TOPOLOGICAL EQUIVALENCE OF LINEAR REPRESENTATIONS FOR CYCLIC GROUPS: II

IAN HAMBLETON AND ERIK K. PEDERSEN

Abstract. In the two parts of this paper we prove that the Reidemeister torsion invariants determine topological equivalence of G–representations, for G a finite cyclic group.

1. Introduction

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically equivalent* to V' (denoted $V \sim_t V'$) if there exists a homeomorphism $h\colon V \to V'$ which is G-equivariant. If V, V' are topologically equivalent, but not linearly isomorphic, then such a homeomorphism is called a non-linear similarity. These notions were introduced and studied by de Rham [23], [24], and developed extensively in [1], [2], [14], [15], and [5]. In the two parts of this paper, referred to as [**I**] and [**II**], we complete de Rham's program by showing that Reidemeister torsion invariants and number theory determine non-linear similarity for finite cyclic groups.

A G-representation is called *free* if each element $1 \neq g \in G$ fixes only the zero vector. Every representation of a finite cyclic group has a unique maximal free subrepresentation.

Theorem. *Let G be a finite cyclic group and V_1, V_2 be free G–representations. For any G–representation W, the existence of a non-linear similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is entirely determined by explicit congruences in the weights of the free summands V_1, V_2, and the ratio $\Delta(V_1)/\Delta(V_2)$ of their Reidemeister torsions, up to an algebraically described indeterminacy.*

The notation and the indeterminacy are given in Section 2 and a detailed statement of results in Theorems A–E. This part of the paper contains the foundational results and calculations in bounded algebraic K- and L-theory needed to prove the main results on non-linear similarity. The study of non-linear similarities $V_1 \oplus W \sim_t V_2 \oplus W$ increases in difficulty with the number of isotropy types in W. We introduce a new method using excision in bounded surgery theory, based on the *orbit type filtration*, to organize and deal with these difficulties. We expect that this technique will be useful for other applications. Our most general results about non-linear similarity for arbitrary cyclic groups are Theorem C and its extensions (see Sections 9 and 10).

In Sections 3 and 13 we study the group $R_{\mathrm{Top}}(G)$ of G–representations modulo *stable* topological equivalences (see [2] where $R_{\mathrm{Top}}(G) \otimes \mathbf{Q}$ is computed). As an application of our general results, we determine the structure of the torsion in $R_{\mathrm{Top}}(G)$, for G any cyclic group (see Theorem 13.1), and in Theorem D we give the calculation of $R_{\mathrm{Top}}(G)$ for $G = C(4q)$,

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for q odd, correcting [5, Thm. 2]. One interesting feature is that Corollary 2.4 and Theorem D indicate a connection between the orders of the *ideal class groups* for cyclotomic fields and topological equivalence of linear representations.

Contents

2. Statement of Results

For the reader's convenience, we recall some notation from Part **I**, and then give the main results of both parts. Theorems A and B are proved in Part **I** and Theorems C and D are proved in Part **II**. The proof of Theorem E is divided between the two parts.

Let $G = C(4q)$, where $q > 1$, and let $H = C(2q)$ denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t \rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbf{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbf{R}_-$.

A *free* G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$, denote the complex numbers $\mathbf{C}$ with action $t \cdot z = \zeta^a z$ for all $z \in \mathbf{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \bmod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The Reidemeister torsion invariant of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k}(t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\} \ .$$

Let $V_2 = t^{b_1} + \cdots + t^{b_k}$ be another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent. This just means that the products of the weights $\prod a_i \equiv \prod b_i \bmod 4q$.

Then the Whitehead torsion of any G-homotopy equivalence is determined by the element

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$$

since $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Q}G)$ is monic [17, p.14].

Let W be a finite-dimensional G-representation. A necessary condition for a non-linear similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is the existence of a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ such that $f * id\colon S(V_2 \oplus U) \to S(V_1 \oplus U)$ is freely G-normally cobordant to the identity map on $S(V_1 \oplus U)$, for *all* free G–representations U (see [**I**], Section 3). If V_1 and V_2 satisfy this condition, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant*. This condition for non-linear similarity can be decided by explicit congruences in the weights of V_1 and V_2 (see [30, Thm. 1.2]).

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. It represents a unit in the group ring $\mathbf{Z}G$, explicitly described for $G = C(2^r)$ by Cappell and Shaneson in [3, §1] using a pull-back square of rings. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbf{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbf{Z}/2; A)$ of $\mathbf{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbf{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbf{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod(t^{a_i}-1)\prod((-t)^{a_i}-1)}{\prod(t^{b_i}-1)\prod((-t)^{b_i}-1)} = \prod \frac{(t^2)^{a_i}-1}{((t^2)^{b_i}-1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbf{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))\ .$$

This calculation takes place over the ring $\Lambda_{2q} = \mathbf{Z}[t]/(1 + t^2 + \cdots + t^{4q-2})$, but the result holds over $\mathbf{Z}G$ via the involution–invariant pull-back square

$$\begin{array}{ccc} \mathbf{Z}G & \to & \Lambda_{2q} \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \to & \mathbf{Z}/2q[\mathbf{Z}/2] \end{array}$$

Consider the exact sequence of modules with involution:

$$K_1(\mathbf{Z}H) \to K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G) \tag{2.1}$$

and define $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = K_1(\mathbf{Z}H \to \mathbf{Z}G)/_{\{\pm G\}}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. Such an exact sequence of $\mathbf{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))\ .$$

Our first result deals with isotropy groups of index 2, as is the case for all the non–linear similarities constructed in [1].

Theorem A. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, with $a_i \equiv b_i \equiv 1 \bmod 4$. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,
(ii) $\operatorname{Res}_H V_1 \cong \operatorname{Res}_H V_2$, *and*
(iii) *the element* $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$ *is in the image of the coboundary* $\delta\colon H^0(\mathbf{k}) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$.

Remark 2.2. The proof of this result is in Part **I**, but note that Condition (iii) simplifies for G a cyclic 2-group since $H^0(\mathbf{k}) = 0$ in that case (see [**I**], Lemma 9.1). Theorem A should be compared with [1, Cor.1], where more explicit conditions are given for "first-time" similarities of this kind under the assumption that q is odd, or a 2-power, or $4q$ is a "tempered" number. See also Theorem 9.2 for a more general result concerning similarities without $\mathbf{R}_+$ summands. The case $\dim V_1 = \dim V_2 = 4$ gives a reduction to number theory for the existence of 5-dimensional similarities (see [**I**], Remark 7.2).

Our next result uses a more elaborate setting for the invariant. Let

$$\Phi = \begin{pmatrix} \mathbf{Z}H & \to & \widehat{\mathbf{Z}}_2 H \\ \downarrow & & \downarrow \\ \mathbf{Z}G & \to & \widehat{\mathbf{Z}}_2 G \end{pmatrix}$$

and consider the exact sequence

$$0 \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to K_1(\widehat{\mathbf{Z}}_2 H \to \widehat{\mathbf{Z}}_2 G) \to K_1(\Phi) \to \widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G) \to 0\ . \tag{2.3}$$

Again we can define the Whitehead group versions by dividing out trivial units $\{\pm G\}$, and get a double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\operatorname{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))\ .$$

There is a natural map $H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H)) \to H^1(\operatorname{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$. We will use the same notation

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\operatorname{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

to denote the image of our Reidemeister torsion invariant. The non-linear similarities handled by the next result have isotropy of index ≤ 2.

Theorem B. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,
(ii) $\operatorname{Res}_H V_1 \cong \operatorname{Res}_H V_2$, *and*
(iii) *the element* $\{\Delta(V_1)/\Delta(V_2)\}$ *is in the image of the double coboundary*

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\operatorname{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))\ .$$

This result can be applied to 6-dimensional similarities.

Corollary 2.4. *Let $G = C(4q)$, with q odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then G has no 6-dimensional non-linear similarities.*

Remark 2.5. For example, the class number condition is satisfied for $q \leq 11$, but not for $q = 29$. The proof of the Corollary 2.4 is given in Section 11 assuming Theorem B, which is proved in Part **I**. This result corrects [5, Thm.1(i)], and shows that the computations of $R_{\mathrm{Top}}(G)$ given in [5, Thm. 2] are incorrect.

Our final example of the computation of bounded transfers is suitable for determining stable non–linear similarities inductively, with only a minor assumption on the isotropy subgroups. To state the algebraic conditions, we must again generalize the indeterminacy for the Reidemeister torsion invariant to include bounded K–groups (see Section 5). In this setting $\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G) = \widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ and $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$. We consider the analogous double coboundary

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$$

and note that there is a map $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ induced by the inclusion on the control spaces. We will again use the same notation

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$$

for the image of the Reidemeister torsion invariant in this new domain.

Theorem C. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) *$S(V_1)$ is s-normally cobordant to $S(V_2)$,*
(ii) *$\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \ ,$$

where $0 \subseteq W_{max} \subseteq W$ is a complex subrepresentation of real dimension ≤ 2, with maximal isotropy group among the isotropy groups of W with 2-power index.

Remark 2.6. The existence of a similarity $V_1 \oplus W \sim_t V_2 \oplus W$ implies that $S(V_1)$ and $S(V_2)$ are s-normally cobordant. In particular, $S(V_1)$ must be freely G-normally cobordant to $S(V_2)$ and this (unstable) normal invariant condition is enough to give us a surgery problem. Crossing with W defines the bounded transfer map

$$trf_W \colon L^h_{2k}(\mathbf{Z}G) \to L^h_{2k+\dim W}(\mathcal{C}_{W,G}(\mathbf{Z}))$$

introduced in [10]. The vanishing of the surgery obstruction is equivalent to the existence of a similarity (see [**I**], Theorem 3.5). The computation of the bounded transfer in L-theory leads to condition (iii), and an expression of the obstruction purely in terms of bounded K-theory. To carry out this computation we may need to stabilize in the free part, and this uses the s-normal cobordism condition.

Remark 2.7. Note that $W_{max} = 0$ in condition (iii) if W has no isotropy subgroups of 2-power index. Theorem C suffices to handle stable topological similarities, but leaves out cases where W has an odd number of $\mathbf{R}_-$ summands (handled in Theorem 9.2 and the results of Section 10). Simpler conditions can be given when $G = C(2^r)$ (see [**I**], Section 9).

The double coboundary in (iii) can also be expressed in more "classical" terms by using the short exact sequence

$$0 \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \tag{2.8}$$

derived in Corollary 6.9. We have $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$, where K is the isotropy group of W_{max}, and $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$. The indeterminacy in Theorem C is then generated by the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))$$

used in Theorem B and the coboundary

$$\delta\colon H^0(K_{-1}(\mathbf{Z}K)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))$$

from the Tate cohomology sequence of (2.8).

Finally, we apply these results to $R_{\mathrm{Top}}(G)$. Since its rank is known (see [2] or Section 4), it remains to determine its torsion subgroup. In Section 3, we will define a filtration

$$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G) \tag{2.9}$$

on the real representation ring $R(G)$, inducing a filtration on $R_{\mathrm{Top}}(G) = R(G)/R_t(G)$. Here the subgroup

$$R_t(G) = \{(V_1 - V_2) \mid V_1 \oplus W \sim_t V_2 \oplus W \text{ for some } W\}$$

is generated by stable topological similarity. Note that $R(G)$ has the following nice basis: $\{t^i, \delta, \epsilon \mid 1 \leqq i \leqq 2q-1\}$, where $\delta = [\mathbf{R}_-]$ and $\epsilon = [\mathbf{R}_+]$ (although we do not have $i \equiv 1 \bmod 4$ for all the weights).

Let $R^{free}(G) = \{t^a \mid (a, 4q) = 1\} \subset R(G)$ be the subgroup generated by the free representations. To complete the definition, we let $R^{free}(C(2)) = \{\mathbf{R}_-\}$ and $R^{free}(e) = \{\mathbf{R}_+\}$. Then inflation and fixed sets of representations defines an isomorphism

$$R(G) = \bigoplus_{K\subseteq G} R^{free}(G/K)$$

and this direct sum splitting can be intersected with $R_t(G)$ to define $R^{free}_t(G)$. We let $R^{free}_{\mathrm{Top}}(G) = R^{free}(G)/R^{free}_t(G)$. Since inflation and fixed sets preserve topological similarities, we obtain an induced splitting

$$R^{free}_{\mathrm{Top}}(G) = \bigoplus_{K\subseteq G} R^{free}_{\mathrm{Top}}(G/K)\ .$$

By induction on the order of G, we see that it suffices to study the summand $R^{free}_{\mathrm{Top}}(G)$.

Let $\widetilde{R}^{free}(G) = \ker(\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$, and then project into $R_{\mathrm{Top}}(G)$ to define

$$\widetilde{R}^{free}_{\mathrm{Top}}(G) = \widetilde{R}^{free}(G)/R^{free}_t(G)\ .$$

In Section 4 we prove that $\widetilde{R}^{free}_{\mathrm{Top}}(G)$ is *precisely* the torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$. Here is a specific computation (correcting [5, Thm. 2]).

Theorem D. *Let $G = C(4q)$, with $q > 1$ odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then $\widetilde{R}^{free}_{\mathrm{Top}}(G) = \mathbf{Z}/4$ generated by $(t - t^{1+2q})$.*

For any cyclic group G, we use normal cobordism and homotopy equivalence to define a filtration

$$R_t^{free}(G) \subseteq R_n^{free}(G) \subseteq R_h^{free}(G) \subseteq R^{free}(G)$$

leading by direct sum to the filtration of $R(G)$ mentioned above. Both $\widetilde{R}^{free}(G)/\widetilde{R}_h^{free}(G)$ and $\widetilde{R}_h^{free}(G)/\widetilde{R}_n^{free}(G)$ are torsion groups which can be explicitly determined by congruences in the weights (see Section 12 and [30, Thm.1.2]). The subquotient $\widetilde{R}_n^{free}(G)/R_t^{free}(G)$ always has exponent two (see Section 13).

We conclude this list of sample results with a calculation of $R_{\mathrm{Top}}(G)$ for cyclic 2-groups (see [**I**] for the proof).

Theorem E. *Let $G = C(2^r)$, with $r \geqq 4$. Then*

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \langle \alpha_1, \alpha_2, \ldots, \alpha_{r-2}, \beta_1, \beta_2, \ldots, \beta_{r-3} \rangle$$

subject to the relations $2^s\alpha_s = 0$ for $1 \leqq s \leqq r-2$, and $2^{s-1}(\alpha_s + \beta_s) = 0$ for $2 \leqq s \leqq r-3$, together with $2(\alpha_1 + \beta_1) = 0$.

The generators for $r \geqq 4$ are given by the elements

$$\alpha_s = t - t^{5^{2^{r-s-2}}} \quad \text{and} \quad \beta_s = t^5 - t^{5^{2^{r-s-2}+1}}.$$

We remark that $\widetilde{R}_{\mathrm{Top}}^{free}(C(8)) = \mathbf{Z}/4$ generated by $t - t^5$. In [**I**], Theorem 11.6 we use this information to give a complete topological classification of linear representations for cyclic 2-groups.

Acknowledgement. The authors would like to express their appreciation to the referee for many constructive comments and suggestions. In particular, the referee pointed out an improvement to the statement of Theorem E. If we let $\psi_i = \alpha_{i+1} + \beta_{i+1}$ for $1 \leq i \leq r-4$, then $\widetilde{R}_{\mathrm{Top}}^{free}(C(2^r))$ is a direct sum of cyclic groups generated by $\alpha_1, \ldots, \alpha_{r-2}, \beta_1$ and $\psi_1, \ldots, \psi_{r-4}$ where the order of the cyclic group generated by a basis element with subscript i is 2^i. This basis displays the group structure explicitly.

3. A splitting of $R_{\mathrm{Top}}(G)$

In this section we point out an elementary splitting of $R_{\mathrm{Top}}(G)$, and some useful filtrations. For G any finite group, we denote by $R(G)$ the *real* representation ring of G. Elements in $R(G)$ can be given as formal differences $(V_1 - V_2)$ of G–representations, and $(V_1 - V_2) \sim 0$ if and only if there exists a representation W such that $V_1 \oplus W \cong V_2 \oplus W$.

Notice that for K any normal subgroup of G, taking fixed sets gives a retraction of the inflation map

$$\inf_K \colon R(G/K) \to R(G)$$

defined by pulling back a G/K representation using the composition with the quotient map $G \to G/K$. More explicitly,

$$\mathrm{Fix}_K \colon R(G) \to R(G/K)$$

is defined by $\mathrm{Fix}_K(V_1 - V_2) = (V_1^K - V_2^K)$ for each normal subgroup $K \lhd G$. Then $\mathrm{Fix}_K \circ \inf_K = id \colon R(G/K) \to R(G/K)$.

Definition 3.1. A G–representation V is *free* if $V^K = \{0\}$ for all non–trivial normal subgroups $1 \neq K \lhd G$.

This is the same as the usual definition (no non–identity element of G fixes any non–zero vector) for cyclic groups. We let

$$R^{free}(G) = \bigcap\{\ker \mathrm{Fix}_K \mid 1 \neq K \lhd G\} \tag{3.2}$$

denote the subgroup of $R(G)$ such that there is a representative of the stable equivalence class $(V_1 - V_2)$ with V_1, V_2 free representations.

Proposition 3.3. *There is a direct sum splitting*

$$R(G) = \bigoplus_{K \lhd G} R^{free}(G/K)$$

indexed by the normal subgroups K in G.

Proof. Let $V(K)$ denote the G–invariant subspace given by the sum of all the irreducible sub–representations of V with kernel exactly K. This is a free G/K representation. The decomposition above is given by mapping $(V_1 - V_2)$ to the elements $(V_1(K) - V_2(K))$. □

Inside $R(G)$ we have the subgroup of stably topologically similar representations

$$R_t(G) = \{(V_1 - V_2) \mid V_1 \oplus W \sim_t V_2 \oplus W \text{ for some } W\} \tag{3.4}$$

and the quotient group is $R_{\mathrm{Top}}(G)$ by definition. We define $R_t^{free}(G) = R^{free}(G) \cap R_t(G)$. Since $R_t(G)$ is preserved by inflation and taking fixed sets, we obtain

Corollary 3.5. *There is a direct sum decomposition*

$$R_{\mathrm{Top}}(G) = \bigoplus_{K \lhd G} R_{\mathrm{Top}}^{free}(G/K)$$

where the summands are the quotients $R^{free}(G/K)\big/R_t^{free}(G/K)$.

