ON THE WALL FINITENESS OBSTRUCTION FOR THE TOTAL SPACE OF CERTAIN FIBRATIONS

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Abstract. The problem of computing the Wall finiteness obstruction for the total space of a fibration $p : E \to B$ in terms of that for the base and homological data of the fiber has been considered by D. R. Anderson and by E. K. Pedersen and L. R. Taylor. We generalize their results and show how the problem is related to the algebraically defined transfer map $\phi^* : K_0(\mathbf{Z}\pi_1(B)) \to K_0(\mathbf{Z}\pi_1(E)), \phi = p_* : \pi_1(E) \to \pi_1(B)$, whenever the latter is defined.

0. Introduction

Let $p : E \to B$ be a Serre fibration with fiber F. Assume that B is finitely dominated and that F has the homotopy type of a finite complex. Let $\tilde{w}(B) \in \widetilde{K}_0(\mathbf{Z}\pi_1(B))$ be Wall's finiteness obstruction for B [19]. Since E is also finitely dominated [10], one also has $\tilde{w}(E) \in \widetilde{K}_0(\mathbf{Z}\pi_1(E))$. We study the relationship between $\tilde{w}(E)$ and $\tilde{w}(B)$.

Assume given a factorization of p_*

$$\pi_1(E) \xrightarrow{s} \pi \xrightarrow{\phi} \pi_1(B)$$

with s onto, and $\ker(\phi) = \nu$ of type (FP) (i.e. $\mathbf{Z}$, viewed as a $\mathbf{Z}\nu$ module with trivial ν action, admits a finite resolution by finitely generated, projective $\mathbf{Z}\nu$ modules). We compute $s_*(\tilde{w}(E))$ under the assumption that a certain covering $\overline{F}$ of a component of F has finitely generated integral homology. The description involves the transfer map induced by ϕ and the integral representations $H_i(\overline{F}; \mathbf{Z})$ of the group π, see Theorem 4.1 for details. Taking ν trivial, i. e.

$$\pi = \mathrm{Im}(p_* : \pi_1(E) \to \pi_1(B)),$$

one recovers the main result of Pedersen and Taylor [13]. Theorem 4.1 has e.g. the following.

Corollary A. *Let ν be a group for which $B\nu$ is a finite complex. If $p : E \to B$ has fiber $F = B\nu$ and $\pi_1(F) \to \pi_1(E)$ is injective then*

$$\tilde{w}(E) = \phi^*(\tilde{w}(B))$$

where $\phi^* : \widetilde{K}_0(\mathbf{Z}\pi_1(B)) \to \widetilde{K}_0(\mathbf{Z}\pi 1(E))$ is the transfer map induced by $\phi = p_* : \pi_1(E) \to \pi_1(B)$.

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Note that this covers, e. g., the case $F = T^k = S^1 \times \cdots \times S^1$. For that case, and not assuming $\pi_1(F) \to \pi_1(E)$ injective, we prove, in 7, the following.

Realizability Theorem for Torus Fibrations. *Let*

$$1 \to \nu \to \pi \to \overline{\pi} \to 1$$

be a short exact sequence of finitely presented groups with ν abelian, and let $\tilde{w} \in \widetilde{K}_0(\mathbf{Z}\overline{\pi})$ be given. There exists a fibration $p : E \to B$ with the properties:
(i) $1 \to \mathrm{Im}(\pi_1(F) \to \pi_1(E)) \to \pi_1(E) \to \pi_1(B) \to 1$ is isomorphic to the given sequence,
(ii) B is finitely dominated and $\tilde{w}(B) = \tilde{w}$,
(iii) $F = T^k$,
if and only if
(iv) For some action of on $\mathbf{Z}\overline{\pi}$ on $\mathbf{Z}^k$ one has an epimorphism of $\mathbf{Z}\overline{\pi}$ modules $\mathbf{Z}^k \to \nu$.

Our interest in the problem originally came from S^1-fibrations. If $F = S^1$ and $\pi_l(F) \to \pi_l(E)$ has image C_k generated by t of order k then we let $N = 1 + t + t^2 + +t^k$. There is then the pull back diagram of rings

$$\begin{array}{ccc} \mathbf{Z}\pi_1(E) & \xrightarrow{\phi} & \mathbf{Z}\pi_1(B) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \overline{r}} \\ \mathbf{Z}\pi_1(E)/(N) & \xrightarrow[\psi]{} & (\mathbf{Z}/k)\pi_1(B) \end{array}$$

where (N) is the ideal generated by N. We prove

Corollary B. *If $p : E \to B$ has fiber S^1 and $\pi_1(S^1)$ maps onto a cyclic group of order k then*

$$r_* \tilde{w}(E) = \psi^* \overline{r}_* \tilde{w}(B)$$

where ψ^* is the transfer map induced by ψ

In case $F = S^{2l-1}/C_k$ is a lens space and $\pi_1(F) \to \pi_1(E)$ is injective, the above diagram still makes sense. Also there is an automorphism β of $(\mathbf{Z}/k)\pi_1(B)$ coming from the $\pi_1(B)$ action on C_k, see §3.

Corollary C. *If $p : E \to B$ has $F = S^{2l-1}/C_k$ and $\pi_1(F) \to \pi_1(E)$ injective then*

$$r_* \tilde{w}(E) = \sum_{i=0}^{l-1} \psi^* (\beta^*)^i \overline{r}_* \tilde{w}(B).$$

For any finite group G acting freely on S^{2l-1} we get a result similar to the above Corollary C, see Theorem 6.2.

Comparison with earlier (and contemporary) results. The main result of [1] computes $\tilde{w}(E)$ when E is a flat bundle. However, it involves, in general, the chains on $\widetilde{F}$, see Theorem 9 of [1]. In the case when the chains can be replaced by the homology, see Lemma 7 of [1], our Theorem 4.2 generalizes the result to any fibration. The same generalization has been obtained by Ehrlich, [6],[8].

The main result of [3] is our Theorem 4.1 for the case of flat bundles and with $\pi = \pi_1(B)$.

For orientable S^1-bundles with $\pi_1(S^1) \to \pi_1(E)$ injective and $\pi_1(E)$ abelian Anderson [2] shows that $\tilde{w}(E) = O$. This result can easily be recovered from Corollary A by showing (algebraically) that ϕ^* vanishes under these conditions.

Pedersen and Taylor [13] proved the special case of Theorem 4.1 when ν vanishes. Actually Theorem 4.1 should be viewed as the main result of [13] extended to its natural generality.

Ehrlich, in his thesis [6] (see also [8]), proves several interesting results. The main ones are sufficient conditions for $\tilde{w}(E) = 0$ and sufficient conditions for having $\pi_1(E) = \pi_1(F) \times \pi_1(B)$ and $\tilde{w}(E) = \tilde{w}(F \times B)$. For orientable S^1 fibrations he proves $\tilde{w}(E) = 0$ provided the image of $\pi_1(S^1)$ intersects $[\pi_1(E), \pi_1(E)]$ trivially. If $\pi_1(S^1) \to \pi_1(E)$ is injective one may recover this result from Corollary A by showing easily that ϕ^* vanishes. If $\pi_1(S^1) \to \pi_1(E)$ is not injective our results, viz. Corollary B, only allow one to conclude that $r_* \tilde{w}(E)$ vanishes.

Finally we note that Ehrlich [7] has a geometrically defined map

$$p^* : \widetilde{K}_0(\pi_1(B)) \to \widetilde{K}_0(\pi_1(E))$$

which he calls transfer. It follows from the results here and those of [7] that p^* is expressible in terms of ϕ^* when $\phi = p_* : \pi_1(E) \to \pi_1(B)$ does give rise (algebraically) to a transfer map.

1. General notation

The map $p : E \to B$ is a Serre fibration; B and E are connected, finitely dominated spaces; the fiber F has the homotopy type of a finite complex. We assume chosen a base point in F and let F_0 be the component containing it.

