THE WALL FINITENESS OBSTRUCTION FOR A FIBRATION

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1. We wish to study the Wall finiteness obstruction for the total space of a fibration $F \to E \to B$. Such a study was first done by V. J. Lal [5], who neglected the action of $\pi_1(B)$ on the fibre. This was noticed by D. R. Anderson [2], who produced formulae for the finiteness obstruction of E in case the fibration is a flat bundle [2, 1]. Our main theorem gives a partial calculation of the general case: If B and E are finitely dominated and $H_*(F)$ is a finitely generated $\mathbf{Z}$-module, we compute the image of the Wall obstruction of E in $K_0(\mathbf{Z}\pi_1(B))$. In case $\pi_1(E) \to \pi_1(B)$ is monic, we may refine this to compute the actual obstruction for E in terms of information on F and B. In this case the assumption that E is finitely dominated may be replaced by F being the homotopy type of a finite complex.

We begin by establishing some terminology. Let X be a path connected space. We let $S_*(X)$ denote the singular chain complex of the universal cover of X. There is an action of $\pi_1(X)$ on $S_*(X)$ making it a free $\mathbf{Z}\pi_1 X$ module. Given a ring Λ and a ring homomorphism $\mathbf{Z}\pi_1 X \to \Lambda$, we say that X is Λ-*dominated* iff $S_*(X) \otimes_{\mathbf{Z}\pi_1 X} \Lambda$ is chain homotopy equivalent to a complex P_* of Λ-modules, where each P_i is a finitely generated projective Λ-module and all P_i are zero except for finitely many.

If X is Λ-dominated, we can form $\sigma(X;\Lambda) = \sum_{i=-\infty}^{\infty}(-1)^i[P_i]$ in $K_0(\Lambda)$. Although many choices were necessary to form $\sigma(X,\Lambda)$, it is easy to check that $\sigma(X;\Lambda)$ is independent of all these choices. Also one sees that $\sigma(X;\Lambda)$ is a homotopy invariant and natural with respect to homomorphisms $\Lambda \to \Lambda_1$.

We say that X is *finitely dominated* if X is $\mathbf{Z}\pi_1 X$-dominated (where $\mathbf{Z}\pi_1 X \to \mathbf{Z}\pi_1 X$ is the identity) and $\pi_1 X$ is finitely presented. In this case we write $\sigma(X;\mathbf{Z}\pi_1 X) = \sigma(X)$.

Wall [7, 8] has proved that a CW complex is dominated by a finite CW complex iff it is finitely dominated and that it has the homotopy type of a finite CW complex iff $\sigma(X) \in K_0(\mathbf{Z}\pi_1 X)$ vanishes in $\widetilde{K}_0(\mathbf{Z}\pi_1 X)$. We call the image of $\sigma(X)$ in $\widetilde{K}_0(\mathbf{Z}\pi_1 X)$ the Wall finiteness obstruction for X.

We say that X is *homologically finite* if X is $\mathbf{Z}$-dominated, where $\mathbf{Z}\pi_1 X \to \mathbf{Z}$ is the natural map. It is easy to see that this is the case iff $\sum_{i=0}^{\infty} H_i(X;\mathbf{Z})$ is a finitely generated abelian group.

To describe the action of the fundamental group on the homology of the fibre we need a functor $G(\pi)$, π a group, and a pairing $G(\pi) \times K_0(\mathbf{Z}\pi_1 X) \to K_0(\mathbf{Z}\pi)$. We let $G(\pi)$ be

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the Grothendieck group of the abelian category of $\mathbf{Z}\pi$-modules which are finitely generated abelian groups. We show that $G(\pi)$ is a ring, and we define a pairing $G(\pi) \times K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi)$ making $K_0\mathbf{Z}\pi)$ into a module over $G(\pi)$. We actually show that $G(\pi)$ is isomorphic to Swan's Grothendieck group of integral representations of π [6].

