Math 371 - Shifting theorems

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November 29, 2005

Let us recall the Dirac delta function.
It is a function $\delta(t)$ which is 0 everywhere but at $t = 0$ it is so large that $\int_a^b (\delta(t) dt = 1$ when $a < 0$ and $b > 0$.
This is of course impossible, but we can approximate by a function which is 0 except for a short interval around 0, and so large in that short interval that the integral over a larger interval is 1

We now calculate the Laplace transform

$$\int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}$$

This is because the value of e^{-st} at $t = a$ is e^{-sa}.
So while Dirac's delta function may not quite make sense it has a perfectly good Laplace transform.
Let us look at the example from last lecture but hit the spring with a hammer at time $t = 1$ instead of applying a constant force of 1.
The equation will now be

$$y'' + y = \delta(t-1) \qquad y(0) = 0 \qquad y'(0) = 1$$

Our classical method to solve this does not work but apply the Laplace transform
We get the same thing on the left side and the Laplace transform of the delta function on the right side

$$s^2 Y - 1 + Y = e^{-s}$$

We can now calculate the Laplace transform $Y = \mathcal{L}(y)$

$$Y = \frac{1}{1+s^2} + \frac{1}{1+s^2} e^{-s}$$

The question now becomes how to recover y from the Laplace transform Y.

That sets the stage for the next theorem, the t-shifting theorem.

Second shift theorem

Assume we have a given function $f(t)$, $t \geq 0$.

We want to physically move the graph to the right to obtain a shifted function:

$$g(t) = \begin{cases} 0 & \text{for} \quad t < a \\ f(t-a) & \text{for} \quad t \geq a \end{cases}$$

What happens to the Laplace transform

Theorem

$$\mathcal{L}(g) = e^{-as}\mathcal{L}(f)$$

To be able to work better with shifting, define a function, the unit step function, by $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$.
We now have $\mathcal{L}(u) = \mathcal{L}(1) = \frac{1}{s}$
This is because the Laplace transform only depends of on the values for $t > 0$.
Shifting by a, can now be described as $g(t) = u(t-a)f(t-a)$
We can therefore reformulate the theorem above as

Theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\mathcal{L}(f)$$

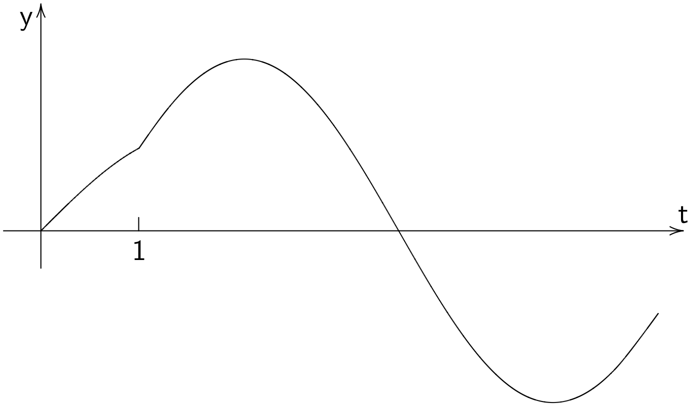
Now we are ready to solve the problem above. We had

$$Y = \frac{1}{1+s^2} + \frac{1}{1+s^2}e^{-s}$$

so

$$\mathcal{L}(y) = Y = \mathcal{L}(\sin(t)) + \mathcal{L}(u(t-1)\sin(t-1))$$

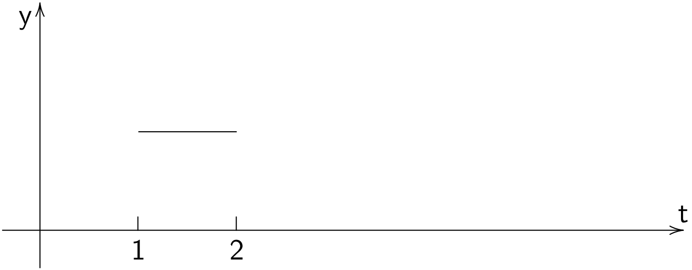
This is because multiplying by e^{-s} corresponds to shifting by 1. Thus the solution is $y = \sin(t) + u(t-1)\sin(t-1)$



In view of this it becomes important to move back and forth between a function and its Laplace transform

Next we consider a typical square wave function as often used in engineering

$$y(t) = \begin{cases} 0 & \text{for} \quad t < 1 \\ 1 & \text{for} \quad 1 < t < 2 \\ 0 & \text{for} \quad 2 < t \end{cases}$$



It does not matter what value the function has at the jump points 1 and 2.
Let us find a good way to compute the Laplace transform avoiding excessive work.