Let $K \subseteq \pi_1(E)$ be a normal subgroup of $\pi_1(E)$, contained in the image of $\pi_1(F)$. Let $\phi : \pi = \pi_1(E)/K \to \rho = \pi_1(B)$ be induced by p, and let $y = \ker(\phi)$. Then we say that the fibration p realizes the exact sequence

$$1 \to \nu \to \pi \xrightarrow{\phi} \rho$$

modulo K.

The covering $\overline{F} \to F_0$ is the one corresponding to the obvious map $\pi_1(F) = \pi_1(F_0) \to \nu$.

Next some homological algebraical conventions. Our ground ring is $\mathbf{Z}$, the integers. More general ground rings could be considered but we leave that to the reader. For a group π the integral group ring is denoted $\mathbf{Z}\pi$. All modules are left modules unless otherwise stated. A *finite resolution* means a resolution of finite length by finitely generated modules. Thus, conceivably, a module may be of finite homological dimension without admitting a finite projective resolution.

For any ring R we let $\mathcal{P}(R)$, $\mathcal{P}_d(R)$, $\mathcal{P}_{<\infty}(R)$ denote, respectively, the category of finitely generated projective R modules, the category of R modules admitting finite projective resolutions of length at most d, and the category of R modules admitting finite projective resolutions.

For any admissible category $\mathfrak{A}$ we let $K_0(\mathfrak{A})$ be the Grothendieck group in the sense of [4]. Also, we let $K_0(R) = K_0(\mathcal{P}(R))$ and $\widetilde{K}_0(R) = K_0(R)/\mathbf{Z}$ with $\mathbf{Z}$ generated by $[R]$.

For any ring homomorphism $\phi : R \to S$, any R-module P, and any S-module M, let $\phi_*(P) = S \otimes_R M$ as S module, and $\phi^*(M) = M$ viewed as R module via ϕ. Then ϕ_* sends $\mathcal{P}(R)$ to $\mathcal{P}(S)$ and induces $K_0(\phi) = \phi_* : K_0(R) \to K_0(S)$. If ϕ^* sends $\mathcal{P}(S)$ to $\mathcal{P}_{<\infty}(R)$ then we also use the name ϕ^* for the composition

$$K_0(S) = K_0(\mathcal{P}(S)) \to K_0(\mathcal{P}_{<\infty}(R)) = K_0(\mathcal{P}(R)) = K_0(R)$$

and we call ϕ^* the *transfer map induced by* ϕ.

Recall also that $K_0(\mathbf{Z}\pi)$, for any group ring $\mathbf{Z}\pi$, is a right module over the integral representation ring $G(\pi)$.

If X is a finitely dominated space, Wall [19, Theorem F] defines an element $\tilde{w}(X) \in \widetilde{K}_0(\mathbf{Z}\pi_1(X))$ which vanishes if and only if X has the homotopy type of a finite complex. We shall denote by $w(X)$ the corresponding coset in $K_0(\mathbf{Z}\pi 1(X))$. By abuse of notation we write $w(X;\pi)$ for the image of $w(X)$ under the map $s_* : K_0(\mathbf{Z}\pi_1(X)) \to K_0(\mathbf{Z}\pi)$ when a homomorphism $s : \pi_1(X) \to \pi$ is given.

2. The main proposition

All the results of this paper are based on the following simple observation.

Proposition 2.1. *Let $p : E \to B$ be a fibration as in §1, realizing the exact sequence*

$$1 \to \nu \to \pi = \pi_1(E)/K \xrightarrow{\phi} \rho = \pi_1(B)$$

modulo the subgroup K of $\pi_1(E)$. If F is homotopy equivalent to a complex of dimension k then there exist an integer n, a chain-complex of $\mathbf{Z}\pi$ modules

$$0 \to P_{n+k} \to P_{n+k-1} \to \cdots \to P_n \to 0$$

with $P \in \mathcal{P}(\mathbf{Z}\pi)$, and an $M \in \mathcal{P}(\mathbf{Z}\rho)$ such that

(i) $(-1)^n[M] \in w(B) \subseteq K_0(\mathbf{Z}\rho)$,
(ii) $\sum(-1)^i[P_j] \in w(E,\pi) \subseteq K_0(\mathbf{Z}\pi)$,
(iii) $H_i(P_*) = \phi^* M \otimes_{\mathbf{Z}} H_i(\overline{F};\mathbf{Z})$.

Here π acts diagonally on the product, and the action on $H_i(\overline{F};\mathbf{Z})$ is induced by an action up to homotopy of π on $\overline{F}$, explained below.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
F & = & F & \supseteq & F_0 & \longleftarrow & \overline{F} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \longleftarrow & \overline{E} & \supseteq & \overline{E}_0 & \longleftarrow & \hat{E} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} & & \downarrow{\scriptstyle \overline{p}_0} & & \downarrow{\scriptstyle \hat{p}} \\
B & \underset{q}{\longleftarrow} & \widetilde{B} & = & \widetilde{B} & = & \widetilde{B}
\end{array}$$

Here $\overline{p}$ is the pull back of p through the universal covering map q. If $\overline{\pi}$ is the image of $\pi_1(E)$ in $\pi_1(B) = \rho$ then F and $\overline{E}$ both have $[\rho : \overline{\pi}]$ components; the ones containing the chosen base point are denoted F_0 and $\overline{E}_0$. Then $\overline{p}_0$ is a fibration with fiber F_0. Also $\pi_1(\overline{E}_0)$ is the image of $\pi_1(F)$ in $\pi_1(E)$ so when we take $\hat{E}$ to be the covering of $\overline{E}_0$ corresponding to the subgroup K we easily see that the fiber of $\hat{p}$ is the covering $\overline{F}$ described in §1.

We now describe the action up to homotopy of π on $\overline{F}$. (This must be well known but we lack a direct reference.) We view $\xi \in \pi$ as a covering transformation on $\hat{E}$ in the usual way; similarly with the image of $\overline{\xi}$ of ξ on $\rho = \pi_1(B)$. Also let $*$ be the base point of $\hat{E}$and $\widetilde{B}$ and let $\overline{F} = \overline{F}_*$, where in general $\overline{F}_b = (\hat{p})^{-1}(b)$, $b \in \widetilde{B}$. Finally, we let w_ξ be a path from $\overline{\xi}(*)$ to $*$ in $\widetilde{B}$ and let $\lambda([w_\xi])$ be the induced (homotopy class of maps) $\overline{F}_{\overline{\xi}(*)} \to \overline{F}_*$. Then the action of ξ on $\overline{F}$ is given by the upper row in the commutative diagram.

$$\begin{array}{ccccc}
\overline{F}_* & \xrightarrow{\xi|\overline{F}_*} & \overline{F}_{\overline{\xi}(*)} & \xrightarrow{\lambda([w_\xi])} & \overline{F}_* \\
\downarrow & & \downarrow & & \\
\hat{E} & \xrightarrow{\xi} & \hat{E} & & \\
\downarrow & & \downarrow & & \\
\widetilde{B} & \xrightarrow{\overline{\xi}} & \widetilde{B} & &
\end{array}$$

It is routine to verify that this does define an action up to homotopy of π on $\overline{F}$. Moreover, for any reasonable subspace A of B the Serre spectral sequence for $(\hat{E}, \hat{E}_A)$, where $\hat{E}_A = (q\hat{p})^{-1}(A)$, has an obvious π-action for which

$$E^2_{**} = \phi^* H_*(\widetilde{B}, \widetilde{A}) \otimes_{\mathbf{Z}} H_*(\overline{F}; \mathbf{Z})$$

with diagonal π-action. Note that $\widetilde{B}$ is simply connected so that one has untwisted coefficients.

