SOME MILDLY WILD CIRCLES IN S^n ARISING FROM ALGEBRAIC K-THEORY

STEVEN C. FERRY AND ERIK KJÆR PEDERSEN

Abstract. We study wild embeddings of S^1 in S^n which are tame in a sense introduced by Quinn. We show that if π is a finitely presented group with $H_1(\pi) = H_2(\pi) = 0$, then any finiteness obstruction $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$ can be realized on the complement of such embeddings. For trivially symmetric σ, the embeddings constructed are shown to be isotopy homogeneous.

0. Introduction

The purpose of this paper is to study wild embeddings of S^1 in S^n which are tame in a sense introduce by Quinn [21]. Here is the definition (Figure 1): Let X and Y be compact ANR's with $X \subset Y$ and let $r : N \to X$ be a retraction from a closed neighborhood N of X in Y to X. We say that X is *end-tame* in Y if $r|(N - X) : N - X \to X$ is a controlled tame end, that is, if for each neighborhood U of X and $\varepsilon > 0$ there exists a neighborhood V of the end and a homotopy $h : (Y - X) \times I \to Y - X$ such that

(1) $h = \mathrm{id}$ on $(Y - X) \times \{0\} \cup (Y - U) \times I$.
(2) h takes $(U - X) \times I$ into $U - X$.
(3) $h((Y - X) \times \{1\}) \subset Y - V$.
(4) The tracks of $r \circ h$ have diameter less than ε.

Remarks. (i) Notice that end-tame embeddings need not be tame in the usual sense. If H^n is a homology sphere, then by work of Cannon and Edwards (see [4]) $\Sigma^2 H^n$ is homeomorphic to S^{n+2}. The suspension circle is end-tame but its complement is not simply connected.

(ii) The definition above is a controlled version of Siebenmann's tameness condition for ends, [27]. There is an equivalent dual definition (see [25, p.452]) which is a controlled version of Siebenmann's homotopy compression axiom. See [31], [32].

We are interested in understanding the possible homotopy types of $X - Y$ when Y is end-tamely embedded in X. It follows quickly from the definition of end-tame that if X and Y are compact ANR's with Y end-tamely embedded in X, then $X - Y$ is finitely dominated, i. e. there exits a finite simplicial complex K and maps $d : K \to X - Y$ and $u : X - Y \to K$ such that $d \circ u$ is homotopic to the identity. Wall [35] has shown that in such a situation $X - Y$ has the homotopy type of a finite complex if and only if a certain finiteness obstruction $\sigma = \sigma(X - Y) \in \widetilde{K}_0(\mathbb{Z}\pi_1(X - Y))$ vanishes. It is then reasonable to ask which finiteness obstructions are actually realized.

Key words and phrases. Lower K-theory, wild embedding, stratified space.

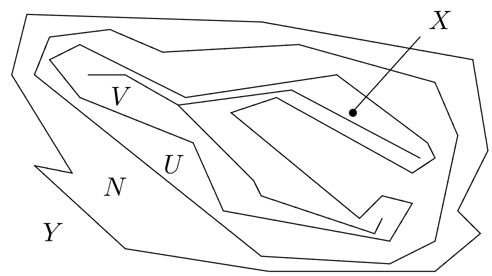


Fig. 1

We show in this paper that among end-tame embeddings of S^1 in S^n, all possible finiteness obstructions are realized. Here is our main result.

Theorem 1. *For each $n \geq 7$ and each finitely presented perfect groups π with $H_1(\pi) = H_2(\pi) = 0$, and $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$ there is an end-tame embedding $i : S^1 \to S^n$, such that $\pi + 1(S^n - i(S^1)) = \pi$, and such that $S^n - i(S^1))$ is finitely dominated with $\sigma(S^n - S^1)) = \sigma$.*

By the term *integral homology k-sphere*, we will mean any complex K which has the integral homology of a k-dimensional sphere. Alexander Duality tells us that the complement of any S^1 in S^n is an n-dimensional complex which is an integral homology $(n-2)$-sphere. Thus our proof requires the construction of a number of finitely dominated integral homology spheres which have non-vanishing finiteness obstructions.

Theorem 2. *For each $n \geq 5$ and each finitely presented perfect group π with $H_1(\pi) = H_2(\pi) = 0$ and $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$.*

(i) *there is a finitely dominated integral homology n-sphere X with $\pi_1(X) = \pi$ and $\sigma(X) = \sigma + (-1)^n\sigma^*$ such that $X \times S^1$ is homotopy equivalent to a closed $(n+1)$-manifold.*

(ii) *There is a finitely dominated integral homology n-sphere X with $\pi_1(X) = \pi$ and $\sigma(X) = \sigma$.*

Remark. X may be taken to be a two-ended $(n+1)$-manifold with tame ends in the sense of Siebenmann. (See the definition preceding Lemma 1.13.) Better yet, X may be taken to be $E - Z$, where $Z\mathbb{R}^n$ is a $1-\mathrm{LCC}$ embedded compactum and E is an I-regular neighborhood of Z. For material on I-regular neighborhoods, see the definitions following Proposition 1.4.

Definition. We will say that an embedding $i : S^1 \to S^n$ is *isotopy homogeneous* if every isotopy $h_t : i(S^1) \to i(S^1)$ with $h_0 = \mathrm{id}$ extends to an isotopy $\overline{h}_t : S^n \to S^n$ with $\overline{h}_0 = \mathrm{id}$.

Remark. It is easy to produce isotopy homogeneous wild embeddings of S^1 in S^n. If α is a wild arc in $D^{n=1}$, $S^1 \times [\alpha]$ is an isotopy homogeneous wild S^1 in $S^1 \times (D^{n-1}/\alpha) \cong S^1 \times D^{n-1}$. Sewing this along the boundary to $D^2 \times S^{n-1}$ creates the desired embedding in S^n.

Theorem 3. *For each $n \geq 7$ and each finitely presented perfect group π with $H_1(\pi) = H_2(\pi) = 0$, and $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$, there is an isotopy homogeneous embedding $S^1 \to S^n$, such that $\pi_1(S^n - i(S^1)) = \pi$, and such that $S^n - i(S^1)$ is finitely dominated with $\sigma(S^n - i(S^1)) = \sigma + (-1)^n\sigma^*$.*

Remarks. (i) Theorem 2(i) is nonvacuous, since $\pi = SL_2(F_{23})$ is a finite group with the properties above such that there is an element $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$ with $\sigma + (-1)^n\sigma \neq 0$.

(ii) Since the finiteness obstruction of a Poincar'e duality space over $\mathbb{Z}\pi$ is self dual, Theorem 0 implies:

Corollary. *There exist embeddings $i : S^1 \to S^n$ such that $S^n - i(S^1)$ is not a Poincar'e duality space over $\mathbb{Z}\pi_1(S^n - i(S^1))$, even though $\pi_1(S^n - i(S^1))$ is finite and $S^n = i(S^1)$ is finitely dominated.*

Let X be a compact ANR contained in a closed manifold M^n, $n \geq 6$. Let $r : U \to X$ be a retraction of a neighborhood of X in M to X. We say that $M = X$ *admits a controlled boundary* if there exist a manifold with boundary N, a homeomorphism $h : \overset{\circ}{N} \to M - X$, and extension of $r \circ h$ to a map $\bar{r} : h^{-1}(U) \cup \partial N \to X$. If X is end-tame in M and satisfies an appropriate stability condition on π_1, then Quinn's Controlled End Theorem [22] shows that $M - X$ admits a controlled boundary if and only if an obstruction in $H_0; \mathcal{S}(\mathbb{Z}\pi))$ is zero, where $\mathcal{S}(\mathbb{Z}\pi)$ is a spectrum with

$$\pi_1 \mathcal{S}(\mathbb{Z}\pi) = \mathrm{Wh}(\mathbb{Z}\pi) \qquad \text{and} \qquad \pi_{-i}\mathcal{S}(\mathbb{Z}\pi) = \widetilde{K}_{-i}(\mathbb{Z}\pi)$$

for $i \geq 0$. Note that the Atiyah-Hirzebruch spectral sequence shows that only the nonpositive homotopy groups of $\mathcal{S}(\mathbb{Z}\pi)$ are important in computing $H_0(X; \mathcal{S}(\mathbb{Z}\pi))$. The situation here is completely analogous to that arising in Siebenmann's thesis. The end obstruction is exactly the controlled finiteness obstruction of a sufficiently nice manifold neighborhood of the end.