We will also need a certain filtration of $R(G)$. First we define

$$R_h^{free}(G) = \{(V_1 - V_2) \mid S(V_1) \simeq_G S(V_2) \ \text{ for } V_1 \text{ and } V_2 \text{ free}\} \tag{3.6}$$

where $\simeq_G$ denotes G–homotopy equivalence. This is a subgroup of $R^{free}(G)$, in fact a sub–Mackey functor since it has induction and restriction for subgroups of G. We define

$$R_h(G) = \bigoplus_{K \lhd G} R_h^{free}(G/K) \tag{3.7}$$

If there exists a G–homotopy equivalence $f \colon S(V_1) \to S(V_2)$ such that

$$S(V_1 \oplus U) = S(V_1){*}S(U) \xrightarrow{f*1} S(V_2){*}S(U) = S(V_2 \oplus U)$$

is freely G–normally cobordant to the identity for all free G–representations U, then we say that $S(V_1)$ and $S(V_2)$ are s-normally cobordant, and we write $S(V_1) \doteq_G S(V_2)$. Define

$$R_n^{free}(G) = \{\alpha \in R_h^{free}(G) \mid \exists V_1, V_2 \text{ with } \alpha = (V_1 - V_2) \ \text{ and } S(V_1) \doteq_G S(V_2)\} \tag{3.8}$$

and note that $R_n^{free}(G)$ is also a subgroup of $R^{free}(G)$. Indeed, if $(V_1 - V_2)$ and $(V_1' - V_2')$ are in $R_n^{free}(G)$, there exist G-homotopy equivalences $f \colon S(V_1) \to S(V_2)$ and $f' \colon S(V_1') \to S(V_2')$ with $f{*}1$ and $f'{*}1$ normally cobordant to the identity under any stabilization. But $f{*}f' \simeq_G (1{*}f') \circ (f{*}1)$, so we just glue together the normal cobordisms for $f{*}1$ (after stabilizing by $U = V_1'$) and for $1{*}f'$ (after stabilizing by $U = V_2$) along the common boundary $id\colon S(V_2 \oplus V_1')$. As above, we define

$$R_n(G) = \bigoplus_{K \triangleleft G} R_n^{free}(G/K) \tag{3.9}$$

Since $R_t^{free}(G) \subseteq R_n^{free}(G)$, we have defined a filtration

$$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G) \tag{3.10}$$

of $R(G)$, natural with respect to restriction of representations. All the terms except possibly $R_n(G)$ are also natural with respect to induction of representations.

Remark 3.11. It follows from the proof of [30, 3.1] that $S(V_1 \oplus U_0)$ is s-normally cobordant to $S(V_2 \oplus U_0)$, for some free G-representation U_0, if and only if $S(V_1)$ is s-normally cobordant to $S(V_2)$. It follows that we could have used the latter condition to define $R_n^{free}(G)$.

4. A rational computation

In this section we use [**I**], Theorem 3.5 and the splitting of the last section to describe the torsion subgroup of $R_{\mathrm{Top}}^{free}(G)$. We also give a new proof of Cappell and Shaneson's result computing $R_{\mathrm{Top}}(G) \otimes \mathbf{Q}$ for all finite groups G.

First we consider cyclic groups. Let $G = C(2^r q)$ be a cyclic group, where $q \geqq 1$ is odd. The Odd Order Theorem [14], [15] gives $R(G) = R_{\mathrm{Top}}(G)$ if $r \leqq 1$, and we recall the definition

$$\widetilde{R}_{\mathrm{Top}}^{free}(G) = \widetilde{R}^{free}(G)/R_t^{free}(G)$$

where $\widetilde{R}^{free}(G) = \ker\{\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd})\}$. Here $\widetilde{R}^{free}(C(2q)) = \widetilde{R}^{free}(C(q)) = 0$, for $q \geqq 1$ odd.

Theorem 4.1. *For G cyclic, the kernel and cokernel of*

$$\mathrm{Res}\colon R_{\mathrm{Top}}^{free}(G) \to R_{\mathrm{Top}}^{free}(G_{odd}) = R^{free}(G_{odd})$$

are 2–primary torsion groups.

Corollary 4.2. *Let $G = C(2^r q)$, q odd, be a finite cyclic group.*

(i) *The torsion subgroup of $R_{\mathrm{Top}}^{free}(G)$ is $\widetilde{R}_{\mathrm{Top}}^{free}(G)$.*
(ii) *The rank of $R_{\mathrm{Top}}^{free}(G)$ is $\varphi(q)/2$ (resp. 1) if $q > 1$ (resp. $q = 1$).*
(iii) *We have the formula*

$$rank(R_{\mathrm{Top}}(G)) = \begin{cases} (r+1)\left\{\sum_{1 \neq d | q} \varphi(d)/2 + 1\right\}, & \text{for } q > 1 \\ (r+1) & \text{for } q = 1. \end{cases}$$

where $\varphi(d)$ is the Euler function.

Proof. The first part follows directly from Theorem 4.1. For parts (ii) and (iii) count the free representations of G_{odd}. ☐

Now let F be the composite of all subfields of $\mathbf{R}$ of the form $\mathbf{Q}(\zeta + \zeta^{-1})$ where $\zeta \in \mathbf{C}$ is an odd root of unity.

Lemma 4.3. *For G cyclic, the composition of the natural map $R_F(G) \to R(G)$ and* $\mathrm{Res}\colon R(G) \to R(G_{odd})$ *induces a p–local isomorphism $R_F^{free}(G) \to R^{free}(G_{odd})$, for any odd prime p.*

Proof. According to a result of Brauer [25, Thm.24] $R_F(G_{odd}) = R(G_{odd})$, and any representation of G can be realized over the field $\mathbf{Q}(\zeta_{|G|})$. In addition, the restriction map

$$\mathrm{Res}\colon R_F(G) \to R_F(G_{odd})$$

is a p–local surjection (since $\mathrm{Res}_{G_{odd}} \circ \mathrm{Ind}_{G_{odd}}$ is just multiplication by 2^r). But the rank of $R_F(G)$ given in [25, 12.4] equals the rank of $R(G_{odd})$, so we are done. □

Corollary 4.4. *For G cyclic and any odd prime p, the natural map $R_F(G) \to R_{\mathrm{Top}}(G)$ induces a p–local isomorphism.*

Proof. This follows from the Lemma 4.3 and Theorem 4.1. □

In [2], Cappell and Shaneson obtained the following result by a different argument. It computes the rank of $R_{\mathrm{Top}}(G) \otimes \mathbf{Q}$ for any finite group G.

Theorem 4.5. *Let F be the composite of all subfields of $\mathbf{C}$ of the form $\mathbf{Q}(\zeta + \zeta^{-1})$ where $\zeta \in \mathbf{C}$ is an odd root of unity. Then for any finite group G, the natural map $R_F(G) \to R(G)$ induces an isomorphism $R_F(G) \otimes \mathbf{Z}_{(p)} \cong R_{\mathrm{Top}}(G) \otimes \mathbf{Z}_{(p)}$ for any odd prime p.*

Proof. The result holds for cyclic groups G by Corollary 4.4, and we apply induction theory to handle general finite groups.

First we observe that the Mackey functors $R(G) \otimes \mathbf{Z}_{(p)}$ and $R_F(G) \otimes \mathbf{Z}_{(p)}$ are generated by induction from p–elementary subgroups (see [25, Thm.27] and note that $\Gamma_F = \{\pm 1\}$). Therefore, by [17, 11.2] they are p–elementary computable in the sense of Dress induction theory [9]. We may therefore assume that G is p–elementary.

Now if G is p–elementary, it is a product of a p–group and a cyclic group prime to p. Any irreducible complex representation of G is then induced from a linear representation on a subgroup [25, Thm.16].

It follows that $R_{\mathrm{Top}}(G)$ is generated by generalized induction (i.e. inflation followed by induction) from cyclic subquotients. Consider the following commutative diagram

$$\begin{array}{ccccc}
R_F(G) & \longrightarrow & R(G) & \longrightarrow & R_{\mathrm{Top}}(G) \\
{\scriptstyle \mathrm{Ind}}\uparrow\downarrow{\scriptstyle \mathrm{Res}} & & \uparrow\downarrow & & \uparrow\downarrow \\
\bigoplus_C R_F(C) & \longrightarrow & \bigoplus_C R(C) & \longrightarrow & \bigoplus_C R_{\mathrm{Top}}(C)
\end{array}$$

It follows from Corollary 4.4 that the top composite map $R_F(G) \otimes \mathbf{Z}_{(p)} \to R_{\mathrm{Top}}(G)) \otimes \mathbf{Z}_{(p)}$ is surjective.

The sum of the (ordinary) restriction maps to cyclic subgroups induces a rational injection on $R_F(G)$ (see [27, 2.5,2.10]). Since $R_F(G)$ is torsion–free, it follows again from Corollary 4.4 that the map $R_F(G) \otimes \mathbf{Z}_{(p)} \to R_{\mathrm{Top}}(G)) \otimes \mathbf{Z}_{(p)}$ is injective. □

The proof of Theorem 4.1. Since an F–representation is free if and only if it is the sum of Galois conjugates of free G–representations, we can decompose $R_F(G)$ as in Section 3, and conclude that $R_F^{free}(G_{odd}) = R^{free}(G_{odd})$. It remains to show that $\widetilde{R}_{\mathrm{Top}}^{free}(G)$ is a torsion group with 2–primary exponent. For this we use the filtration of §3.

For $\widetilde{R}^{free}(G)/\widetilde{R}_h^{free}(G)$ this is easy since the k–invariant gives a homomorphism (via joins of free G–spheres) to $(\mathbf{Z}/2^r)^\times$ and this is a 2–group. The next quotient is also 2–primary torsion, by results of [30]: a sufficiently large 2–power join of a G–homotopy equivalence between two free G–spheres, which are linearly equivalent over G_{odd}, becomes s–normally cobordant to the identity. The point is that the normal invariant is detected by a finite number of 2–power congruences conditions among the Hirzebruch L–classes of the tangent bundles of the lens spaces, and this can be satisfied after sufficiently many joins.

Finally, the last quotient $\widetilde{R}_n^{free}(G)/\widetilde{R}_t^{free}(G)$ is shown to be 2–primary torsion in the next proposition. □

Proposition 4.6. *Let $G = C(2^r q)$, q odd, and assume*

$$\sigma \in \ker(\mathrm{Res}\colon L_{2k}^h(\mathbf{Z}G) \to L_{2k}^h(\mathbf{Z}G_{odd})).$$

Then there exists a complex representation W with $W^G = 0$ such that $trf_W(2^r\sigma) = 0$.

Proof. We will take $W = \mathbf{R}_- \oplus \mathbf{R}_- \oplus W_0$, where W_0 is the sum of all the irreducible 2–dimensional representations of G with isotropy of 2–power index. Note that the $\mathbf{R}_-$-transfer is just the compact I_- transfer of one-sided codimension 1 surgery followed by adding rays to infinity, so whenever the I_-transfer is 0, the $\mathbf{R}_-$-transfer will have to be 0. This was discussed in more detail in [**I**], Section 4.

Step 1: If G has odd order, there is nothing to prove. Otherwise, let $H \subset G$ be of index 2. If $\mathrm{Res}_H(\sigma) = 0$, then $trf_{\mathbf{R}_-}(\sigma) \in \mathrm{Im}(L_1^h(\mathbf{Z}G^-) \to L_1^h(\mathbf{Z}H \to \mathbf{Z}G^-))$. But $L_1^h(\mathbf{Z}G^-)$ has exponent 2 [13, 12.3], so $trf_{\mathbf{R}_-}(2\sigma) = 0$. Then take $W = \mathbf{R}_- \oplus \mathbf{R}_-$. Note that this case applies to $G = C(2q)$, so we can always get started.

Step 2: We may assume that $r \geqq 2$. If $\mathrm{Res}_H(\sigma) \neq 0$ note that

$$\mathrm{Res}_H(2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)) = 2\,\mathrm{Res}_H(\sigma) - 2\,\mathrm{Res}_H(\sigma) = 0.$$

By induction $\mathrm{Res}_H W_0$ works for $\mathrm{Res}_H(\sigma)$: say

$$trf_{\mathrm{Res}_H W_0}(2^{r-1}\mathrm{Res}_H(\sigma)) = 0$$

and $W_0^H = 0$. Let $\dim W_0 = m$ and consider the commutative diagram

$$\begin{array}{ccccc}
L_{2k}(\mathbf{Z}G) & \xrightarrow{\mathrm{Res}} & L_{2k}(\mathbf{Z}H) & \xrightarrow{\mathrm{Ind}} & L_{2k}(\mathbf{Z}G) \\
\downarrow{\scriptstyle trf_{W_0}} & & \downarrow{\scriptstyle trf_{\mathrm{Res}_H W_0}} & & \downarrow{\scriptstyle trf_{W_0}} \\
L_{2k+m}(\mathcal{C}_{W_0,G}(\mathbf{Z})) & \xrightarrow{\mathrm{Res}} & L_{2k+m}(\mathcal{C}_{\mathrm{Res}\, W_0,H}(\mathbf{Z})) & \xrightarrow{\mathrm{Ind}} & L_{2k+m}(\mathcal{C}_{W_0,G}(\mathbf{Z})).
\end{array}$$

From this we get

$$2^{r-1} \cdot trf_{W_0}(\mathrm{Ind}_H \mathrm{Res}_H(\sigma)) = 0\ .$$

The first step implies that $trf_{\mathbf{R}_-^2}(2\sigma_1) = 0$, where $\sigma_1 = 2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)$. Let $W = \mathbf{R}_- \oplus \mathbf{R}_- \oplus W_0$, so that we have W complex and $W^G = 0$. Note that $trf_W = trf_{\mathbf{R}_-^2} \circ trf_{W_0} = trf_{W_0} \circ trf_{\mathbf{R}_-^2}$. But

$$2^r \cdot trf_W(\sigma) = 2^{r-1} trf_W(2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)) + 2^{r-1} trf_W(\mathrm{Ind}_H \mathrm{Res}_H(\sigma))$$

and both terms vanish (because $r \geqq 2$ and by the property of W_0 respectively). □

A similar argument to that in Step 2 above gives:

Proposition 4.7. *If* $\mathrm{Res}_H(trf_W(x)) = 0$ *for* $x \in L_0^h(\mathbf{Z}G)$*, then* $4 \cdot trf_{W\times\mathbf{R}_-}(x) = 0$.

Proof. Since

$$4 \cdot trf_{W\times\mathbf{R}_-}(\sigma) = 2 \cdot trf_W trf_{\mathbf{R}_-}(2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)) + 2 \cdot trf_{\mathbf{R}_-} trf_W(\mathrm{Ind}_H \mathrm{Res}_H(\sigma))$$

we conclude as above that both terms vanish. □

5. Excision in bounded surgery theory

A small additive category with involution $\mathcal{A}$ is a small additive category together with a contravariant endofunctor $*$ such that $*^2 = 1_{\mathcal{A}}$. Ranicki defines algebraic L–theory $L_*^h(\mathcal{A})$ for such categories and corresponding spectra $\mathbb{L}^h(\mathcal{A})$ with $L_*^h(\mathcal{A}) = \pi_*(\mathbb{L}^h(\mathcal{A}))$ [21]. The obstruction groups for bounded surgery are obtained this way for appropriately chosen additive categories. We shall also need a simple version of such groups. For this, the additive category must come equipped with a system of stable isomorphisms and a subgroup $s \subset K_1(\mathcal{A})$, such that any composition resulting in an automorphism defines an element in s. The point here is that whenever two objects are stably isomorphic, there is a canonically chosen stable isomorphism, canonical up to automorphisms defining elements of s. In this situation Ranicki refines the definition of $\mathbb{L}^h(\mathcal{A})$ to give the simple L–theory spectrum $\mathbb{L}^s(\mathcal{A})$, by requiring appropriate isomorphisms to give elements of $K_1(\mathcal{A})$ belonging to the subgroup s. More generally, we also get the $\mathbb{L}^a(\mathcal{A})$-spectra for any involution invariant subgroup a with $s \subset a \subset K_1(\mathcal{A})$, coinciding with $\mathbb{L}^h(\mathcal{A})$ when $a = K_1(\mathcal{A})$.

Example 5.1. Let $\mathcal{A}$ be the category of free $\mathbf{Z}G$-modules with a G-invariant $\mathbf{Z}$-basis, and $\mathbf{Z}G$-module morphisms. Two objects are stably isomorphic if and only if they have the same rank. The preferred isomorphisms are chosen to be the ones sending a $\mathbf{Z}$-basis to a $\mathbf{Z}$-basis, so automorphisms define elements of $\{\pm G\} \subset K_1(\mathbf{Z}G)$. In this situation one obtains Wall's L^s-groups.

The theory of projective L–groups fits into the scheme as follows: one defines $\mathbb{L}^p(\mathcal{A}) = \mathbb{L}^h(\mathcal{A}^\wedge)$, where $\mathcal{A}^\wedge$ is the idempotent completion of $\mathcal{A}$. The objects of $\mathcal{A}^\wedge$ are pairs (A, p) with A an object of $\mathcal{A}$ and $p^2 = p$. The morphisms $\phi\colon (A,p) \to (B,q)$ are the $\mathcal{A}$-morphisms $\phi\colon A \to B$ with $q\phi p = \phi$. Again it is possible to "partially" complete $\mathcal{A}$. If $K_0(\mathcal{A}) \subset \mathbf{k} \subset K_0(\mathcal{A}^\wedge)$ is an involution invariant subgroup , we define $\mathcal{A}^{\wedge k}$ to be the full subcategory of $\mathcal{A}^\wedge$ with objects defining elements of $\mathbf{k} \subset K_0(\mathcal{A}^\wedge)$. This way we may define $\mathbb{L}^k(\mathcal{A}) = \mathbb{L}(\mathcal{A}^{\wedge k})$. Similarly to the above, for $\mathbf{k} = K_0(\mathcal{A}) \subset K_0(\mathcal{A}^\wedge)$, $L_*^k(\mathcal{A})$ is naturally isomorphic to $L_*^h(\mathcal{A})$. The quotient $\widetilde{K}_0(\mathcal{A}) = K_0(\mathcal{A}^\wedge)/K_0(\mathcal{A})$ is called the reduced projective class group of $\mathcal{A}$.

Example 5.2. If $\mathcal{A}$ is the category of free $\mathbf{Z}G$-modules then $\mathcal{A}^\wedge$ is isomorphic to the category of projective $\mathbf{Z}G$-modules and the $L_*^p(\mathcal{A})$ are Novikov's original L^p-groups.