Assume that B is dominated by an n-dimensional complex. By [19] (especially Theorem F and the discussion preceding Lemma 3.1), there exist an $(n-1)$- dimensional complex A and an $(n-1)$-connected map $\psi : A \to B$; and $M = \pi_n(\psi)$ satisfies (i). Replacing B by the mapping cylinder of ψ, $M(\psi)$, and E by its pull back via $M(\psi) \to B$, we can assume that ψ is an inclusion. Then $M = H_n(\widetilde{B}, \widetilde{A})$. Let $\psi_E : E_A \to E$ be the inclusion of $p^{-1}(A) = E_A$. By Theorem 1 of [10], E_A has the homotopy type of an $(n+k1)$-dimensional complex. Also E is dominated by an $(n+k)$-dimensional complex. And ψ_E is $(n-1)$-connected. We can now modify E_A (and ψ_E) by gluing on finitely many cells in the dimensions $n, n+1, \dots, n+k-1$, thereby obtaining an $(n+k-1)$-connected extension $\tau : Y \to E$ of ψ_E. Then, by Wall's definition,

$$(-1)^{n+k}[H_{n+k}(\widetilde{E}, \widetilde{Y})] \in w(E)$$

and, consequently,

$$(-1)^{n+k}[H_{n+k}(\hat{E}, \hat{Y})] \in w(E, \pi).$$

Here the ˜ indicates universal coverings and the ˆ the covering corresponding to the subgroup K.

One now defines the chain complex P_* as follows. For $n \le i < n+k$ one has $P_i = C_i(\hat{Y}, \hat{E}_A)$, the cellular chains of $(\hat{Y}, \hat{E}_A)$ with their usual free $\mathbf{Z}\pi$ module basis; also the boundary map $P_i \to P_{i-1}$ is the usual one. For $i = n+k$ we let $P_{n+k} = H_{n+k}(\hat{E}, \hat{Y})$. Finally, we take $P_{n+k} \to P_{n+k-1}$ to be the composition

$$H_{n+k}(\hat{E}, \hat{Y}) \xrightarrow{\partial} H_{n+k-1}(\hat{Y}, \hat{E}_A) \subseteq C_{n+k-1}(\hat{Y}, \hat{E}_A)$$

where ∂ is the boundary map of the triple (and we recall that $(\hat{Y}, \hat{E}_A)$ has cells only up to dimension $n+k-1$).

Property (ii) is then clear and (iii) follows from a simple computation using the long exact sequence for $(\hat{E}, \hat{Y}, \hat{E}_A)$ and the Serre spectral sequence for $\hat{p} : (\hat{E}, \hat{E}_A) \to (\widetilde{B}, \widetilde{A})$.

3. The transfer map on K_0

Let $\phi : R \to S$ be a ring homomorphism with the property that $\phi^* S$, i. e. S viewed as R module via ϕ, belongs to $\mathcal{P}_{<\infty}(R)$. Then, see e. g. [4], $\phi^* M \in \mathcal{P}_{<\infty}(R)$ for each $M \in \mathcal{P}(S)$ and the formula

$$\phi^*([M]) = \sum (-1)^i [P_i],$$

where $P_* \to \phi^* M$ is a finite resolution with $P_i \in \mathcal{P}(R)$, defines a group homomorphism $\phi^* : K_0(S) \to K_0(R)$, called the transfer map induced by ϕ.

If ϕ is a map of group rings $\mathbf{Z}\pi \to \mathbf{Z}\rho$ induced by a group homomorphism (also called) $\phi : \pi \to \rho$ then π acts on $\nu = \ker(\phi)$ by conjugation. This induces a π-action on $H_i(\nu; \mathbf{Z})$ which, in turn, factors over an action of $\overline{\pi} = \mathrm{Im}(\phi)$ (because inner automorphisms of ν induce the identity map on $H_i(\nu; \mathbf{Z})$). Thus $H_i(\nu; \mathbf{Z})$ represents an element $[H_i(\nu; \mathbf{Z})]$ of $G(\overline{\pi})$ = the Grothendieck ring of $\mathbf{Z}\overline{\pi}$ modules which are finitely generated over $\mathbf{Z}$. Recall also, from [13], that $K_0(\mathbf{Z}\overline{\pi})$ is a right module over $G(\overline{\pi})$ the action being given by $\otimes_{\mathbf{Z}}$ and the isomorphism $K_0(\mathbf{Z}\overline{\pi}) \cong K_0(\mathcal{P}_1(\mathbf{Z}\overline{\pi}))$. This action is used in

Proposition 3.1. *Let*

$$1 \to \nu \to \pi \xrightarrow{\phi} \rho$$

be an exact sequence of groups. The transfer map $\phi^* : K_0(\mathbf{Z}\rho) \to K_0(\mathbf{Z}\pi)$ *is defined if the following two conditions hold:*

(i) v *is of type (FP),*

(ii) The index $[\rho : \mathrm{Im}(\phi))]$ *is finite. Moreover, if this is the case and* ϕ *is onto, then*

(iii) $\phi_*\phi^*(x) = x \cdot \sum(-1)^i[H_i(\nu; \mathbf{Z})], \quad x \in K_0(\mathbf{Z}\rho).$

The proof is deferred to the end of this section.

There is one other situation of interest to us where a transfer map is defined. Let

$$1 \to C_k \to \pi \to \overline{\pi} \to 1$$

be an exact sequence of groups with C_k cyclic of order k, generator t. Let $N = 1+t++t^{k-1} \in \mathbf{Z}\pi$. The left ideal (N) is ,then two-sided and one has the commutative diagram of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\phi} & \mathbf{Z}\overline{\pi} \\ {\scriptstyle r}\downarrow & & \\ \mathbf{Z}\pi/(N) & \xrightarrow[\psi]{} & (\mathbf{Z}/k)\overline{\pi} \end{array}$$

with r and $\overline{r}$ projections. Let the $\overline{\pi}$-action on C_k be given by the homomorphism $\varepsilon : \overline{\pi} \to \mathrm{Aut}(C_k) = (Z/k)^*$. One easily checks that the formula

$$\alpha(r(\xi)) = r(1 + t + t^2 + \cdots + t^{n-1})r(\xi), \qquad \xi \in \pi,$$

where $n > 0$ has $\xi t \xi^{-1} = t^n$, gives a (well-defined) automorphism a of the ring $\mathbf{Z}\pi/(N)$. Note that when C_k is central, α is the identity automorphism.

Similarly there is an automorphism β of $(\mathbf{Z}/k)\overline{\pi}$ defined by

$$\beta(\overline{r}(n\overline{\xi})) = \overline{r}(n\overline{\xi}), \qquad \overline{\xi} \in \overline{\pi},$$

where $\varepsilon(\overline{\xi})$ is represented by $n > O$. Again this is the identity when ε is trivial. Clearly $\psi\alpha = \beta\psi$.

Proposition 3.2. *The transfer* ψ^* *is defined and*

(i) $\psi^*\psi_*(y) = y - \alpha^*(y), y \in K_0(\mathbf{Z}\pi/(N))),$

(ii) $\psi_*\psi^*(x) = x - \beta^*(x), x \in K_0((\mathbf{Z}/k)\overline{\pi}).$

Especially $\psi^*\psi_* = \psi_*\psi^* = 0$ *when* C_k *is central.*

Proof of Proposition 3.1. If $P_* \to \mathbf{Z}$ is a finite projective resolution over $\mathbf{Z}\nu$ then $\mathbf{Z}\pi \otimes_{\mathbf{Z}\nu} P_* \to \mathbf{Z}\pi \otimes_{\mathbf{Z}\nu} \mathbf{Z} = \mathbf{Z}\overline{\pi}$, where $\overline{\pi} = \mathrm{Im}(\phi)$, is a finite projective resolution over $\mathbf{Z}\pi$. Since, assuming (ii), $\mathbf{Z}\rho$ is a direct sum of finitely many copies of $\mathbf{Z}\overline{\pi}$, as $\mathbf{Z}\pi$ module, we see that (i) and (ii) do imply that ϕ^* is defined.

To prove (iii) we choose a finite projective resolution Q_* for $\phi^* M$. Then

$$\phi_* \phi^*([M]) = \sum (-1)^i [\mathbf{Z}\rho \otimes_{\mathbf{Z}\pi} \mathbf{Q}_i]$$

so all we need, is the following pair of lemmas.