Given a Serre fibration $F \to E \xrightarrow{p} B$ with B connected and F homologically finite, we can define $\chi(p) \in G(\pi)$ as $\sum_{i=0}^{\infty}(-1)^i[H_i(F;\mathbf{Z})]$. $\chi(p)$ can be seen to be independent of a choice of basepoint for B. We have

Theorem 1.1. *Let $F \to E \to B$ be a Serre fibration with E and B connected. Suppose B is finitely dominated and F homologically finite. Then E is $\mathbf{Z}\pi_1 B$-dominated (using the homomorphism $\mathbf{Z}\pi_1 E \to \mathbf{Z}\pi_1 B$ induced by p) and*

$$\sigma(E;\mathbf{Z}\pi_1 B) = \chi(p)\cdot\sigma(B).$$

We obtain a number of corollaries: Let π denote the image of $\pi_1 E$ in $\pi_1 B$. Since $H_*(F)$ is homologically finite, π has finite index in $\pi_1 B$, and hence the restriction map $\mathrm{Res} : K_0(\mathbf{Z}\pi_1 B) \to K_0(\mathbf{Z}\pi)$ is defined. π acts on F by taking each component to itself. Define $\chi_0(p) \in G(\pi)$ to be $\sum_{i=0}^{\infty}(-1)^i[H_i(F_0,\mathbf{Z})]$, where F_0 denotes one component of F. We then have

Corollary 1.2. *Hypothesis as above. Then E is $\mathbf{Z}\pi$-dominated and*

$$\sigma(E;\pi) = \chi(p)\cdot\mathrm{Res}\,\sigma(B).$$

Remarks. If $\pi_1 E \to \pi_1 B$ is injective, then E is finitely dominated and the corollary computes the Wall finiteness obstruction of E. If $\pi_1 B$ acts trivially on $H_*(F,\mathbf{Z})$ then $\chi(p)$ becomes the classical Euler characteristic of F and $\cdot$ denotes the usual $\mathbf{Z}$-module structure on K_0.

As an example. let $S^k \to E_k \to B$ be a spherical fibration with $\omega : \pi_1 B \to \mathbf{Z}_2$ denoting the first Stiefel-Whitney class. Let Θ denote the kernel of ω and let $F_* : K_0(\mathbf{Z}\pi_1 B) \to K_0(\mathbf{Z}\pi_1 B)$ be the result of restricting to Θ and then inducting up to $\mathbf{Z}\pi_1 B$. Then

Corollary 1.3. *Assume B is finitely dominated. Then:*

(1) *If ω is trivial, then*

$$\sigma(E_k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2\sigma(B) & \text{if } k \text{ is even.} \end{cases}$$

(2) *If ω is onto, then*

$$\sigma(E_k) = \begin{cases} F_*\sigma(B) & \text{if } k \text{ is even,} \\ 2\sigma(B) - F_*\sigma(B) & \text{if } k \text{ is odd.} \end{cases}$$

(In case $k = 0$ or 1, $\sigma(E_k)$ should be replaced by $\sigma(E_k,\mathbf{Z}\pi_1 B)$.)

Proof. Case 1 follows immediately from the remarks above. To see case 2, notice that the action of $\pi_1 B$ on the homology of the fibre factors through $\mathbf{Z}_2$ and $\chi(p)$ is the image of $1 + (-1)^k g$ under the natural map $G(\mathbf{Z}_2) \to G(\pi_1 B)$, where g represents $\mathbf{Z}$ with nontrivial action of $\mathbf{Z}_2$. We denote this by $\mathbf{Z}^t$. If P is a projective $\mathbf{Z}(\pi_1 B)$-module representing $\sigma(B)$, we need to compute $P \otimes Z^t$, where the action of $\pi_1 B$ on $\mathbf{Z}$ is through ω. However, in $K_0(\mathbf{Z}(\pi_1 B))$ we have

$$\begin{aligned}[P \otimes \mathbf{Z}^t] &= [P \otimes (\mathbf{Z}^t + \mathbf{Z})] - [P] \\ &= [P \otimes (\mathbf{Z}[\mathbf{Z}_2])] - [P] \\ &= F_*[P] - [P],\end{aligned}$$

and the result follows. □

This corollary gives us a general method for constructing examples of the type considered by Anderson [2].