Suppose M is a metric space with the finite group G acting by isometries, R a ring with involution. In [10, 3.4] we defined an additive category $\mathcal{G}_{M,G}(R)$ with involution as follows:

Definition 5.3. An object A is a left $R(G)$-module together with a map $f\colon A \longrightarrow F(M)$, where $F(M)$ is the set of finite subsets of M, satisfying

(i) f is G-equivariant.
(ii) $A_x = \{a \in A \mid f(a) \subseteq \{x\}\}$ is a finitely generated free sub R-module
(iii) As an R-module $A = \bigoplus_{x \in M} A_x$
(iv) $f(a+b) \subseteq f(a) \cup f(b)$
(v) For each ball $B \subset M$, $\{x \in B \mid A_x \neq 0\}$ is finite.

A morphism $\phi\colon A \longrightarrow B$ is a morphism of RG-modules, satisfying the following condition: there exists k so that the components $\phi^m_n\colon A_m \longrightarrow B_n$ (which are R-module morphisms) are zero when $d(m,n) > k$. Then $\mathcal{G}_{M,G}(R)$ is an additive category in an obvious way. The full subcategory of $\mathcal{G}_{M,G}(R)$, for which all the object modules are required to be *free* left RG-modules, is denoted $\mathcal{C}_{M,G}(R)$.

Given an object A, an R-module homomorphism $\phi\colon A \longrightarrow R$ is said to be locally finite if the set of $x \in M$ for which $\phi(A_x) \neq 0$ is finite. We define $A^* = \mathrm{Hom}^{lf}_R(A,R)$, the set of locally finite R-homomorphisms. We want to make $*$ a functor from $\mathcal{G}_{M,G}(R)$ to itself to make $\mathcal{G}_{M,G}(R)$ a category with involution. We define $f^*\colon A^* \longrightarrow FM$ by $f^*(\phi) = \{x \mid \phi(A_x) \neq 0\}$ which is finite by assumption. A^* has an obvious right action of G turning it into a right RG module given by $\phi g(a) = \phi(ga)$, and f^* is equivariant with respect to the right action on M given by $xg = g^{-1}x$. To make $*$ an endofunctor of $\mathcal{G}_{M,G}(R)$ we need to replace the right action by a left action. We do this by the standard way in surgery theory by letting g act on the left by letting g^{-1} act on the right. In the unoriented case, given a homomorphism $w\colon G \longrightarrow \{\pm 1\}$, we let g act on the left of A^* by $w(g)\cdot g^{-1}$ on the right. The involution $*$ induces a functor on the subcategory $\mathcal{C}_{M,G}(R)$, so that $\mathcal{C}_{M,G}(R)$ is also a category with involution.

Example 5.4. Let $\rho_W\colon G \to O(W)$ be an orthogonal action of G on a finite dimensional real vector space W. We take $M = W$ with the action through ρ_W, and orientation character $\det(\rho_W)$. This will be called the *standard orientation* on $\mathcal{C}_{W,G}(\mathbf{Z})$.

Remark 5.5. We will need to find a system of stable isomorphisms for the category $\mathcal{C}_{M,G}(R)$ to be able to do simple L–theory. To do this we choose a point x in each G-orbit, and an RG_x-basis for A_x, where G_x is the isotropy subgroup of x. We then extend that by equivariance to an R-basis of the module. Having a basis allows defining an isomorphism in the usual fashion. In each case we need to describe the indeterminacy in the choices coming from the choice of R-basis and points in the orbit. For our particular choices of M it will be easy to determine the subgroup s, so we will not formulate a general statement.

We will study the L–theory of the categories $\mathcal{C}_{M,G}(R)$ using excision. Let N be a G-invariant metric subspace of M. Denoting $\mathcal{C}_{M,G}(R)$ by $\mathcal{U}$, let $\mathcal{A}$ be the full subcategory on modules A so that $A_x = 0$ except for x in some bounded neighborhood of N. The category $\mathcal{A}$ is isomorphic to $\mathcal{C}_{N,G}(R)$ in an obvious fashion. The quotient category $\mathcal{U}/\mathcal{A}$, which we shall denote by $\mathcal{C}^{>N}_{M,G}(R)$, has the same objects as $\mathcal{U}$ but two morphisms are identified if the difference factors through $\mathcal{A}$, or in other words, if they differ in a bounded neighborhood of N. This is a typical example of an additive category $\mathcal{U}$ which is $\mathcal{A}$-filtered in the sense of Karoubi. We recall the definition.

Definition 5.6. Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{U}$. Denote objects of $\mathcal{A}$ by the letters A through F and objects of $\mathcal{U}$ by the letters U through W. We say that $\mathcal{U}$ is $\mathcal{A}$-filtered, if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$, so that

(i) For each U, the decompositions form a filtered poset under the partial order that $E_\alpha \oplus U_\alpha \le E_\beta \oplus U_\beta$, whenever $U_\beta \subseteq U_\alpha$ and $E_\alpha \subseteq E_\beta$.
(ii) Every map $A \to U$, factors $A \to E_\alpha \to E_\alpha \oplus U_\alpha = U$ for some α.
(iii) Every map $U \to A$ factors $U = E_\alpha \oplus U_\alpha \to E_\alpha \to A$ for some α.
(iv) For each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$ i.e. to $U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)$

The main excision results were proved in [18], [6], [7], [22]. We give a slight generalization of the L–theory results. Let $\mathbb{K}$ denote the Quillen K–theory spectrum, and $\mathbb{K}^{-\infty}$ its non-connective delooping (with the K_{-i}-groups as homotopy groups).

Theorem 5.7. *Let $\mathcal{U}$ be an $\mathcal{A}$-filtered additive category with involution. Consider the map $i\colon K_0(\mathcal{A}^\wedge) \to K_0(\mathcal{U}^\wedge)$ induced by inclusion, and let $k = i^{-1}(K_0(\mathcal{U}))$. There are fibrations of spectra*

$$\mathbb{K}(\mathcal{A}^{\wedge k}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A})$$

and

$$\mathbb{K}^{-\infty}(\mathcal{A}) \to \mathbb{K}^{-\infty}(\mathcal{U}) \to \mathbb{K}^{-\infty}(\mathcal{U}/\mathcal{A})$$

If $\mathcal{U}$ and $\mathcal{A}$ admit compatible involutions there is a fibration of spectra

$$\mathbb{L}^k(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A}) \ .$$

More generally, if

(i) *$a \subset K_i(\mathcal{A})$,$b \subset K_i(\mathcal{U}^\wedge)$, and $c \subset K_i((\mathcal{U}/\mathcal{A})^\wedge)$, for $i \leqq 1$,*
(ii) *$a = i^{-1}(b)$ and $b \to c$ is onto,*
(iii) *a, b, and c contain $K_0(\mathcal{A})$, $K_0(\mathcal{U})$ and $K_0(\mathcal{U}/\mathcal{A})$ respectively, if $i = 0$, and*
(iv) *a, b and c contain the indeterminacy subgroup given by the system of stable isomorphisms in the case $i = 1$,*

then we have a fibration of spectra

$$\mathbb{L}^a(\mathcal{A}) \to \mathbb{L}^b(\mathcal{U}) \to \mathbb{L}^c(\mathcal{U}/\mathcal{A})$$

Proof. The K–theory statements are implicitly contained in [18]. A simpler, more modern proof and explicit statements are given in [6]. The first L–theory statement was proved in [7], and the other L–theory statements follow by the following argument: we have an exact sequence

$$K_0(\mathcal{A}^{\wedge k}) \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}) \to 0,$$

where the map from $K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A})$ is onto because the categories have the same objects. Letting I denote the image of $K_1(\mathcal{U}/\mathcal{A})$ in $K_0(\mathcal{A}^{\wedge k})$, we consider the diagram of short exact sequences:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & I & \longrightarrow & K_0(\mathcal{A}^{\wedge k}) & \longrightarrow & K_0(\mathcal{U}) & \longrightarrow & K_0(\mathcal{U}/\mathcal{A}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & a & \longrightarrow & b' & \longrightarrow & K_0(\mathcal{U}/\mathcal{A}) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & K_0(\mathcal{A}^{\wedge}) & \longrightarrow & K_0(\mathcal{U}^{\wedge}) & \longrightarrow & K_0((\mathcal{U}/\mathcal{A})^{\wedge}) & &
\end{array}
$$

The vertical arrows are either equalities or inclusions. We define b' simultaneously as the pullback of $a/I \to b \to c$ and as the pushout of $0 \to K_0(\mathcal{A}^{\wedge k})/I \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}) \to 0$. We have

$$a/K_0(\mathcal{A}^{\wedge k}) \cong b'/K_0(\mathcal{U}),$$

so using the Ranicki–Rothenberg fibrations of spectra [22]

$$
\begin{array}{ccccc}
\mathbb{L}^k(\mathcal{A}) & \longrightarrow & \mathbb{L}^a(\mathcal{A}) & \longrightarrow & \mathbb{H}(a/K_0(\mathcal{A}^{\wedge k})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}^h(\mathcal{U}) & \longrightarrow & \mathbb{L}^{b'}(\mathcal{U}) & \longrightarrow & \mathbb{H}(b'/K_0(\mathcal{U}))
\end{array}
$$

we get a fibration

$$\mathbb{L}^a(\mathcal{A}) \to \mathbb{L}^{b'}(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A}).$$

We now repeat this argument using the isomorphisms $b/b' \cong c/K_0(\mathcal{U}/\mathcal{A})$ to obtain the desired fibration of spectra. Since L^h-groups may be understood as simple L-groups with all of K_1 as allowed torsions, the above bootstrapping argument extends to fibrations of the L-spectra stated, using the isomorphism

$$K_1(\mathcal{U}/\mathcal{A})/\ker(\partial) \cong \mathrm{image}(\partial)$$

where ∂ is the boundary map $\partial\colon K_1(\mathcal{U}/\mathcal{A}) \to K_0(\mathcal{A}^{\wedge})$. □

In Section 10 we need to use bounded surgery groups with geometric anti–structure generalizing the definition of [**I**], Section 4 (see [10]). The new ingredient is a counterpart to the automorphism $\theta\colon H \to H$ at the metric space level.

Let $\theta_H\colon H \longrightarrow H$ be a group automorphism so that the data (H, θ_H, w, b) gives a geometric anti–structure on RH. Let $\theta_M\colon M \longrightarrow M$ be an isometry with the properties $\theta_M(g \cdot m) = \theta_H(g) \cdot \theta_M(m)$, $\theta_M^2(m) = bm$, and $\theta_H^2(g) = bgb^{-1}$.

Given an object $A \in \mathcal{G}_{M,H}(R)$, we have the functor $*$ from $\mathcal{G}_{M,H}(R)$ to itself so that $\mathcal{G}_{M,H}(R)$ is a category with involution. We may then twist the involution $*$ by composing with the functor sending (A, f) to (A^θ, f^θ) where A^θ is the same R-module, but g acts on the left by multiplication by $\theta(g)$ and $f^\theta = \theta_M^{-1} \cdot f$. This defines the bounded anti–structure on $\mathcal{G}_{M,H}(R)$ and on the subcategory $\mathcal{C}_{M,H}(R)$ of free RH modules.

Example 5.8. Bounded geometric anti–structures arise geometrically as above. The bounded $\mathbf{R}_-$ transfer sits in the long exact sequence

$$(5.8)\quad LN_n(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}), w) \to L^h_{n+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \to LN_{n-1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \to L^h_{n-1}(\mathcal{C}_{W,G}(\mathbf{Z}), w) \to \dots$$

where $w = \det(\rho_W)$ is the standard orientation (see Example 5.4). The bounded LN–group

$$LN_n(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \cong L_n(\mathcal{C}_{W,H}(\mathbf{Z}), \alpha, u)$$

where $\theta_W(x) = t\cdot x$ and $\theta_H(h) = tht^{-1}$ for a fixed $t \in G - H$.

Conversely, given a bounded geometric antistructure $(\theta_H, \theta_M, b, w)$, we can define $G = \langle H, t \mid t^{-1}ht = \theta_H(h), t^2 = b\rangle$ and $t\cdot m = \theta_M(m)$. Then $\mathcal{C}_{M,G}(R)$ induces $(\theta_H, \theta_M, b, w)$ as above, showing that all geometric antistructures arise by twisting and restricting to an index two subgroup.

The L-theory of these bounded geometric anti–structures also has a useful vanishing property which we now wish to formulate. We first give a basic construction.

Definition 5.9. If $\mathcal{A}$ is an additive category, then the *opposite category* $\mathcal{A}^{op}$ is the category with the same objects as $\mathcal{A}$ but $\hom_{\mathcal{A}^{op}}(A,B) = \hom_{\mathcal{A}}(B,A)$. The product category $\mathcal{A}\times\mathcal{A}^{op}$ is an additive category with involution given by $*\colon (A,B) = (B,A)$ on objects and $*\colon (\alpha,\beta) = (\beta,\alpha)$ on morphisms.

Clearly $K_i(\mathcal{A}^{op}) = K_i(\mathcal{A})$, so we can identify $K_i(\mathcal{A}\times\mathcal{A}^{op}) = K_i(\mathcal{A})\times K_i(\mathcal{A})$.

Lemma 5.10. *Let $b \subseteq K_i(\mathcal{A})$ for some $i \leqq 1$, and $q = b\times b \subseteq K_i(\mathcal{A}\times\mathcal{A}^{op})$. Then $L^q_n(\mathcal{A}\times\mathcal{A}^{op}) = 0$ for all n.*

Proof. Let $\mathcal{P}(\mathcal{A})$ denote the category with the same objects as $\mathcal{A}$, but with morphisms given by $\mathcal{A}$–isomorphisms. Then it suffices to prove that the quadratic category $\mathcal{Q}(\mathcal{A}\times\mathcal{A}^{op}) \simeq \mathcal{P}(\mathcal{A})$ via the hyperbolic map (see [29, p.122]). This shows that $L^h_*(\mathcal{A}\times\mathcal{A}^{op}) = 0$ and other decorations follow trivially from the Ranicki Rothenberg exact sequences (note that the Tate cohomology $H^*(q) = 0$). The result for lower L–groups follows by replacing $\mathcal{A}$ by $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$.

Suppose $(\nu_1,\nu_2)\colon (A,B)\to(B,A)$ is a non–singular quadratic form representing an element in $\mathcal{Q}(\mathcal{A}\times\mathcal{A}^{op})$. This means that the bilinearization $\nu_1+\nu_2$ is an isomorphism, and we are allowed to change (ν_1,ν_2) by terms of the form $(\alpha,\beta)-(\beta,\alpha)$. We have

$$(\nu_1,\nu_2) + (\nu_2,0) - (0,\nu_2) = (\nu_1+\nu_2, 0)$$

and the right hand side is a hyperbolic form. □

We encounter the $\mathcal{A}\times\mathcal{A}^{op}$ situation in the following setting:

Example 5.11. Let $M = M_1\cup M_2$ be a metric space given as the union of two sub–metric spaces M_1 and M_2, where we denote $M_1\cap M_2$ by N. Suppose that

(i) G acts by isometries on M, such that each $g\in G$ preserves or switches M_1 and M_2 in this decomposition,
(ii) $H = \{g\in G \mid g(M_1) = M_1\}$ is an index two subgroup of G,
(iii) for every $k>0$ there exists an $l>0$ such that, if $x\in M_1$ (resp. $x\in M_2$) with $d(x,N)>l$, then $d(x,M_2)>k$ (resp. $d(x,M_1)>k$).

The category $\mathcal{C}_{M,H}(R)$ has a bounded geometric antistructure (α, u) given by $(\theta_H, \theta_M, b, w)$ as in Example 5.8, with $\theta_M(m) = t \cdot m$ and $\theta_H(h) = tht^{-1}$ for a fixed $t \in G - H$. Next, observe that the category

$$\mathcal{C}^{>N}_{M,H}(R) = \mathcal{C}^{>N}_{M_1,H}(R) \times \mathcal{C}^{>N}_{M_2,H}(R)$$

because of our separation condition (iii). Moreover, the functor $T\colon \mathcal{C}^{>N}_{M_1,H}(R) \to \mathcal{C}^{>N}_{M_2,H}(R)^{op}$ defined by $T(A, f) = (A^*, \theta_M^{-1} \circ f^*)$ on objects and $T(\phi) = \phi^*$ on morphisms is an equivalence of categories. We are thus in the $\mathcal{A} \times \mathcal{A}^{op}$ situation described above and $L^h_*(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$ by Lemma 5.10.

For any bounded geometric antistructure, notice that the action of θ_M on M takes H–orbits to H–orbits since $\theta_M(g \cdot m) = \theta_H(g) \cdot \theta_M(m)$. Let $M_{(H,\theta)}$ denote the subset consisting of H–orbits in M which are *fixed* by the θ–action. Then $M_{(H,\theta)} = \{m \in M \mid \theta_M(m) \in H \cdot m\}$. Note that $M_{(H,\theta)}$ is a H–invariant subspace of M.

Theorem 5.12. *Suppose that $(\mathcal{C}_{M,H}(R), \alpha, u)$ has a bounded geometric antistructure (α, u) given by $(\theta_H, \theta_M, b, w)$, such that:*

- (i) *$M = O(K)$, where K is a finite H–CW complex and M has the cone of the given H–action on K,*
- (ii) *θ_M is induced by a simplicial map on K,*
- (iii) *$M_{(H,\theta)} \subseteq O(L) := N$ for some H–invariant subcomplex $L \subset K$, and*
- (iv) *for some $i \leqq 1$, $I \subset K_i(\mathcal{C}^{>N}_{M,H}(R))$ is a subgroup with $H^*(I) = 0$.*

Then $L^I_n(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$ for all n.

Corollary 5.13. *Let $G = C(2^r q)$, q odd, be a cyclic group and $H \subset G$ the subgroup of index 2. Let W be a G–representation, and $N = \bigcup\{W^K \mid [G : K] \text{ is odd}\}$. Then $L^I_n(\mathcal{C}^{>N}_{W,H}(\mathbf{Z}), \alpha, u)(q) = 0$ on the top component, where (α, u) is the antistructure given above.*

The proof of Theorem 5.12. We extend the given H–action on M to a simplicial action of $G = \langle H, t \mid t^{-1}ht = \theta_H(h), t^2 = b\rangle$ as described above. The proof is by induction on cells, so suppose that K is obtained from L by attaching exactly one G–equivariant k–cell $D^k \times G/G_0$. Since $M_{(H,\theta)} \subset O(L) = N$, it follows that $G_0 \subset H$ and we may write $G/G_0 = H/G_0 \sqcup tH/G_0$. Now we define $M_1 = O(L \cup (D^k \times H/G_0))$ and consider the category $\mathcal{A} = \mathcal{C}^{>O(L)}_{M_1,H}(R)$. By construction, we have

$$\mathcal{C}^{>N}_{M,H}(R) = \mathcal{A} \times \mathcal{A}^{op}$$

which has trivial L–theory by Lemma 5.10. Since the Tate cohomology of the K_1 decoration I vanishes, we get $L^I_n(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$. □

6. Calculations in bounded K-theory

We begin to compute the bounded transfers trf_W by considering the bounded K–theory analogue. In this section, $G = C(2^r q)$ is cyclic of order $2^r q$, with $r \geqq 2$ and $q \geqq 1$ odd. By [**I**], Theorem 3.8 we can restrict our attention to those W where the isotropy subgroups have 2–power index. Let $G_i \subseteq G$ denote the subgroup of index $[G : G_i] = 2^i$ for $i = 0, 1, \ldots, r$. As above, we reserve the notation $H < G$ for the subgroup of index 2.