Lemma 3.3. *The homology of* $\mathbf{Z}\rho \otimes_{\mathbf{Z}\pi} \mathbf{Q}_*$ *is* $M \otimes_{\mathbf{Z}} H_*(\nu; \mathbf{Z})$ *where* ρ *acts diagonally on the latter tensor product.*

Lemma 3.4. *If a finite chain complex* P_* *over a ring* R *has each* P_i *and each* $H_i(P_*)$ *in* $\mathcal{P}_{<\infty}(R)$ *then*

$$\sum (-1)^i [P_i] = \sum (-1)^i [H_i(P_*)]$$

in $K_0(\mathcal{P}_{<\infty}(R)) \cong K_0(R)$.

In fact it follows from [13] that $M \otimes_{\mathbf{Z}} H_i(\nu; \mathbf{Z}) \in \mathcal{P}_{<\infty}(\mathbf{Z}\rho)$ and that it represents $[M] \cdot [H_i(\nu; \mathbf{Z})]$ under the isomorphism $K_0(\mathcal{P}_{<\infty}(\mathbf{Z}\rho)) \to K_0(\mathbf{Z}\rho)$.

Proof of Lemma 3.3. Choose a finite right $\mathbf{Z}\nu$ projective resolution $R_* \to \mathbf{Z}$ and fix an element $\xi \in \pi$ with projection $\bar{\xi} \in \rho$. Let $i(\xi)$ be the automorphism of ν given by $\eta \to \xi\eta\xi^{-1}, \eta \in \nu$; also denote the induced ring homomorphism $\mathbf{Z}\nu \to \mathbf{Z}\nu$ by $i(\xi)$. Let $f_* : R_* \to R_*$ be a lifting of $1_{\mathbf{Z}}$ by a right $i(\xi)$ linear map. Also let $l(\xi)$ denote left multiplication by ξ. Then $f_* \otimes l(\xi) : R_* \otimes_{\mathbf{Z}} \mathbf{Z}\pi \to R_* \otimes_{\mathbf{Z}} \mathbf{Z}\pi$ respects the differential $d \otimes 1$ and the augmentation $\varepsilon \otimes 1 : R_0 \otimes_{\mathbf{Z}} \mathbf{Z}\pi \to \mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}\pi$. Also it factors to give a map of resolutions

$$\begin{array}{ccc} R_* \otimes_{\mathbf{Z}\nu} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z} \otimes_{\mathbf{Z}\nu} \mathbf{Z}\pi = \mathbf{Z}\rho \\ \Big\downarrow {\scriptstyle f_* \otimes_{\mathbf{Z}\nu} l(\xi)} & & \Big\downarrow {\scriptstyle l(\bar{\xi})} \\ R_* \otimes_{\mathbf{Z}\nu} \mathbf{Z}\pi & \longrightarrow & \mathbf{Z} \otimes_{\mathbf{Z}\nu} \mathbf{Z}\pi = \mathbf{Z}\rho \end{array}$$

Now the desired homology groups are precisely $\mathrm{Tor}_*^{\mathbf{Z}\pi}(\mathbf{Z}\rho, M)$ which can also be computed from the chain complex $R_* \otimes_{\mathbf{Z}\nu} \mathbf{Z}\pi \otimes_{\mathbf{Z}\pi} M \cong R_* \otimes_{\mathbf{Z}\nu} M$ in which case the action of $\bar{\xi}$ is induced by the map $f_* \otimes_{\mathbf{Z}\nu} l(\xi) \otimes_{\mathbf{Z}\pi} 1_M$. This map corresponds to $f_* \otimes_{\mathbf{Z}\nu} l(\bar{\xi})$ under the above isomorphism. Since ν acts trivially on M and M is free over $\mathbf{Z}$ it follows that the homology is as stated.

Proof of Lemma 3.4. By induction on the length of the chain complex P_* using the exact sequence

$$0 \to P' \to P \to P'' \to 0$$

where $P_i'' = 0$ for $i \neq n$, $P'' = P_n$, and $P_i = 0$ for $i > n$. We shall leave the details to the reader.

Proof of Proposition 3.2. For any $P \in \mathcal{P}(\mathbf{Z}\pi//(N))$ define $f : \alpha^* P \to P$ by $f(x) = r(t-1)x, x \in P$. The obvious formula

$$r(t-1)\alpha(r(\xi)) = r(\xi)r(t-1), \qquad \xi \in \pi,$$

shows that f is a morphism of left $\mathbf{Z}\pi/(N)$ modules. Furthermore, f is monic; since f respects direct sums one only has to verify this for the case $P = \mathbf{Z}\pi/(N)$ where it is trivial. Finally, it is obvious that the cokernel of f is $(\mathbf{Z}/k)\overline{\pi} \otimes_{\mathbf{Z}\pi/(N)} P$. Thus $f : \alpha^* P \to P$ is a projective resolution of $(\mathbf{Z}/k)\overline{\pi} \otimes_{\mathbf{Z}\pi/(N)} P$ over $\mathbf{Z}\pi/N$. For $P = \mathbf{Z}\pi/(N)$ this show that ψ^* exists, and for general $P \in \mathcal{P}(\mathbf{Z}\pi/(N))$ it proves formula (i).

To prove (ii) we take any $M \in \mathcal{P}((\mathbf{Z}/k)\overline{\pi})$ and any finite resolution $P_* \to \psi^* M$ over $\mathbf{Z}\pi/(N)$. Then

$$\psi_*\psi^*([M]) = \sum (-1)^i [(\mathbf{Z}/k)\overline{\pi} \otimes_{\mathbf{Z}\pi/(N)} P_i]$$

and one gets the desired result by proving that the homology of the chain complex $(\mathbf{Z}/k)\overline{\pi} \otimes_{\mathbf{Z}\pi/(N)} P_*$ is M in degree 0, $\beta^*(M)$ in degree 1 and 0 elsewhere. This in turn amounts just to a computation of $\mathrm{Tor}_*^{\mathbf{Z}\pi/(N)}((\mathbf{Z}/k)\overline{\pi}, M)$ which is left to the reader.

Remarks. 1. It is easily seen that condition (ii) in Proposition 3.1 is also necessary for ϕ^* to be defined. In fact if is factored as

$$\pi \xrightarrow{\overline{\phi}} \overline{\pi} \xrightarrow{i} \rho$$

with $\overline{\phi}$ epic, i monic then ϕ^* exists iff $\overline{\phi}^*$ exists and $[\rho : \overline{\pi}] < \infty$, in which case $\phi^* = \overline{\phi}^* i^*$. Conceivably ϕ^* may be defined without $\nu = \ker(\phi)$ being of type (FP), although clearly ν must be of finite homological dimension.

2. If ν has a classifying space $B\nu$ which is a finite complex then clearly 3.1(ii) holds, see e.g. [15].

3. In general $\phi^* : K_0(S) \to K_0(R)$, when defined may not induce a homomorphism $\widetilde{K}_0(S) \to \widetilde{K}_0(R)$, i. e. it may not map $[S]$ to an integral multiple of $[R]$. This is the reason that we sometimes consider the Wall obstruction a subset of $K_0(\mathbf{Z}\pi)$ rather than an element of $\widetilde{K}_0(\mathbf{Z}\pi)$. For the situation in Proposition 3.1 one does have an induced $\phi^* : \widetilde{K}_0(\mathbf{Z}\rho) \to \widetilde{K}_0(\mathbf{Z}\pi)$ when ν is of type (FF), i.e. $\mathbf{Z}$ has a finite *free* resolution over $\mathbf{Z}\nu$. For the situation in Proposition 3.2 one does have an induced map $\psi^* : \widetilde{K}_0((\mathbf{Z}/k)\overline{\pi}) \to \widetilde{K}_0(\mathbf{Z}\pi/(N))$, the reason being that $\alpha^* = 1$ on $\mathbf{Z} \subseteq K_0(\mathbf{Z}\pi/(N))$

4. Fibrations with ν of type (FP)

Theorem 4.1. *Let $p : E \to B$ be a fibration as in §1. Let K be a subgroup of $\pi_1(E)$ such that p realizes the group extension (modulo K)*

$$1 \to \nu \to \pi \xrightarrow{\phi} \rho.$$

Assume that

(i) ν is of type (FP),

(ii) $H_i(\overline{F}; \mathbf{Z})$ is finitely generated over $\mathbf{Z}$ for all i.

