THE REALIZABILITY OF LOCAL LOOP SPACES AS MANIFOLDS

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ABSTRACT. As an extension of earlier work, we show that every P-local loop space, where P is a set of primes, is homotopy equivalent to the P-localization of a compact, smooth, parallelizable manifold. A similar result is also proved for P-complete loop spaces.

1. INTRODUCTION

In [BKNP02], it was shown that every quasifinite loop space is homotopy equivalent to a parallelizable, compact, smooth manifold. In this paper, we prove a stronger local version of this result.

Let R be a commutative ring. We call a nilpotent space X R**-finite** if $H_*(X;R)$ is totally finitely generated as an R-module, and R**-local** if $[Y,X]=0$ for every $H_*(-;R)$-acyclic space Y. For any such R, there exists a localization functor L_R : Top $\to$ Top. For a set of primes P, call a nilpotent space X P**-local** (resp. P**-complete**) if it is local with respect to $R=\mathbf{Z}_{(P)}$ (resp. $R=\mathbf{Z}/P:=\prod_{p\in P}\mathbf{Z}/p$.) We abbreviate the corresponding localization functors by $L_{(P)}$ and L_P. A nilpotent space X is P-complete if and only if $X\simeq\prod_{p\in P}L_pX$. By the universal coefficient theorem, a finite loop space is $\mathbf{Z}_{(P)}$-finite and $\mathbf{Z}/P$-finite for any set of primes P.

The main result of this paper can be cast in a local and in a complete setting:

THEOREM 1.1. *Let P be a collection of primes, and let X be a P-local, $\mathbf{Z}_{(P)}$-finite loop space. Then X is homotopy equivalent to the P-localization of a compact, smooth, parallelizable manifold.*

THEOREM 1.2. *Let P be a collection of primes, and let X be a nilpotent P-complete space. Assume that there exist integers $\{d_1,\dots,d_r\}$ such that for all primes $p\in P$, L_pX is a p-compact group with degrees d_i. Then X is homotopy equivalent to the P-completion of a compact, smooth, parallelizable manifold.*

Setting $P=$ the set of all primes, we recover from Thm 1.1 the main result of [BKNP02] that every quasifinite (i. e. $\mathbf{Z}$-finite) loop space is homotopy equivalent to a compact, smooth, parallelizable manifold.

On the other extreme, we have as a special case of Thm. 1.2:

COROLLARY 1.3. *Every p-compact group is the p-completion of a compact, smooth, parallelizable manifold.*

A natural question to ask is whether Thm. 1.1 can be reduced to the global setting of [BKNP02] by showing that every space satisfying the conditions of Thm.

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1.1 can be represented as the P-localization (resp. P-completion) of a quasifinite loop space. This is in fact not the case, which is illustrated by the following theorem [Cla63]

THEOREM 1.4 (Clark). *Every nontrivial quasifinite loop space X with finite fundamental group has $H^3(X;\mathbf{Q}) \neq 0$.*

On the other hand, simple p-compact groups typically have trivial third cohomology.

Theorem 1.1 can be further strengthened.

It is sufficient to assume that X is a p-compact group after completing at every prime $p \in P$. For P-local spaces, we have

PROPOSITION 1.5. *If X is a P-local space such that L_pX is a p-compact group for each $p \in P$, then X has a unique structure as a loop space such that all $X \to L_pX$ are loop maps.*

An interesting problem not addressed in this paper is finding explicit manifold models for simple p-compact groups in nontrivial cases (meaning neither of Lie type nor a sphere).

2. Lifts of loop space structures

Theorem 1.2 reduces to Theorem 1.1 by the implication (2)⇒(4) in the following lemma, together with the fact that giving a loop structure on a P-complete space X is equivalent to giving (independently) loop structures on every L_pX, $p \in P$

LEMMA 2.1. *Let X be a P-complete loop space with finite mod p homology for every $p \in P$. Then the following are equivalent:*

1. *X is the P-completion of a $\mathbf{Z}$-finite CW complex;*
2. *There exist positive integers $\{d_1, \dots, d_r\}$ such that the p-compact group L_pX has those degrees for all p;*
3. *There exist positive integers $\{d_1, \dots, d_r\}$ such that as loop spaces,*

$$L_{\mathbf{Q}}X \simeq \prod_{i=1}^{r} K(\mathbf{A}_P, 2d_i - 1),$$

where $\mathbf{A}_P = \left(\prod_{p\in P} \mathbf{Z}_p\right) \otimes \mathbf{Q}$ is the ring of finite adeles of $\mathbf{Z}_{(P)}$. (If P is finite, this is of course just $\prod_{p\in P} \mathbf{Q}_p$.)