Any real, orthogonal G–representation W can be decomposed uniquely into isotypical direct summands indexed by the subgroups $K \subseteq G$, where in each summand G operates with isotropy group K away from the origin. Since we assume that W has isotropy of 2–power index, we can write

$$W = W[0] \oplus W[1] \oplus \cdots \oplus W[r]$$

where $W[i]$ is isotypic with isotropy group G_i. Thus $W[0] = \mathbf{R}^k$ is a trivial G–representation, and $W[1]$ is a sum of $\mathbf{R}_-$ factors. We say that W is *complex* if $\dim W[0]$ and $\dim W[1]$ are even (in this case, W is the underlying real representation of a complex representation). If W is complex, then $W_{max} \subseteq W$ denotes a complex sub-representation of real dim ≤ 2 with maximal proper isotropy subgroup. If $W = W[0]$ then $W_{max} = 0$. Then W_{max} is either irreducible or $W_{max} = \mathbf{R}_- \oplus \mathbf{R}_-$.

We study bounded K–theory by means of equivariant filtrations of the control space. The basic sequence is (see [10]):

$$\cdots \to K_{i+1}(\mathcal{C}^{>U}_{V,G}(\mathbf{Z})) \xrightarrow{\partial_{i+1}} K_i(\mathcal{C}_{U,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{V,G}(\mathbf{Z})) \to K_i(\mathcal{C}^{>U}_{V,G}(\mathbf{Z})) \to \dots$$

valid for $U \subseteq V$ a closed G–invariant subspace. If W_1 is a complex representation with $\dim W_1 = 2$ and isotropy group $K \neq G$, let $U = \bigcup \ell_\alpha$ be the union of $[G : K]$ rays from the origin in W_1, which are freely permuted by G/K. Then $W_1 \setminus \bigcup \ell_\alpha$ is a disjoint union of open fundamental domains for the free G/K–action. If $W = W_1 \oplus W_2$, we call $W_2 \subset \bigcup \ell_\alpha \times W_2 \subset W$ the *orbit type* filtration of W.

Recall that t denotes a generator of G, and thus acts as an isometry on the control spaces M we use in the bounded categories $\mathcal{C}_{M,G}(\mathbf{Z})$. Let t_* denote the action of t on bounded K–theory induced by its action on the control space.

Lemma 6.1. *Let $W = W_1 \oplus W_2$, where W_1 is a complex 2–dimensional sub-representation of W with minimal isotropy subgroup $K \neq G$. Then*

$$K_{i+1}(\mathcal{C}^{>\bigcup \ell_\alpha \times W_2}_{W,G}(\mathbf{Z})) \cong K_{i-1-k}(\mathbf{Z}K)$$
$$K_i(\mathcal{C}^{>W_2}_{\bigcup \ell_\alpha \times W_2,G}(\mathbf{Z})) \cong K_{i-1-k}(\mathbf{Z}K),$$

where $k = \dim W_2$. The boundary map $\partial_{i+1} = 1 - t_*$ in the long exact sequence of the orbit type filtration for W.

Proof. The bounded category $\mathcal{C}^{>W_2}_{\bigcup \ell_\alpha \times W_2,G}(\mathbf{Z})$ of germs away from W_2 has effective fundamental group K, as defined in [10, 3.13]. It therefore has the same K–theory as $\mathcal{C}_{\mathbf{R}^{k+1}}(\mathcal{C}_{pt}(\mathbf{Z}K))$. The other case is similar. The identification of ∂_{i+1} with $1 - t_*$ is discussed in detail in the proof of Proposition 6.7. □

Since $K_{-j}(\mathbf{Z}K) = 0$ for $j \geqq 2$ by [8], this Lemma gives vanishing results for bounded K–theory as well.

Lemma 6.2. *Suppose that W is complex, and $W^G = 0$. Then the inclusion map induces an isomorphism*

$$K_i(\mathcal{C}_{W_{max} \times \mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W \times \mathbf{R}_-,G}(\mathbf{Z}))$$

for $i \leqq 1$.

Proof. This is an argument using the orbit type filtration. Let $W = W_{max} \oplus W_2$, and suppose that $W_2 \neq 0$ or equivalently $\dim W_2 \geqq 2$, since W is complex. Write $W = W' \oplus W''$ with $W' \subseteq W_2$, $\dim W' = 2$ and $\mathrm{Iso}(W') = K$ minimal. We choose W'' containing W_{max}, and by induction we assume the result holds for W''.

Then applying the first part of Lemma 6.1, we get the calculations

$$K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-, G}(\mathbf{Z})) = K_{i-2-|W''|}(\mathbf{Z}K) \tag{6.3}$$

and

$$K_i(\mathcal{C}^{>\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = K_{i-3-|W''|}(\mathbf{Z}K)\ . \tag{6.4}$$

Since $\dim W'' \geqq 2$, we get the vanishing results $K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 2$ by [8], and $K_i(\mathcal{C}^{>\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 3$. From the filtration sequence, it follows that $K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 2$, and therefore

$$K_i(\mathcal{C}_{W'' \times \mathbf{R}_-, G}(\mathbf{Z})) \xrightarrow{\approx} K_i(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))$$

for $i \leqq 1$. We are done, by induction. □

Corollary 6.5. $K_i(\mathcal{C}^{>W_{max} \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ *for* $i \leqq 2$.

Proof. We continue the notation from above, and look at part of the filtration sequence

$$K_i(\mathcal{C}^{>W_{max} \times \mathbf{R}_-}_{W'' \times \mathbf{R}_-, G}(\mathbf{Z})) \to K_i(\mathcal{C}^{>W_{max} \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) \to K_i(\mathcal{C}^{>W_{max} \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))\ .$$

The first term is zero for $i \leqq 2$ by induction on dimension, and the third term is zero for $i \leqq 2$ as above. □

We can obtain a little sharper result with some additional work. First a useful observation:

Lemma 6.6. *Let $\mathcal{A}$ be an additive category (with involution). Then the map $\mathbf{R} \to \mathbf{R}$ sending x to $-x$ induces minus the identity on K-theory (and L-theory) of $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$.*

Proof. The category $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$ is filtered by the full subcategory whose objects have support in a bounded neighborhood of 0. This subcategory is equivalent to $\mathcal{A}$ and the quotient category may be identified with $\mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A}) \times \mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A})$ via the projection maps in an obvious way.

Consider the diagram

$$\begin{array}{ccccc}
\mathcal{A} & \longrightarrow & \mathcal{C}_{\mathbf{R}}(\mathcal{A}) & \longrightarrow & \mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A}) \times \mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A}) \\
\| & & \downarrow & & \downarrow \\
\mathcal{A} & \longrightarrow & \mathcal{C}_{[0,\infty)}(\mathcal{A}) & \longrightarrow & \mathcal{C}^{>0}_{[0,\infty)}
\end{array}$$

where the vertical map is induced by $x \mapsto |x|$. In the lower horizontal row, K and L–theory of the middle term is trivial, so the boundary map will be an isomorphism. The lower row splits off the upper row in two different ways, one induced by including $[0,\infty) \subset \mathbf{R}$ and the other by sending $x \in [0,\infty)$ to $-x \in \mathbf{R}$. Under these two splittings we may identify K or L–theory of the quotient $\mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A}) \times \mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A})$ with $\mathcal{C}^{>0}_{[0,\infty)} \times \mathcal{C}^{>0}_{[0,\infty)}$ and under this identification, the flip map of $\mathbf{R}$ corresponds to interchanging the two factors. On K–theory (or L–theory) we conclude that the exact sequence is of the form

$$0 \to A_* \to A_* \times A_* \xrightarrow{+} A_* \to 0 \ .$$

The flip action on the last term is trivial, and on the middle term it interchanges the two factors, so the inclusion must send a to $(a, -a)$. Hence the flip action on the first term must be $a \mapsto -a$. □

As above, t_* denotes the induced action of a generator $t \in G$ on K–theory, and $\varepsilon\colon G \to \{\pm 1\}$ the non-trivial action of G on $\mathbf{R}_-$.

Proposition 6.7. *Let W be a complex 2-dimensional G–representation with $W^G = 0$, and isotropy subgroup K. Then under the isomorphisms of (6.3) and (6.4) the complex*

$$K_{i+1}(\mathcal{C}^{>\bigcup \ell_\alpha \times \mathbf{R}_-}_{W\times \mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\partial_a} K_i(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\partial_b} K_{i-1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

with $\partial_b \circ \partial_a = 0$ is isomorphic to

$$K_{i-2}(\mathbf{Z}K) \xrightarrow{\partial'_a} K_{i-2}(\mathbf{Z}K) \xrightarrow{\partial'_b} K_{i-2}(\mathbf{Z}H)$$

where $\partial'_a = 1 - t_* = 2$ and $\partial'_b = \operatorname{Ind}_H \circ(1 - t_*) = 0$.

Proof. The orbit type filtration is based on a G–equivariant simplicial model for W, where G acts through the projection to G/K. The third term in the complex is $K_{i-1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})) = K_{i-2}(\mathbf{Z}H)$ and the identification of the boundary maps follows from the definition of the germ categories.

To compute ∂_a, we use the isomorphisms between the domain of ∂_a and $K_{i-2}(\mathbf{Z}K)$ obtained by noticing that every element is induced from an element of

$$K_{i+1}(\mathcal{C}^{>\partial C\times \mathbf{R}}_{C\times \mathbf{R},K}(\mathbf{Z})),$$

where C is the region between two adjacent half–lines of $\bigcup \ell_\alpha$. This follows since the regions in the complement of $\bigcup \ell_\alpha \times \mathbf{R}_-$ are disjoint and the boundedness condition ensures there is no interference. Similarly the isomorphism of the range of ∂_a with $K_{i-2}(\mathbf{Z})$ is obtained by noticing that every element is induced from $\mathcal{C}^{>\mathbf{R}}_{h\times \mathbf{R},K}$ where h is just one half–line in $\bigcup \ell_\alpha$, and we can think of $\partial C = h \cup th$. To compute the boundary we first take the standard boundary to $\mathcal{C}_{\partial C\times \mathbf{R},K}(\mathbf{Z})$ which is an isomorphism, and then map away from $0 \times \mathbf{R}$. It follows from the proof of Lemma 6.6 above that this map is of the form $a \mapsto (a, -a)$ in K or L–theory. In this picture, the support of one of the boundary components is along h and the other along th. We need to use the group action to associate both elements to the same ray. Since t flips the $\mathbf{R}$–factor, we get a change of sign before adding, so $t_* = \varepsilon(t) = -1$ and ∂_a sends a to $2a$.

To compute ∂_b, we start with an element in the source of ∂_b which as above is identified via induction with $K_i(\mathcal{C}^{>\mathbf{R}}_{h\times \mathbf{R},K,G}(\mathbf{Z}))$. The boundary first sends this isomorphically to $\mathcal{C}_{\mathbf{R},K}(\mathbf{Z})$, then by induction to $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$, and then via the natural map to the range of ∂_b, which is

$\mathcal{C}^{>0}_{\mathbf{R}_-,G}$. We have a commutative square

$$\begin{array}{ccc} K_1(\mathcal{C}^{>\mathbf{R}}_{h\times\mathbf{R},K}(\mathbf{Z})) & \xrightarrow{\text{Ind}} & K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial_b} \\ K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z})) & \xrightarrow{\text{Ind}} & K_0(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})) \end{array}$$

where $\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z}) = \mathcal{C}_{\mathbf{R},K}(\mathbf{Z}) \times \mathcal{C}_{\mathbf{R},K}(\mathbf{Z})$. Under this identification, the natural map to the germ category $K_0(\mathcal{C}_{\mathbf{R},K}(\mathbf{Z})) \to K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z}))$ is just $a \mapsto (a, -a)$, and the induction map

$$\text{Ind}\colon K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z})) \to K_0(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

is given by $(a, b) \mapsto \text{Ind}_H(a + t_*b)$. In this case the action of t on $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is the identity since any element is invariant under the action of G, hence under the action of t. It follows that $\text{Ind}(a, -a) = \text{Ind}_H(a - t_*a) = 0$ as required. □

Lemma 6.8. *Let W be a complex 2-dimensional G–representation with proper isotropy group K. The boundary map*

$$K_{i+1}(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

is zero for $i \leqq 1$.

Proof. If $i \leqq -1$ the domain of this boundary map is zero, so the result is trivial. For $i = 1$ we use the injection $\text{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \text{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Z}}_2))$, which follows from the vanishing of $SK_1(\mathbf{Z}G)$. But $\text{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Z}}_2)) = \text{Wh}(\widehat{\mathbf{Z}}_2G)/\text{Wh}(\widehat{\mathbf{Z}}_2H)$ and the group $\text{Wh}(\widehat{\mathbf{Z}}_2G) = \prod_{d|q} \widehat{\mathbf{Z}}_2[\zeta_d]G_2$, where $G_2 \subset G$ is the 2-Sylow subgroup and q is the odd part of the order of G. Now Oliver [17, Thm. 6.6] constructed a short exact sequence

$$1 \to \text{Wh}(\widehat{\mathbf{Z}}_2[\zeta_d]G_2) \to \widehat{\mathbf{Z}}_2[\zeta_d]G_2 \to \langle -1\rangle \times G_2 \to 1$$

by means of the integral 2-adic logarithm. This sequence is natural with respect to inclusion of subgroups, so we may use it to compare $\text{Wh}(\widehat{\mathbf{Z}}_2G)$ and $\text{Wh}(\widehat{\mathbf{Z}}_2H)$. Since each corresponding term injects, and the middle quotient is $\mathbf{Z}$-torsion free, we conclude that $\text{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Z}}_2))$ is also $\mathbf{Z}$-torsion free. Since $K_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z}) = K_0(\mathbf{Z}K)$ is torsion (except for $\mathbf{Z} = K_0(\mathbf{Z})$ which is detected by projection to the trivial group), the given boundary map is zero.

For $i = 0$, we use the surjection $K_0(\widehat{\mathbf{Q}}K) \to K_{-1}(\mathbf{Z}K)$, and compute with $\widehat{\mathbf{Q}}$ coefficients and $i = 1$. We will list the steps, and leave the details to the reader. First, compute that $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}G)/K_1(\widehat{\mathbf{Q}}H)$ surjects onto $K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$, by means of a braid containing the cone point inclusions into $\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})$ and $\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}})$. Second, prove that $K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$ fits into a short exact sequence

$$0 \to K_1(\mathcal{C}_{\bigcup \ell_\alpha,H}(\widehat{\mathbf{Q}})) \to K_1(\mathcal{C}_{\bigcup \ell_\alpha,G}(\widehat{\mathbf{Q}})) \to K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \to 0$$

by means of a braid containing the inclusion $\bigcup \ell_\alpha \subseteq \bigcup \ell_\alpha \times \mathbf{R}_-$. Finally, compute the first two terms $K_1(\mathcal{C}_{\bigcup \ell_\alpha,G}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}G)/K_1(\widehat{\mathbf{Q}}K)$, and $K_1(\mathcal{C}_{\bigcup \ell_\alpha,H}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}H)/K_1(\widehat{\mathbf{Q}}K)$ by comparing the groups under the inclusion $H < G$. We conclude that

$$K_1(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \cong K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$$

under the inclusion map, and hence $\partial = 0$. $\square$

Corollary 6.9. *Let W be a complex G–representation with $W^G = 0$. Then the inclusion induces an isomorphism $K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ for $i \leqq 0$, and an injection for $i = 1$. If $K_{-1}(\mathbf{Z}K) = 0$ for the maximal proper isotropy groups K of W, then $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$.*

Proof. We may assume that $\dim W = 2$, and apply the filtering argument again. By (6.3) and (6.4) we get $K_i(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ for $i \leqq 0$, and $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a quotient of $K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$. Since the composition

$$K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

is zero by Lemma 6.8, the result follows for $i \leqq 0$.

Similarly, $K_2(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a quotient of $K_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_0(\mathbf{Z}K)$, because the boundary map $K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$ to $K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$ is multiplication by 2, and hence injective. Then we make the same argument, using Lemma 6.8.

If we also assume $K_{-1}(\mathbf{Z}K) = 0$, then $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ so we get the isomorphism $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\approx} K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. $\square$

7. The double coboundary

The composite Δ of maps from the $L^h - L^p$ and $L^s - L^h$ Rothenberg sequences

$$\begin{array}{ccc} H^i(\widetilde{K}_0(\mathbf{Z}G)) & \longrightarrow & L^h_i(\mathbf{Z}G) \\ & \searrow^{\Delta} & \downarrow \\ & & H^i(\mathrm{Wh}(\mathbf{Z}G)) \end{array}$$

(see [13]) has an algebraic description by means of a "double coboundary" homomorphism

$$\delta^2 \colon H^i(\widetilde{K}_0(\mathbf{Z}G)) \to H^i(\mathrm{Wh}(\mathbf{Z}G))$$

In this section, we will give a brief description due to Ranicki [19] of this homomorphism (see also [20, §6.2] for related material on "interlocking" exact sequences in K and L–theory).

Let X be a space with a $\mathbf{Z}/2$ action $T\colon X \to X$, and define homomorphisms

$$\Delta\colon H^i(\pi_n(X)) \to H^i(\pi_{n+1}(X))$$

by sending $g\colon S^n \to X$ to

$$h \cup (-1)^i Th\colon S^{n+1} = D^{n+1}_+ \cup_{S^n} D^{n+1}_- \to X$$

for any null-homotopy $h\colon D^{n+1} \to X$ of the map $g + (-1)^{i+1}Tg\colon S^n \to X$.

