi_1 EF$ describing the action of A on the fiber. Then there exists a universal fibration $P : Z \to W$ so that given any fibration $p : E \to M$ with fiber F and isomorphisms $h_1 : \pi_1 E \to A$, $h_2 : \pi_1(M) \to B$ with $fh_1 = h_2 \pi_1(p)$ and $\phi \cdot h_1$ the classifying map $\pi_1 E \to \pi_1 EF$, there exists a unique homotopy class of fibrewise maps of p to P inducing the stated isomorphisms.*

Remark. This theorem was originally proved in [11, p. 123] for the case S^1 and S^0 fibrations.

Remark. Completely analogous theorems hold if we replace the word fibration with topological bundle, PL-bundle or any other kind of bundle theory, the proof being verbatim the same. Of course in the case of e. g. topological bundles one replaces $\pi_0 H(F)$ by $\pi_0(\mathrm{Homeo}(F))$, $G_1(F)$ by $\mathrm{Im}(\pi_1(\mathrm{Homeo}\, F) \to \pi_1 F)$ etc.

5. Application to geometrically defined transfers

In this section we apply the results of Section 4 to various geometrically defined transfer maps discussed in the introduction. The transfer map of Wall groups is defined for any topological bundle with fibre a topological manifold [11], whereas the K_0 and Wh transfers are only defined for respectively fibrations and PL-bundles with finite base space. The first order of the day is to remedy this situation.

Proposition 5.1. *Given a Hurewicz fibration $F \to E \to B$ with F finitely dominated and $\pi_1 B$ finitely presented. There exists a (transfer) homomorphism $p^* : \widetilde{K}_0(\mathbf{Z}\pi_1 B) \to \widetilde{K}_0(\mathbf{Z}\pi_1 E)$ uniquely determined by the following properties:*

(1) *p^* is natural with respect to pullbacks.*
(2) *If B is finitely dominated, p^* agrees with the transfer map of* [3] *mentioned in the introduction.*

Proof. If B is finitely dominated we let p^* agree with Ehrlich's definition. If B is not finitely dominated, we may find a finite complex K and a map $K \to B$ inducing isomorphism on the fundamental group and such that $\pi_2 K \to \pi_2 B/\ker(\pi_2 B \to \pi_1 F)$ is onto. This may be done by first constructing K using a given presentation and then take a wedge product with finitely many 2-spheres. We obtain a pullback fibration

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ h^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ K & \longrightarrow & B \end{array}$$

and $h^*(E) \to E$ as well as $K \to B$ induce isomorphisms on π_1. We may now define p^* to be the composite

$$\widetilde{K}_0(\mathbf{Z}\pi_1 B) \xrightarrow{\cong} \widetilde{K}_0(\mathbf{Z}\pi_1 K) \xrightarrow{p_K^*} \widetilde{K}_0(\mathbf{Z}\pi_1 h^* E) \xrightarrow{\cong} \widetilde{K}_0(\mathbf{Z}\pi_1 E).$$

To see this is well defined let (K_i, h_i), $i = 1, 2$, be as above. We then form

$$\begin{array}{ccc} K_1 \vee K_2 & \longrightarrow & B \\ \downarrow & \nearrow & \\ X & & \end{array}$$

where X is obtained from $K_1 \vee K_2$ by attaching 2-cells to $K_1 \vee K_2$ to make generators of $\pi_1(K_1)$ equal to the corresponding elements in $\pi_1(K_2)$. This makes X satisfy the conditions above and by naturality of Ehrlich's transfer the transfers defined using (K_1, h_1) respectively (K_2, h_2) will coincide. Naturality is proved similarly using naturality of the transfer defined by Ehrlich. □

The corresponding extension of the Wh-transfer of arbitrary PL-bundles is proved entirely similarly once it is noticed that the transfer map defined by Anderson in [2] actually is natural with respect to pullback.

Applying the transfer maps to the universal fibrations (PL-bundles, Top-bundles) we obtain the following.

Theorem 5.2. *Given fibrations (*PL*-bundles,* Top*-bundles), the transfer maps in* $\widetilde{K}_0$ *(*Wh*,* L_**) discussed in the introduction only depend on the fundamental group data of the fibration (*PL*-bundle,* Top*-bundle).*

Remarks 5.3. *In the case of* S^1*-bundles the above theorem is proved for the* L*-groups in* [11] *using this method. In* [3] *Ehrlich proved the theorem for fibrations with* π_1 *of the basespace acting trivial on the fibre for the* $\widetilde{K}_0$*-transfer. This theorem is a new proof of that, but with no assumptions on the action of* π_1 *(as was conjectured in* [3]*).*

We obtain a number of corollaries of this theorem. We shall concentrate on applications to the $\widetilde{K}_0$-transfer and only mention the modifications needed in other cases.

Corollary 5.4. *Consider a fibration* $F \to X \to Y$ *with finitely dominated fibre* F *and fundamental group data*

$$\begin{array}{ccc} \pi_1 F/A & & \\ \downarrow & & \\ \rho & \longrightarrow & \pi_1 EF \\ \downarrow & & \\ \pi & & \end{array} \tag{*}$$

and let $B \subseteq G_1 F$ *be some* $\pi_1 BF$ *invariant subgroup of* $G_1 F$ *with* $A \subseteq B$*. Then there exists a fibration with fundamental group data*

$$\begin{array}{ccc} \pi_1 F/B & & \\ \downarrow & & \\ \rho/(B/A) & \longrightarrow & \pi_1 EF \\ \downarrow & & \\ \pi & & \end{array} \tag{**}$$

and the associated transfer map is $\widetilde{K}_0(\mathbf{Z}\pi) \to \widetilde{K}_0(\mathbf{Z}\rho) \to \widetilde{K}_0(\mathbf{Z}(\rho/(B/A)))$*.*