4. *X is the P-completion of a $\mathbf{Z}_{(P)}$-finite, P-local loop space.*

Proof. (1)⇒(2): Let $X = L_PF$ for a finite CW-complex F.

$$H^*(L_pX;\mathbf{Z}_p) \otimes \mathbf{Q} \cong H^*(F;\mathbf{Z}_p) \otimes \mathbf{Q} \cong H^*(F;\mathbf{Q}) \otimes \mathbf{Q}_p.$$

This shows that $H^*(L_pX;\mathbf{Z}_p) \otimes \mathbf{Q}$ is an exterior algebra, and the degrees are independent of p.

(2)⇒(3)*:* By [ABGP03, Thm. 2.1], $L_pX \simeq L_p\left(\prod_{i=1}^r \mathbf{S}^{2d_i-1}\right)$ for $p > h = \max_i\{d_i\}$. Thus

$$\begin{aligned} L_{\mathbf{Q}}X &\simeq \prod_{p\le h} L_{\mathbf{Q}}X_p \times L_{\mathbf{Q}}\left(\prod_{p>h}\prod_{i=1}^r L_p\mathbf{S}^{2d_i-1}\right) \\ &\simeq \prod_{p\le h} L_{\mathbf{Q}}L_p\prod_{i=1}^r \mathbf{S}^{2d_i-1} \times \prod_{i=1}^r L_{\mathbf{Q}}\prod_{p>h} L_p\mathbf{S}^{2d_i-1} \\ &\simeq \prod_{i=1}^r L_{\mathbf{Q}}L_P\mathbf{S}^{2d_i-1}. \end{aligned}$$

Furthermore,

$$\pi_k L_{\mathbf{Q}}L_P\mathbf{S}^{2d-1} = \mathbf{Q}\otimes\prod_{p\in P}\pi_k(\mathbf{S}^{2d-1}) = \begin{cases} 0; & k\neq 2d-1 \\ \mathbf{A}_P; & k = 2d-1. \end{cases}$$

Thus $L_{\mathbf{Q}}BX$ is a space with homotopy groups concentrated in even dimensions, and hence a product of Eilenberg-Mac Lane spaces

$$L_{\mathbf{Q}}BX \simeq \prod_{i=1}^r K(\mathbf{A}_P, 2d_i).$$

which proves (3).

(3)⇒(4)*:* Define $BK = \prod_{i=1}^r K(\mathbf{Q}, 2d_i)$, and let $BK \to L_{\mathbf{Q}}BX$ be the product of the maps induced by the unit ring map $\mathbf{Q}\to A_P$. Let BF be the homotopy pullback of $BK \to L_{\mathbf{Q}}BX \leftarrow BX$. It remains to show that $K = \Omega BK$ is $\mathbf{Z}_{(P)}$-finite. Since $\mathbf{Q}$ and $\prod_{p\in P}\mathbf{Z}_p$ generate A_P, the Mayer-Vietoris sequence for the homotopy groups of a fiber product splits, and hence

$$\pi_n(BF) \cong \bigoplus_{\{i|2d_i=n\}}\mathbf{Z}_{(P)} \oplus \bigoplus_{p\in P}\mathrm{Tor}(\mathbf{Z}, \pi_n(BX_p)).$$

Thus $\pi_n(F)$ is finitely generated as a $\mathbf{Z}_{(P)}$-module, hence so is $H^n(F;\mathbf{Z}_{(P)})$ for all n. There exists a d such that $H^i(F;\mathbf{Z}_p) = H^i(X;\mathbf{Z}_p) = 0$ for all $p\in P$ and $i\ge d$. Since $H^i(F;\mathbf{Z}_p)\cong H^i(F;\mathbf{Z}_{(p)})\otimes_{\mathbf{Z}_{(p)}}\mathbf{Z}_p$ and $H^i(F;\mathbf{Z}_{(p)})$ is finitely generated, it follows that $H^i(F;\mathbf{Z}_{(p)}) = 0$ for those i and all $p\in P$. Thus $H^i(F;\mathbf{Z}_{(P)}) = 0$ for those i as well.