In case $X \cong S^1$ and the π_1 system is a product, as in the examples constructed in this paper, this says that the obstruction to putting a controlled boundary on $M - X$ lies in $\widetilde{K}_0(\mathbb{Z}\pi) \oplus \widetilde{K}_{-1}(\mathbb{Z}\pi)$. We show that we can realize some of the $\widetilde{K}_{-1}(\mathbb{Z}\pi)$ obstructions by embeddeb S^1's in S^n.

Theorem 4. *For each $n \geq 7$ and each and each finitely presented group π with $H_1(\pi) = H_2(\pi) = 0$.*

- (i) *For each $\tau \in \widetilde{K}_{-1}(\mathbb{Z}\pi)$, there is an embedding $i : S^1 \to S^n$, such that $\pi_1(S^n - i(S^1)) = \pi$, $S^n - i(S^1)$ has the homotopy type of a finite complex, and such that the controlled end obstruction of $S^n - i(S^1)$ over S^1 is $\tau + (-1)^n\tau^*$.*
- (ii) *If π is finite and $n = 4k+3$, $k \geq 1$, then given any $\tau \in \widetilde{K}_{-1}(\mathbb{Z}\pi)$ with $\tau = (-1)^n\tau^*$, there is an embedding $i : S^1 \to S^n$, such that $\pi_1(S^n - i(S^1)) = \pi$, $S^n - i(S^1)$ has the homotopy type of a finite complex, and such that the controlled end obstruction of $S^n - i(S^1)$ over S^1 is τ.*

Moreover, the embeddings in (i) and (ii) are isotopy homogeneous.

Our proof uses many ingredients from other people's work. In outline, here is how it goes: Following [19, 18, 8, 35], form an acyclic three-dimensional compactum Z with fundamental group π and finiteness obstruction σ and embed it in $\mathbb{R}^n$, $n \geq 6$. Let E be an I-regular neighborhood of Z as in [31, 30, 32]. The space $E - Z$ is finitely dominated and is a cyclic cover of a closed topological manifold M^n having the homotopy type of $(E - Z) \times S^1$. We observe that the end obstructions of $E - Z$ are $(-1)^{n-1}\sigma^*$ (near Z) and σ (at the end of E). The finiteness obstruction of $E - Z$ is therefore $\sigma + (-1)^{n-1}\sigma^*$. Alexander duality shows that $E - Z$ is a homology sphere over Z, proving Theorem 2(i).

If M^n is the manifold constructed above and $p : M \to S^1$ is the map induced by $(E - Z) \times S^1 \to S^1$, consider $p \circ \text{proj} : M \times S^1 \to S^1 \times S^1$. The cover $(E - Z) \times \mathbb{R}^1 \to M \times S^1$ is the pullback of the cover $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ over $o \circ \text{proj}$, so there is an induced covering projection $q : (E - Z) \times \mathbb{R}^1 \to \mathbb{R}^2$. The radial compactification of $\mathbb{R}^2$ by S^1 induces a compactification X of $(E - Z) \times \mathbb{R}^1$ by adding a circle at ∞. We note that X is also the suspension of the two-point compactification of $E - Z$. Results of Cannon [4] show that X is therefore homeomorphic to S^{n+1}. See [2] for a similar compactification trick.). The 'suspension circle' is wild with complement having the homotopy type of $E-Z$. The suspension circle is homogeneously embedded because all points on the circle have local coordinate neighborhoods U such that $(U, U \cap S^1) \cong ((0,1) \times T, (0,1) \times \{t\})$, where T is an ANR homology manifoldwith a singularity at $\{t\}$. (T varies from point to point.)

This proves Theorem 0. The situation is quite analogous to that arising from 'honest' double suspension.

Theorems 0 and 2(ii) are proved by replacing $E - Z$ with $E - N$, where N is a suitable chosen finite acyclic polyhedron near Z. Again, one compactifies $E - N$ by adding two points and suspends the results. Cannon's theorem still shows that the resulting space is S^{n+1} with a wildly embedded S^1. Since N is finite, the finiteness obstruction of $E - N$ is σ. $S^n - S^1$ has the homotopy type of $E - N$, proving Theorem 2(ii). Since the 'homotopy link', see [25] for a definition, of S^1 in S^n is ∂N near some points of S^1 and $E - Z$ near others, this $S^1 \subset \mathbb{R}^{n+1}$ is not homogeneously embedded.

1. The Proof of Theorem 2

The proof of our Theorem 2 is similar to Kervaire's proof [19] that finitely presented groups π with $H_1(\pi) = H_2(\pi) = 0$ are precisely the fundamental groups of (manifold) integral homology spheres in dimensions ≥ 5. A key ingredient is the theorem of Hopf (see [3, p.1]) which states that if K is a CW complex with fundamental group π, then there is an exact sequence:

$$\pi_2(K) \xrightarrow{\rho} H_2(K) \xrightarrow{H}_2 (\pi) \to 0$$

where ρ is the Hurewicz map. For instance, one sees immediately from this sequence that $H_2(\pi)$ mist be 0 if $H_2(K) = 0$ for any CW complex K with fundamental group π, confirming the necessity of the H_2-condition when π is the fundamental group of a homology n-sphere with $n > 2$.

Lemma 1.1. *Let π be a finitely presented perfect group with $H_1(\pi) = H_2(\pi) = 0$, and let $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$. Then there is a* CW *complex $\overline{X}$ which is dominated by a three-dimensional complex such that $\pi_1(\overline{X}) = \pi$, $H_*(\overline{X} = H_*(pt)$ and $\sigma(\overline{X}) = \sigma$.*

Proof. Let K' be a finite two-dimensional complex with $\pi_1(K') = \pi$. Since K' is two-dimensional, $H_2(K')$ is a finitely generated free abelian group. By the theorem of Hopf quoted above, the Hurewicz map $\rho : \pi_2(K') \to H_2(K')$ is surjective, so we can pick a basis for $H_2(K')$ and attach three-cells to K' to form a three dimensional CW complex K'' with $H_*(K'') = H_*(pt)$.

Given $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$, represent σ by a finitely generated projective module P For some N, we can write $F^N = P \oplus Q$, where F^N is a free module on N generators over $\mathbb{Z}\pi$. Let A be the matrix of the projection $F^N = P \oplus Q \to Q \subset F^N$, Let

$K = K'' \vee \bigvee_{i=1}^{N} S^2$. Note that $\pi_2(K, K'') \cong F^N$ with the ith S^2 corresponding to the ith basis element in F^N. This is easily seen by passing to the universal cover and using the Hurewicz theorem. (The homotopy group is considered to be a $\mathbb{Z}\pi$-module via the usual action of π_1 on higher homotopy groups or, equivalently, via the action of the covering translation on the homotopy groups of the cover.) Define $\alpha : (K, K'') \to (K, K'')$ to represent A geometrically by defining α to be the identity on K'' and to send the ith copy of S^2 to the element of $\pi_3(K, K'')$ corresponding to the image of the ith basis element of F^N under A. Note that since $A^2 = A$, we have $\alpha \sim \alpha \circ \alpha$ rel K''. Let X be the infinite mapping telescope of α pictured below, where all of the K_i's are copies of K (Figure 2).