The maps Δ lead to a universal description of double coboundary maps, as follows. Let $f\colon X \to Y$ be a $\mathbf{Z}/2$–equivariant map of spaces with $\mathbf{Z}/2$ action, and consider the long exact sequence

$$\cdots \to \pi_n(X) \xrightarrow{f} \pi_n(Y) \to \pi_n(f) \to \pi_{n-1}(X) \to \pi_{n-1}(Y) \to \dots$$

We define $I_n = \ker(f\colon \pi_n(X) \to \pi_n(Y))$ and $J_n = \mathrm{Im}(\pi_n(Y) \to \pi_n(f))$, and get an exact sequence

$$0 \to \pi_n(X)/I_n \to \pi_n(Y) \to \pi_n(f) \to I_{n-1} \to 0$$

which can be spliced together from the short exact sequences

$$\begin{aligned} &0 \to \pi_n(X)/I_n \to \pi_n(Y) \to J_n \to 0 \\ &0 \to J_n \to \pi_n(f) \to I_{n-1} \to 0\ . \end{aligned} \tag{7.1}$$

Then it follows directly from the definitions that the double coboundary

$$\delta^2\colon H^i(I_{n-1}) \xrightarrow{\delta} H^{i+1}(J_n) \xrightarrow{\delta} H^i(\pi_n(X)/I_n)$$

from the Tate cohomology sequences induced by (7.1) is given by the composite

$$\delta^2\colon H^i(I_{n-1}) \xrightarrow{inc_*} H^i(\pi_{n-1}(X)) \xrightarrow{\Delta} H^i(\pi_n(X)) \xrightarrow{proj_*} H^i(\pi_n(X)/I_n)\ .$$

If we can pick the map $f\colon X \to Y$ appropriately, say with $I_n = 0$ and $I_{n-1} = \pi_{n-1}(X)$, this gives an algebraic description of Δ.

In later sections we will use the relative Tate cohomology groups $H^i(\Delta)$, which are just (by definition) the relative Tate cohomology groups [20, p.166] of the map $\pi_n(Y) \to \pi_n(f)$ in the long exact sequence above. These groups fit into the commutative braids given in [19] which will be used in the proofs of Theorems A-C.

We now give some examples, with G denoting a finite cyclic group as usual. These arise from homotopy groups of certain fibrations of algebraic K-theory spectra.

Example 7.2. There is an exact sequence [13]

$$0 \to \mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}_2G) \to \mathrm{Wh}(\mathbf{Z}G{\to}\widehat{\mathbf{Z}}_2G) \to \widetilde{K}_0(\mathbf{Z}G) \to 0$$

of $\mathbf{Z}/2$ modules and the associated double coboundary in Tate cohomology equals

$$\Delta\colon H^i(\widetilde{K}_0(\mathbf{Z}G)) \to H^i(\mathrm{Wh}(\mathbf{Z}G))\ .$$

The point here is that $\ker(\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}_2G)) = 0$ [17], and the map $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\widehat{\mathbf{Z}}_2G)$ is zero by a result of Swan [26]. We could also use the exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}G) \oplus \mathrm{Wh}(\mathbf{Q}G) \to \mathrm{Wh}(\widehat{\mathbf{Q}}G)) \to \widetilde{K}_0(\mathbf{Z}G) \to 0$$

to compute the same map Δ.

Example 7.3. There is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Z}} \oplus \mathbf{Q})) \to \\ \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \to \widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \end{aligned}$$

where $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$ and $\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G)$. The injectivity on the left follows because $K_2(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$ is a quotient of $K_2(\widehat{\mathbf{Q}}G)$, mapping trivially through $K_1(\mathbf{Z}G)$ into $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ (since $SK_1(\mathbf{Z}G) = 0$ [17]). We therefore get an algebraic description of $\delta^2\colon H^i(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^i(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$ as used in the statement of Theorem B.

Example 7.4. There is an exact sequence

$$0 \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\widehat{\mathbf{Z}}\oplus\mathbf{Q})) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \to \widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \tag{7.4}$$

for any complex G–representation W with $W^G = 0$. We therefore get an algebraic description of $\delta^2\colon H^i(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^i(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ as used in the statement of Theorem C.

Lemma 7.5. *For complex G–representations $W_1 \subseteq W$ with $W^G = 0$, there is a commutative diagram*

$$\begin{array}{ccc} H^i(\widetilde{K}_0(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))) & \xrightarrow{\delta^2} & H^i(\mathrm{Wh}(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))) \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} \\ H^i(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) & \xrightarrow{\delta^2} & H^i(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \end{array}$$

where the vertical maps are induced by the inclusion $W_1 \subseteq W$.

For our applications, the main point of the double coboundary description is that it permits these maps induced by cone point inclusions to be computed using bounded K–theory, instead of bounded L–theory.

The double coboundary maps also commute with restriction to subgroups of G.

Proposition 7.6. *Let $G_1 < G$ be a subgroup of odd index, and $H_1 < G_1$ have index 2, then there are twisted restriction maps*

$$H^i(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) \xrightarrow{\mathrm{Res}} H^i(\widetilde{K}_0(\mathbf{Z}H_1\to\mathbf{Z}G_1^-))$$

and

$$H^i(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-)) \xrightarrow{\mathrm{Res}} H^i(\mathrm{Wh}(\mathbf{Z}H_1\to\mathbf{Z}G_1^-))$$

such that the diagram

$$\begin{array}{ccc} H^i(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) & \xrightarrow{\delta^2} & H^i(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-)) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ H^i(\widetilde{K}_0(\mathbf{Z}H_1\to\mathbf{Z}G_1^-)) & \xrightarrow{\delta_1^2} & H^i(\mathrm{Wh}(\mathbf{Z}H_1\to\mathbf{Z}G_1^-)) \end{array}$$

commutes.

Proof. The vertical maps are twisted restriction maps given by composing the twisting isomorphisms $H^i(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) \cong H^{i+1}(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G))$ and $H^i(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-) \cong H^{i+1}(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G)$, discussed in [**I**], Section 4, with the restriction maps induced by the inclusion $\mathbf{Z}G_1 \to \mathbf{Z}G$ of rings with involution. Since $G_1 < G$ has odd index, $H_1 = H\cap G_1$ and the composition $\mathrm{Res}_{G_1}\circ\mathrm{Ind}_H$ lands in the image of Ind_{H_1} by the double coset formula. □

This can be generalized to the double coboundary maps used in the statement of Theorem C, under certain conditions.

Proposition 7.7. *Let $G_1 < G$ be an odd index subgroup, and $H_1 < G_1$ have index 2. Suppose that W only has proper isotropy subgroups of 2–power index. Then there are a twisted restriction maps $H^i(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^i(\widetilde{K}_0(\mathcal{C}_{\operatorname{Res} W\times\mathbf{R}_-,G_1}(\mathbf{Z})))$ and $H^i(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^i(\mathrm{Wh}(\mathcal{C}_{\operatorname{Res} W\times\mathbf{R}_-,G_1}(\mathbf{Z})))$ which commute with the corresponding double coboundary maps δ^2_W and $\delta^2_{\operatorname{Res} W}$.*

8. Calculations in bounded L–theory

Suppose that $\sigma = \sigma(f) \in L^h_0(\mathbf{Z}G)$ is the surgery obstruction arising from a normal cobordism between $S(V_1)$ and $S(V_2)$, as in the statement of [**I**], Theorem 3.5. In this section, we establish two important properties of $trf_W(\sigma)$ in preparation for the proof of Theorem C. Unless otherwise mentioned, all bounded categories will have the standard orientation (see Example 5.4).

For a complex G–representation W the standard orientation is trivial, and the cone point inclusion $0 \in W$ induces the map

$$c_*\colon L^h_n(\mathbf{Z}G) \to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))$$

Note that the presence of an $\mathbf{R}_-$ factor introduces a non-trivial orientation at the cone point

$$c_*\colon L^h_{n+1}(\mathbf{Z}G, w) \to L^h_{n+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

where $w\colon G \to \{\pm 1\}$ is the non-trivial projection. The properties are:

Theorem 8.1. *Suppose W is a complex G–representation with no $\mathbf{R}_+$ summands. If $trf_{W\times\mathbf{R}_-}(\sigma) \in L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a torsion element, then*

(i) *there exists a torsion element $\hat\sigma \in L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ such that*

$$c_*(\hat\sigma) = trf_{W\times\mathbf{R}_-}(\sigma) \in L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

(ii) *there exists a torsion element $\hat\sigma \in L^p_{2k+1}(\mathbf{Z}G, w)$ such that*

$$c_*(\hat\sigma) = trf_{W\times\mathbf{R}_-}(\sigma) \in L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

where $\dim W = 2k$.

We remark that the condition "$trf_{W\times\mathbf{R}_-}(\sigma)$ is a torsion element" follows from the assumption $\operatorname{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \operatorname{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$ in Theorem C, as an immediate consequence of Proposition 4.7. Before giving the proof, we need some preliminary results. When the L–theory decoration is not explicitly given, we mean $L^{\langle -\infty\rangle}$.

Lemma 8.2. *Let W be a complex G–representation. Then*

$$L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \otimes \mathbf{Q} = 0$$

for $k \geqq 0$

Proof. We argue by induction on $\dim W$, starting with

$$L_{2k+1}(\mathbf{Z}G) = \mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}G))$$

which is all 2–torsion. It is enough to prove the result for the top component $L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$, and therefore by [**I**], Theorem 3.8 we may assume that the isotropy groups of W all have 2–power index. Since we are working with $L^{\langle -\infty\rangle}$, we may ignore $\mathbf{R}_+$ summands of W.

Let $W = W' \oplus W''$ where $\dim W' = 2$ and W' has minimal isotropy group K. We assume the result for W'', and let $\bigcup \ell_\alpha \subset W'$ be a G-invariant set of rays from the origin, dividing W' into fundamental domains for the free G/K-action.

Then

$$L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'', G}(\mathbf{Z})) = L_{n-1}(\mathcal{C}_{W'',G}(\mathbf{Z})) = L_{n-1-|W''|}(\mathbf{Z}K),$$

which is torsion for n even, and

$$L_n(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) = L_{n-2}(\mathcal{C}_{W'',G}(\mathbf{Z})) = L_{n-2-|W''|}(\mathbf{Z}K),$$

which is torsion for n odd. Moreover, we have a long exact sequence

$$\cdots \to L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'', G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) \to \dots$$

We claim that the first map in this sequence is rationally injective. The previous map in the long exact sequence is

$$L_{n+1}(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) \longrightarrow \partial L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'', G}(\mathbf{Z})),$$

which may be identified (using the isomorphisms above and Proposition 6.7) with a geometrically induced map

$$L_{n-1-|W''|}(\mathbf{Z}K) \xrightarrow{1-u} L_{n-1-|W''|}(\mathbf{Z}K)$$

"multiplication by $1 - w(t)$", where $\bar{t} = w(t)t^{-1}$ from the action of t in the antistructure used to define the L-group.

In the oriented case, $w(t) = +1$ and this boundary map is zero. Therefore, we conclude that

$$L_{2k+1}(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'', G}(\mathbf{Z})) \xrightarrow{\approx} L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z}))$$

is a rational isomorphism and so

$$L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) = L_{2k-|W''|}(\mathbf{Z}K)\ .$$

Finally, we will substitute this computation and our inductive assumption into the long exact sequence

$$\cdots \to L_{2k+1}(\mathcal{C}_{W'',G}(\mathbf{Z})) \to L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \to L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z})) \to$$

and find that

$$0 \to L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \to L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z})).$$

However, the second map in this sequence can be identified with the inclusion map

$$\mathrm{Ind}_K \colon L_{2k}(\mathcal{C}_{W'',K}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z}))$$

and the composition $\mathrm{Res}_K \circ \mathrm{Ind}_K$ is multiplication by $[G : K]$, which is a rational isomorphism. Therefore Ind_K is injective and $L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) = 0$. □

Another computation we will need is

Lemma 8.3. $L_n(\mathcal{C}^{>\mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) \otimes \mathbf{Q} = 0$ *for W a complex G-representation.*

Proof. We may assume that $W^G = 0$ and argue by induction on the dimension of W. We write $W = W' \oplus W''$, where $\mathrm{Iso}(W') = K$ is minimal (2–power index isotropy subgroups may be assumed as usual). We have 2 long exact sequences (all L–groups are tensored with $\mathbf{Q}$):

(i) from the inclusion $W'' \oplus \mathbf{R}_- \subset W \oplus \mathbf{R}_-$. For short, let $A = \mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$, $B = \mathcal{C}^{>\mathbf{R}_-}_{W''\times\mathbf{R}_-,G}(\mathbf{Z})$, and then $A/B = \mathcal{C}^{>W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$. We need the piece of the L-group sequence:

$$... \to L_n(B) \to L_n(A) \to L_n(A/B)... \tag{8.4}$$

and note that $L_n(B) = 0$ by induction.

(ii) we study $L_n(A/B)$ by looking at the usual rays $\bigcup \ell_\alpha \subset W'$ which divide the 2-dim representation W' into G/K chambers. Let

$$D_0 = \mathcal{C}^{>W''\times\mathbf{R}_-}_{\bigcup \ell_\alpha\times W''\times\mathbf{R}_-,G}(\mathbf{Z})$$

and

$$D_1 = \mathcal{C}^{>\bigcup \ell_\alpha\times W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$$

Then we need the L–group sequence

$$\to L_n(D_0) \to L_n(A/B) \to L_n(D_1) \to L_{n-1}(D_0) \tag{8.5}$$

But the groups $L_n(D_i)$ are the ones we have been computing by using the chamber structure. In particular,

$$L_n(D_0) = L_{n-2}(\mathcal{C}_{W'',K}(\mathbf{Z}))$$

and

$$L_n(D_1) = L_{n-3}(\mathcal{C}_{W'',K}(\mathbf{Z})).$$

But since K is the minimal isotropy group in W, it acts trivially on W'' and these L-groups are just

$$L_n(D_0) = L_{n-2-|W''|}(\mathbf{Z}K)$$

and

$$L_n(D_1) = L_{n-3-|W''|}(\mathbf{Z}K).$$

Now the boundary map

$$L_n(D_1) \to L_{n-1}(D_0) \tag{8.6}$$

in the sequence is just multiplication by $1-w(t) = 2$ ($w(t) = -1$ since $\mathbf{R}_-$ is non-oriented), and this is a rational isomorphism. Therefore, $L_n(A/B) = 0$ by (8.5) and substituting back into (8.4), the $L_n(A) = 0$.

□

Now a more precise result in a special case:

Lemma 8.7. *Let W be a complex 2–dimensional G-representation with $W^G = 0$. Then* $L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$

Proof. We have an exact sequence

$$\cdots \to L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to$$
$$L^I_3(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z})) \to \dots$$

arising from the orbit type filtration. Here

$$\begin{aligned} I &= \mathrm{Im}(K_2(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \\ &= \ker(K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z}))) \\ &= \ker(K_{-1}(\mathbf{Z}K) \xrightarrow{2} K_{-1}(\mathbf{Z}K)) \end{aligned}$$

by Proposition 6.7, where K is the isotropy subgroup of W. But $K_{-1}(\mathbf{Z}K)$ is torsion–free, so $I = \{0\} \subseteq K_2$. Now substitute the computations

$$\begin{aligned} L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z})) &= L^p_1(\mathbf{Z}K) = 0 \\ L^I_3(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) &= L^p_0(\mathbf{Z}K) \\ L^s_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z})) &= L^p_0(\mathbf{Z}K) \end{aligned}$$

into the exact sequence. The boundary map

$$L^I_3(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha \times \mathbf{R}_-,G}(\mathbf{Z}))$$

is multiplication by 2 so $L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$. □

Remark 8.8. The same method shows that $L^s_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = L^p_3(\mathbf{Z}K) = \mathbf{Z}/2$ for $\dim W = 2$ as above, assuming that $K \neq 1$.

Our final preliminary result is a Mayer–Vietoris sequence:

Lemma 8.9. *Let W be a complex G–representation with $W^G = 0$. Let $W = W_1 \oplus W_2$ be a direct sum decomposition, where $W_1 = W_{max}$.*

(i) *There is a long exact sequence*

$$\cdots \to L^I_{n+1}(\mathcal{C}^{>W_1\times\mathbf{R}_-\cup W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\partial_{n+1}} L^s_n(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to$$
$$L^s_n(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L^s_n(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^I_n(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

of bounded L-groups, where

$$I = \mathrm{Im}\left[K_2(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_2(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\right]$$

is the decoration subgroup.

(ii) *For $n \equiv 3 \bmod 4$, the boundary map $\partial_{n+1} = 0$.*

Proof. The Mayer–Vietoris sequence in $L^{\langle -\infty \rangle}$

$$\cdots \to L_{n+1}(\mathcal{C}^{>U\cup V}_{M,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>U\cap V}_{M,G}(\mathbf{Z})) \to$$
$$L_n(\mathcal{C}^{>U}_{M,G}(\mathbf{Z})) \oplus L_n(\mathcal{C}^{>V}_{M,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>U\cup V}_{M,G}(\mathbf{Z})$$

where U, V are nice G–invariant subspaces of the control space M, follows from an excision isomorphism

$$L_n(\mathcal{C}^{>U}_{U\cup V,G}(\mathbf{Z})) = L_n(\mathcal{C}^{>U\cap V}_{V,G}(\mathbf{Z}))$$

and standard diagram-chasing. We apply this to $M = W \times \mathbf{R}_-$, $U = W_1 \times \mathbf{R}_-$ and $V = W_2 \times \mathbf{R}_-$, where $U \cap V = \mathbf{R}_-$. The decorations follow from Section 5 and Corollary 6.5.

To see that $\partial_0 = 0$, note that this boundary map is the composition of

$$\partial\colon L^I_0(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-,G}(\mathbf{Z}))$$

and an excision isomorphism

$$L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-,G}(\mathbf{Z})) \cong L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))\ .$$

But $L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ by Lemma 8.7. □

Corollary 8.10. *Suppose W is a complex G–representation with no $\mathbf{R}_+$ summands. Then the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L^s_{2k+1}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$, where $\dim W = 2k$.*

Proof. First recall from [**I**], Theorem 3.6 that $trf_{W\times\mathbf{R}_-}(\sigma) \in L^{c_1}_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ where $c_1 = \mathrm{Im}(\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ under the cone point inclusion. Therefore, the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ "away from $\mathbf{R}_-$" lands in $L^s_{2k+1}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. We may therefore apply the Mayer–Vietoris sequence from Lemma 8.9.