(4)⇒(1)*:* Let F_P be such a P-local, $\mathbf{Z}_{(P)}$-finite space, and let Q be the complementary set of primes. Choose F_Q to be a Q-local, $\mathbf{Z}_{(Q)}$-finite space such that $L_{\mathbf{Q}}F_Q \cong L_{\mathbf{Q}}F_P$, and let F be the pullback of $F_Q \to L_{\mathbf{Q}}F_P \leftarrow F_P$. Since F is $\mathbf{Z}_{(P)}$-finite and $\mathbf{Z}_{(Q)}$-finite, it is $\mathbf{Z}$-finite. □

Lemma 2.2. *Let X be a $\mathbf{Q}$-finite rational loop space, or the rationalization of a P-complete space satisfying the equivalent conditions of Lemma 2.1 for a set of primes P. Then it has a unique deloop, i.e. for every space Y such that $\Omega Y \simeq X$ we have $Y \simeq BX$. The unique loop structure on X is commutative.*

Proof. If X is a $\mathbf{Q}$-finite rational loop space, $H^*(X;\mathbf{Q})$ is an exterior algebra on finitely many odd-dimensional generators. Hence for any Y such that $\Omega Y \simeq X$, $H^*(Y;\mathbf{Q})$ is an even polynomial algebra. Any two such rational spaces are equivalent since all k-invariants necessarily vanish. The unique deloop is a product of even-dimensional Eilenberg-Mac Lane spaces, therefore X is abelian.

A similar argument works for rationalizations of P-complete $\mathbf{Z}/P$-finite loop spaces. In that case, X is homotopy equivalent to a product of $K(\mathbf{A}_P, \text{odd})$. □

LEMMA 2.3. *Let P be a set of primes and let X satisfy the equivalent conditions of Lemma 2.1. Let $f\colon Y \to X$ be a P-completion map for a P-local space Y. Then Y is $\mathbf{Z}_{(P)}$-finite and carries a unique loop structure such that $Y \to X$ is a loop map.*

Proof. Consider the arithmetic pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & L_P Y \simeq X \\ \downarrow & & \downarrow \\ L_{\mathbf{Q}} Y & \xrightarrow{L_{\mathbf{Q}} f} & L_{\mathbf{Q}} L_P Y \simeq L_{\mathbf{Q}} X. \end{array}$$

The $\mathbf{Z}_{(P)}$-finiteness follows from the argument given in Prop. 2.1(3) ⇒(4). As $X \to L_{\mathbf{Q}} X$ is given as a loop map, loop structures on Y such that f is a loop map are in one-to-one correspondence with loop structures on $L_{\mathbf{Q}} Y$ such that $L_{\mathbf{Q}} f$ is a loop map.

Giving a loop structure on $L_{\mathbf{Q}} Y$ is equivalent to giving a Hopf algebra structure on $H^*(Y; \mathbf{Q})$ extending the cohomology algebra structure. Since any loop structure on $L_{\mathbf{Q}} Y$ is commutative, such a Hopf algebra structure is completely determined by the vector space of primitives in $H^*(Y; \mathbf{Q})$. Now $L_{\mathbf{Q}} f$ induces an isomorphism of $\mathbf{A}_P$-algebras

$$f^*\colon H^*(X; \mathbf{Z}_P) \otimes \mathbf{Q} \xrightarrow{\cong} H^*(Y; \mathbf{Q}) \otimes \mathbf{A}_P.$$

This is an isomorphism of Hopf algebras if and only if

$$f^*(P(H^*(X; \mathbf{Z}_P) \otimes \mathbf{Q})) \cong P(H^*(Y; \mathbf{Q})) \otimes \mathbf{A}_P.$$

This shows that the only possible choice for the vector space of primitives in $H^*(Y; \mathbf{Q})$ is $(f^* P(H^*(X; \mathbf{Z}_P) \otimes \mathbf{Q})) \cap H^*(Y; \mathbf{Q})$, showing existence and uniqueness of the loop space structure. □

Proof of Prop. 1.5. We have a loop structure on $L_p X$ for every $p \in P$, or equivalently on the P-completion of X. We can lift this uniquely to X by Lemma 2.3, proving the Proposition. □

3. Double 1-tori

Recall [BK72, Chapter IV] that associated to any commutative ring R and group G, there is a notion of R-completion of G compatible with R-localization of nilpotent spaces. We will denote by $\mathbf{Z}_R$ the R-localization of the integers and by $\mathbf{Q}_R \cong \mathbf{Z}_R \otimes \mathbf{Q}$ the R-localization of the rationals. Thus $\mathbf{Z}_{\mathbf{Z}/p} = \mathbf{Z}_p$ and $\mathbf{Z}_{\mathbf{Z}_{(p)}} = \mathbf{Z}_{(p)}$. The localizations G_R carry a natural profinite topology. Similarly, for R-local spaces X, the chain complex $C_*(X)$ naturally carries a profinite topology. If A is an abelian group, we denote by $H_*(X; A)$ the homology of the chain complex $C_*(X) \hat{\otimes} A$ and by $H^*(X; A)$ the cohomology of the cochain complex $\operatorname{Hom}^{\text{cont.}}(C_*(X); A)$. We abbreviate $H_*(X) = H_*(X; \mathbf{Z}_R)$ and $H^*(X) = H^*(X; \mathbf{Z}_R)$.