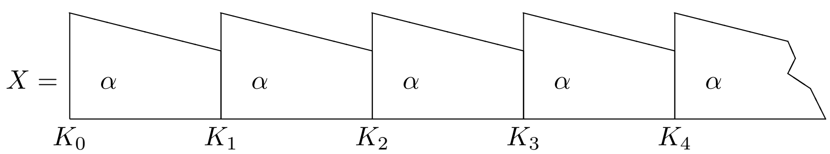


Fig. 2

We claim that the inclusion $i : K \to X$ of K into the top of X is a finite domination. To see this, first note that the obvious retraction $r : K \to K''$ splits the long exact homology sequence of the pair (K, K'') and we have $H_j(\widetilde{K}) \cong H_j(\widetilde{K}'') \oplus H_j(\widetilde{K}, \widetilde{K}'')$ for all j. Of course, this is just $H_j(\widetilde{K}'')$ for $j \neq 2$ and $H_2(\widetilde{K}'') \oplus F^n$ for $j = 2$. Here, the tildes denote universal covers. Since X is the direct limit of the system (K_i, α), we have $H_j(\tilde{X}) \cong H_j(\widetilde{K}'')$, for $j \neq 2$ and $H_2(\widetilde{K}'') \oplus Q$. Define a map $u : X \to K$ by letting $u = \alpha$ on each K_i and using the homotopy $\alpha \sim \alpha \circ \alpha$ to extend over the mapping cylinders. It is clear that $u_* : H_J(\widetilde{X}) \to H_j(\widetilde{K})$ is the identity for $j \neq 2$ and that it is the inclusion $H_2(\widetilde{K}'') \oplus Q \to H_2(\widetilde{K}'') \oplus F^N$ for $j = 2$. It follows that $\phi = i \circ u : X \to X$ is a homotopy equivalence and that i is a domination with the right inverse $u \circ \phi^{-1}$. Since the kernel of $\tilde{i} : H_*(\widetilde{K}) \to H_*(\widetilde{X})$ is isomorphism to P, it follows from [35] that $\sigma(X) = [P] = \sigma$.

The only problem is that X is not acyclic. If $C_*(\widetilde{K}')$, $C_*(\widetilde{X})$ are the chain complexes of $\widetilde{K}''$ and $\widetilde{X}$, it is clear that the inclusion $C_*(\widetilde{K}'') \oplus Q_* \to C_*(\widetilde{X})$ is a homology equivalence and, therefore, a chain equivalence, where Q_* is the chain complex with a single copy of Q in dimension 2. We have, therefore,

$$\begin{aligned} H_*(X) &\cong H_*(C_*(Z) \otimes_{\mathbb{Z}\pi} \mathbb{Z}) \cong H_*((C_*(\widetilde{K}'') \oplus Q_*) \otimes_{\mathbb{Z}\pi} \mathbb{Z}) \\ &\cong H_*(K'') \oplus (Q_* \otimes_{\mathbb{Z}\pi} \mathbb{Z}) \cong H_*(pt) \oplus (Q_* \otimes_{\mathbb{Z}\pi} \mathbb{Z}). \end{aligned}$$

Since $Q \otimes_{\mathbb{Z}\pi} \mathbb{Z}$ is finitely generated and projective over $\mathbb{Z}$, it is free on finitely many generators. By construction, these homology classes are in the image of the Hurewicz homomorphism, so we can attach finitely many three-cells to $K \subset X$ to form $\overline{X} \supset X$ with $H_*(\overline{X}) \cong H_*(pt)$. the sum theorem for the finiteness obstruction [27] tells us that $\sigma(\overline{X}) = \sigma(X) = \sigma$.

$\overline{X}$ is homotopy dominated by a three-dimensional complex, since any finitely dominated complex is dominated by a finite subcomplex of itself (any subcomplex containing the image of the domination will do) and every finite subcomplex of $\overline{X}$ is contained in a finite subcomplex which has the homotopy type (by collapsing mapping cylinders) of a three-dimensional complex. □

Lemma 1.2. *Let K^k be a finite polyhedron, $k \geq 3$, and let Q^{2k} be a* PL *manifold. Let $f : K \to Q$ be a map such that $f_* : \pi_1(K) \to \pi_1(Q)$ is surjective. then there is a k-dimensional complex $\overline{K}$ and a simple homotopy equivalence $d : K \to \overline{K}$ such that there is an embedding $\bar{f} : \overline{K} \to Q$ such that $\bar{f} \circ c = f$.*

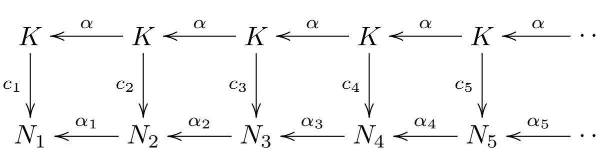
Proof. This is a special case of a theorem of Hudson and Stallings. See [33] or Theorem 12.1 of [15]. □

Definition 1.3. An embedding $i : X \to Y$ is said to be $1 - \mathrm{LCC}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that each map $\alpha : S^1 \to X - Y$ with $\mathrm{diam}(\alpha(S^1)) < \delta$ extends to a map $\bar{\alpha} : D^2 \to X - Y$ with $\mathrm{diam}(al\bar{p}ha(D^2)) < \varepsilon$. This is the basic condition for embeddings of compacta into manifolds in codimension ≥ 3.

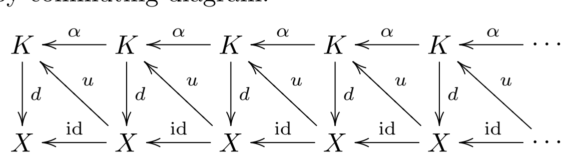
Proposition 1.4. *Let π be a finitely generated perfect group with $H_1(\pi) = H_2(\pi) = 0$. Let $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$ and let $n \geq 6$ be given. Then there is a $1 - \mathrm{LCC}$ embedded three-dimensional compactum $Z \subset \mathbb{R}^n$ which has the shape of a* CW *complex X such that $\pi_1(X) = \pi$, $H_*(X) = H_*(pt)$ and $\sigma(X) = \sigma$*

Proof. Compare [8]). Choose a CW complex X as in the conclusion of Lemma 1.1. That is choose X dominated by a three-dimensional complex with $\pi_1(X) = \pi$, $H_*(X) = H_*(pt)$, and $\sigma(X) = \sigma$¿ Let $d : K \to X$, $u : X \to K$ be maps with $u \circ d \sim \mathrm{id}_X$ such that K is three-dimensional. Then let $\alpha : K \to K$ be the map $u \circ d$. Note that $\alpha^2 \sim \alpha$. The construction in Lemma 1.1 produces K and α such that $\alpha_* : \pi_1(K) \to \pi_1(K)$ is an isomorphism.

We first wish to construct a homotopy commuting diagram:



where the c_i's are the homotopy equivalences, the N_i are regular neighborhoods of polyhedra K_i in $\mathbb{R}^n$, and the maps α_i are inclusions. This is easily accomplished using Lemma 1.2. Map K into $\mathbb{R}^n$. By Lemma 1.2 (or generalposition if $n \geq 7$)) there is a simple homotopy equivalence c_i from K to a polyhedron K_1 which embeds in $\mathbb{R}^n$. Let N_1 be a regular neighborhood of K_1 in $\mathbb{R}^n$. Now, let $\alpha_1' : K \to N_1$ be the map induced by α. By Lemma 1.2 or general position, as before, there exists a finite polyhedron K_2, a simple homotopy equivalence $c_2 : K \to K_2$, and an embedding $K_2 \to N_1$ in the homotopy class determined by α. Let N_2 be a regular neighborhood of K_2 in N_1 and continue in this fashion. The diagram we have constructed shows that $Z = \cap N_i$ is shape equivalent to $\varprojlim(K, \alpha)$. We have another homotopy commuting diagram:



which shows that $\varprojlim(K, \alpha)$ is shape equivalent to X. Thus Z is shape equivalent to X. The reader unaccustomed to shape theory should take the existence of such diagrams as the definition of shape equivalence. Štanko's approximation theorem [34] guarantees that we can approximate this embedding arbitrarily closely to a

$1-\mathrm{LCC}$ embedding. (Actually, we could guarantee a $1-\mathrm{LCC}$ result by choosing each N_i during the construction to be a sufficiently small regular neighborhood of K_i.) □

ADDENDUM 1.5. *Each of the N_i constructed above contains a finite acyclic complex L_i such that $\pi_1(L_i) \to \pi_1(N_i)$ is an isomorphism.*

Proof. The polyhedron N_i is a regular neighborhood of K_i, which is simple homotopy equivalent to K. Recall that K contains the acyclic polyhedron K'' constructed in the first paragraph of the proof of Lemma 1.1. By construction, $\pi_1(K'') \to \pi_1(K)$ is an isomorphism.

There is therefore a map $K'' \to N$, which induces an isomorphism on π_1. By Lemma 1.2, there is a polyhedron $L_i \subset N_i$ which is simple-homotopy equivalent to K''. □

Next we need to recall some facts from Siebenmann's theory of I-regular neighborhoods.