Then

$$w(E, \pi) \supseteq \phi^*(w(B)) \cdot \sum (-1)^[H_i(\overline{F}; \mathbf{Z})]$$

(as subsets of $K_0(\mathbf{Z}\pi)$).

Remarks. 1. Corollary A of the introduction follows immediately, since with $K = 1$ and $\pi_1(F) \to \pi_1(E)$ injective, one gets $\overline{F}$ = the universal covering of F. Note that the reduced transfer exists because ν is of type (FF).

2. The main theorem of [13] is the case obtained by taking

$$\pi = \mathrm{Im}(\pi_1(E) \to \pi_1(B)),$$

except that Pedersen and Taylor allow more general fibers; compare the next remark.

3. As the reader will easily be able to see, one could replace the assumption that F be finite by the assumption that F be finitely dominated. One would then add to the formula for $w(E)$ a term of the form $\chi(A) \cdot i_* w(F)$ where $i : \pi_1(F) \to \pi_1(E) \to \pi$. Here A is as in the proof of Proposition 2.1. Since A is an awkward variable in the result we leave such a generalization to the reader.

Recall from Ehrlich, [6] or [8], that $G_\nu(\pi)$ is the Grothendieck group of $\mathbf{Z}\pi$ modules L which admit finite resolutions $P_* \to L$ over $\mathbf{Z}\pi$ with each $i^*P_j \in \mathcal{P}(\mathbf{Z}\nu), i : \nu \to \pi$ the inclusion. Also, ibidem, there is a pairing (where $\overline{\pi} = \pi/i(\nu)$)

$$K_0(\mathbf{Z}\overline{\pi}) \otimes G_\nu(\pi) \to K_0(\mathbf{Z}\pi)$$

given by $\otimes_{\mathbf{Z}}$ with diagonal π-action. Using this one has

Theorem 4.2. *Let $p : E \to B$ be a fibration as in §1, realizing the extension $1 \to \nu \to \pi \to \rho$ modulo K. Assume that*

(i) Each $H_i(\overline{F}; \mathbf{Z})$ represents an element of $G_\nu(\pi)$.

Then

$$w(E, \pi) \supseteq j^*(w(B)) \cdot \sum (-1)^i [H_i(\overline{F}; \mathbf{Z})]$$

where $j : \overline{\pi} \to \rho$ is the inclusion.

Remarks. 1. This has also been proved by Ehrlich [6],[8].

2. Condition 4.2(i) does imply 4.1(i), but not 4.1(ii). In fact take $F = S^1 \vee S^2$ and $E = F \times B$. Then $H_*(\overline{F}; \mathbf{Z})$ satisfies 4.2(i) but not 4.1(ii). Also, 4.1(i) and 4.1(ii) do not imply 4.2(i), as will be shown by an example below. Thus neither of the two theorems implies the other. It can be shown, purely algebraically, that when both theorems apply the results coincide.

Example 4.3. *Let π be generated by a and b with relations $[a, [a, b]] = [b, [a, b]] = 1$. Let ν be the infinite cyclic subgroup generated by the central element $t = [a, b]$. Then $\overline{\pi} = \pi/\nu = \mathbf{Z} \oplus \mathbf{Z}$. We shall show that no π module can be finitely generated and projective over $\mathbf{Z}\nu$. In fact if P is such a module then P is actually free over $\mathbf{Z}\nu$, see e. g. [18]. Choosing a $\mathbf{Z}\nu$ basis for P, the actions of a and b are given by matrices A and B over $\mathbf{Z}\nu$. The commutator of A and B must be a diagonal matrix with all entries $= t$. Taking determinants we get the contradiction $t^n = 1$.*

Proof of Theorem 4.1. In view of Proposition 2.1 and Lemma 3.4 we only need to show

Lemma 4.4. *If $M \in \mathcal{P}_{<\infty}(\mathbf{Z}\pi)$ and M is $\mathbf{Z}$-torsion free, while L is a $\mathbf{Z}\pi$ module, finitely generated over $\mathbf{Z}$, then $M \otimes_{\mathbf{Z}} L \in \mathcal{P}_{<\infty}(\mathbf{Z}\pi)$ and $[M \otimes_{\mathbf{Z}} L] = [M][L]$ in $K_0(\mathbf{Z}\pi) = K_0(\mathcal{P}_{<\infty}(\mathbf{Z}\pi))$.*

Remark. This generalizes Proposition 2.3 of [13].

Proof. The short exact sequence of $\mathbf{Z}\pi$ modules

$$O \to T \to L \to F \to O$$

where T is $\mathbf{Z}$-torsion and F is $\mathbf{Z}$-torsion free gives a short exact sequence

$$0 \to M \otimes_{\mathbf{Z}} T \to M \otimes_{\mathbf{Z}} L \to M \otimes_{\mathbf{Z}} F \to 0$$

of $\mathbf{Z}\pi$ modules. The two sequences show that it suffices to treat the cases $L = T$ and $L = F$ separately. For $L = T$ one chooses, as in [13], a resolution

$$O \to Q_1 \to Q_0 \to T \to 0$$

of T by $\mathbf{Z}\pi$ modules, free and finitely generated over $\mathbf{Z}$. For $L = F$ one takes $Q_1 = 0$ and $Q_0 = F$. Also let $P_* \to M$ be a finite projective resolution for M. Then $P_* \otimes_{\mathbf{Z}} Q_* \to M \otimes_{\mathbf{Z}} L$ is a finite projective resolution for $M \otimes_{\mathbf{Z}} L$ (it is here one needs M $\mathbf{Z}$-torsion free) so

$$\begin{aligned}[M \otimes_{\mathbf{Z}} L] &= \sum (-1)^{i+j}[P_i \otimes_{\mathbf{Z}} Q_j] \\ &= \sum (-1)^i [P_i] \cdot \sum (-1)^j [Q_j].\end{aligned}$$

Since $\sum (-1)^j [Q_j] \in G_{\mathbf{Z}}(\pi)$ is the element corresponding to $[L]$ under the isomorphism $G(\pi) \to G_{\mathbf{Z}}(\pi)$, see [13], this finishes the proof.

Proof of Theorem 4.2. This follows immediately from Proposition 3.1 and Lemma 3.4 since the hypotheses guarantee that $M \otimes_{\mathbf{Z}} H_i(\overline{F}; \mathbf{Z}) \in \mathcal{P}_{<\infty}(\mathbf{Z}\pi)$.

5. Fibrations with $\overline{F} = S^{2l-1}, \nu$ cyclic

Theorem 5.1. *Assume that p is a fibration as in §1, that $\nu = C_k$ is cyclic of order k, that F is connected and that $\overline{F} \equiv S^{2l-1}$. Then, with the notation of §3, one has*

$$r_* \tilde{w}(E, \pi) = \psi^* \left(\sum_{i=0}^{l-1} (\beta^*)^i \overline{r}_* (\tilde{w}(B)) i \right)$$

in $\widetilde{K}_0(\mathbf{Z}\pi/(N))$.

Remarks. 1. Corollaries B and C of the introduction are immediate consequences.

2. Note that the pull back diagram of §3 gives an exact sequence

$$K_1((\mathbf{Z}/k)\pi_1(B)) \xrightarrow{\partial} \widetilde{K}_0(\mathbf{Z}\pi) \xrightarrow{(r_*, \phi_*)} \widetilde{K}_0(\mathbf{Z}\pi/(N)) \oplus \widetilde{K}_0(\mathbf{Z}\pi_1(B)).$$

Corollaries B and C compute $r_*\tilde{w}(E)$ (for the relevant fibrations) and [13] (or our Theorem 4.1) computes $\phi\tilde{w}(E)$. We have no information on the relationship between $\tilde{w}(E)$ and the map ∂.