For an R-local finite loop space X, define the **level** of X to be the smallest l such that $H^{2l+1}(X; \mathbf{Q}_R) \neq 0$. Note that since the cohomology ring is a finite dimensional Hopf algebra, it is generated by odd degree exterior classes. Define the **rank** of X to be the number of exterior generators in $H^*(X; \mathbf{Q}_R)$.

In [BKNP02], the pivotal tool for studying the finiteness obstruction and the surgery obstruction for a loop space X as well as for mixing homotopy types is

Definition 3.1. Let R be a ring and X be an R-finite, R-local, nilpotent, connected space. We say that X **admits a double 1-torus** if there is a diagram of horizontal and vertical fibrations of nilpotent spaces

$$\begin{array}{ccccc} L_R\mathbf{S}^1 & = & L_R\mathbf{S}^1 & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle i_0} & & \downarrow \\ L_R\mathbf{S}^1 \times C & \xrightarrow{i} & X & \xrightarrow{p} & Z \\ \downarrow & & \downarrow{\scriptstyle p_0} & & \| \\ C & \xrightarrow{i_1} & Y & \xrightarrow{p_1} & Z \end{array}$$

where $C = \mathbf{Z}/2$ if $\frac{1}{2}$ is not in R, and trivial otherwise, satisfying the following additional conditions:

(1) Z is R-finite,
(2) Y is stably reducible,
(3) $X \to Y$ is orientable,
(4) $\pi_1(p_0)$ is an isomorphism, and
(5) $Y \overset{p_1}{\twoheadrightarrow} Z$ induces an isomorphism $\pi_1(Y) \times C \cong \pi_1(Z)$.

Such an object was called rationally splitting in [BKNP02] if it rationally splits off a Hopf fibration. This condition needs to be relaxed for the present purposes.

Definition 3.2. A 1-torus $L_R\mathbf{S}^1 \to X \xrightarrow{p_0} Y$ is called *rationally splitting of level l* if p_0 rationally has a retract of the form $L_{\mathbf{Q}}L_R\mathbf{S}^1 \to L_{\mathbf{Q}}L_R\mathbf{S}^{2l+1} \to L_Q L_R\mathbf{C}P^l$.

Thus, a rational splitting in the sense of [BKNP02] is a rational splitting of level 1.

Proposition 3.3. *Given a sequence of positive integers $(n_0, n_1, \ldots, n_k)$ where $k \geq 1$. Then there exists a bundle of manifolds $S^1 \to M \to N$ satisfying the following conditions:*

(1) *N is stably parallelizable*
(2) *M is parallelizable*
(3) *Rationally the bundle is a product of the standard bundle $L_{\mathbf{Q}}\mathbf{S}^1 \to L_{\mathbf{Q}}\mathbf{S}^{2n_0+1} \to L_{\mathbf{Q}}\mathbf{C}P^{n_0}$ and trivial bundles $* \to L_{\mathbf{Q}}\mathbf{S}^{2n_i+1} \xrightarrow{=} L_{\mathbf{Q}}\mathbf{S}^{2n_i+1}$ for $1 \leq i \leq k$.*
(4) *There exists an $N_0 \in \mathbf{N}$, depending only on the integers n_i, such that*

$$L_pM \simeq \prod_{i=0}^{k} L_p\mathbf{S}^{2n_i+1} \quad \text{for all } p \geq N_0.$$

Proof. We start out with the manifold $X = \mathbf{C}P^{n_0} \times \prod_{i=1}^{k} \mathbf{S}^{2n_i+1}$. The Spivak normal fibration is rationally trivial, so a multiple of the top class of $\Sigma^d X$ is spherical for sufficiently large d. Making the Thom collapse map $\mathbf{S}^m \to \Sigma^d X$ transverse to

X establishes a rational surgery problem

$$\begin{array}{ccc} \nu_N & \longrightarrow & X \times \mathbf{R}^d \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & X \end{array}$$

with f of some non-zero degree on the top homology. Since X is an integral Poincaré duality space, the only possible surgery obstruction to turning f into a homotopy equivalence is the signature obstruction [TW79] when $m = \dim X$ is divisible by 4. However, the signature of X is zero since $k \geq 1$, and Hirzebruch's signature theorem implies that the signature of N is also zero since its normal bundle is trivial (condition (1)).