DEFINITION 1.6. If U and V are neighborhoods of Z in some space Y, with $V \subset U$, we write $V \searrow Z$ in U if for each neighborhood W of Z there is an ambient isotopy of Y supported on U throwing V into W while fixing some small neighborhood of Z. A neighborhood E of Z in Y in an *I-regular neighborhood* if $E = \cup E_i$ where $E_0 \subset E_1 \subset E_2 \subset \cdots$ and $E_i \searrow Z$ in E_{i+1}.

THEOREM. *(*[31, §1]*)* (Existence) *A* $1-\mathrm{LCC}$ *embedded compactum Z in the interior of a manifold $M^n m \geq 5$, has an I-regular neighborhood if and only if Z has the shape of a* CW *complex.*

(Uniqueness) *If E and E' are two I-regular neighborhoods of Z in Y, then there is an isotopy of open embeddings $g_t : E \to Y$, $0 \leq t \leq 1$, fixing a small neighborhood of Z such that $g_1(E) = E'$.*

PROPOSITION 1.7. *If Z is a compact subset of a manifold Y and E is an I-regular neighborhood of Z, then Z is shape equivalent to $E \cup Z$.*

Proof. Let $E_0 \subset E_1 \subset E_2 \subset E+3$ be neighborhoods of Z in Y such that $E_i \searrow Z$ in E_{i+1} for $i = 0, 1, 2$. Let $h_t : Y \to Y$ be an isotopy supported on E_2 such that $h(E_1) \subset E_0$. Then $h(E_1) \searrow Z$ in E_1 and conjugation by h shows that $h^i(E_1) \searrow Z$ in $h^{i-1}(E_1)$ for all $-\infty < i < \infty$. It follows that $F = \bigcup_{i=0}^{\infty} h^{-1}(E_1)$ is an I-regular neighborhood of Z contained in E_2.

Given any neighborhood U of Z, let $k : Y \to Y$ be a homeomorphism supported on E_3 with $k(F) \subset k(E_2) \subset U$. Then $k(F)$ is also an I-regular neighborhood of Z and the uniqueness theorem applies to $k(F) \subset F$ (forget about the rest of Y for a moment) to show that the inclusion of $k(F)$ into F is a homotopy equivalence. Thus, Z has a basis of neighborhoods $\{U_i\}$ in Y such that $U_0 \supset U_1 \supset U_2 \supset \cdots$ and each $U_{i+1} \to U_i$ is a homotopy equivalence. As in the proof of Proposition 1.4, this shows that Z is shape equivalent to each U_i. Since $U_i \sim E$ for all i, the proof is complete. □

COROLLARY 1.8. *If Z is compact, $Z \subset Y$ and E is an I-regular neighborhood of Z, then for each neighborhood U of Z there is an I-regular neighbrohood F of Z contained in U having the form $F = \bigcup_{i=0}^{\infty} h^{-1}(E_1)$. Here E_1 is an open subset of E and $h : E \to E$ is a homeomorphism with compact support homotopic to the* *identity which fixes some neighborhood W of Z in Y. (Here Y is not assumed to be a manifold.)*

Proof. The first part follows immediately from the proof of Proposition 1.7. Since W contains an I-regular neighborhood E' of Z, $E' \to E$ is a homotopy equivalence, and $h|E' = \mathrm{id}$, we must have $h \sim \mathrm{id}$. □

Notation. If Z is compact and Y is locally compact and E is an I-regular neighborhood of Z in Y, we will denote the point at ∞ in the one-point compactification $\overline{E}$ of E by ε^+. By a neighborhood of ε^+, we will mean a subset of $\overline{E}$ having the form $E - C$, where $C \subset E$ is compact. By abuse of notation, we will also use ε^+ to denote the end (or collection of ends) of E.

Proposition 1.9. *Let $Z \subset Y$ be compact and let E ne an I-regular neighborhood of Z in Y of the form $E = \bigcup_{i=0}^{\infty} h^{-1}(E_1)$, as above (Figure 3). Here $E_1 \subset E$ is an open neighborhood of Z, and $h : E \to E$ is a homeomorphism fixed on a neighborhood of Z such that $h(E_1) \searrow Z$ in E_1. Set $G = \bigcup_{i=-\infty}^{\infty} h^{-i}(E_1 - h(E_1))$. Then there is an isotopy of open embeddings $g_t : G \to E - Z$ such that $g_1(G) = \bigcup_{i=0}^{\infty} h^{-i}(E_1) - Z$ and such that g is fixed on $E - E_1$.*

Proof. Let $U_1 \supset U_2 \supset U_3 \supset \cdots$ be a basis of neighborhoods for Z in Y. Using the fact that $h^{i+1}(E_1) \searrow Z$ in $h^i(E_1)$ for each i, inductively construct homeomorphisms $g_i : G \to E$ such that $g_0 = \mathrm{id}$, $g_i(h^i(E_1)) \subset U_i$, and $g_i = g_{i-1}$ on $G - h^{i-1}(E_1)$. Then $g = \lim g_i$ is a homeomorphism from G onto $E - Z$. The construction produces g isotopic through open embeddings to the identity. □

Proposition 1.10. *Let $Z \subset Y$ be compact and let E ne an I-regular neighborhood of Z in Y. Then if U is any neighborhood of ε^+, there is an isotopy of open embeddings $\bar{g}_t : E - Z \to E - Z$ such that $\bar{g}_1$ throws $E - Z$ into U and such that g_t is fixed on some small neighborhood of ε^+.*

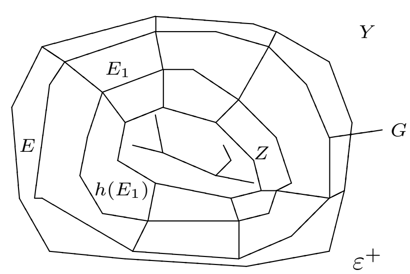


Fig. 3

Proof. By Corollary 1.8 and the uniqueness theorem for I-regular neighborhoods, we may assume that $E = \bigcup_{i=0}^{\infty} h^{-i}(E_1)$ where $E_1 \subset E$ is open and $h = \mathrm{id}$ in a neighborhood W of Z. Letting $F = \bigcup_{i=-\infty}^{\infty} h^{-i}(E_1 - h(E_1))$, choose N so that $h^{-i}(E_1 - h(E_1)) \subset U$ for all $i \geq N$. Let G be an I-regular neighborhood of Z which is contained in W and let H be an I-regular neighborhood of Z such that $h^{-N}(E_1) \subset H \subset h^{-N+2}(E_1)$. Such an H exists by the first paragraph of the proof of proposition 1.7. We may assume that $G \subset H$.

Let $g_t : G \to H$ be an isotopy of open embeddings with $g_0 = \mathrm{id}$ and such that g_t throws G onto H. Consider $g_t|g_1^{-1}(H - U)$. By the isotopy extension theorem

[9], this extends to an isotopy $\bar{g}_t : E \to E$ which has compact support. Then $\bar{g}_t$ throws W onto an open set containing $E - U$ and therefore throws F into U. Since proposition 1.9 produces an isotopy k_t, throwing $E - Z$ onto F (fixing a neighborhood of ε^+) replacing $\bar{g}_t$ by its concatenation with k_t produces the desired isotopy. The rather schematic diagram in Figure 4 may help the reader to keep this straight. □

Remark 1.11. The last proposition shows that if E is an I-regular neighborhood of Z in Y, then E is also an I-regular neighborhood of $\overline{Y} - E$ in $\overline{Y}$ where by $\overline{Y}$ we mean the one-point compactification of Y.

Definition 1.12. Siebenmann [27] defines an end ε of a manifold M to be *tame* if $\pi_1(\varepsilon$ is stable and there exist arbitrarily small neighborhoods of ε which are homotopy dominated by finite complexes. The first condition means that if $U_1 \supset U_2 \supset \cdots$ is a sequence of neighborhoods of ε^+, then the system $\{\pi_1(U_i), j_{i*}\}$ is equivalent as a system to a system of isomorphisms. Here $j_{i*} : \pi_1(U_i) \to \pi_1(U_{i-1})$ is the map induced by inclusion. If ε is stable, then the *end obstruction* $\sigma(\varepsilon)$ of σ is a well-defined element of $\widetilde{K}_0(\mathbb{Z}\pi)$ which vanishes if and only if M admits a boundary at the end ε.