As usual, we may assume that W has only 2–power isotropy subgroups, and we will argue by induction on $\dim W$ and on the orbit type filtration of W. The result is true for $\dim W = 2$ (by Lemma 8.7) or for W a free representation (by a direct calculation following the method of Lemma 8.3). For sets of 2–power index subgroups (ordered by inclusion, with repetitions allowed), we say that $\{K'_1, \dots, K'_s\} < \{K_1, \dots, K_t\}$ if $s < t$ or $s = t$, $K'_i = K_i$ for $1 \leqq i < j \leqq s$, and $K'_j \subsetneqq K_j$. Let $\mathrm{Iso}(W) = \{K_1, \dots, K_t\}$ denote the set of isotropy subgroups of W ordered by inclusion. Define an ordering by $W' < W$ if (i) $\dim W' < \dim W$ or (ii) $\dim W' = \dim W$ but $\mathrm{Iso}(W') < \mathrm{Iso}(W)$.

Let $W = W_1 \oplus W_2$, where W_1 has maximal isotropy subgroup K, and suppose the result holds for all $W' < W$. If $\dim W \equiv 2 \bmod 4$, we get an injection

$$L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

from Lemma 8.9, where $W_2 < W$ and $W_1 < W$. If $\dim W \equiv 0 \bmod 4$, we get an injection

$$L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L^s_3(\mathcal{C}^{>W_2\times U\times\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z}))$$

again from Lemma 8.9, where U is any 2–dimensional free G–representation. Notice that $W_2 \oplus U < W$ in the ordering above. Consider the following commutative diagram (with $\dim W \equiv 2 \bmod 4$):

$$\begin{array}{ccccc}
L^h_0(\mathbf{Z}G) & \xrightarrow{trf_{W_2\times\mathbf{R}_-}} & L^{c_1}_1(\mathcal{C}_{W_2\times\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & L^s_1(\mathcal{C}^{>\mathbf{R}_-}_{W_2\times\mathbf{R}_-,G}(\mathbf{Z})) \\
& \searrow^{trf_{W\times\mathbf{R}_-}} & \downarrow{\scriptstyle trf_{W_1}} & & \downarrow{\scriptstyle trf_{W_1}} \\
& & L^{c_3}_3(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))
\end{array}$$

This diagram, and our inductive assumption shows that the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. Similarly, by reversing the roles of W_1 and W_2 in the diagram, we see that the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. Therefore the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. If $\dim W \equiv 0 \bmod 4$, we replace W by $W\oplus U$ and make a similar argument, using the observation that $W_2 \oplus U < W$ to justify the inductive step. □

The proof of Theorem 8.1. Part (i) follows from Corollary 8.10: we know that the image of our obstruction $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero "away from $\mathbf{R}_-$", so comes from a torsion element in $\hat\sigma \in L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ (by Lemma 8.3, $L_{2k+2}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion).

For part (ii) observe that $L^h_{2k+1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})) = L^p_{2k}(\mathbf{Z}H)$ is torsion–free (except for the Arf invariant $\mathbf{Z}/2$ which is detected by the trivial group, so does not matter). Hence the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ vanishes in $L^h_{2k+1}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. The exact sequence

$$\cdots \to L^{\mathbf{U}}_{2k+1}(\mathbf{Z}G,w) \to L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^h_{2k+1}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})),$$

with $\mathbf{U} = \ker(\widetilde K_0(\mathbf{Z}G) \to \widetilde K_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$, now shows that there exists a torsion element

$$\hat\sigma \in \operatorname{Im}(L^{\mathbf{U}}_{2k+1}(\mathbf{Z}G,w) \to L^p_{2k+1}(\mathbf{Z}G,w))$$

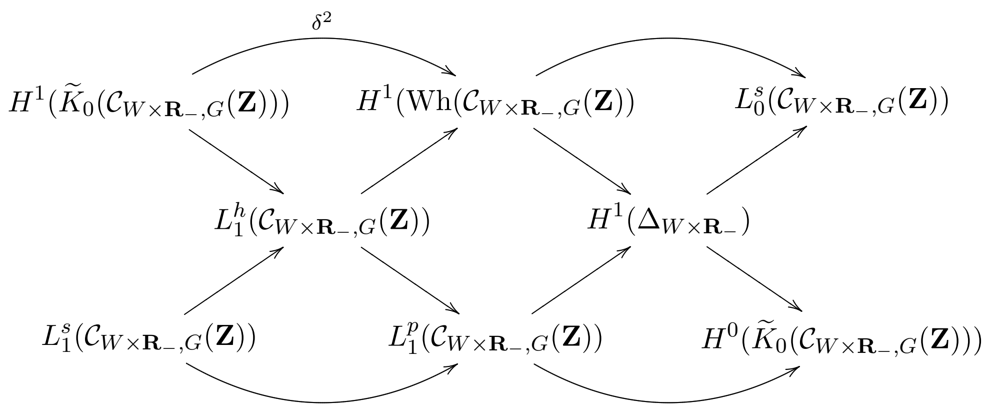
with $c_*(\hat\sigma) = trf_{W\times\mathbf{R}_-}(\sigma) \in L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. □

Remark 8.11. In Theorem 10.7 we use the sharper result $\hat\sigma \in L^{\mathbf{U}}_{2k+1}(\mathbf{Z}G,w)$ to improve our unstable similarity results by removing extra $\mathbf{R}_-$ factors. The L-group decoration $\mathbf{U}$ is independent of W, since $K_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to K_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$ is injective by Corollary 6.9, and hence $\mathbf{U} = \operatorname{Im}(\widetilde K_0(\mathbf{Z}H) \to \widetilde K_0(\mathbf{Z}G))$.

9. The proof of Theorem C

By replacing V_i by $V_i \oplus U$ if necessary, where U is free and 2–dimensional, we may assume that $\dim V_i \oplus W \equiv 0 \bmod 4$. This uses the s–normal cobordism condition. By the top component argument (see [**I**], Theorem 3.8) and Proposition 7.7, we may also assume that W has only 2–power isotropy. We have a commutative diagram analogous to [**I**] (8.1).

(9.1)



where $H^1(\Delta_{W\times\mathbf{R}_-})$ denotes the relative group of the double coboundary map.

This diagram for W can be compared with the one for $W_1 = W_{max}$ via the inclusion maps, and we see that the K-theory terms map isomorphically by Lemma 6.2. Now by Theorem 8.1 there exists a torsion element

$$\hat{\sigma} \in \operatorname{Im}(L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})))$$

which hits $trf_{W\times\mathbf{R}_-}$ under the cone point inclusion. In particular, $\hat{\sigma}$ vanishes "away from $\mathbf{R}_-$". The main step in the proof of Theorem C is to show that the torsion subgroup of $L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ essentially injects into the relative group $H^1(\Delta_{W_1\times\mathbf{R}_-})$ of the double coboundary.

We have the exact sequence

$$L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^Y(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$$

and $Y = \ker(\operatorname{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \operatorname{Wh}(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is zero, by Corollary 6.9. But the first term $L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a torsion group, by Lemma 8.3, and the next term in the sequence $L_1^{\mathbf{Y}}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion–free by [**I**], Lemma 7.1 so the middle map $L_1^{\mathbf{Y}}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \rightarrowtail L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is an injection.

Since $L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = \mathbf{Z}/2$ by Remark 8.8 we conclude that the previous group $L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion–free modulo this $\mathbf{Z}/2$, which injects into $L_1^{\langle -1\rangle}(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = \mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}K))$. Since $\hat{\sigma}$ vanishes away from $\mathbf{R}_-$, this extra $\mathbf{Z}/2$ may be ignored.

By substituting this computation into the exact sequence

$$L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to H^1(\Delta_{W_1\times\mathbf{R}_-})$$

of (9.1), we see that the torsion subgroup of

$$\ker(L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})))$$

injects into the relative group $H^1(\Delta_{W_1\times\mathbf{R}_-})$ of the double coboundary. Therefore $\hat{\sigma} \in L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ vanishes if and only if the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\operatorname{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})))$$

This completes the proof of Theorem C.

A very similar argument can be used to give an inductive criterion for non–linear similarity without the $\mathbf{R}_+$ summand (generalizing Theorem A). In the statement we will use the analogue to $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$, namely

$$\mathbf{k}_W = \ker(\operatorname{Wh}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \widetilde{K}_0(\mathbf{Z}G)).$$

By Corollary 6.9

$$\ker(\operatorname{Wh}(\mathbf{Z}G) \to \operatorname{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \cong \operatorname{Im}(\operatorname{Wh}(\mathbf{Z}H) \to \operatorname{Wh}(\mathbf{Z}G)) \cong \operatorname{Wh}(\mathbf{Z}H)$$

so we have a short exact sequence

$$0 \to \operatorname{Wh}(\mathbf{Z}G)/\operatorname{Wh}(\mathbf{Z}H) \to \operatorname{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \mathbf{k}_W \to 0\ .$$

It follows from our K–theory calculations that $\mathbf{k}_W = \mathbf{k}_{W_{max}}$ and that $\mathbf{k}_W = \mathbf{k}$ whenever $K_{-1}(\mathbf{Z}K) = 0$ for all $K \in \operatorname{Iso}(W)$.

Theorem 9.2. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) *$\operatorname{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \operatorname{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the coboundary*

$$\delta \colon H^0(\mathbf{k}_{W_{max}}) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H)),$$

where $0 \subseteq W_{max} \subseteq W$ is a complex subrepresentation of real dimension ≤ 2 with maximal isotropy group among the isotropy groups of W with 2-power index.

Proof. In this case, we have a torsion element $\hat{\sigma} \in L_1^h(\mathcal{C}_{W_1 \times \mathbf{R}_-,G}(\mathbf{Z}))$ which maps to our surgery obstruction $trf_{W \times \mathbf{R}_-}$ under the cone point inclusion. As in the proof of Theorem C, the torsion subgroup of $L_1^h(\mathcal{C}_{W_1 \times \mathbf{R}_-,G}(\mathbf{Z}))$ injects into $H^1(\operatorname{Wh}(\mathcal{C}_{W_1 \times \mathbf{R}_-,G}(\mathbf{Z}))) \cong H^1(\operatorname{Wh}(\mathcal{C}_{W_1 \times \mathbf{R}_-,G}(\mathbf{Z})))$. Therefore $\hat{\sigma} \in L_1^h(\mathcal{C}_{W_1 \times \mathbf{R}_-,G}(\mathbf{Z}))$ vanishes if and only if the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the coboundary

$$\delta \colon H^0(\mathbf{k}_{W_{max}}) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$$

as required. □

10. Iterated $\mathbf{R}_-$ transfers

We now apply Theorem 5.12 to show that iterated $\mathbf{R}_-$ transfers do not lead to any new similarities.

Theorem 10.1. *Suppose that W is a complex G–representation with no $\mathbf{R}_+$ summands. Then $V_1 \oplus W \oplus \mathbf{R}_-^l \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_-^l \oplus \mathbf{R}_+$ for $l \geqq 1$ implies $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$.*

The first step in the proof is to show injectivity of certain transfer maps. For any homomorphism $w \colon G \to \{\pm 1\}$, we will use the notation $(\mathcal{C}_{W,G}(\mathbf{Z}), w)$ to denote the antistructure where the involution is $g \mapsto w(g)g^{-1}$ at the cone point. The standard orientation (5.4) has $w = \det(\rho_W)$, but we will need others in this section. Let $\phi = \det(\rho_{\mathbf{R}_-})$ for short, and notice that $\phi \colon G \to \{\pm 1\}$ is non–trivial.

Lemma 10.2. *The transfer map $trf_{\mathbf{R}_-} \colon L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+2}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$ is injective, where w is the non–trivial orientation.*

Proof. Since $L^p_{2k+2}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w) \cong L^p_{2k+2}(\mathbf{Z}H \to \mathbf{Z}G)$, we just have to check that the previous map in the L^p version of (5.8) is zero (using the tables in [13]). □

Next, a similar result for the $\mathbf{R}_-^2$ transfer.

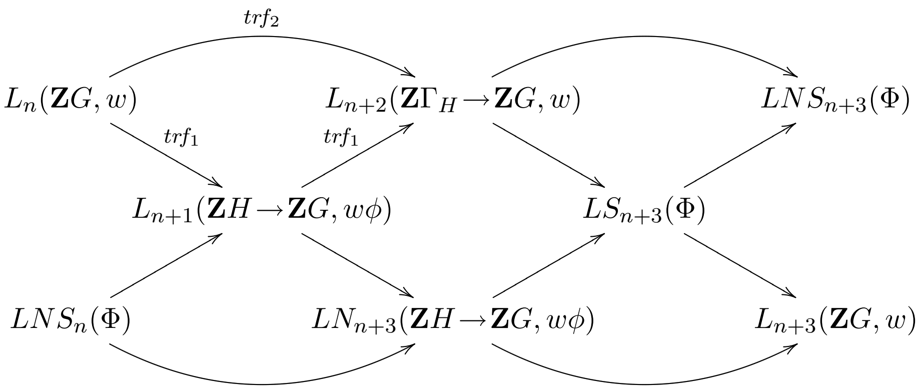
Proposition 10.3. *The bounded transfer*

$$trf_{\mathbf{R}_-^2} \colon L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+3}(\mathcal{C}_{\mathbf{R}_-^2,G}(\mathbf{Z}), w)$$

is injective, where w is the non–trivial orientation.

Proof. We can relate the iterated $\mathbf{R}_-$ transfer $trf_1 \circ trf_1$ to $trf_2 = trf_{\mathbf{R}^2_-}$ by means of the braid diagram:

(10.4)



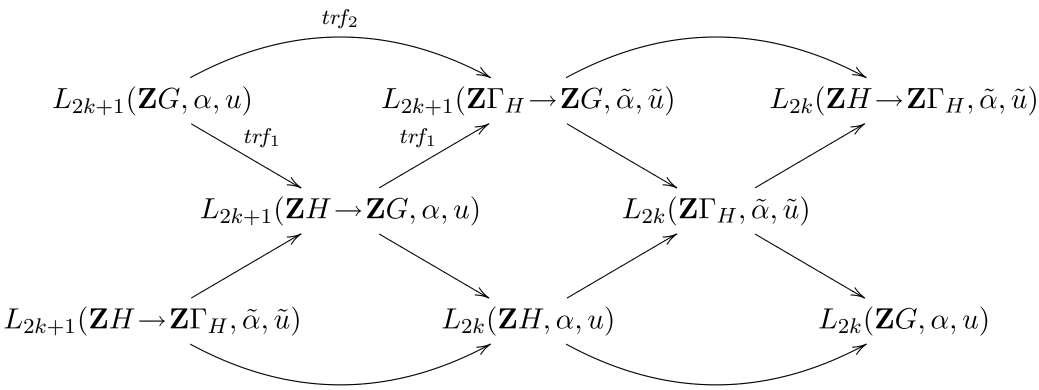
where the new groups $LNS_*(\Phi)$ are the relative groups of the transfer

$$trf_1 = trf_{\mathbf{R}_-} : L^p_k(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \to L^p_{k+1}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) \ .$$

The diagram Φ of groups (as in [28, Chap.11]) contains $\Gamma_H \cong \mathbf{Z} \times H$ as the fundamental group of $S(\mathbf{R}_- \oplus \mathbf{R}_-)/G$ (see [11, 5.9]).

The groups $LNS_*(\Phi)$ have a geometric bordism description in terms of triples $L^{n-2} \subset N^{n-1} \subset M^n$ where each manifold has fundamental group G, N is characteristic in M, and L is characteristic in N, with respect to the index two inclusion $H < G$. There is also an algebraic description for $LNS_*(\Phi)$ in terms of "twisted antistructures" [12] as for the other groups $LN(\mathbf{Z}H \to \mathbf{Z}G, w\phi)$ [28, 12C] and for $LS(\Phi, w)$ [20, 7.8.12]. Substituting these descriptions into the braid gives:

(10.5)



The antistructure $(\mathbf{Z}G, \alpha, u)$ is the twisted antistructure obtained by scaling with an element $a \in G - H$. The antistructure $(\mathbf{Z}\Gamma_H, \tilde{\alpha}, \tilde{u})$ is the one defined by Ranicki [20, p.805], then scaled by $\tilde{a} \in \Gamma_H$, where $\tilde{a}$ maps to a under the projection $\Gamma_H \to G$. Since $\Gamma_H \cong \mathbf{Z} \times H$, we have an exact sequence (see [16, Theorem 4.1])

$$\to L^p_n(\mathbf{Z}H, \alpha, u) \to L^p_n(\mathbf{Z}\Gamma_H, \tilde{\alpha}, \tilde{u}) \to L^{\langle -1 \rangle}_{n-1}(\mathbf{Z}H, \alpha, u) \xrightarrow{1-w(a)} L^p_{n-1}(\mathbf{Z}H, \alpha, u)$$

and it follows that $L^p_n(\mathbf{Z}H \to \mathbf{Z}\Gamma_H, \tilde{\alpha}, \tilde{u}) \cong L^{\langle -1\rangle}_{n-1}(\mathbf{Z}H, \alpha, u)$. It is not difficult to see that $L^{\langle -1\rangle}_{2k}(\mathbf{Z}H, \alpha, u)$ is torsion–free (except for the Arf invariant summand) by a similar argument to [**I**], Theorem 5.2, using the L^p to $L^{\langle -1\rangle}$ Ranicki-Rothenberg sequence. We first check that $L^p_{2k}(\mathbf{Z}H, \alpha, u)$ is torsion–free (again except for the Arf invariant summand) from the tables in [13, 14.21].

Now observe that $L^p_n(\mathbf{Z}G, \alpha, u) \cong L^p_n(\mathbf{Z}G, w)$, and

$$L^p_{2k+1}(\mathbf{Z}H \to \mathbf{Z}G, \alpha, u) \cong L^p_{2k+2}(\mathbf{Z}H \to \mathbf{Z}G)\ .$$

The transfer $trf_{\mathbf{R}_-}\colon L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+2}(\mathbf{Z}H \to \mathbf{Z}G)$ is injective by Lemma 10.2, and since $w(a) = -1$, the map

$$L^{\langle -1\rangle}_{2k}(\mathbf{Z}H, \alpha, u) \xrightarrow{1-w(a)} L^p_{2k}(\mathbf{Z}H, \alpha, u)$$

is also injective (except for the Arf invariant summand). Therefore $L^p_{2k+1}(\mathbf{Z}G, w)$ must inject into the relative group $L^p_{2k+1}(\mathbf{Z}\Gamma_H \to \mathbf{Z}G, \tilde{\alpha}, \tilde{u}) \cong L^p_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$. □

The final step in the proof is to consider the following commutative square of spectra:

$$\begin{array}{ccc} \mathbb{L}(\mathcal{C}_{pt}(\mathbf{Z}G), w) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{V,G}(\mathbf{Z}), w) \\ \downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}_-}} \\ \mathbb{L}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{V\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \end{array}$$

for any G-representation V with no $\mathbf{R}_+$ summands.