Corollary 1.4. *Let π be a finitely presented group, $\omega : \pi \to \mathbf{Z}_2$ a nontrivial map and Θ the kernel of ω. Assume there is an element $\alpha \in K_0(\mathbf{Z}\pi)$ such that α is in the kernel of the restriction map $K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\Theta)$ and such that $2\alpha \neq 0$. By Wall [7] we may find a complex B with $\pi_1 B = \pi$ and $\sigma(B) = \alpha$. Given k, it is easy to find a vector bundle of dimension $k+1$ with first Stiefel-Whitney class given by ω. Let E_k denote the corresponding sphere-bundle; then*

$$\sigma(E_k) \begin{cases} = 0, & k \text{ even}, \\ \neq 0, & k \text{ odd}. \end{cases}$$

As an example we may take $\pi = \mathbf{Z}_6 + \mathbf{Z}_3$. Since by Fröhlich [4], $\widetilde{K}_0(\mathbf{Z}[\mathbf{Z}_3 \oplus \mathbf{Z}_3])$ has order 3 while $\widetilde{K}_0(\mathbf{Z}(\mathbf{Z}_6 \oplus \mathbf{Z}_3))$ has order 81, any element α in $\ker(\mathrm{Res})$ will satisfy the conditions of the corollary.

2. In this section we define algebraic functors $G(\pi)$ and $K_0'(\mathbf{Z}\pi)$ and relate these to well-known algebraic objects, namely $G_{\mathbf{Z}}(\pi)$, Swan's Grothendieck group of integral representations, and $K_0(\mathbf{Z}\pi)$.

We define $G(\pi)$ to be the Grothendieck construction on $\mathbf{Z}\pi$-modules that are finitely generated abelian groups, i. e., as generators we take isomorphism classes of $\mathbf{Z}\pi$-modules that are finitely generated abelian groups, and if $0 \to A \to B \to C \to 0$ is an exact sequence of such modules we have the relation $[A] + [C] = [B]$. Similarly we define $K_0'(\mathbf{Z}\pi)$ to be the Grothendieck group of isomorphism classes of finitely generated $\mathbf{Z}\pi$-modules of homological dimension ≤ 1.

Lemma 2.1. *Let T be a $\mathbf{Z}\pi$-module which is finite. Then there is an exact sequence of $\mathbf{Z}\pi$-modules $0 \to F_2 \to F_1 \to T \to 0$ where F_1 and F_2 are finitely generated free abelian groups.*

Proof. Since T is a finite group, $\mathrm{Aut}(T)$ is also finite. Hence π acts on T through a quotient group π' which is finite. We can find a free $\mathbf{Z}\pi'$-module F_1 and an epimorphism $F_1 \to T$. Let F_2 be the kernel. Then F_1 and F_2 are finitely generated free abelian groups, and $0 \to F_2 \to F_1 \to T \to 0$ is an exact sequence of $\mathbf{Z}\pi'$- and hence $\mathbf{Z}\pi$-modules. □

Lemma 2.2. *The natural maps $G_{\mathbf{Z}}(\pi) \to G(\pi)$ and $K_0(\mathbf{Z}\pi) \to K_0'(\mathbf{Z}\pi)$ are isomorphisms.*

Proof. $G_{\mathbf{Z}}(\pi)$ is the Grothendieck group of $\mathbf{Z}\pi$-modules which are finitely generated free abelian groups. Let A represent an element of $G(\pi)$. Since A is an abelian group, we have a short exact sequence $0 \to T \to A \to F \to 0$ where T is torsion and F is free. This will be an exact sequence of $\mathbf{Z}\pi$-modules. By Lemma 2.1 there is an exact sequence $0 \to F_2 \to F_1 \to T \to 0$, where F_1 and F_2 represent elements of $G_{\mathbf{Z}}(\pi)$. We thus have $[A] = [F] + [F_1] - [F_2]$ in $G(\pi)$ and we may define a map $\Sigma : G(\pi) \to G_{\mathbf{Z}}(\pi)$ by

$$\Sigma[A] = [F] + [F_1] - [F_2].$$

We need only see that Σ is well defined, because by the above remarks, if $I : G_{\mathbf{Z}}(\pi) \to G(\pi)$ denotes the natural map, it will be clear that $I\Sigma = id$ and $\Sigma I = id$. As for well-definedness, it is enough to show that $\Sigma[T]$ is well defined and to show that if $0 \to A_1 \to A_2 \to A_3 \to 0$ is exact, then

$$\Sigma[A_2] = \Sigma[A_1] + \Sigma[A_2]$$

Suppose we have $0 \to H_2 \to H_1 \to T \to 0$, an exact sequence of $\mathbf{Z}\pi$-modules with H_1 and H_2 finitely generated free abelian groups. Let K be the pullback

$$\begin{array}{ccc} K & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ H_1 & \longrightarrow & T \end{array}$$

