A proof of Taylor's Inequality.

We first prove the following proposition, by induction on n. Note that the proposition is similar to Taylor's inequality, but looks weaker.

Let $T_{n,f}(x)$ denote the n-th Taylor polynomial of $f(x)$,

$$T_{n,f}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{(n)!}(x-a)^n,$$

and

$$R_{n,f}(x) = f(x) - T_{n,f}(x),$$

the n-th Taylor remainder of $f(x)$.

Proposition 1. *If $f(x)$ is n-times differentiable and $f^{(n)}(x) \leq M$ for all $x \in [a, a+d]$, then*

$$f(x) \leq T_{n-1,f}(x) + \frac{M}{n!}(x-a)^n,$$

for all $x \in [a, a+d]$.

Proof. The proof is by induction on n.
Base case (n=1) Note that $T_{0,f}(x) = f(a)$ is a constant.
Assume $f'(x) \leq M$ for all $x \in [a, a+d]$. Then integrating from a to x, we get

$$\begin{aligned} \int_a^x f'(t)dt &\leq \int_a^x M dt \\ f(x) - f(a) &\leq M(x-a) \\ f(x) &\leq f(a) + M(x-a) \\ &= T_{0,f}(x) + \tfrac{M}{1!}(x-a)^1 \end{aligned}$$

Induction step Assume the statement of the proposition for n (inductive hypothesis), and show it for $n+1$. In other words, we want to show that:

> *if $f(x)$ is $(n+1)$-times differentiable and $f^{(n+1)}(x) \leq M$ for all $x \in [a, a+d]$, then*
>
> $$f(x) \leq T_{n,f}(x) + \frac{M}{(n+1)!}(x-a)^{n+1},$$
>
> *for all $x \in [a, a+d]$.*

Assume f is $(n+1)$-times differentiable and $f^{(n+1)}(x) \leq M$ for all $x \in [a, a+d]$. Consider the function $g(x) = f'(x)$. Then $g(x)$ is n-times differentiable and for all $x \in [a, a+d]$ we have $g^{(n)}(x) \leq M$. Applying the inductive hypothesis to $g(x)$ we get that

$$g(x) \leq T_{n-1,g}(x) + \frac{M}{n!}(x-a)^n,$$

for all $x \in [a, a+d]$, i.e.

$$g(x) \leq g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots \frac{g^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{M}{n!}(x-a)^n.$$

We can rewrite this now in terms of $f(x)$ as

$$f'(x) \leq f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{M}{n!}(x-a)^n,$$

and now integrating between a and x we get

$$\int_a^x f'(t)dt \leq \int_a^x \left(f'(a) + f''(a)(t-a) + \frac{f'''(a)}{2!}(t-a)^2 + \cdots \frac{f^{(n)}(a)}{(n-1)!}(t-a)^{n-1} + \frac{M}{n!}(t-a)^n \right) dt.$$

This yields

$$f(x) - f(a) \leq f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{(n)!}(x-a)^n + \frac{M}{(n+1)!}(x-a)^{n+1},$$

i.e.

$$f(x) \leq T_{n,f}(x) + \frac{M}{(n+1)!}(x-a)^{n+1},$$

as desired. □

Note that the arguments in the proposition could have been done with $\geq -M$ instead of $\leq M$, showing that:

> *if $f(x)$ is n-times differentiable and $f^{(n)}(x) \geq -mM$ for all $x \in [a, a+d]$, then*
>
> $$f(x) \geq T_{n-1,f}(x) + \frac{-M}{n!}(x-a)^n,$$
>
> *for all $x \in [a, a+d]$.*

Combining both results, we get the full result to the right of a:

If $f(x)$ is n-times differentiable and $|f^{(n)}(x)| \leq M$ for all $x \in [a, a+d]$, then

$$|f(x) - T_{n-1,f}(x)| \leq \frac{M}{n!}(x-a)^n,$$

for all $x \in [a, a+d]$.

Finally, for $x \in [a-d, a]$, on the left of a, the argument can be repeated using $(a-x)$ instead of $(x-a)$, and combining with the case $x \in [a, a+d]$, yield Taylor's inequality.