Proof of Theorem 5.1. We start by proving

Lemma 5.2. *For the exact sequence*

$$1 \to C_k \to \pi \xrightarrow{\phi} \overline{\pi} \to 1$$

and any $L \in \mathcal{P}(\mathbf{Z}\overline{\pi})$, *one has*

$$\begin{aligned}\operatorname{Tor}_i^{\mathbf{Z}\pi}(\mathbf{Z}\pi/(N), \phi^* L) &= 0, i \textit{ odd} ,\\ &\psi^*(\beta^*)^{i/2}\overline{r}_*(L).i \textit{ even} ,\end{aligned}$$

as left $\mathbf{Z}\pi/(N)$ *modules.*

Proof. We define a resolution of right $\mathbf{Z}\pi$ modules

$$\cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \xrightarrow{d_1} F_0 \to r^*(\mathbf{Z}\pi/(N)) \to 0$$

as follows. The module F_i is $\mathbf{Z}\pi$ and one has, for $x \in \mathbf{Z}\pi$,

$$\begin{aligned} d_i x &= Nx, i \text{ odd} ,\\ &= (t1)x, i \text{ even} .\end{aligned}$$

Exactness is easily proved. To find the $\mathbf{Z}\pi/(N)$ structure on Tor_* we fix an element ξ of π, and let $l(\xi)$ be left multiplication by ξ on $r^*(\mathbf{Z}\pi/(N))$. We must lift $l(\xi)$ to a map of resolutions. Let $\xi t \xi^{-1} = t^n$ with $n > 0$ and define $T \in \mathbf{Z}\pi$ to be $1 + t + t^2 + \cdots + t^{n-1}$. Then the maps $l_i : F_i \to F_i$ given by

$$l_i(x) = T^{[i/2]}\xi x$$

define the required lifting. Since t acts trivially on $\phi^*(L)$ one gets $\operatorname{Tor}_{\text{odd}} = 0$ and

$$\operatorname{Tor}_{2i}^{\mathbf{Z}\pi}(\mathbf{Z}\pi/(N), \phi^*(L)) = L/kL$$

additively. Moreover $r(\xi) \in \mathbf{Z}\pi/(N)$ acts on $\operatorname{Tor}_{2i} = L/kL$ as left multiplication by

$$\overline{r}(n^i\phi(\xi)) = \beta^i\overline{r}(\phi(\xi)) = \beta^i\psi r(\xi).$$

And this precisely means that

$$\operatorname{Tor}_{2i}^{\mathbf{Z}\pi}(\mathbf{Z}\pi/(N), \phi^*(L)) = \psi * (\beta^*)^i\overline{r}_*(L) = \psi^*(\beta_*)\overline{r}_*(L).$$

To prove the theorem we consider M and P_* as in Proposition 2.1 and we let $Q_* = \mathbf{Z}\pi \otimes_{\mathbf{Z}\pi} P_*$. Then $r_*\tilde{w}(E, \pi) = \sum(-1)^i[Q_i]$. We want to compute this by means of Lemma 3.4. The Künneth spectral sequence converging to $H_*(Q_*)$ has

$$E^2_{*i} = \operatorname{Tor}_*^{\mathbf{Z}\pi}(\mathbf{Z}\pi/(N), H_i(P_*)).$$

Since $\overline{F} = S^{21-1}$ this becomes

$$\begin{aligned} E^2_{st} &= 0 \text{ for } s \text{ odd or } t \neq n, 21-1+n, \\ &= \psi^*(\beta^*)^i \overline{r}_*(M), \text{ for } s = 2i, t = n, \\ &= \psi^*(\beta^*)^i \overline{r}_*(M^{(t)}), \text{ for } s = 2i, t = 21-1+n. \end{aligned}$$

Here $M^{(t)} = M \otimes_{\mathbf{Z}} H_{2l-1}(S^{21-1})$ with diagonal $\mathbf{Z}\overline{\pi}$-action. Now the vanishing of $H_i(Q_*)$ for $i > 21-1+n$ shows that the differential d^{2l} gives an isomorphism

$$\psi^*(\beta^*)^{l+i}\overline{r}_*(M) \cong \psi^*(\beta^*)\overline{r}_*(M^{(t)}), \qquad i > 0. \tag{5.3}$$

Also, it fits into a short exact sequence

$$0 \to \psi^*(\beta^*)^l \overline{r}_*(M) \to \psi^* \overline{r}_*(M^{(t)}) \to H_{2l-1+n}(Q_*) \to 0. \tag{5.4}$$

Finally,

$$\psi^*(\beta^*)^i \overline{r}_*(M) \cong H_{2i+n}(Q_*) \text{ for } 0 \leq i < l \tag{5.5}$$

and

$$H_{2i-1+n}(Q_*) = 0, \qquad i < 1. \tag{5.6}$$

By (5.4)-(5.6) each $H_i(Q_*) \in \mathcal{P}_{<\infty}(\mathbf{Z}\pi/(N))$ so Lemma 3.4 applies. Also (5.3) holds even for $i = 0$ (cancel ψ^*, noticing that ψ, is onto, and apply $\psi^*(\beta^*)^{-1}$ to the case $i = 1$). Therefore, (5.4) implies that $[H_{2l-1+n}(Q_*)] = 0$ in $\widetilde{K}_0(\mathcal{P}_{<\infty}(\mathbf{Z}\pi/(N)))$. The desired result follows.

6. Fibrations with $F = S^{2l-1}/G$

. Let G be a finite group of order k acting freely on S^{2l-1}, $l > 1$. We consider fibrations with $F = S^{2l-1}/G$ and $\pi_1(F) = G \to \pi = \pi_1(E)$ injective. Let the corresponding sequence of fundamental groups be

$$1 \to G \to \pi \xrightarrow{\phi} \overline{\pi} \to 1. \tag{$\mathfrak{E}$}$$

Also let $\mathbf{Z}^{(t)}$ be $H_{2l_1}(S^{2l-1})$ with the resulting $\mathbf{Z}\overline{\pi}$ module structure. If $M \in \mathcal{P}(\mathbf{Z}\overline{\pi})$ we write $M^{(t)}$ for $M \otimes_{\mathbf{Z}} Z^{(t)}$ with diagonal action.

Definition 6.1. *Define a relation $\gamma = \gamma(\mathfrak{E}, \mathbf{Z}^{(t)})$ from $\widetilde{K}_0(\mathbf{Z}\overline{\pi})$ to $\widetilde{K}_0(\mathbf{Z}\pi)$ as follows. Let $x = [M] \in \widetilde{K}_0(\mathbf{Z}\overline{\pi})$. Then $\gamma(x)$ is defined if there exists an exact sequence of $\mathbf{Z}\pi$ modules with each $P_i \in \mathcal{P}(\mathbf{Z}\pi)$*

$$0 \to \phi^* M^{(t)} \to P_{2l-1} \to P_{2l-2} \to \cdots \to P_0 \to \phi^* M \to 0.$$

In that case $\gamma(x)$ consists of all $\gamma = \sum(-1)^i[P_i]$ obtained from such a sequence.

Let $r : \mathbf{Z}\pi \to \mathbf{Z}\pi/(N)$, where $N = \sum_{g \in G} g$, be the projection.