Lemma 10.6. *This is a pull–back square of $L^{\langle -\infty\rangle}$ spectra.*

Proof. The fibres of the vertical $\mathbf{R}_-$ transfers are

$$\mathbb{LN}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \cong \mathbb{L}(\mathcal{C}_{pt}(\mathbf{Z}H, \alpha, u))\ ,$$

and $\mathbb{LN}(\mathcal{C}_{V\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \cong \mathbb{L}(\mathcal{C}_{V,H}(\mathbf{Z}), \alpha, u)$ respectively by (5.8), and the fibre of the cone point inclusion

$$\mathbb{L}(\mathcal{C}_{pt}(\mathbf{Z}H), \alpha, u) \to \mathbb{L}(\mathcal{C}_{V,H}(\mathbf{Z}), \alpha, u)$$

is $\mathbb{L}(\mathcal{C}^{>0}_{V,H}(\mathbf{Z}), \alpha, u)$. But Theorem 5.12 shows that $\mathbb{L}(\mathcal{C}^{>0}_{V,H}(\mathbf{Z}), \alpha, u)$ is contractible, and therefore the cone point inclusion is a homotopy equivalence. It follows that the fibres of the horizontal maps are also homotopy equivalent. □

The proof of Theorem 10.1. It is enough to prove that $trf_{W\times\mathbf{R}^{l+1}_-}(\sigma) = 0$, for $l \geqq 1$, implies $trf_{W\times\mathbf{R}_-}(\sigma) = 0$ in the top component of $L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. We may assume that $\mathrm{Iso}(W)$ only contains subgroups of 2–power index, and let $\dim W = 2k$. We may also assume that l is even, by crossing with another $\mathbf{R}_-$ if necessary. Now by Theorem C, $trf_{W\times\mathbf{R}^{2s+1}_-}(\sigma) = 0$ implies $trf_{W\times\mathbf{R}^{2s-1}_-}(\sigma) = 0$, provided that $s \geqq 2$. It therefore remains to study $l = 2$.

The pullback squares provided by Lemma 10.6 can be combined as follows. Consider the diagram of $L^{\langle -\infty\rangle}$-spectra

$$\begin{array}{ccccc}
\mathbb{L}(\mathcal{C}_{pt}(\mathbf{Z}G), w) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{W\times\mathbf{R}^2_-,G}(\mathbf{Z}), w) \\
\downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}_-}} \\
\mathbb{L}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{W\times\mathbf{R}^3_-,G}(\mathbf{Z}), w\phi)
\end{array}$$

whose outer square ($V = W \times \mathbf{R}^2_-$) and the left-hand square ($V = \mathbf{R}_-$) are both pullback squares, and hence so is the right-hand square. Next consider the diagram of $L^{\langle -\infty\rangle}$-spectra

$$\begin{array}{ccc}
\mathbb{L}(\mathcal{C}_{pt}(\mathbf{Z}G), w) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \\
\downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}_-}} \\
\mathbb{L}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{W\times\mathbf{R}^2_-,G}(\mathbf{Z}), w) \\
\downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}_-}} \\
\mathbb{L}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) & \xrightarrow{c_*} & \mathbb{L}(\mathcal{C}_{W\times\mathbf{R}^3_-,G}(\mathbf{Z}), w\phi)
\end{array}$$

The lower square was just shown to be a pull-back, and the upper square is another special case of Lemma 10.6 (with $V = W \times \mathbf{R}_-$). Therefore the outer square is a pull-back, and this is the one used for the case $l = 2$.

Now we apply homotopy groups to these pull-back squares (using the fact that $L^{\langle -\infty\rangle} \equiv L^{\langle -1\rangle}$ to obtain the lower squares in the commutative diagram:

$$\begin{array}{ccccc}
L^p_{2k+2}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & L^p_{2k+1}(\mathbf{Z}G, w) & \xrightarrow{c_*} & L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w) \\
\downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow \\
L^{\langle -1\rangle}_{2k+2}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G, w) & \xrightarrow{c_*} & L^{\langle -1\rangle}_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle trf_{\mathbf{R}^2_-}} & & \downarrow{\scriptstyle trf_{\mathbf{R}^2_-}} \\
L^{\langle -1\rangle}_{2k+4}(\mathcal{C}^{>0}_{W\times\mathbf{R}^3_-,G}(\mathbf{Z})) & \longrightarrow & L^{\langle -1\rangle}_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) & \xrightarrow{c_*} & L^{\langle -1\rangle}_{2k+3}(\mathcal{C}_{W\times\mathbf{R}^3_-,G}(\mathbf{Z}), w)
\end{array}$$

The vertical maps in the top row squares are induced by the change of K-theory decoration. We need some information about the maps in this diagram.

(i) The upper left-hand vertical map is an isomorphism, since $K_{-1}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$.

(ii) The lower left-hand vertical map is an isomorphism, by Lemma 10.6.

(iii) The transfer map

$$trf_{\mathbf{R}^2_-} : L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$$

is injective, from Proposition 10.3.

(iv) The map

$$L^p_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) \to L^{\langle -1\rangle}_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$$

is injective, by [**I**], Corollary 6.13.

(v) The composite of the middle two vertical maps in the diagram is injective, by combining parts (iii)-(iv).

Suppose that $trf_{W\times\mathbf{R}^3_-}(\sigma)=0$. We have proved that the transfer map

$$trf_{\mathbf{R}^2_-}\colon L^p_{2k+1}(\mathbf{Z}G,w)\to L^{\langle -1\rangle}_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}),w)$$

is injective in part (v). Since $trf_{W\times\mathbf{R}_-}(\sigma)=c_*(\hat\sigma)$ for some $\hat\sigma\in L^p_{2k+1}(\mathbf{Z}G,w)$, it follows by a diagram chase that $trf_{W\times\mathbf{R}_-}(\sigma)=0$ and we are done. □

We also have a version without $\mathbf{R}_+$ summands.

Theorem 10.7. *Suppose that W is a complex G–representation with $W^G=0$. Then $V_1\oplus W\oplus\mathbf{R}^l_-\sim_t V_2\oplus W\oplus\mathbf{R}^l_-$ for $l\geqq 3$ implies $V_1\oplus W\oplus\mathbf{R}^3_-\sim_t V_2\oplus W\oplus\mathbf{R}^3_-$.*

Proof. This follows from a similar argument, using Theorem 9.2 instead of Theorem C. The injectivity results of Lemma 10.2 and Proposition 10.3 also hold for trf_1 and trf_2 on $L^{\mathbf{U}}_{2k+1}(\mathbf{Z}G,w)$, where $\mathbf{U}=\operatorname{Im}(\widetilde K_0(\mathbf{Z}H)\to\widetilde K_0(\mathbf{Z}G))$ as in Remark 8.11. This L-group decoration fits in exact sequences with $L^p_n(\mathbf{Z}H)$ so the previous calculations for injectivity apply again. The details are left to the reader. □

11. The proof of Corollary 2.4

Suppose that $G=C(4q)$, q odd, and that $V_1\oplus\mathbf{R}_-\oplus\mathbf{R}_+\sim_t V_2\oplus\mathbf{R}_-\oplus\mathbf{R}_+$ with $\dim V_i=4$. We will use Theorem B and the assumption about odd class numbers to prove that $V_1\cong V_2$. This is the only case we need to discuss to prove Corollary 2.4. If we started instead with $V_1\oplus\mathbf{R}^2_-\sim_t V_2\oplus\mathbf{R}^2_-$, then stabilizing with $\mathbf{R}_+$ and applying Theorem 10.1 would reduce to the case above.

We may assume that $q>1$ is a prime, and that the free representations have the form $V_1=t+t^i$ and $V_2=t^{1+2q}+t^{i+2q}$ where $(i,2q)=1$ (not all weights are $\equiv 1 \bmod 4$). The Reidemeister torsion invariant is $\Delta(V_1)/\Delta(V_2)=U_{1,i}\in \operatorname{Wh}(\mathbf{Z}G)$ in the notation for units of $\mathbf{Z}G$ used in [5, p. 732]. With respect to the involution ${}^-\colon t\mapsto -t^{-1}$, this element defines a class $u_{1,i}=\{U_{1,i}\}\in H^1(\operatorname{Wh}(\mathbf{Z}G^-))$. Since the map $L^p_1(\mathbf{Z}G^-)\to L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is injective, it is enough to show that the image of $u_{1,i}$ is non–trivial in $H^1(\Delta)$ (see diagram [**I**] (6.7)).

Let $A=\mathbf{Z}[\zeta_{4q}]$ be the ring of cyclotomic integers., and $B=\mathbf{Z}[\zeta_{2q}]$. We will study the top component $\operatorname{Wh}(\mathbf{Z}G)(q)$ by comparing it to $K_1(\mathcal{M})$, where $\mathcal{M}=A\times B\times B$ is the top component of an involution invariant maximal order in $\mathbf{Q}G$ containing $\mathbf{Z}G$. Notice that the two copies of B are interchanged under the involution, so $H^i(K_1(\mathcal{M}))=H^i(A^\times)$. We have the exact sequence

$$0\to\operatorname{Wh}(\mathbf{Z}G)(q)\to K_1(\mathcal{M})\to K_1(\widehat{\mathcal{M}})/\operatorname{Wh}(\widehat{\mathbf{Z}}G)(q)\to D(\mathbf{Z}G)(q)\to 0\ .$$

where $D(\mathbf{Z}G)=\ker(\widetilde K_0(\mathbf{Z}G)\to\widetilde K_0(\mathcal{M}))$. Note that $H^i(D(\mathbf{Z}G))=H^i(\widetilde K_0(\mathbf{Z}G))$ since A has odd class number.

Let $i_*\colon H^1(\operatorname{Wh}(\mathbf{Z}G^-)(q))\to H^1(A^\times)$ denote the map on the Tate cohomology induced by the inclusion $i\colon \operatorname{Wh}(\mathbf{Z}G)(q)\to A^\times$.

Lemma 11.1. *For $(j,2q)=1$ and $j\equiv 1 \bmod 4$, the image $i_*(u_{1,j})=\langle -1\rangle\in H^1(A^\times)$.*

Proof. Let $\gamma_j=\frac{t^j-1}{t-1}\in A^\times$, and compute $\bar\gamma_j/\gamma_j$ with respect to the non–oriented involution. We get $\bar\gamma_j/\gamma_j=u_{1,j}/u_{1,1}$, so $i_*(u_{1,j})=i_*(u_{1,1})$ for all i. Now let $v=\frac{t+1}{t-1}$. Since $t=-t^{1+2q}$, this is a cyclotomic unit in A with $v\bar v=-1$. But $v/\bar v=-u_{1,1}$, so $i_*(u_{1,j})=\langle -1\rangle$ in $H^1(A^\times)$. □

Next we need a computation:

Lemma 11.2. $0 \neq \langle -1 \rangle \in H^1(A^\times)$.

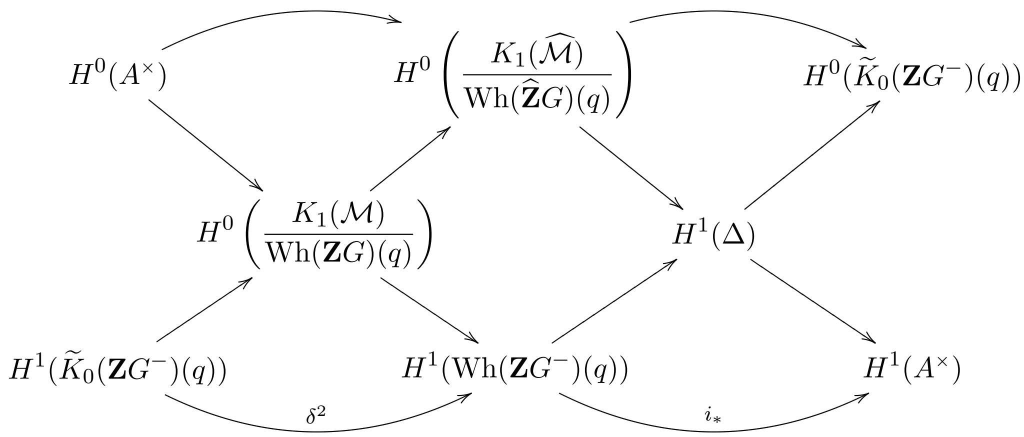
Proof. Let $E = \mathbf{Q}(\zeta_{4q})$ be the quotient field of A. Since A has odd class number, $H^i(E^\times/A^\times) \cong H^i(\hat{E}^\times/\hat{A}^\times)$. Next, observe that the extension E/F is evenly ramified at q, where F is the fixed field of the involution $^-\colon E \to E$. It follows that the map $H^0(\hat{E}_q^\times) \to H^0(\hat{E}_q^\times/\hat{A}_q^\times)$ is zero. We finish by considering the commutative diagram

$$\begin{array}{ccccc} H^0(E^\times) & \longrightarrow & H^0(E^\times/A^\times) & \xrightarrow{\ \delta\ } & H^1(A^\times) \\ \downarrow & & \downarrow \approx & & \downarrow \\ H^0(\hat{E}^\times) & \longrightarrow & H^0(\hat{E}^\times/\hat{A}^\times) & \rightarrowtail & H^1(\hat{A}^\times) \end{array}$$

and the prime element $w = \zeta_q - \zeta_q^{-1} \in E$ over q. Since $\bar{w}/w = -1$, ω defines an element in $H^0(E^\times/A^\times)$ and $\delta(w) = \langle -1 \rangle \in H^1(A^\times)$. On the other hand, w maps under the middle isomorphism to a non–zero element in $H^0(\hat{E}_q^\times/\hat{A}_q^\times)$ so $\delta(w) \neq 0$. □

Now we can complete the proof of Corollary 2.4 by considering the commutative braid:

(11.3)



containing the double coboundary and its relative group $H^1(\Delta)$. By commutativity of the lower right triangle, the image of our obstruction element $u_{1,j}$ is non–zero in $H^1(A^\times)$ and therefore non–zero in $H^1(\Delta)$.

12. The normal invariant

In this section we collect some results about normal cobordisms of lens spaces with cyclic fundamental group G. Recall that a necessary condition for the existence of a similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is that $S(V_1)$ and $S(V_2)$ must be s-normally cobordant. In [4, §1] it is asserted that their formula (A') gives necessary and sufficient conditions for homotopy equivalent lens spaces to be s–normally cobordant, when $G = C(2^r)$. However in [30, 1.3] it is shown that the given conditions (A') are sufficient but not necessary. We only use the sufficiency here, and study the subgroups $\widetilde{R}^{free}(G)$ and $\widetilde{R}_h^{free}(G)$ of $R(G)$ defined in Section 3. Recall that $\widetilde{R}^{free}(G) = \ker(\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$.

Lemma 12.1. *Let $G = C(2^r q)$ be a finite cyclic group, and G_2 be the 2–Sylow subgroup. If $q > 1$ there is an exact sequence*

$$0 \to \mathbf{Z}/2 \to \widetilde{R}^{free}(G)/\widetilde{R}^{free}_h(G) \to \widetilde{R}^{free}(G_2)/\widetilde{R}^{free}_h(G_2) \to 0$$

given by the restriction map Res_{G_2}. The kernel is generated by any element $\alpha \in \widetilde{R}^{free}(G)$ with k–invariant $k(\alpha) \in (\mathbf{Z}/|G|)^\times/\{\pm 1\}$ in the coset $k \equiv 1 \bmod 2^r$ and $k \equiv -1 \bmod q$.

Proof. There is a short exact sequence

$$0 \to \widetilde{R}^{free}_h(G) \to \widetilde{R}^{free}(G) \to (\mathbf{Z}/2^r q)^\times/\{\pm 1\}$$

given by the k–invariant. Moreover, the k–invariants of elements in $\widetilde{R}^{free}(G)$ lie in

$$\ker\big((\mathbf{Z}/2^r q)^\times/\{\pm 1\} \to (\mathbf{Z}/q)^\times/\{\pm 1\}\big) \cong \{lq+1 \,|\, l \in \mathbf{Z}\} \subseteq (\mathbf{Z}/2^r q)^\times$$

which injects under reduction $\bmod 2^r$ into $(\mathbf{Z}/2^r)^\times$ if $q > 1$. Restriction of the k–invariant to G_2 detects only its value in $(\mathbf{Z}/2^r)^\times/\{\pm 1\}$, so the kernel has order 2. Since the restriction map $\widetilde{R}^{free}G \to \widetilde{R}^{free}(G_2)$ is surjective, we have the required exact sequence. □

The situation for the normal invariant is simpler. Recall that we write $V \doteq V'$ or $(V - V') \doteq 0$ if there exists a homotopy equivalence $f\colon S(V)/G \to S(V')/G$ of lens spaces, such that f is s–normally cobordant to the identity.

Lemma 12.2. *Let $G = C(2^r q)$ be a finite cyclic group. Then*

$$\ker\big(\operatorname{Res}\colon R^{free}(G) \to R^{free}(C(q)) \oplus R^{free}(C(2^r))\big) \subseteq R^{free}_n(G)\ .$$

Proof. We may assume that $r \geq 2$, since $\operatorname{Res}\colon R^{free}(C(2q)) \to R^{free}(C(q))$ is an isomorphism, and consider an element

$$\sum t^{a_i} - \sum t^{b_i} \in \ker\big(R^{free}(G) \to R^{free}(C(q)) \oplus R^{free}(C(2^r))\big).$$

With a suitable ordering of the indices, $a_i \equiv b_i \bmod 2^r$ and $a_i \equiv b_{\tau(i)} \bmod q$ for some permutation τ. Since the normal invariants of lens spaces are detected by a cohomology theory, we can check the condition at each Sylow subgroup separately. It follows that $\sum t^{a_i} \doteq \sum t^{b_i}$. □

Example 12.3. For $G = C(24)$ we have $\widetilde{R}^{free}(G) = \{t - t^5, t - t^7, t - t^{11}\}$ and the subgroup $\ker \operatorname{Res}_{C(8)} \cap \widetilde{R}^{free}(G) = \{t - t^7, t^5 - t^{11}\}$.

Lemma 12.4. *Let $G = C(2^r q)$ be a finite cyclic group, and G_2 be the 2–Sylow subgroup. If $r \geq 2$ there is an isomorphism*

$$\widetilde{R}^{free}_h(G)/\widetilde{R}^{free}_n(G) \to \widetilde{R}^{free}_h(G_2)/\widetilde{R}^{free}_n(G_2)$$

given by the restriction map Res_{G_2}.