We can write

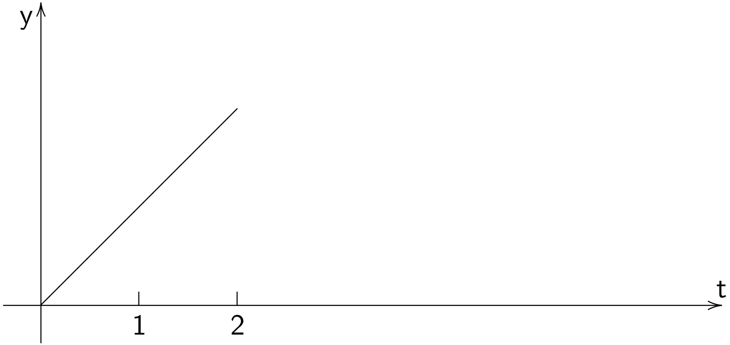
$$y(1) = u(t-1) - u(t-2)$$

But $u(t-1)$ is just 1 shifted by 1 and similarly $u(t-2)$ is 1 shifted by 2, so the Laplace transform is

$$\frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

Let us expand this example a little

Consider



This time the function is given as

$$y(t) = \begin{cases} t & \text{for} \quad t < 2 \\ 0 & \text{for} \quad 2 < t \end{cases}$$

How do we compute the Laplace transform
We start out realizing this function can be written as

$$y(t) = t - tu(t-2)$$

This is because $u(t-2)$ kills off t for $t < 2$ giving us t and for $t > 2$ we get $t - t$ which is 0 as we want.
Now we can rewrite this

$$y(t) = t - (t-2)u(t-2) - 2u(t-2)$$

The advantage is that now y is a combination of shifted functions and we immediately can read off the Laplace transform to be

$$\mathcal{L}(y) = Y = \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - 2\frac{1}{s}e^{-2s}$$

Let us finish by solving a differential equation

$$y'' - y = r(t) \qquad y(0) = 1 \qquad y'(0) = -1$$

Where $r(t)$ is the function mentioned above which is 1 for t between 1 and 2 and 0 otherwise.
We calculate that $\mathcal{L}(y') = sY - 1$
$\mathcal{L}(y'') = s\mathcal{L}(y') + 1 = s^2Y - s + 1$
This means the equation becomes

$$s^2Y - s + 1 - Y = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

Isolating the terms with Y we see that

$$s^2Y - Y = s - 1 + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

Dividing by (s^2-1) we get

$$Y = \frac{1}{s+1} + \frac{1}{s(s^2-1)}(e^{-s} - e^{-2s})$$

Using that $\frac{1}{s(s^2-1)} = \frac{1}{2}(\frac{-2}{s} + \frac{1}{s-1} + \frac{1}{s+1})$ we now get

$$Y = \frac{1}{s+1} + \frac{1}{2}(\frac{-2}{s} + \frac{1}{s-1} + \frac{1}{s+1})(e^{-s} - e^{-2s})$$

Let us unravel this. The 1. term is the Laplace transform of e^{-t}. Forgetting the exponential functions the second term is the Laplace transform of

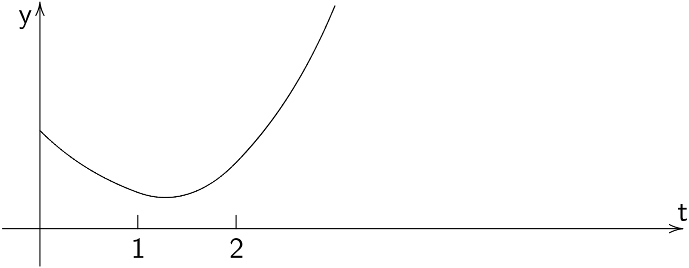
$$\frac{1}{2}(-2 + e^{t} + e^{-t}).$$

The exponential terms mean that this function has to be shifted by 1 and 2 respectively and the terms have to be subtracted.

Shifting by 1 means replacing t by $t-1$ everywhere and multiply by $u(t-1)$
Similarly shifting by 2 means replacing t by $t-2$ everywhere and multiply by $u(t-2)$.
So the complete answer is

$$y = e^{-t}+\frac{1}{2}u(t-1)(-2+e^{t-1}+e^{-(t-1)})-\frac{1}{2}u(t-2)(-2+e^{t-2}+e^{-(t-2)})$$

Here is a graph of the solution.
Notice how there is a big change at $t=1$, but almost no change at $t=2$.



Let us find a formula for integration

Theorem

$$\mathcal{L}\left(\int_0^t f\right) = \frac{\mathcal{L}(f)}{s}$$

This is very easy to see from the formula for the Laplace transform of the derivative, using that $\int_0^0 f = 0$.

Example

Find the inverse Laplace transform of

$$\frac{1}{s^2(s+1)}$$

We can of course use partial fractions to write $\frac{1}{s^2(s+1)}$ as a sum of fractions for which we know the inverse Laplace transform
But it is much easier to proceed as follows:

- $\mathcal{L}^{-1}(\frac{1}{s+1})$ is e^{-t}
- This now has to be integrated twice to get the $\frac{1}{s^2}$ factor.
- Integrating once gives $-e^{-t}+1$
- Integrating again gives $e^{-t}-1+t$ and that is the answer

We can of course check the answer by calculating the Laplace transform

$$\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

If we have made no mistakes this should be equal to

$$\frac{1}{s^2(1+s)}$$