Then $0 \to H_2 \to K \to F_1 \to 0$ and $0 \to F_2 \to K \to H_1 \to 0$ are exact, and K is a finitely generated free abelian group. Hence in $G_{\mathbf{Z}}(\pi)$, $[H_1] - [H_2] = [F_1] - [F_2]$, so $\Sigma[T]$ is well defined.

We prove our equation for $0 \to A_1 \to A_2 \to A_3 \to 0$ in several steps. First suppose that A_1 and A_2 are free abelian groups. Consider

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where K is the kernel of $A_2 \to F$. Then K is a finitely generated free abelian group, so in $G_{\mathbf{Z}}(\pi)$ we have $[A_2] = [K] + [F]$, so $\Sigma[A_3] = [F] + \Sigma[T] = [F] + [K] - [A_1] = [A_2] - [A_1]$, since $0 \to A_1 \to K \to T \to 0$ is exact.

The next case to consider is the case that A_1, A_2 and A_3 are all torsion groups. As in the proof of Lemma 2.1 we may find a finite group π' so that the action of π on each A_i goes through a map $\pi \to \pi' \to \mathrm{Aut}(A_i)$. Then we can produce a diagram of $\mathbf{Z}\pi'$-modules:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where F_i and K_i are finitely generated free abelian groups that are all $\mathbf{Z}\pi'$- and hence $\mathbf{Z}\pi$-modules. It is now easy to check that our equation holds in $G_{\mathbf{Z}}(\pi)$.

Now suppose A_1 is a torsion group, and consider

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_2 & = & F_3 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Our equation holds for all three columns and two of the rows, so our equation holds for the last row.

In the general case, consider

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here F_1 and F_2 are free abelian groups, and S is just the quotient of T_2 by T_1 and hence is torsion. Our equation holds for all columns and two of the three rows, so it holds for the third row.

For the proof that $K_0(\mathbf{Z}\pi) \to K_0'(\mathbf{Z}\pi)$ is an isomorphism we refer the reader to Bass [3, p. 407].

□

Now Swan defines pairings $G_{\mathbf{Z}}(\pi) \times G_{\mathbf{Z}}(\pi) \to G_{\mathbf{Z}}(\pi)$ and $G_{\mathbf{Z}}(\pi) \times K_0(\pi) \to K_0(\mathbf{Z}\pi)$ by tensor product over the integers, the π-action in each case being just $g(a \otimes b) = ga \otimes gb$.

For all of this, see Bass [3, p. 563–565]. The tensor product over $\mathbf{Z}$ clearly defines a pairing $G(\pi) \times G(\pi)$ to $G(\pi)$, so that I becomes a map of rings.

PROPOSITION 2.3. *The tensor product over $\mathbf{Z}$ defines a pairing $G(\pi) \times K_0(\mathbf{Z}\pi) \to K_0'(\mathbf{Z}\pi)$.*

Proof. Let P be a projective $\mathbf{Z}\pi$-module and A a $\mathbf{Z}\pi$-modules which is a finitely generated abelian group. We need to show that $A \otimes_{\mathbf{Z}} P$ is a $\mathbf{Z}\pi$-module of homological dimension ≤ 1. However, P, being a projective $\mathbf{Z}\pi$-module, is a free abelian group (not necessarily finitely generated), so tensoring over $\mathbf{Z}$ with P is exact. Given the short exact sequence of $\mathbf{Z}\pi$-modules $0 \to T \to A \to F \to 0$, where T is a torsion and F a free abelian group, we have that $0 \to T \otimes_{\mathbf{Z}} P \to A \otimes_{\mathbf{Z}} P \to F \otimes_{\mathbf{Z}} P \to 0$ is exact and $F \otimes_{\mathbf{Z}} P$ is projective. To show $h.d.(A \otimes_{\mathbf{Z}} P) \leq 1$ it thus suffices to show $h.d.(T \otimes_{\mathbf{Z}} P) \leq 1$. But use Lemma 2.1 to find an exact sequence $0 \to F_2 \to F_1 \to T$ with F_i $\mathbf{Z}\pi$-modules that are free abelian groups. Then $0 \to F_2 \otimes_{\mathbf{Z}} P \to F_1 \otimes_{\mathbf{Z}} P \to T \otimes_{\mathbf{Z}} P \to 0$ is exact and $F_i \otimes_{\mathbf{Z}} P$ is projective, so $h.d.(T \otimes_{\mathbf{Z}} P) \leq 1$. □