Lemma 1.13. *A* CW *complex is finitely dominated if and only if there is a homotopy $h : K \times K \to K$ such that the closure of $h_1(K)$ is compact.*

Proof. ($\Rightarrow$) If K is finitely dominated, let $d : L \to K$ be a finite domination with right inverse $u : K \to L$. By definition, there is a homotopy $h : K \times I \to K$ such that $h_0 = \mathrm{id}$ and $h_1(K) = d \circ u$. Let $K' \subset K$ be a finite subcomplex containing $d(L)$ ad let $i : K' \to K$ be the inclusion. Then $i : K' \to K$ is a domination with right inverse $d \circ u : K \to K$".

($\Leftarrow$) If $h : K \times I \to K$ is a homotopy as specified above, let $L \subset K$ be a finite subcomplex containing $h_1(K)$. Then $i : L \to K$ is a domination with right inverse h_1. □

In words, K is finitely dominated if and only if K can be deformed homotopically into a compact subset of itself.

Proposition 1.14. *Let $Z^k \subset \mathbb{R}^n$ be compact, connected, k-dimensional and* $1-$LCC *embedded, $k + 3 \leq n$. Suppose that Z has the shape of a* CW *complex X with $\sigma(X) = \sigma$. Then the end ε^+ is tame in the sense of Siebenmann and its end obstruction is σ.*

Proof. Since shape theory reduces to homotopy theory on CW complexes, $X \approx Z \approx E \Rightarrow X \sim E$. (Here we have used $\approx$ to denote shape equivalence and $\sim$, as usual, to denote homotopy equivalence.)

To show that Siebenmann's end obstruction is defined at ε^+, we must show that π_1 is stable at ε^+ and that ε^+ has a cofinal system of neighborhoods which are finitely dominated. We have shown that ε^+ has a basis $F_0 \supset F_1 \supset F_2 \supset \cdots$ of neighborhoods such that each F_i is homeomorphic to $E - Z$ and such that each inclusion is a homotopy equivalence. Since Z is $1-$LCC embedded in codimension 3, $\pi_1(E-Z) \to \pi_1(E) \to \pi_1(X)$ is an isomorphism and ε^+ has stable finitely presented π_1. To see that $E-Z$ is finitely dominated, let U be a small neighborhood of ε^+. By Proposition 1.9, there is an isotopy $g_t : E - Z \to E - Z$ throwing $E - Z$ into U and such that g_t is fixed on some small neighborhood of ∞. By Remark 1.11, there is therefore an I-regular neighbrohoodE' of ε^+ on which g_t is fixed for all t. Applying Proposition 1.9 again gives an isotopy $h_t : E' \cap E \to E' \cap E$ which pulls $E' \cap E$ back from ε^+. Piecing g_t and h_t together gives an isotopy $\bar{g}_t : E - Z \to E = Z$ which throws $E - Z$ into a compact subset of itself. This shows that $E - Z$ is finitely dominated and that Siebenmann's end obstruction is well-defines at ε^+.

By definition, the end obstruction at ε^+ is the finiteness obstruction $\sigma(E - C)$, where C is any finite subcomplex of E such that $E-C \to E$ induces an isomorphism on π_1. Since $\pi_1(E) = \pi = \pi_1(\varepsilon^+)$, we can take $C = \emptyset$ and we are done. □

We are now in a position to complete the proof of Theorem 2. Let Z be a three-dimensional acyclic compactum in $\mathbb{R}^n$, $n \geq 6$, as constructed in Proposition 1.4. Compactify $\mathbb{R}^n$ to S^n and let E be an I-regular neighborhood of Z in S^n. By Alexander duality and Proposition 1.7, $Z' = S^n - E$ is acyclic. By Alexander duality, again, $E - Z = S^n - (Z \cup Z')$ has the integral homology of the $(n-1)$-sphere. The space $E - Z$ is the desired integral homology sphere. We show that $(E - Z) \times S^1$ has the homotopy type of a closed manifold. Recall that $E - Z \simeq \bigcup_{i=-\infty}^{\infty} h^{-i}(E_1 - h(E_1))$ for some open subset E_1 of E, where $h : E - Z \to E - Z$ is a free $\mathbb{Z}$-action homotopic to $\mathrm{id}_{(E-Z)}$.

Let $\mathbb{Z}$ act on $\mathbb{R}^1$ by $x \to x+1$ and form the balanced product $(E-Z) \times_{\mathbb{Z}} \mathbb{R}^1$. We have fibrations $\mathbb{R}^1 \to (E-Z) \times_{\mathbb{Z}} \mathbb{R}^1 \to (E-Z)/\mathbb{Z}$ and $(E-Z) \to (E-Z) \times_{\mathbb{Z}} \mathbb{R}^1 \to S^1$. Thus $(E - Z)/bz$ is a manifold homotopy equivalent to $(E - Z) \times_{\mathbb{Z}} \mathbb{R}^1$ and $(E - Z) \times_{\mathbb{Z}} \mathbb{R}^1$ fibers over S^1 with fiber $(E - Z)$ and holonomy h. Since h is homotopic to the identity, this shows that $(E - Z) \times_{\mathbb{Z}} \mathbb{R}^1$ is a closed manifold homotopy equivalent to $(E - Z) \times S^1$.

It remains to compute the finiteness obstruction of $E-Z$. Since the end invariant of $E - Z$ at ε^+ is σ, Siebenmann's duality theorem [27, p. 119] tells is that the end invariant of $E - Z$ at Z is $(-1)^{n-1}\sigma^*$. If C is a large finite subcomplex of $E - Z$, $(E - Z) - C$ has two infinite components, one with finiteness obstruction σ and one with finiteness obstruction $(-1)^{n-1}\sigma^*$. By the sum theorem for finiteness obstructions, the finiteness obstruction of $E - Z$ itself is therefore $\sigma + (-1)^{n-1}\sigma^*$.

To prove Theorem 2(ii), recall that $Z = \bigcap_{i=0}^{\infty} N_i$ where each N_i is a compact codimension-0 submanifold of $\mathbb{R}^n$ which contains a finite acyclic subcomplex L_i such that $L_i \to N_i$ induces an isomorphism on π_1. Choosing i large, so that $L_i \subset E$, $E - L_i$ is the desired integral homology sphere with finiteness obstruction equal to σ.

Remark 1.15. The proofs of 1.7-1.14 are well-known to experts. See, for example, [8, Prop. 6.2]. Our purpose in providing these proofs has been to disentangle the proofs from technical considerations of shape theory. While we have used shape-theoretic language throughout, the reader who keeps in mind the homotopy commuting diagrams we have constructed should be able to manufacture his own proofs based on the existence and uniqueness theorems for I-regular neighborhoods. In fact, the reader who is willing to give up a few dimensions can avoid much of the preceding, including the I-regular neighborhood theory, by using the following construction:

Construct a three-dimensional complex K and a homotopy idempotent $\alpha : K \to K$ as above. Embed K in $\mathbb{R}^n$, $n \geq 8$, and take a regular neighborhood $N_1 = N(K)$. Embed K in N_1 by an embedding homotopic to α and take a regular neighborhood N_2. Continue this process, constructing N_i, $i = 1, 2, 3, \ldots$

Let $W_i = N_i = \operatorname{int}(N_{i+1})$. By regular neighborhood theory, all of the W_i's are PL homeomorphic. Since $\alpha \sim \alpha^2$, regular neighborhood theory shows that W_i is homeomorphic to $N \cup \operatorname{int}(N_{i+2}) = W_i \cup W_{i+1}$. If we write W for a single PL manifold homeomorphic to all of the W_i's, we see that there is an open subset of Euclidean space which is homeomorphic to a bi-infinite union of copies of W. This is the space $E - Z$ of the preceding and it is not difficult to use elementary tools of PL topology to verify that Proposition 1.14 holds.

2. The Proof of Theorems 1 and 3

Definition 2.1. Let X be a two-ended space and let $\hat{X}$ be the two-point compactification of X. The $\hat{X} \times \mathbb{R}^1$ is two-ended, and we denote its two-point compactification by $DS(X)$. (The notation suggests the relation to double suspension if $X = Y \times \mathbb{R}^1$. The space $DS(X)$ is like a double suspension of the homotopy type of X.)