Since $H^2(N, \mathbf{Q}) = \mathbf{Q}$ there is a nonzero integral class represented by a map $N \to B\mathbf{S}^1$, classifying a bundle $S^1 \to M \to N$. The Serre spectral sequence for this fibration immediately shows that requirement (3) is satisfied. To see (2), that M is parallelizable, we proceed as in [BKNP02], using the criterion of [Dup70, Sut76]. If the dimension of M is even, the parallelizability of a stably parallelizable manifold is determined by the vanishing of the Euler characteristic of M which is clearly 0 in this case. If the dimension is 1, 3, or 7, M is automatically parallelizable; for all other odd dimensions, M is parallelizable if and only if its Kervaire semi-characteristic $\kappa(X)$ vanishes. It is defined as

$$\kappa(X) = \frac{1}{2}\left(\sum_{i \geq 0} \dim H_i(X; \mathbf{F}_2)\right) \pmod 2.$$

In this case $\dim N = m = 2l$ is even, and an easy rational computation shows that the Euler characteristic of N is 0. Thus the total dimension of $H_*(N; \mathbf{F}_2)$ is even. We may assume the Serre spectral sequence of $S^1 \to M \to N$ with $\mathbf{Z}/2$-coefficients collapses, since we otherwise could have composed $N \to B\mathbf{S}^1$ with the degree two map $B\mathbf{S}^1 \to B\mathbf{S}^1$ without affecting the rational types. It thus follows from the spectral sequence that the total dimension of $H_*(M; \mathbf{F}_2)$ is divisible by four, and hence the Kervaire semi-characteristic is zero.

To prove (4), let $\{e_i \mid 0 \leq i \leq k\}$ denote a basis of the indecomposables of $H^*(M; \mathbf{Z})$ modulo torsion and consider the lifting problem

$$\begin{array}{ccc} & & \prod_{i=0}^{k} \mathbf{S}^{2n_i+1} \\ & \nearrow & \downarrow \\ M & \xrightarrow[\prod e_i]{} & \prod_{i=0}^{k} K(\mathbf{Z}, 2n_i + 1) \end{array}$$

where the vertical map is the Hurewicz map, whose fiber will be denoted by F. The obstructions to this lifting problem lie in $H^{i+1}(M; \pi_i(F))$. The homotopy groups $\pi_i(F)$ are all finite since the dimensions $2n_i + 1$ are odd. If $N_0 = l.c.m.\{\#\pi_i(F) \mid 1 \leq i \leq \dim M\}$ then a lift exists after inverting N_0. Being parallelizable, M is in particular orientable and hence $\dim M = n_0 n_1 \cdots n_k$; thus N_0 only depends on the integers n_i. The lift is an $H\mathbf{F}_p$-isomorphism for $p > N_0$, and thus induces an equivalence $L_pM \to \prod L_p\mathbf{S}^{2n_i+1}$. □

PROPOSITION 3.4. *Let $P = P_1 \cup P_2$ be a partition of a set of primes, and let G be a simply connected P-local space such that $G_{(P_1)}$ is a P_1-local finite loop space of level l and rank bigger than one, and $G_{(P_2)}$ is the P_2-localization of the total space of a bundle $\mathbf{S}^1 \to M \to N$ satisfying the properties of Prop. 3.3 and such that $L_{\mathbf{Q}}M \simeq L_{\mathbf{Q}}G$. Assume that for each $p \in P_1$, L_pG has finite center, and that either $2 \notin P_1$ or*
($\star$)
$L_2G/Z(L_2G)$ is not equivalent to $L_2\,\mathrm{SO}(3)^l \times L_2\,\mathrm{SO}(5)^\epsilon$ for $l \geq 0$ and $\epsilon = 0,\ 1$.

Then G admits a rationally splitting double 1-torus of level l.

Proof. The argument is a variation, and simplification, of the proof of [BKNP02, Prop. 5.3, Prop. 4.3, and Prop. 4.1].

Note that by [ABGP03, Thm. 2.1] and Prop. 3.3(4), there exists an $N \gg 0$ such that L_pG is homotopy equivalent to a product of $p-$complete odd-dimensional spheres for all $p \geq N$, $p \in P$. Thus, by property (4) of Prop. 3.3, L_pG is homotopy equivalent to L_pM for all but finitely many $p \in P_1$. Hence we may assume without loss of generality that P_1 is finite.