Proof. We may find a map from the wedge of S^2's $\vee_i S_i^2 \to BF$ so that $\pi_2(\vee_i S_i^2) \to (\vee_i S_i^2) \to \pi_2 BF \to \pi_1 F$ has image B. Consider the classifying map $X \to BF$ of the given fibration with fundamental group data $(*)$. Using Van Kampen's theorem it is easy to see that the fibration with classifying map $X \vee \vee_i S_i^2 \to BF$ has fundamental group data $(**)$. The total space of this fibration is the union of the total spaces of $X \to BF$ and $\vee_i S_i^2 \to BF$ along F, so we may use a Van Kampen type theorem of Siebenmann [10] to compute the finiteness obstruction of the total space, thus computing the transfer map for one fibration and hence by Theorem 5.2 for any fibration with fundamental group data $(**)$ and the result follows. □

Example 5.5. *Let $S^1 \to Y \to X$ be a fibration such that the fundamental group data $\mathbf{Z}/n\mathbf{Z} \to \pi_1 Y \to \pi_1 X$, $\pi_1 Y \to \mathbf{Z}/2\mathbf{Z}$ is reduction mod n of some exact sequence $\mathbf{Z} \to \rho \to \pi_1 X$ with the action of ρ on $\mathbf{Z}$ given by $\rho \to \pi_1 Y \to \mathbf{Z}/2\mathbf{Z}$. The transfer map associated to the fibration $S^1 = K(\mathbf{Z}, 1) \to K(\rho, 1) \to K(\pi_1 X, 1)$ is described algebraically in* [9]. *(A projective $\mathbf{Z}\pi_1 X$-module has homological dimension 1 as $\mathbf{Z}\rho$ module and may thus be considered an element of $\widetilde{K}_0(\mathbf{Z}\rho)$.) By the above corollary the finiteness obstruction of Y is determined as the image of the finiteness obstruction of X in the composition $\widetilde{K}_0(\mathbf{Z}\pi_1 X) \to \widetilde{K}_0(\mathbf{Z}\rho) \to \widetilde{K}_0(\mathbf{Z}\pi_1 Y)$. This gives a new proof of Ehrlich's result that if the fundamental group sequence of an orientable S^1-fibration is pseudo abelian, then the finiteness obstruction of the total space is 0. This follows since one easily sees that a pseudo abelian extension is the reduction mod n of a pseudo abelian integral extension, and an easy calculation shows that the transfer map of a pseudo abelian integral extension is 0* [9].

Remark. The above example holds in the Wh case as well (using [8]).

6. An algebraic application

In view of the results of the preceding sections, it becomes of some interest to compute $G_1 F$ for some more spaces. In the original paper [4], Gottlieb proves that $G_1 F = C\pi_1 F$ for H-spaces and $K(\pi, 1)$'s. He also proves that if $\chi(F) \neq 0$ then $G_1(F) = 0$. In this section we will compute $G_1(X)$ for X a (π, n) polarized space, i. e. a finitely dominated CW-complex X with an isomorphism $\pi_1(X) \cong \pi$ and a homotopy equivalence $\widetilde{X} \simeq S^{n-1}$ (we assume n even and π acting orientation preserving on $\widetilde{X}$). Let $k(X) \in H^n(K(\pi, 1); \mathbf{Z})$ be the first k-invariant of X. It is proved in [6] that $\pi_0 H(X, *) \to \operatorname{Aut} \pi$ and $\pi_0 H(X) \to \operatorname{Out}(\pi)$ are monic and the image are those automorphisms of π that preserve this k-invariant.

Lemma 6.1. *Let X be a polarized space as above. Then $G_1(X) = C\pi$.*

Proof. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1 X/G_1 X & \longrightarrow & \pi_0(H(X, *)) & \longrightarrow & \pi_0(H(X)) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1 X/C\pi & \longrightarrow & \operatorname{Aut} \pi & \longrightarrow & \operatorname{Out} \pi & \longrightarrow & 1 \end{array}$$

Since $\pi_0(H(X, *)) \to \operatorname{Aut} \pi$ is monic it follows that $G_1(X) = C\pi$. □

Corollary 6.2. *Let X be a polarized space with fundamental group π and k-invariant $k(X) \in H^n(K(\pi, 1); \mathbf{Z})$. Let $1 \to \pi \to \rho\xi \to 1$ be an exact sequence and assume that the action of ξ on π preserves $k(X)$. The there exists a fibration $X \to E \to B$ with fundamental group sequence isomorphic to the above exact sequence.*

Proof. By assumption, the homomorphism $\xi \to \mathrm{Out}(\pi)$ factors through $\pi_0 H(X)$. The obstruction to the existence of a fibration of Proposition 3.2 must be 0 since it is equal to the obstruction to the existence of the given exact sequence. □

Corollary 6.3. *Let $1 \to \pi \to p \to \xi \to 1$ be an exact sequence of finitely presented groups, π a group with periodic cohomology of period n (n even). Assume that the action of ξ on π preserves some generator of $H^n(K(\pi, 1); \mathbf{Z})$. Then any projective $\mathbf{Z}[\xi]$-modules M admits a resolution of finitely generated projectives over $\mathbf{Z}[\rho]$ which is periodic of period M.*

Proof. Let k be some generator of $H^n(K(\pi, 1); \mathbf{Z})$ which ξ preserves. We may then [6] find a polarized space with k as k-invariant. The result now follows from Corollary 6.2 by considerations analogous to Corollary 7.2 of [9] since Corollary 6.2 provides the relevant realizability result. □

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