Proposition 2.2. *Let X be a two-ended n-manifold with tame ends in the sense of Siebenmann. Then the two-point compactification $\hat{X}$ of X is an* ANR.

Proof. Assume, first, that $n \geq 5$. If ε is an end of X, $\sigma(\varepsilon \times S^1) = \sigma(\varepsilon) \times \chi(S^1) = 0$, so $X \times S^1$ is two-ended and admits a compact manifold boundary. The boundaries admit collar neighborhoods, so the two-point compactification of $X \times S^1$ is locally contractible and is therefore an ANR. Since $\hat{X}$ is a retract of the two-point compactification of $X \times S^1$, it is also an ANR. If $n \leq 5$, the desired result is obtained by crossing with T^{6-n} instead of S^1. □

Remark 2.3. Far more precisely theorems of this sort are known. See [20, p. 258 ff.] for details.

Lemma 2.4. *Let M^n be a homology n-manifold and let Z be a compact subset of M^n which is acyclic in the sense that $\check{H}^*(Z) \cong \check{H}^*(pt)$. Then $M/.Z$ is a homology manifold.*

Proof. By Alexander Duality,

$$\check{\mathrm{H}}^k(Z) \cong H_{n-k}(M, M-K) \cong H_{n-k}(M/Z, M/Z - [Z]).$$

Thus, $H_k(M/Z, M/Z - [Z])$ is $\mathbb{Z}$ for $k = 0$ and 0 otherwise, as required. □

Recall that the homology sphere X with finiteness obstruction $\sigma(X) = \sigma$ constructed in Theorem 2(ii) may be taken to be $E - L$, where E is an I-regular neighborhood of an acyclic $1-$LCC embedded codimension three compactum $Z \subset S^{n+1}$ and L is a PL embedded acyclic polyhedron. Both ends of E are tame, so the next proposition applies.

Proposition 2.5. *Let Z_1 and Z_2 be disjoint acyclic compacta in S^n such that $X = S^n - (Z_1 \cup Z_2)$ has two tame ends. Then $DS(X) \cong S^{n+1}$.*

Proof. By Proposition 2.2, the two-point compactification $\hat{X}$ of X is an ANR, and by lemma 2.4, $\hat{X} = (S^{n+1}/Z_1)/Z_2$ is a homology manifold which has manifold points. By [22] and [23], $\hat{X}$ is the cell-like image of a topological manifold. We now invoke a theorem of Cannon:

Theorem. *(*[4, Theorem 10.2]*). Let $n \geq 4$ and let $f : M^n \to Y^{\mathrm{ANR}}$ be a CE map such that the nonmanifold part of Y is contained in a topological polyhedron and codimension ≥ 3, then $Y \times \mathbb{R}^1$ is a manifold.*

Applying this to $\hat{X}$, we see that $\hat{X} \times \mathbb{R}^1$ is a manifold. $\hat{X}$ is simply connected, so $\hat{X} \times \mathbb{R}^1$ admits a boundary which is, by the generalized Poincar'e Conjecture, a sphere. The two-point compactification of $\hat{X} \times \mathbb{R}^1$ is therefore a sphere, as desired.

The complement of the 'suspension circle' in the two-point compactification of $\hat{X} \times \mathbb{R}^1$ is $X \times \mathbb{R}^1$. Thus to complete the proof, we need only show that this suspension circle is end-tamely embedded. Fortunately, this is easy. Recall that X has two tame ends. so for every compact subset D of X we can find a homotopy (an isotopy, even!) $h_t : X \to X$ from X to itselfsuch that $h_0 = \text{id}$, $h_t = \text{id}$ on D, and such that $h_t(X)$ has compact closure. If $\rho : \mathbb{R}^1 \to \mathbb{R}^1$ is an isotopy of open embeddings which pulls $\mathbb{R}^1$ back onto $(-N, N)$ for some large N while fixing $[-N+1, N-1]$, then $h \times \rho$ pulls $X \times \mathbb{R}^1$ back from the suspension circle by a small motion. □

Remark 2.6. Quinn's notion of tameness differs from Siebenmann's where π_1-stability is concerned. The examples constructed in proving Theorem 1 do, however, have the appropriate stability property on π_1. The map $h \times p$ constructed above is an isotopy, rather than just a homotopy.

In proving Theorem 3, we find it useful to introduce a second compactification of X.

Definition 2.7. Let X be a two-ended space with a proper map $p : X\mathbb{R}^1$. Then $p \times \text{id} : X \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{R}^1$. If $r : \mathbb{R}^2 \to \text{int}(D^2)$ is a radial homeomorphism we compactify $X \times \mathbb{R}^1$ to form a space $R(X)$ by adding a copy of S^1 at ∞ so that a neighborhood of a point $\theta \in S^1$ is given by $(U \cap S^1) \cup r \circ (p \times \text{id})^{-1}(U)$, where U is an open neighborhood of θ in D^2.

Proposition 2.8. *There is a homeomorphism $h : DS(X) \to R(X)$ which takes the S^1 at ∞ to the S^1 at ∞.*

Proof. We give a formula, leaving the verification that it defines a homeomorphism to the reader.

$$h(x,t) = r \circ (s, t\sqrt{p^2(x)+1}).$$

Here the coordinates are given in $X \times \mathbb{R}^1$. The homeomorphism h extends to the compactification by continuity. The idea behind this formula is to bend horizontal lines in $DS(X)$ up to hyperbola of varying slopes in $R(X)$ (Figure 5). □

Remark 2.9. Since $DS(X)$ is independent of p, so is $R(X)$. That is, the radial compactification is independent of the choice of proper function p. Note that $DS(X)$ is locally isotopy homogeneous everywhere except at the last two suspension points. Of course, this means that the same holds for $R(X)$. We exploit this in the next proposition.

Proposition 2.10. *Let $p : M^n \to S^1$ be a map, $n \geq 5$, such that the induced infinite cyclic cover $\widetilde{m}$ of M is finitely dominated. Then the S^1 at infinity in $DS(\widetilde{M}) \cong R(\widetilde{M})$ is isotopy homogeneous.*

Proof. By the product formula for the fibering obstruction, [10, 29], $p \circ \text{proj}$ is homotopic to the projection map of a locally trivial fiber bundle. Thus, there exist a closed n-manifold F^n having the homotopy type of $\widetilde{M} \times S^1$, a homeomorphism

$\phi : F \to F$, and a homeomorphism $h : M \times S^1 \to T(\phi)$ such that the diagram

$$\begin{array}{ccc} M \times S^1 & \xrightarrow{h} & T(\phi) \\ & {\scriptstyle p\circ \mathrm{proj}_M} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_{S^1}} & \\ & S^1 & \end{array}$$

homotopy commutes. For aficionados, F is the twist-gluing (see[29]) of $\widetilde{M}$ by the identity map. Passing to cyclic covers, h is covered by a homeomorphism $\hat{h}$ so that the diagram below commutes

$$\begin{array}{ccc} \widetilde{M} \times S^1 & \xrightarrow{\hat{h}} & F \times \mathbb{R}^1 \\ & {\scriptstyle p\circ \mathrm{proj}_{\widetilde{M}}} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_{\mathbb{R}^1}} & \\ & \mathbb{R}^1 & \end{array}$$

Defining the map $q : F \times \mathbb{R}^1 \to S^1$ to make the diagram

$$\begin{array}{ccc} \widetilde{M} \times S^1 & \xrightarrow{\tilde{h}} & F \times \mathbb{R}^1 \\ & {\scriptstyle (p\circ \mathrm{proj}_{\widetilde{M}}, \mathrm{proj}_{S^1})} \searrow \quad \swarrow {\scriptstyle (\mathrm{proj}_{\mathbb{R}^1}, q)} & \\ & \mathbb{R}^1 \times S^1 & \end{array}$$

where $\tilde{h}$ is bounded over $\mathbb{R}^2$. Here $\widetilde{F}$ is the infinite cyclic cover of F induced by q. Since $\tilde{h}$ is bounded over $\mathbb{R}^2$, $\tilde{h}$ extends to a homeomorphism of the radial compactification of $\tilde{p} \circ \mathrm{proj}_{\widetilde{M} \times \mathrm{proj}_{\mathbb{R}^1}} : \widetilde{M} \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{R}^1$ and $\mathrm{proj}_{\mathbb{R}^1} \times \tilde{q} \circ \mathrm{proj}_{\widetilde{F}} : \widetilde{F} \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{R}^1$. But the second of these is homeomorphic to $DS(\widetilde{F})$, so it is locally isotopically homogeneous away from two points on the circle at ∞. Thinking of this compactification as $DS(\widetilde{M})$ shows it to be locally isotopically homogeneous away from two different points at ∞. It follows easily that the circle at ∞ is isotopically homogeneous. (The reader should recall that if $\mathcal{U}$ is an open cover of a closed manifold M, then any isotopy of M can be factored as a concatenation of isotopies supported on some element of $\mathcal{U}$. See [9].) $\square$

Remark. The referee has suggested an alternative construction for constructing a wildly embedded homogeneous circle in S^n with nontrivial end obstruction. The result differs from out Theorems 1 and 3 in that the complement obtained is homotopy equivalent to S^{n-2} and therefore has trivial finiteness obstruction.