Let $r = \mathrm{rk}(L_{(P_1)}G)$ and $r'' = \max\{\mathrm{rk}_p\, Z(L_pG) \mid p \in P_1\}$ be the maximal p-rank of the centers of the p-compact groups L_pG. It was proved in [BKNP02, Lemma 5.2] that $r'' < r$ unless $p = 2$ and G has type 3^k. Let $T'' = (\mathbf{S}^1)^{r''}$ and choose a torus

$$T'' \to L_pT'' \xrightarrow{i''_p} L_pG$$

containing the center of L_pG for all $p \in P_1$ [DW95]. Extend this to a maximal torus

$$T = T' \times T'' \to L_pT' \times L_pT'' \xrightarrow{i_p = i'_p \cdot i''_p} L_pG.$$

Let $(t_1, \ldots, t_{r'})$ be coordinates on T' and $(t_{r'+1}, \ldots, t_r)$ on T''.

For $p \in P_1$, the map $(L_pG)/(L_pT) \to L_pBT$ induced by the subgroup inclusion i_p induces an isomorphism

$$H^2(L_pBT; \mathbf{Q}_p) \xrightarrow{\simeq} H^2((L_pG)/(L_pT); \mathbf{Q}_p).$$

Let $q_{j,p}$ denote the image of L_pBt_j

Choose maps g, h, defining the rational homotopy equivalence

$$g \times h\colon G \xrightarrow{\simeq_{\mathbf{Q}}} L_{\mathbf{Q}}M \to L_{\mathbf{Q}}\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}Y,$$

where Y is a product of odd-dimensional spheres such that $L_{\mathbf{Q}}N \simeq L_{\mathbf{Q}}\mathbf{C}P^l \times L_{\mathbf{Q}}Y$. Since $[X, L_{\mathbf{Q}}\mathbf{S}^{2l+1}] \cong H^{2l+1}(X)$, g also denotes a class outside the image of $H^{2l+1}(N; \mathbf{Q}) \to H^{2l+1}(M; \mathbf{Q}) \cong H^{2l+1}(G; \mathbf{Q})$. Under the map

$$H^{2l+1}(G; \mathbf{Q}) \xrightarrow{\simeq} H^{2l+2}(BL_{\mathbf{Q}}G; \mathbf{Q}) \to H^{2l+2}(BL_{\mathbf{Q}}L_pG; \mathbf{Q}_p)$$
$$\xrightarrow{i_p^*} H^{2l+2}(L_pBT; \mathbf{Q}_p) \cong (\mathbf{Q}_p[t_1, \ldots, t_r])_{l+1}\,,$$

g maps to nonzero homogeneous polynomials f_p of degree $l+1$ for all $p \in P_1$.

For every $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{Z}^r$, define $\mathbf{S}^1 \cong S_\alpha \to T$ by $t \to \alpha_1 t_1 + \cdots + \alpha_r t_r$, where t is a coordinate on $\mathbf{S}^1$. Then S_α is a subgroup of T if and only if the set α is coprime; moreover, $i_p(L_pS_\alpha) \subseteq L_pG$ intersects the center of L_pG trivially if $p \nmid \alpha_j$

for some $1 \leq j \leq r'$. For $p \in P_1$, denote by $q_p \in H^2((L_pG)/(L_pS_\alpha); \mathbf{Q}_p)$ the image of the coordinate $t \in H^2(BS_\alpha; \mathbf{Q})$ under the map classifying the fibration

$$L_pS_\alpha \to L_pG \to (L_pG)/(L_pS_\alpha).$$

We thus have a commutative diagram

$$\begin{array}{ccccc}
H^*(L_pBT; \mathbf{Q}_p) & & \longrightarrow & & H^*((L_pG)/(L_pT); \mathbf{Q}_p) \\
 & t_i & \longmapsto & q_{i,p} & \\
\downarrow \alpha & \downarrow & & \downarrow & \downarrow \\
 & \alpha_i t & \longmapsto & \alpha_i q_p & \\
H^*(L_pBS_\alpha; \mathbf{Q}_p) & & \longrightarrow & & H^*((L_pG)/(L_pS_\alpha); \mathbf{Q}_p).
\end{array}$$

In $H^{2l+2}(-; \mathbf{Q}_p)$, we also have

$$\begin{array}{ccc}
f_p & \longmapsto & 0 \\
\downarrow & & \downarrow \\
f_p(\alpha_1 t, \dots, \alpha_r t) & \longmapsto & 0 \\
\| & & \| \\
f_p(\alpha_1, \dots, \alpha_r) t^{l+1} & & f_p(\alpha_1, \dots, \alpha_r) q_p^{l+1}.
\end{array}$$

In order for S_α to induce a splitting 1-torus on L_pG for a given $p \in P_1$, we need α to satisfy the following two conditions:

(1) $\{\alpha_1, \dots, \alpha_{r'}\}$ are coprime; and
(2) $f_p(\alpha_1, \dots, \alpha_r) \neq 0$.