3. In this section we prove our main algebraic result. It says that complexes with "nice" homology must be Λ-dominated and that we can compute the Wall obstruction from the homology. A complex is Λ-dominated iff is is chain homotopy equivalent to a complex of finitely generated projective Λ-modules with only finitely many nonzero terms. Of course, X is Λ-dominated iff $S_*(X) \otimes_{\mathbf{Z}\pi_1 X} \Lambda$ is Λ-dominated. A Λ-dominated complex C_* has an invariant $\sigma(C_*) \in K_0(\Lambda)$ defined the usual way as an alternating sum, $\sigma(S_*(X) \otimes_{\mathbf{Z}\pi_1 X} \Lambda) = \sigma(X, \Lambda)$.

PROPOSITION 3.1. *Let C_* be a projective chain complex over Λ with $C_* = 0$, $* < 0$. Suppose that $H_*(C)$ is a finitely generated Λ-module of homological dimension ≤ 1 and $H_*(C) = 0$ for $*$ sufficiently large. Then C_* is Λ-dominated and*

$$\sigma(C_*) = \Sigma(-1)^i [H_i(C)] \in K_0'(\Lambda)$$

under the natural isomorphism $K_0(\Lambda) \cong K_0'(\Lambda)$.

Proof. Consider the two short exact sequences

$$0 \to Z_i \to C_i \to B_i \to 0,$$
$$0 \to B_{i+1} \to Z_i \to H_i \to 0.$$

C_* projective just means that each C_i is projective (perhaps not finitely generated). By induction we can prove that each B_i and Z_i are projective, so $C_i \cong Z_i \oplus B_i$.

Since $H_i(C)$ is finitely generated and of homological dimension ≤ 1, we can find finitely generated projective modules Q_{i+1} and P_i with $0 \to Q_{i+1} \to P_i \to H_i \to 0$ exact. If $H_i = 0$, choose $P_i = Q_{i+1} = 0$. Let $D_i = P_i \oplus Q_{i+1}$. It is easy to find a map $D_* \to C_*$ inducing an isomorphism in homology, and hence a chain homotopy equivalence.

Obviously C_* is Λ-dominated. To show $\sigma(C_*) = \Sigma(-1)^i [H_i(C)]$ is now a standard argument. □

To get the most out of proposition 3.1 we must observe

PROPOSITION 3.2. *Let* $0 \to A_* \to B_* \to C_* \to 0$ *be a short exact sequence of projective chain complexes over* Λ. *If any two of these complexes are* Λ*-dominated, then so is the third, and* $\sigma(B_*) = \sigma(A_*) + \sigma(C_*)$.

Proof. If A_* and C_* are Λ-dominated, we can splice together the complexes for A_* and C_* to get one for B_*. The argument is elementary, and the equation follows.

If A_* and B_* are Λ-dominated, C_* is equivalent to the algebraic mapping cone of $A_* \to B_*$. Let D_* be the mapping cone. Then $0 \to B_* \to D_* \to A_{*-1} \to 0$ is exact, so D_* is Λ-dominated, and hence so is C_*. Again the equation follows. The other case is similar. □

4.

Proof of Main Theorem. We get our principal geometric insight from a filtration that we put on the base space of our fibration. We then look at the induced filtration on the total space and compute using the results of Sections and .