Given π as in the statement of Theorem 3, construct a contractible open manifold M^n with π as fundamental group at ∞. Given $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$, realize σ as the torsion of a proper h=cobordism (W, M, M'). Double W along M' and 1-point compactify, forming a space $\widehat{DW}$. Now attach a copy of $\hat{M} \times I$ to $\widehat{DW}$ to form X^{n+1} as in Figure 6.

One now verifies using Cannon-Edwards that this space is a manifold homeomorphic to $S^n \times S^1$. The circle at ∞ is shown to be isotopy homogeneous by an application of Theorem 1.1 of [25] (see [16] for improvements) - it is easily seen from the construction that the holinks at points of the circle at ∞ are I-regular neighborhoods of ∞ in $\hat{M}$, $\hat{M}'$, and $\widehat{DW}$, and are therefore homotopy equivalent. The end obstruction at this S^1 is $\sigma + (-1)^{n+1}\sigma^*$, as in our Theorem 3, since $\sigma(M) = 0$, $\sigma(W) = \sigma$, and $\sigma(M') = \sigma + (-1)^n\sigma^*$. This proves:

THEOREM 3A. *If M^n is a contractible 1-ended open manifold with $\pi_1(\varepsilon) = \pi$, then for each $\sigma \in \widetilde{K}_0(\mathbb{Z}\pi)$ there is an isotopy homogeneous embedding of $i : S^1 \to S^1 \times S^n$ such that the end obstruction $\sigma(S^1 \times S^n - i(S^1))$ is $\sigma + (-1)^{n+1}\sigma^*$.*

Of course, doing surgery on a standard S^1 away from the wild ones gives an isotopy homogeneous S^1 in S^{n+1} whose complement is homotopy equivalent to the standard $(n-1)$-sphere. The authors do not know if there are isotopy homogeneous embeddings with end obstructions which are not trivially symmetric.

3. REALIZING K_{-1} OBSTRUCTIONS

In this section we will prove Theorem 4. Our technique is to start with an embedding of S^1 into S^n as constructed in Section 2. We then modify the complement by a controlled homotopy equivalence and pate the new complement back in. We start with a series of lemmas whose purpose is to show that th result of modifying an embedding of S^1 into S^n in this way is another embedding of S^1 into S^n.

DEFINITION 3.1. Let Y be a locally compact but noncompact space with $r : U \to X$ a map from a neighborhood of infinity in Y to a compact space X. By $Y \cup_r X$, we will mean the space whose points are points of X and Y and whose topology is given by the basis

$$U = \{V | V \text{ is open in } Y\} \cup \{r^{-1}(V) \cap V') \cup V | V \text{ is open in } \\ X \text{ and } V' \text{ is an open neighborhood of } \infty \text{ in } Y\}.$$

Note that $Y \cup_r X$ is a compact Hausdorff space. the compactification of Y by this process is called a *teardrop construction.*

DEFINITION 3.2. Let Y and Y' be noncompact spaces with maps $r : U \to X$ $r' : U' \to X$ from neighborhoods of ∞ in Y and Y' to X. The spaces (Y, r) and (Y', r') are *teardrop equivalent* over X if ht ere is a homotopy equivalence $h : Y \to Y'$ with homotopy inverse $g : Y' \to Y$ such that h, g, and the homotopy $g \circ h \sim \text{id}$, $h \circ g \sim \text{id}$ extend by the identity to $Y \cup_r X$ and $Y' \cup_r X$.

LEMMA 3.3. *If (Y, r) and (Y', r') are teardrop equivalent, then for each $x \in X$, $(Y \cup_r X, Y \cup_r X - \{x\})$ and $(Y' \cup_{r'} X, Y' \cup_{r'} X - \{x\})$ are homotopy equivalent as pairs.*

Proof. Since the maps and homotopies in the definition are the identity on X and preserve the complement of X, the result follows. □

LEMMA 3.4. *If Y and Y' are n-manifolds and (Y, r) and (Y', r') are teardrop equivalent, then $Y \cup_r X$ is a homology n-manifold if and only if $Y' \cup_{r'} X$ is.*

Proof. This follows directly from Lemma 3.3 □

LEMMA 3.5. *If Y and Y' are finite-dimensional* ANR*'s and (Y, r) and (Y', r) are teardrop equivalent, then $Y \cup_r X$ is a finite-dimensional* ANR *if and only if $Y' \cup_{r'} X$ is.*

Proof. Suppose that $Y \cup_r X$ is a finite-dimensional ANR. Then $Y' \cup_{r'} X$ is finite-dimensional, so we need only check that $Y' \cup_{r'} X$ is locally contractible at points of X. In general, if $f : (A, a) \to (B, b)$ is a homotopy equivalence of pairs, then A is locally contractible at a if and only if B is locally contractible at b. This is left as an exercise for the reader. □

DEFINITION 3.6. A metric space is said to have the Disjoint Disks Property (DDP) id for every pair of maps $f, g : D^2 \to X$ and $\varepsilon > 0$ there exist maps $f', g' : D^2 \to X$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$, and $f'(D^2) \cap g'(D^2) = \emptyset$.

LEMMA 3.7. *If Y and Y' are n-dimensional manifolds, $n \geq 5$, and (Y, r) and (Y', r') are teardrop equivalent, then $Y \cup_r X$ has the DDP if and only if $Y' \cup_{r'} X$ does.*

Proof. For simplicity, we will assume that $Y \cup_r X$ is a closed n-manifold. The general case is similar, but involves more epsilonics. To set the scene, let $h : Y \cup_r X \to Y' \cup_{r'} X$ and $k : Y' \cup_{r'} X \to Y \cup_r X$ be maps with $H : h \circ h \sim \mathrm{id}$ and $K : h \circ k \sim \mathrm{id}$ homotopies as in the definition of controlled equivalence. In particular, all maps and homotopies are the identity on X and preserve the complement of X.

$$k \circ h \overset{H}{\sim} \mathrm{id} \qquad Y \cup_r X \underset{k}{\overset{h}{\rightleftarrows}} Y' \cup_{r'} X \qquad h \circ k \overset{K}{\sim} \mathrm{id}$$

□

CLAIM. For each $\varepsilon > 0$ and map $\alpha : D^2 \to Y' \cup_{r'} X$ there is a map $\bar{\alpha} : D^2 \to Y' \cup_{r'} X$ such that $d(\alpha, \bar{\alpha}) < \varepsilon$ and such that $\bar{\alpha} = h \circ k \circ \alpha$ over (over $\equiv$ on the inverse image of) a neighborhood of X,

Proof. Let U be an open neighborhood of X in $Y' \cup_{r'} X$ such that $K \circ \alpha$ is an $\varepsilon/2$-homotopy from α to $h \circ k \circ \alpha$ over U. Such a neighborhood exists by compactness because h, k, and K are the identity on X. Let $V \subset U$ be a close neighborhood of X with a manifold boundary. The maps α, $h \circ k \circ \alpha$, and $K \circ \alpha$ combine to give a map

$$A : ((V - X) \times 0) \cup (\partial V \times I) \cup (Y' - \mathrm{int}(V)) \times I \to Y'$$