Theorem 6.2. *Let $p : E \to B$ be a fibration with $F = S^{2l-1}/G$ and $\pi_1(F) \to \pi_1(E)$ injective. The relation γ from $\widetilde{K}_0(\mathbf{Z}\pi_1(B))$ to $\widetilde{K}_0(\mathbf{Z}\pi_1(E))$ is then everywhere defined. Also $r_* \circ \gamma : \widetilde{K}_0(\mathbf{Z}\pi_1(B)) \to \widetilde{K}_0(\mathbf{Z}\pi_1(E)/(N))$ is single valued and*

$$r_*\tilde{w}(E) = (r_* \circ \gamma)\tilde{w}(B).$$

Proof. Since the given group extension is realized by some fibration having fiber S^{2l-1}/G, it is also realized by a fibration $S^{2l-1}/G \to E_0 \to B_0$ with B_0 a finite complex. By Theorem F of [19] (or rather its proof) there is a complex B' and an $(n-1)$ equivalence (some $n > 2$) of B' with B_0 such that $\tilde{w}(B') =$ any chosen $\tilde{w} \in \widetilde{K}_0(\mathbf{Z}\pi_1(B))$. Proposition 2.1, and the remark after it, then shows that γ is everywhere defined. We next prove

Lemma 6.3. *Under the assumptions of the theorem $\gamma(x)$ consists precisely of a full coset modulo the image of the boundary map*

$$\partial : K_1((\mathbf{Z}/k)\pi_1(B)) \to \widetilde{K}_0(\mathbf{Z}\pi_1(E)).$$

Proof. First note that ∂ comes from the pull back diagram

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\phi} & \mathbf{Z}\overline{\pi} \\ \downarrow & & \downarrow \\ \mathbf{Z}\pi/(N) & \longrightarrow & (\mathbf{Z}/k)\overline{\pi} \end{array} \tag{6.4}$$

where $\pi = \pi_1(E)$ and $\overline{\pi} = \pi_1(B)$, see e. g. [12]. Let $[P] \in \operatorname{Im}\partial$; that means that P fits into some pull back diagram

$$\begin{array}{ccccc} P & \longrightarrow & & & (\mathbf{Z}\overline{\pi})^m \\ \downarrow & & & & \downarrow \\ (\mathbf{Z}\pi/(N))^m & \longrightarrow & ((\mathbf{Z}/k)\overline{\pi})^m & \cong & ((\mathbf{Z}/k)\overline{\pi})^m \end{array}$$

where $\cong$ indicates an isomorphism of $(\mathbf{Z}/k)\overline{\pi}$ modules. We compare this with the direct sum of m copies of (6.4) and notice that $P \to (\mathbf{Z}\overline{\pi})^m$ and $(\mathbf{Z}\pi)^m \to (\mathbf{Z}\overline{\pi})^m$ have isomorphic kernels. Now Proposition 2.1 implies the existence of an exact sequence

$$0 \to ((\mathbf{Z}\overline{\pi})^{(t)})^m \to F_{2l-1} \to \cdots \to F_l \to (\mathbf{Z}\pi)^m \to (\mathbf{Z}\overline{\pi})^m \to 0$$

with each F_ν a free finitely generated $\mathbf{Z}\pi$ module. Thus one also has an exact sequence

$$0 \to ((\mathbf{Z}\overline{\pi})^{(t)})^m \to F_{2l-1} \to \cdots \to F_1 \to P \to (\mathbf{Z}\overline{\pi})^m - \to 0.$$

If

$$0 \to \phi^*M^{(t)} \to P_{2l-1} \to \cdots \to P_1 \to P_0 \to \phi^*M \to 0$$

is a sequence showing that $\gamma \in \gamma([M])$, then the direct sum of the two sequences shows that $\gamma + [P] \in \gamma([M \oplus (\mathbf{Z}\overline{\pi})^m]) = \gamma([M])$. Thus $\gamma([M])$ contains a full coset of $\operatorname{Im}(\partial)$.

To prove the opposite inclusion we consider the exact sequence

$$K_1((\mathbf{Z}/k)\overline{\pi}) \to \widetilde{K}_0(\mathbf{Z}\pi) \xrightarrow{(r_*,\phi_*)} \widetilde{K}_0(\mathbf{Z}\pi/(N)) \oplus \widetilde{K}_0(\mathbf{Z}\overline{\pi}).$$

We have to show that $r_* \circ \gamma$ and $\phi_* \circ \gamma$ are single valued. If one has two exact sequences

$$0 \to \phi^* M^{(t)} \to P_k \to \cdots \to P_0 \to \phi^* M \to 0,$$
$$0 \to \phi^* M^{(t)} \to P'_k \to \cdots \to P'_0 \to \phi^* M \to 0$$

with $P_i, P'_i \in \mathcal{P}(\mathbf{Z}\pi)$ then the generalized Schanuel lemma, see e.g. [17], gives an isomorphism

$$\phi^* M^{(t)} \oplus \bigoplus P_{2i} \oplus \bigoplus P'_{2i+1} \cong \phi^* M^{(t)} \oplus \bigoplus P'_{2i} \oplus \bigoplus P_{2i+1}. \tag{6.5}$$

If one applies ϕ^* to this and adds a stable inverse of $\phi_*\phi^*(M^{(t)}) = M^t$ one sees that $\phi_* \circ \gamma$ is single valued. If one applies r^* to (6.5) then one may cancel the terms $r_*\phi^*(M^{(t)})$ (because they are $\mathbf{Z}$-torsion modules-this uses the fact that the action of G on $H_{2l-1}(\overline{F}) = H_{2l-1}(S^{2l-1})$ is trivial, so that N acts as multiplication by k on $M^{(t)}$. The resulting isomorphism shows that $r_* \circ \gamma$ is single valued.

7. Proof of the realizability theorem

Let $1 \to \nu \to \pi \to \overline{\pi} \to 1$ be the given extension with action $\alpha : \overline{\pi} \to \mathrm{Aut}(\nu)$. If this is realized by $p : E \to B$ having fiber T^k then ν is the cokernel of $\pi_2(B) \to \pi_1(T^k)$ so condition (iv) is necessary.

Conversely, let $\tilde{\alpha} : \overline{\pi} \to Gl(k, \mathbf{Z})$ be an action on $\mathbf{Z}^k$ for which an epimorphism of $\mathbf{Z}\overline{\pi}$ modules $\mathbf{Z}^k \to \nu$ is given. Also fix an epimorphism $q : F \to \overline{\pi}$ where F is a free group with generators $a_i, i \in I$, I finite.

To construct $E \to B$ we start by taking $B^{(1)} = \vee_{i \in I} S^1$ and define $B^{(1)} \to BGl(k, \mathbf{Z})$ by letting the restriction to the ith S^1 represent $\tilde{\alpha}q(a_i) \in Gl(k, \mathbf{Z}) = \pi_1(BG1(k, \mathbf{Z}))$. Let $P^{(1)} \to B^{(1)}$ be the resulting principal $Gl(k, \mathbf{Z})$ bundle and take $E^{(1)} \to B^{(1)}$ to be the associated bundle with fiber T^k. This is a torus fibration realizing the middle row in the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{Z}^k & \longrightarrow & \mathbf{Z}^k \times_{\tilde{\alpha}q} F & \longrightarrow & F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \\
1 & \longrightarrow & \nu & \longrightarrow & \pi & \longrightarrow & \overline{\pi} & \longrightarrow & 1
\end{array}$$

The map into the given extension $\mathfrak{E}$ exists because the pull back of $\mathfrak{E}$ through q is split (F is free) and because the action $\tilde{\alpha}q$ is given. The respective kernels are denoted K', K, K''.