We can always arrange this by choosing $\alpha_1 = 1$; since f_p is homogeneous of degree $l+1$, $f'_p = f_p(1, \alpha_2, \dots, \alpha_r)$ is a nonzero polynomial for all $p \in P_1$. Since P_1 is finite, $\alpha_2, \dots, \alpha_r$ can be chosen in such a way that $(1, \alpha_2, \dots, \alpha_r)$ is not a zero of f_p for any $p \in P_1$.

Now condition (1) ensures that S_α meets the center of L_pG trivially. Condition (2) implies that $q_p^{l+1} = 0$ and thus

$$H^*((L_pG)/(L_pS_\alpha); \mathbf{Q}_p) \cong \mathbf{Q}_p[q_p]/(q_p^{l+1}) \otimes H^*(L_pG; \mathbf{Q}_p)/(g)$$

which implies that $L_pS_\alpha \to L_pG \to (L_pG)/(L_pS_\alpha)$ has a rational splitting of level $l+1$.

Let

$$S \cong S_\alpha \to L_PG = \prod_{p \in P} L_pG$$

be the map thus constructed for $p \in P_1$, and the fiber inclusion $\mathbf{S}^1 \to M$ followed by p-completion for $p \in P_2$. Denote by $L_pG \to L_pG/L_pS$ the quotient of the subgroup inclusion for $p \in P_1$ and the p-completion of the fibration $M \to N$ for $p \in P_2$. Thus for every $p \in P$, $L_pS \to L_pG \to L_pG/L_pS$ is a rationally splitting 1-torus. Thus,

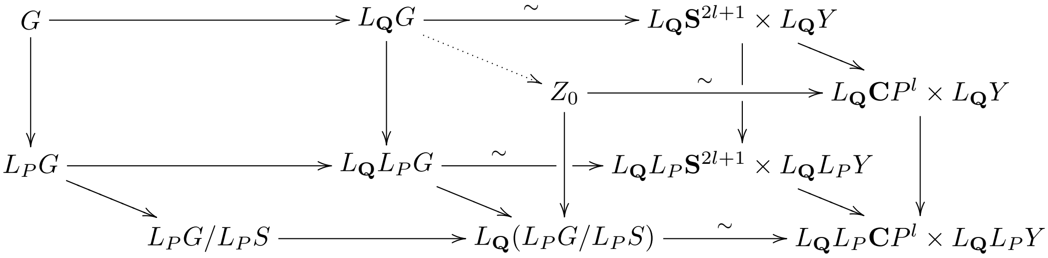
$$\begin{array}{ccccc}
L_pG & \longrightarrow & L_{\mathbf{Q}}L_pG & \xrightarrow[L_p(g \times h)]{\sim} & L_{\mathbf{Q}}L_p\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}L_pY \\
\downarrow & & \downarrow & & \downarrow \\
L_pG/L_pS & \longrightarrow & L_{\mathbf{Q}}(L_pG/L_pS) & \xrightarrow[\bar{h}_p]{\sim} & L_{\mathbf{Q}}L_p\mathbf{C}P^l \times L_{\mathbf{Q}}L_pY.
\end{array}$$

The map $L_p(g\times h)$ induces a map $L_{\mathbf{Q}}L_PG \xrightarrow{\sim} L_{\mathbf{Q}}L_P\mathbf{S}^{2l+1}\times L_{\mathbf{Q}}L_PY$ since it comes from a map $G\to L_{\mathbf{Q}}\mathbf{S}^{2l+1}\times L_{\mathbf{Q}}$. Likewise, for $p\in P_2$, $\overline{h}_p$ is the image of a rationally defined map. This together with the fact that P_1 is finite implies that the diagram above can be lifted to a diagram

$$\begin{array}{ccccc} L_PG & \longrightarrow & L_{\mathbf{Q}}L_PG & \xrightarrow{\sim} & L_{\mathbf{Q}}L_p\mathbf{S}^{2l+1}\times L_{\mathbf{Q}}L_PY \\ \downarrow & & \downarrow & & \downarrow \\ L_PG/L_PS & \longrightarrow & L_{\mathbf{Q}}(L_PG/L_PS) & \xrightarrow{\sim} & L_{\mathbf{Q}}L_P\mathbf{C}P^l\times L_{\mathbf{Q}}L_PY. \end{array}$$

We now address the problem of lifting this to G.