Without loss of generality we may replace B by a homotopy equivalent CW-complex, and consider the fibration induced over this complex. Wall [7] shows, that since B is finitely dominated, B may be chosen so that there is a finite subcomplex $K \subset B$ with $H_*(\widetilde{B}, \widetilde{K}; \mathbf{Z}) = 0$ except for one dimension, and there it is a finitely generated projective $\mathbf{Z}\pi_1 B = \Lambda$-module. Here $\widetilde{B}$ and $\widetilde{K}$ denote the universal covers of B and K respectively. We let $\widetilde{E}$ denote the pullback of

$$\begin{array}{ccc} & & \widetilde{B} \\ & & \downarrow \\ E & \longrightarrow & B \end{array}$$

and $\tilde{p} : \widetilde{E} \to \widetilde{B}$ and $\pi : \widetilde{B} \to B$ be the obvious maps.

The filtration on B is defined as follows: If K has dimension $n-1$, define $B_n = B$; $B_{n-1} = K$; $B_{n-2} = (n-2)$-skeleton of K; …; $B_0 =$ 0-skeleton of K; $B_{-1} =$ empty set.

Notice that $H_*(\widetilde{B}_r, \widetilde{B}_{r-1}, \mathbf{Z})$ is always 0 except in one dimension and that there is a finitely generated projective Λ-module. Here $\widetilde{B}_r = \pi^{-1}(B_r)$.

Consider the Serre spectral sequence of (E_r, E_{r-1}) where $\widetilde{E}_r = \widetilde{P}^{-1}(\widetilde{B}_r)$. The two-term $E^2_{p,q}$ is just $H_p(\widetilde{B}_r, \widetilde{B}_{r-1}, H_q(F, \mathbf{Z}))$, but $H_p(\widetilde{B}_r, \widetilde{B}_{r-1}, \mathbf{Z})$ is 0 or free as an abelian group, so $E^2_{p,q} = H_p(\widetilde{B}_r, \widetilde{B}_{r-1}) \otimes H_q(F, \mathbf{Z})$. There is an action of $\pi_1(B)$ on the Serre spectral sequence, and the action of E^2 is just the diagonal action on the tensor product, where $\pi_1(B)$ acts on $H_q(F, \mathbf{Z})$ as usual. Since $H_p(\widetilde{B}_r, \widetilde{B}_{r-1}) = 0$ except for one value of p, the Serre sequence collapses and we have

$$H_{r+q}(\widetilde{E}_r, \widetilde{E}_{r-1}) \cong H_r(\widetilde{B}_r, \widetilde{B}_{r-1}) \otimes H_q(F).$$

Proposition 2.3 shows that $H_r + q(\widetilde{E}_r, \widetilde{E}_{r-1})$ is a finitely generated Λ-module of homological dimension ≤ 1. Proposition 3.1 shows that the chain complex for the pair $(\widetilde{E}_r, \widetilde{E}_{r-1})$ is Λ-dominated and $\sigma(\widetilde{E}_r, \widetilde{E}_{r-1}) = \chi(p) \cdot (-1)^r [H_r(\widetilde{B}_r, \widetilde{B}_{r-1})]$.

An easy induction argument using proposition 3.2 shows that each $\widetilde{E}_r$ is Λ-dominated and that $\sigma(\widetilde{E}_r) = \chi(p) \cdot \sigma(B_r)$. Since it is a standard fact that the singular chains of $\widetilde{E}$ is isomorphic to $S_*(E) \otimes_{\mathbf{Z}\pi_1 E} \mathbf{Z}\pi_1 B$, we have proved out main theorem:

$$\sigma(E, \Lambda) = \chi(p) \cdot \sigma(B).$$

□

Proof of Corollary. Let $\overline{B}$ be the cover of B corresponding to the subgroup $\pi \subset \pi_1 B$, where π is the image of π in $\pi_1 B$. Then we have a fibration $F_0 \to E \to \overline{B}$. Since B is finitely dominated and $\overline{B}$ is a finite cover, $\overline{B}$ is finitely dominated. Applying the theorem yield $\sigma(E) = \chi(p) \cdot \sigma(\overline{B})$ and $\sigma(\overline{B}) = \operatorname{Res} \sigma(B)$, as is easy to see directly from the definitions. □

We finally notice that if $\pi_1(B)$ acts trivially on $H_*(F)$, multiplication with $\chi(p)$ becomes the usual multiplication with $\chi(F)$, since $\mathbf{Z}$ with trivial action is the unit of $G(\pi_1 B)$ and $K_0(\mathbf{Z}\pi_1 B)$ is a unital module.

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