Let $\mathbf{Z}^k$ have generators $b_1, \ldots, b_k$ (viewed as a multiplicative group) and let $\mu_1, \ldots \mu_m$ be monomials in b_*, b_*^{-1} which generate K'. To get a set of generators for K we choose elements

$r_{m+1}, \dots r_{m+m_l}$ which generate K''. Recall that each element of $\mathbf{Z}^k \times_{\tilde{\alpha}q} F$ (and hence of K) has the form μr with $\mu \in \mathbf{Z}^k$, $r \in F$; then lift r_j to an element $\mu_j r_j \in K$, where μ_j is a monomial in the b_*, b_*^{-1} ($j = m+1, \dots, m_l$). To treat all generators of K alike put $r_j = 1$ for $j = 1, 2, \dots, m$

Then K is generated by $\mu_j r_j$ and K'' is generated by r_j where in both cases j runs from 1 to $m + m_l$. We want to glue onto $E^{(1)} \to B^{(1)}$ a copy of $e^2 \times T^k \to e^2$ (along some map from $S^1 \times T^k \to S^1$ to $E^{(1)} \to B^{(1)}$) in such a way that $\mu_j r_j$ is killed upstairs and r_j is killed downstairs. Using van Kampen's theorem we see that this is possible because of

Lemma 7.1. *There exists a commutative diagram*

$$\begin{array}{ccc} S^1 \vee T^k & & \\ {\scriptstyle i}\downarrow & \overset{g_j}{\searrow} & \\ S^1 \times T^k & \xrightarrow[f_j]{} & E^{(1)} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p^{(1)}} \\ S^1 & \xrightarrow[\rho_j]{} & B^{(1)} \end{array}$$

where ρ_j represents $r_j \in \pi_1(B^{(1)}) = F$ and g_j represents $\mu_j r_j + i$ in $\pi_1(E^{(1)}) \oplus [T^k, E^{(1)}]$. Here, in both instances, i is an inclusion.

Using this lemma we finish the proof as follows. For some $j \geq 0$ assume that we have already constructed a map of torus fibrations

$$\begin{array}{ccc} E^{(1)} & \longrightarrow & E^{(1,j)} \\ {\scriptstyle p^{(1)}}\downarrow & & \downarrow{\scriptstyle p^{(1,j)}} \\ B^{(1)} & \longrightarrow & B^{(1,j)} \end{array}$$

such that the kernel of the epimorphism $\pi_1(E^{(1)}) \to \pi_1(E^{(1,j)})$ is the subgroup generated by $\mu_1 r_1, \dots, \mu_j r_j$ and the kernel of the epimorphism $\pi_1(B^{(1)}) \to \pi_1(B^{(1,j)})$ is the subgroup generated by $r_1, \dots, r_j$.

To obtain $E^{(1,j+1)} \to B^{(1,j+1)}$ one glues $e^2 \times T^k \to e^2$ onto $E^{(1,j)} \to B^{(1,j)}$ along the composition

$$\begin{array}{ccccc} S^1 \times T^k & \xrightarrow{f_{j+1}} & E^{(1)} & \longrightarrow & E^{(1,j)} \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \xrightarrow{\rho_j} & B^{(1)} & \longrightarrow & B^{(1,j)} \end{array}$$

The van Kampen theorem applied to $E^{(1,j+1)} = E^{(1,j)} \cup e^2 \times T^k$ with $E^{(1,j)} \cap e^2 \times T^k = S^1 \times T^k$ presents $\pi_1(E^{(1,j+1)})$ as the push out of the diagram

$$\begin{array}{ccc} & & \pi_1 E^{(1,j)} \\ & \nearrow & \\ \pi_1(S^1 \times T^k) & & \\ & \searrow & \\ & & \pi_1(e^2 \times T^k) \end{array}$$

The first generator of $\pi_1(S^1 \times T^k)$ maps to (the image of) $\mu_{j+l} r_{j+1}$ in $\pi_1(E^{(1,j)})$ and to 1 in $\pi_1(e^2 \times T^k)$. Thus $\mu_{j+1} r_{j+1}$ vanishes under $\pi_1(E^{(1,j)}) \to \pi_l(E^{(1,j+1)})$. Since the other generators of $\pi_1(S^1 \times \mathbf{R}^k)$ map isomorphically onto $\pi_1(e^2 \times T^k)$ no further relations (and no new generators) are introduced by passing from $\pi_l(E^{(1,j)})$ to $\pi_1(E^{(1.j+1)})$. Similarly one checks that $\pi_l(B^{(1,j)}) \to \pi_1(B^{(1,j+1)})$ has kernel generated by the image of r_{r+1} and is onto.

Now $E^{(2)} = E^{(1.m+m_l)} \to B^{(1,m+m_l)} = B^{(2)}$ is a T^k fibration realizing the given group extension.

By Theorem F of [19] (or better its proof) we can wedge onto $B^{(2)}$ a finite wedge of 3-spheres, to obtain a finite complex X which dominates a space B, 2-equivalent to $B^{(2)}$ and with $\tilde{w}(B) = \tilde{w}$. The pull back of $E^{(2)}$ through the obvious map $B \to X \to B^{(2)}$ is then the desired fibration.

Proof of Lemma 7.1. Consider the commutative diagram.

$$\begin{array}{ccccc} [g_j] \in [S^1 \vee T^k, E^{(1)}] & \xleftarrow{\quad i^* \quad} & & & [S^1 \times T^k, E^{(1)}] \\ \downarrow p_*^{(1)} & & & & \downarrow p_*^{(1)} \\ [S^1 \vee T^k, B^{(1)}] & \xleftarrow{\quad i^* \quad} & & & [S^1 \times T^k, B^{(1)}] \\ & \nwarrow^{i^* p_1^*} & & \nearrow^{p_1^*} & \\ & & [S^1, B^{(1)}] & & \end{array}$$

The typical 2-cell of $(S^1 \times T^k, S^1 \vee T^k)$ is glued on by means of the commutator of $\iota_1 : S^1 \to S^1 \vee T^k$ and $\iota_{\nu+1} : S^1 \to S^1 \vee T^k$ where $\iota_{\nu+1}$ maps into the νth factor of T^k. Since $g_j(\iota_1) = \mu_j r_j$ and $g_j(\iota_{\nu+1}) \in \mathbf{Z}^k$ commute in $\pi_1(E^{(1)}) = \mathbf{Z}^k \times_{\tilde{\alpha} q} F$ (because $\tilde{\alpha} q(\mu_j r_j) = \tilde{\alpha}(1) = I \in Gl(k, \mathbf{Z})$) it follows that g_j extends over the 2- skeleton of $(S^1 \times T^k, S^1 \vee T^k)$. Since $\pi_i(E^{(1)}) = 0$ for $i > 2$, g_j then extends to all of $S^1 \times T^k$. Call the extension obtained f_j'. By obstruction theory the two maps i^* in the diagram are monic. Since $p_*^{(1)}(g_j)$ is trivial on T^k, it is in the image of $i^* p_1^*$. It follows that $p_*^{(1)}(f_j') = p_1^*(\rho_j)$ for some ρ_j. Clearly ρ_j represents r_j. Finally the homotopy lifting property allows one to replace f'_j by a homotopic f_j making the square honestly commutative.

From the realizability theorem and Proposition 2.1 we get the following.

Corollary 7.2. *Let*

$$1 \to C_k \to \pi \to \overline{\pi} \to 1$$

be a short exact sequence of finitely presented groups with C_k cyclic of order k and $\overline{\pi}$ acting on C_k via some $\varepsilon : \overline{\pi} \to \{\pm 1\} \subseteq \mathrm{Aut}(C_k)$. Then any $M \in \mathcal{P}(\mathbf{Z}\overline{\pi})$ admits a resolution by finitely generated projectives over $\mathbf{Z}\pi$ which is periodic of period 4 (period 2 if ε is trivial).

In case $\overline{\pi}$ is finite there is a purely algebraic proof of this which runs as follows. The spectral sequence of XVI.7 (3) of [5] gives an isomorphism.

$$\hom_{\mathbf{Z}\overline{\pi}} \simeq \mathrm{Ext}^*_{\mathbf{Z}\pi}(M, -)$$

of functors defined on left $\mathbf{Z}\pi$ modules. Since $H^*(C_k, -)$ is periodic of period 4 (the usual periodicity of period 2 breaks down because of the $\mathbf{Z}\overline{\pi}$ action), so is $\mathrm{Ext}^*_{\mathbf{Z}\pi}(M, -)$. It then follows from [9] that there is an exact sequence

$$0 \to M \oplus P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to M \to O$$

with $P_i \in \mathcal{P}(\mathbf{Z}\pi)$. Finally one argues as in [16] to remove the summand P_4. It is this latter step that requires π finite.

Question. Can one prove enough realizability results to get a similar corollary for any periodic group G (instead of C_l)? Or can one give a direct algebraic proof covering the general case?

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