Consider the diagram



where the bottom two rows are the diagram constructed before, and Z_0 is defined as the pullback.Hence we can also fill in a map $L_QG\to Z_0$. Let $\mathbf{S}^1\to G\to Z$ be the pullback of this diagram. This constitutes a rationally splitting 1-torus of level l. The only axiom that is not quite immediate is the $\mathbf{Z}_{(P)}$-finiteness of Z, but it follows in the same way as in the proof of Prop. 2.1(3)$\Rightarrow$ (4).

Now if $2\in P$ it was shown in [BKNP02, Lemma 5.2] that under the hypothesis ($\star$), this 1-torus can be extended to a double 1-torus, and the proof is finished. □

Corollary 3.5. *Let P be any set of primes, and let G be an n-dimensional P-local loop space of level l and rank bigger than one satisfying ($\star$). Then there exists a finite n-dimensional CW-complex X whose P-localization (resp. P-completion) is homotopy equivalent to G and such that X admits a rationally splitting double 1-torus of level l.*

Proof. It was shown in [BKNP02] how to reduce to the case where G has finite fundamental group. For the reader's convenience, we recall the argument. Choose a map $BG\to L_{(P)}(B\mathbf{S}^1)^k$ which is an isomorphism on $H^2(-;\mathbf{Z}_{(P)})$, and denote its homotopy fiber by BG'. Then G' has finite fundamental group. Moreover, the map given by

$$L_{(P)}(\mathbf{S}^1)^k\to G^k\xrightarrow{\text{mult.}} G,$$

representing a basis of $\pi_1(G)$ modulo torsion, is easily seen to be a section of $G\to L_{(P)}(\mathbf{S}^1)^k$, thus $G\simeq G'\times L_{(P)}(\mathbf{S}^1)^k$, although not as loop spaces. Now if $X\xrightarrow{p} Y$ is a rationally splitting double 1-torus for G', then $X\times(\mathbf{S}^1)^k\xrightarrow{p\times\mathrm{Id}} Y\times(\mathbf{S}^1)^k$ is such for G.

Thus assume that the fundamental group of X is finite. Let X be constructed by mixing G with the manifold constructed in Proposition 3.3 at the complementary set of primes. By Prop. 3.4, the universal cover $\tilde{X}$ has a rationally splitting double 1-torus of level l, and since it does not meet the center of X at any prime $p\in P$, we obtain a splitting double 1-torus on X. The stable reducibility condition is satisfied since that is a p-local condition and it is satisfied at every prime. $\square$

Proof of Thm. 1.1. Let X be as in Theorem 1.1. If X has rank 1, the structure of p-compact groups implies that either $2 \in P$ and $X \in \{L_{(P)}SO(3), L_{(P)}\mathbf{S}^3\}$ or $2 \notin P$ and $X \simeq L_{(P)}\mathbf{S}^{2l-1}$ for some l. In the former case, the claim is obvious. In the latter case, the obvious choice for the manifold model M would be the sphere $\mathbf{S}^{2l-1}$, but if parallelizability is needed, as claimed in the theorem, then the manifold M, constructed by a surgery on twice the $l-1$-dimensional class in $S^{l-1} \times S^l$, will do, since its Kervaire semicharacteristic is obviously zero.

Now let X be as in Theorem 1.1 of rank ≥ 2, and let Z be constructed by mixing as in the corollary above. By the above corollary Z admits a double 1-torus. By [BKNP02, Prop. 3.3], Z is homotopy equivalent to a compact, stably parallelizable, smooth manifold. This manifold is in fact parallelizable on the nose: in even dimensions, this follows from the vanishing of the Euler characteristic of Z; in odd dimensions, it suffices to show that the Euler-Kervaire semi-characteristic vanishes. If $2 \notin P$, this follows from Prop. 3.3, and otherwise from the fact that the $\mathbf{F}_2$-cohomology of the 2-compact group L_2Z is a tensor product of truncated polynomial algebras $\mathbf{F}_2[z]/(z^{2^k})$, as in [BKNP02, Proof of the Main Theorem]. $\square$

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