Proof. We first remark that if $W = t^{a_1} + \cdots + t^{a_n}$ is a free G_2–representation (here we use the assumption that $r \geq 2$), then by choosing integers $b_i \equiv a_i \bmod 2^r$ and $b_i \equiv 1 \bmod q$ we obtain a free G–representation $V = t^{b_1} + \cdots + t^{b_n}$ with $\operatorname{Res}_{G_2}(V) = W$. It follows that

$$\operatorname{Res}_{G_2}\colon \widetilde{R}^{free}_h(G) \to \widetilde{R}^{free}_h(G_2)$$

is surjective, and therefore

$$\operatorname{Res}_{G_2}\colon \widetilde{R}^{free}_h(G)/\widetilde{R}^{free}_n(G)) \to \widetilde{R}^{free}_h(G_2)/\widetilde{R}^{free}_n(G_2)$$

is also surjective.

Now suppose $\alpha \in \widetilde{R}_h^{free}(G)$ and $\mathrm{Res}_{G_2}(\alpha) \in \widetilde{R}_n^{free}(G_2)$. This means that $\mathrm{Res}_{G_2}(\alpha) = (W - W')$ for some free G_2–representation W, W' such that $W \doteq W'$. By the construction of the last paragraph, we can find free G–representations V, V' such that (i) $\mathrm{Res}_{G_2}(V) = W$ and $\mathrm{Res}_{G_2}(V') = W'$, and (ii) $\alpha' = (V - V') \in \widetilde{R}_h^{free}(G)$. It follows that $V \doteq V'$ and so $\alpha' \in \widetilde{R}_n^{free}(G))$, and

$$\alpha - \alpha' \in \ker\big(R_h^{free}(G) \to R_h^{free}(C(q)) \oplus R_h^{free}(C(2^r))\big).$$

By Lemma 12.2, $\alpha - \alpha' \in \widetilde{R}_n^{free}(G)$ and so $\alpha \in \widetilde{R}_n^{free}(G)$. □

Our final result is a step towards determining $R_h(G)/R_n(G)$ more explicitly.

Lemma 12.5. *Let* $b_{i,s} = (t^i - t^{2^{r-s}q-i}) \in \widetilde{R}^{free}(G)$ *for* $1 \le i < 2^{r-s-1}q$, $1 \le s \le r-1$, *and* $(i, 2q) = 1$. *Let* $l(i,s) = s$ *for* $1 \le s \le r-2$, *and* $l(i, r-1)$ *the order of* $i/(2q-i) \in (\mathbf{Z}/2^r)^\times$. *Then* $b_{i,s}$ *is an element of order* $2^{l(i,s)}$ *in* $R(G)/R_h(G)$, *and* $2^{l(i,s)} \cdot b_{i,s} \doteq 0$.

Proof. If $1 \le s \le r-2$ then $k(b_{i,s}) = i/(2^{r-s}q - i) \in (\mathbf{Z}/2^r q)^\times/\{\pm 1\}$ is congruent to $-1 \bmod 4q$. By Lemma 12.1 the linear span of these elements in $R(G)/R_h(G)$ injects into $R(G_2)/R_h(G_2)$, where G_2 is the 2–Sylow subgroup. By Lemma 12.4 the normal invariant is also detected by restriction to G_2, so it is enough to prove the assertions about these elements ($s \le r-2$) when $G = C(2^r)$.

For the first part we must show that the k–invariant of $b_{i,s}$ has order 2^s. We will use the expression

$$\nu_2(r!) = r - \alpha_2(r),$$

where $\alpha_2(r)$ is the number of non-zero coefficients in the 2-adic expansion of r, for the 2-adic valuation of r. Then

$$\nu_2(\binom{2^s}{k}) = s - \nu_2(k)$$

for $1 \le k \le 2^s$. These formulas and the binomial expansion show that $k(b_{i,s})^{2^s} \equiv 1 \bmod 2^r$ for $1 \le s \le r-2$.

Next we consider the normal invariant. For $1 \le s \le r-2$ we will take $q = 1$ and apply the criterion (A') of [4] to show that $2^s \cdot b_{i,s} \approx 0$. This amounts to a re-labelling of our original elements $b_{i,s}$ without changing the order of their k–invariants.

We must now compute the elementary symmetric functions $\sigma_k(2^s \cdot i^2)$ and $\sigma_k(2^s \cdot (2^{r-s} - i)^2)$, where the notation $2^s \cdot i^2$ means that the weight i^2 is repeated 2^s times in the symmetric function. The formula (A') is

$$\sigma_k(2^s \cdot (2^{r-s} + i)^2) - \sigma_k(2^s \cdot i^2) \equiv 2\big((2^{r-s} - i)^{2^s} - i^{2^s}\big)\binom{2^s - 1}{k-1} \bmod 2^{r+3}$$

so on the right-hand side we have

$$\begin{cases} +2^{r+1} \bmod 2^{r+3} & \text{if } k \text{ odd} \\ -2^{r+1} \bmod 2^{r+3} & \text{if } k \text{ even} \end{cases}$$

The formula (A') assumes that the weights are congruent to $1 \bmod 4$, or in our case $i \equiv 1 \bmod 4$. Therefore, if $i \equiv 3 \bmod 4$, we must use the equivalent weights $-i$ and $-(2^{r-s} - i)$.

To compute the left-hand side we use the Newton polynomials s_k and their expressions in term of elementary symmetric functions. We need the property $s_k(2^s \cdot i^2) = 2^s i^{2k}$ and the coefficient of σ_k in s_k which is $(-1)^{k+1}k$. By induction, we see that the left-hand side is just

$$\frac{(-1)^{k+1}}{k}\big(s_k(2^s \cdot (2^{r-s}-i)^2) - s_k(2^s \cdot i^2)\big) \equiv \frac{(-1)^{k+1}2^s}{k}\big((2^{r-s}-i)^{2k} - i^{2k}\big) \bmod 2^{r+3}$$

But by writing

$$(2^{r-s}-i)^{2k} - i^{2k} = (2^{r-s}\theta + i^2)^k - i^{2k}$$

where $\theta = 2^{r-s} - 2i \equiv 2 \bmod 4$, our expression becomes

$$\frac{(-1)^{k+1}2^s}{k}(2^{r-s}k\theta) \equiv (-1)^{k+1}2^{r+1} \bmod 2^{r+3}$$

If $s = r-1$, then $k(b_{i,s}) = i/(2q-i) \equiv -1 \bmod q$, and $k(b_{i,s}) \equiv 1 \bmod 2^r$ whenever $i \equiv q \bmod 2^{r-1}$. This is for example always the case for $G = C(4q)$. Such elements $b_{i,r-1}$ lie in the kernel of the restriction map to $R(C(2^r))$. Moreover, if $b_{i,r-1} \in \ker\big(R(G) \to R(C(2^r))\big)$, then $2b_{i,r-1} \in R_h(G)$ and $2b_{i,r-1} \doteq 0$ by Lemma 12.2. Otherwise, the order of $b_{i,r-1} \in R(G)/R_h(G)$ is the same as the order of its restriction $\operatorname{Res}(b_{i,r-1})$ to the 2–Sylow subgroup, and $\operatorname{Res}(b_{i,r-1}) = \pm \operatorname{Res}(b_{j,s})$ for $l(i, r-1) = s \leq r-2$ and some j. □

Remark 12.6. As pointed out by the referee, Lemma 12.1 and Lemma 12.4 together give a short exact sequence

$$0 \to \mathbf{Z}/2 \to \widetilde{R}^{free}(G)/\widetilde{R}^{free}_n(G) \to \widetilde{R}^{free}(G_2)/\widetilde{R}^{free}_n(G_2) \to 0,$$

assuming that G is not a cyclic 2-group. It is not difficult to see from the proof of Theorem E (given in [**I**]), that for $G_2 = C(2^r)$, and $r \geqq 4$, the term $\widetilde{R}^{free}(G_2)/\widetilde{R}^{free}_n(G_2)$ is the quotient of $\widetilde{R}^{free}_{\mathrm{Top}}(C(2^r))$ by the subgroup $\langle \alpha_1 + \beta_1 \rangle$.

13. The proof of Theorem D

We first summarize our information about $R_{\mathrm{Top}}(G)$, obtained by putting together results from previous sections. If $\alpha \in \widetilde{R}^{free}_{h,\mathrm{Top}}(G)$, we define the *normal invariant order* of α to be the minimal 2–power such that $2^t\alpha \in \widetilde{R}^{free}_{n,\mathrm{Top}}(G)$.

Theorem 13.1. *Let $G = C(2^r q)$, with q odd, and $r \geqq 2$.*

(i) *The torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$ is $\widetilde{R}^{free}_{\mathrm{Top}}(G)$.*

(ii) *The rank of $R^{free}_{\mathrm{Top}}(G)$ is $\varphi(q)/2$ for $q > 1$ (resp. rank 1 for $q = 1$), and the torsion is at most 2–primary.*

(iii) *The subgroup $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ has exponent two, and the Galois action induced by group automorphisms is the identity.*

(iv) *For any $\alpha \in \widetilde{R}^{free}_{h,\mathrm{Top}}(G)$, if the normal invariant order of $\operatorname{Res}_H(\alpha)$ is 2^t, then the normal invariant order of α is 2^{t+1}.*

Remark 13.2. In part (ii), $\varphi(q)$ is the Euler function. The precise number of $\mathbf{Z}/2$ summands in $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ is determined by working out the conditions in Theorem C on the basis elements of $\widetilde{R}^{free}_n(G)$. In cases where the conditions in Theorem C can actually be evaluated, the structure of $R_{\mathrm{Top}}(G)$ will thus be determined completely.

Proof. Parts (i) and (ii) of Theorem 13.1 have already been proved in Corollary 4.2, so it remains to discuss parts (iii) and (iv). In fact, the assertion that $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ has exponent 2 is an immediate consequence of Theorem C. To see this, suppose that $(V_1 - V_2)$ is any element in $\widetilde{R}^{free}_n(G)$. The obstruction to the existence of a stable non–linear similarity $V_1 \approx_t V_2$ is determined by the class $\{\Delta(V_1)/\Delta(V_2)\}$ in the Tate cohomology group $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$, which has exponent 2. Since the Reidemeister torsion is multiplicative, $\Delta(V_1 \oplus V_1) = \Delta(V_1)^2$, and we conclude that $V_1 \oplus V_1 \approx_t V_2 \oplus V_2$ by Theorem C. Finally, suppose that $\alpha \in \widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ and that σ is a group automorphism of G. By induction, we can assume that $\mathrm{Res}_H(\alpha - \sigma(\alpha)) = 0$. We now apply Theorem C to $\beta = \alpha - \sigma(\alpha)$, with W a complex G–representation such that $W^G = 0$, containing all the non-trivial irreducible representations of G with isotropy of 2-power index. Then β is detected by the image of its Reidemeister torsion invariant in $H^1(\Delta_{W\times\mathbf{R}_-})$. But by [**I**], Lemma 8.2, the Galois action on this group is trivial. Therefore $\beta = \alpha - \sigma(\alpha) = 0$.

For part (iv) we recall that the normal invariant for G lies in a direct sum of groups $H^{4i}(G;\mathbf{Z}_{(2)}) \cong \mathbf{Z}/2^r$. Since the map Res_H induces the natural projection $\mathbf{Z}/2^r \to \mathbf{Z}/2^{r-1}$ on group cohomology, the result follows from [1, 2.6]. □

It follows from our results that the structure of $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ is determined by working out the criteria of Theorem C on a basis of $\widetilde{R}^{free}_n(G)$. Suppose that V_1 is stably topologically equivalent to V_2. Then there exists a similarity

$$V_1 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R}^s_+ \sim_t V_2 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R}^s_+$$

where W has no $\mathbf{R}_+$ or $\mathbf{R}_-$ summands, and $l, s \geqq 1$. But by [**I**], Corollary 6.13 we may assume that $s = 1$, and by Theorem 10.1 that $l = 1$, so we are reduced to the situation handled by Theorem C. The algebraic indeterminacy given there is computable, but not very easily if the associated cyclotomic fields have complicated ideal class groups. We carry out the computational details in one further case of interest (Theorem D).

The proof of Theorem D. We have a basis

$$\mathcal{B} = \{t^i - t^{i+2q} \mid (i, 2q) = 1, i \equiv 1 \bmod 4, 1 \leqq i < 4q\}$$

for $\widetilde{R}^{free}(G)$, so it remains to work out the relations given by topological similarity. Notice that $\widetilde{R}^{free}_h(G) = \widetilde{R}^{free}_n(G)$, and that the sum of any two elements in $\mathcal{B}$ lies in $\widetilde{R}^{free}_n(G)$. Moreover, by Corollary 2.4 there are no 6–dimensional similarities for G.

Now suppose that $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$, for some complex G–representation W. Then $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$ since q is odd, and by Theorem C we get a similarity of the form $V_1 \oplus \mathbf{R}^l_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}^l_- \oplus \mathbf{R}_+$. But by Theorem 10.1 this implies that $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$. Therefore, for any element $a_i = (t^i - t^{i+2q}) \in \mathcal{B}$ we have $2a_i \notin R^{free}_t(G)$ but $4a_i \in R^{free}_t(G)$. However, in Section 11 we determined the bounded surgery obstructions for all these elements. Since $i_*(u_{1,j}) = \langle -1 \rangle \in H^1(A^\times)$ for all j with $(j, 2q) = 1$ by Lemma 11.1, there are further stable relations $a_1 + a_j \approx a_1 + a_k$, or $a_1 \approx a_j$ for all j. It follows that a basis for $\widetilde{R}^{free}_{\mathrm{Top}}(G)$ is given by $\{a_1 \mid 4a_1 \approx 0\}$. □

References

[1] S. E. Cappell and J. L. Shaneson, *Non-linear similarity*, Ann. of Math. (2) **113** (1981), 315–355.

[2] ———, *The topological rationality of linear representations*, Inst. Hautes Études Sci. Publ. Math. **56** (1983), 309–336.

[3] ———, *Torsion in L-groups*, Algebraic and Geometric Topology, Rutgers 1983, Lecture Notes in Mathematics, vol. 1126, Springer, 1985, pp. 22–50.

[4] S. E. Cappell, J. L. Shaneson, M. Steinberger, S. Weinberger, and J. West, *The classification of non-linear similarities over* $\mathbb{Z}/2^r$, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 51–57.

[5] S. E. Cappell, J. L. Shaneson, M. Steinberger, and J. West, *Non-linear similarity begins in dimension six*, J. Amer. Math. Soc. **111** (1989), 717–752.

[6] M. Cárdenas and E. K. Pedersen, *On the Karoubi filtration of a category*, K-theory **12** (1997), 165–191.

[7] G. Carlsson and E. K. Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology **34** (1995), 731–758.

[8] D. Carter, *Localizations in lower algebraic K-theory*, Comm. Algebra **8** (1980), 603–622.

[9] A. W. M. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Ann. of Math. (2) **102** (1975), 291–325.

[10] I. Hambleton and E. K. Pedersen, *Bounded surgery and dihedral group actions on spheres*, J. Amer. Math. Soc. **4** (1991), 105–126.

[11] ———, *Non-linear similarity revisited*, Prospects in Topology: (Princeton, NJ, 1994), Annals of Mathematics Studies, vol. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 157–174.

[12] I. Hambleton, L. Taylor, and B. Williams, *An introduction to the maps between surgery obstruction groups*, Algebraic Topology (Aarhus, 1982), Lecture Notes in Mathematics, vol. 1051, Springer, Berlin, 1984, pp. 49–127.

[13] I. Hambleton and L. R. Taylor, *A guide to the calculation of surgery obstruction groups*, Surveys in Surgery Theory, Volume I, Annals of Mathematics Studies, vol. 145, Princeton Univ. Press, 2000, pp. 225–274.

[14] W-C. Hsiang and W. Pardon, *When are topologically equivalent representations linearly equivalent*, Invent. Math. **68** (1982), 275–316.

[15] I. Madsen and M. Rothenberg, *On the classification of G-spheres I: equivariant transversality*, Acta Math. **160** (1988), 65–104.

[16] R. J. Milgram and A. A. Ranicki, *The L-theory of Laurent extensions and genus* 0 *function fields*, J. Reine Angew. Math. **406** (1990), 121–166.

[17] R. Oliver, *Whitehead Groups of Finite Groups*, London Math. Soc. Lecture Notes, vol. 132, Cambridge Univ. Press, 1988.

[18] E. K. Pedersen and C. Weibel, *A nonconnective delooping of algebraic K-theory*, Algebraic and Geometric Topology, (Rutgers, 1983), Lecture Notes in Mathematics, vol. 1126, Springer, Berlin, 1985, pp. 166–181.

[19] A. A. Ranicki, *The double coboundary*, Letter to B. Williams, available at the web page http://www.maths.ed.ac.uk/ aar/surgery/brucelet.pdf, 1981.

[20] ———, *Exact Sequences in the Algebraic Theory of Surgery*, Math. Notes, vol. 26, Princeton Univ. Press, 1981.

[21] ———, *Additive L-theory*, K-theory **3** (1989), 163–195.

[22] ———, *Algebraic L-theory and Topological Manifolds*, Cambridge Tracts in Math., vol. 102, Cambridge Univ. Press, 1992.

[23] G. de Rham, *Sur les nouveaux invariants topologiques de M Reidemeister*, Mat. Sbornik **43** (1936), 737–743, Proc. International Conference of Topology (Moscow, 1935).

[24] ———, *Reidemeister's torsion invariant and rotations of* S^n, Differential Analysis, (Bombay Colloq.), Oxford University Press, London, 1964, pp. 27–36.

[25] J.-P. Serre, *Linear Representations of finite groups*, Graduate Texts in Mathematics, vol. 42, Springer, 1977.

[26] R. Swan, *Induced representations and projective modules*, Ann. of Math. (2) **71** (1960), 267–291.

[27] R. G. Swan and E. G. Evans, *K-theory of finite groups and orders*, Lecture Notes in Mathematics, vol. 149, Springer, 1970.

[28] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, New York, 1970.
[29] ———, *On the classification of Hermitian forms. II. Semisimple rings*, Invent. Math. **18** (1972), 119–141.
[30] C. Young, *Normal invariants of lens spaces*, Canad. Math. Bull. **41** (1998), 374–384.

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DEPARTMENT OF MATHEMATICS & STATISTICS, MCMASTER UNIVERSITY
HAMILTON, ONT., CANADA, L8S 4K1
E-mail address: ian@math.mcmaster.ca

DEPARTMENT OF MATHEMATICAL SCIENCES, SUNY AT BINGHAMTON
BINGHAMTON, NY, 13901
E-mail address: erik@math.binghamton.edu