If

$$\beta : Y' \times I \to ((V - X) \times 0) \cup (\partial V \times I) \cup ((Y' - \mathrm{int}(V)) \times 1)$$

is a retraction which is projection outside of a very small neighborhood of $\partial V \times I$, $\beta \circ A | Y' \times 0$ extends over X by $h \circ k \circ \alpha$ to give the desired map $\bar{\alpha}$, proving the claim. □

Let $f.g : D^2 \to Y' \cup_{r'} X$ be given. Choose maps $\bar{f}, \bar{g} : D^2 \to Y' \cup_{r'} X$ which are $\varepsilon/2$=close to f and g such that $\bar{f} = h \circ k \circ f$ and $\bar{g} = h \circ k \circ g$ over a neighborhood of X as above. Now choose $f'', g'' : D^2 \to Y \cup_r X$ so that f'' is very close to $k \circ f$, g'' is very close to $k \circ g$, $f'' = k \circ f$, and $g'' = k \circ g$ outside of a *very* small neighborhood of X, and so that $f''(D^2) \cap g''(D^2) \cap X = \emptyset$. Let f''' be obtained by piecing together $h \circ f''$ and $\bar{f}$ to obtain a map which is equal to $h \circ f''$ over a small neighborhood of X and which is equal to $\bar{f}$ away from a slightly larger neighborhood of X. A similar construction with g's gives g''' so that

$$d(f, f''') < \varepsilon, \qquad d(g, g''') < \varepsilon \qquad \text{and} \qquad f'''(D^2) \cap g'''(D^2) \cap X = \emptyset.$$

general position in the manifold Y' now gives us maps f', g' approximating f and g with $f'(D^2) \cap g'(D^2) = \emptyset$.

Proposition 3.8. *If Y and Y' are n-dimensional manifolds, $n \geq 5$, and (Y, r) and $Y', r')$ are teardrop equivalent, then $Y \cup_r X$ is an n-manifold if and only if $Y' \cup_{r'} X$ is.*

Proof. Quinn's characterization of topological manifolds ([24]) shows that a connected ANR homology n-manifold which has the DDP and which contains an open subset homeomorphic to $\mathbb{R}^n$ is a topological n-manifold. The result follows from this and the previous lemmas. □

Corollary 3.9. *If Y and Y' are n-dimensional manifolds, $n \geq 5$ and (Y, r) and (Y', r') are teardrop equivalent, then $Y \cup_r X$ is homeomorphic to S^n if and only if $Y' \cup_{r'} X$ is*

We are now ready to prove Theorem 4(i). the proof is quite simple, given the requisite topology. If Y is a locally finite 1-ended CW complex, $n \geq 5$, *and* $r : U \to X$ is a control map from a neighborhood of ∞ to X, we can imitate [7, 28, 5, 1] to form a geometrically defined Whitehead group $\mathrm{Wh}(Y, r)$ consisting of (Y', r') controlled equivalent to Y. As in [28], one uses the equivalence relation generated by locally finite collapses controlled over X. One proceeds to prove analogs of Siebenmann's thesis and the s-cobordism theorem in this category. In particular, if Y is an n-manifold, $n \geq 5$, and $\tau \in \mathrm{Wh}(Y, r)$, then one shows that there is a controlled h-cobordism $(W, \bar{r})$ realizing this τ with $\partial_0 W = Y$ and $\bar{r}|\partial_0 W = r$. As usual, the torsion of $\partial_1 W \to W$ is $(-1)^n \tau^*$ and the torsion of the composite controlled equivalence $\partial_0 W \to \partial_1 W$ is $\tau + (-1)^n \tau^*$. This is rather formal.

Given π as in the statement of Theorem 4, let H^{n-2} be a closed $(n-2)$-dimensional PL homology sphere with fundamental group π. The double suspension $\Sigma^2 H$ is an n-sphere. We will call the suspension circle X. The space $\Sigma^2 H - X$ has the homotopy type of H. The circle X has a mapping cylinder neighborhood in $\Sigma^2 H$, so the controlled end obstruction of $\Sigma^2 H - X$ over X vanishes. If Y^n is an n-dimensional manifold with control map $r : U \to X$, and Y is teardrop equivalent to $\Sigma^2 H - X$ over X, then $Y \cup_r X$ is homeomorphic to S^n. it remains to understand the relation between the torsion of the controlled equivalences and the controlled end obstruction of Y.

One analyzes controlled torsion in a manner analogous to [28]. Thus one first attempts to find controlled splittings of Y near X. This leads to an obstruction in $\mathrm{Wh}(Y \times S^1, r \circ \mathrm{proj})$. If such splittings exist, there is a further obstruction which lies in $\lim^1$ of controlled Whitehead groups. because of the simplicity of the π_1 system in $\Sigma^2 H - X$, the stability theorem (Theorem 2.4 of [22]) applies to show that this $\lim^1$ term vanishes. Thus the controlled Whitehead group of $\Sigma^2 H - X$ over X is isomorphic to $\widetilde{K}_{-1}(\mathbb{Z}\pi)$ and the torsion of a controlled equivalence $f : Y \to \Sigma^2 H - X$ is the controlled end obstruction of Y over X. The $\widetilde{K}_0(\mathbb{Z}\pi)$ factor in the end obstruction is never realized because the realizable obstructions lie in the kernel of the forgetful homomorphism from the controlled finiteness obstruction at infinity to the ordinary finiteness obstruction in Y. This is analogous to Theorem I on page 483 of [28]. Realizing torsions of the form $\tau + (-1)^n \tau^*$ using controlled h-cobordisms as above completes the proof of Theorem 4(i). For details see [22] or [11].

Proving theorem 4(ii) requires some surgery theory. Given (Y, r) with Y an n-manifold, one sets up a surgery theory to study controlled structure sets. One defines structure sets and simple structure sets and obtains the commuting diagram below, where the horizontal sequences are surgery sequences, the vertical sequence is a Rothenberg-Ranicki sequence, $\mathrm{Wh}(Y,r)$ is the controlled Whitehead group of (Y,r) and $L^s_n(Y,r)$ and $L^h_n(Y,r)$ are the appropriate L-groups. In the special case when Y is $\Sigma^2 H - X$ as above, theorem 10.13 of [12] shows that $L^s_n(Y,r) = L^p_n(\mathbb{Z}\pi)$ and $L^h(Y,r) = L^{<-1>}_n(\mathbb{Z}\pi)$. When $n = 4k+3$, Theorem 3.12 of [14] therefore applies to show that the map $L^s_n(Y,r) \to L^h_n(Y,r)$ in the vertical sequence is a monomorphism. It follows that the map $\mathcal{S}^h(Y,r) \to \hat{H}^*(\mathbb{Z}_2, \mathrm{Wh}(Y))$ which sends a structure to its torsion is onto, which is to say that every self-dual torsion can be realized by a controlled homotopy equivalence.

$$\begin{array}{ccccccccc}
\longrightarrow & [\Sigma Y, G/CAT] & \longrightarrow & L^s_{n+1}(Y,r) & \longrightarrow & \mathcal{S}^s(Y,r) & \longrightarrow & [Y,G/CAT] & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & [\Sigma Y, G/CAT] & \longrightarrow & L^h_{n+1}(Y,r) & \longrightarrow & \mathcal{S}^h(Y,r) & \longrightarrow & [Y,G/CAT] & \longrightarrow \\
 & & & \downarrow & \swarrow & & & & \\
 & & & \hat{H}^*(\mathbb{Z}_2, \mathrm{Wh}(Y,r)) & & & & & \\
 & & & \downarrow & & & & & \\
 & & & L^s_n(Y,r) & & & & & \\
 & & & \downarrow & & & & & \\
 & & & L^h_n(Y,r) & & & & &
\end{array}$$

This completes the proof of Theorem 4(ii). To see that the embeddings thus created are isotopy homogeneous, we note that teardrop equivalences preserves Quinn's homotopy stratification condition, [25] p. 443, so Theorem 1.1 of [25] applies to prove isotopy homogeneity.

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Department of Mathematics, SUNY, Binghamton, NY 13901, USA

Department of Mathematics, SUNY, Binghamton, NY, 13901, USA