Conjugation and Inverse Semigroups

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Conjugation in groups

As a binary relation

Let G be a group. For $a, b \in G$, we say a, b are conjugates if there is $x \in G$ such that

$$a \sim b \quad \text{iff there is} \quad x \in S, \text{ s.t. } b = x^{-1}ax$$

Extend this to monoids and semigroups.

G-conjugated
M a monoid;
G subgroup of invertible elements; $a, b \in M$

$$a \sim_G b \quad \text{iff} \quad \exists x \in G, \text{ s.t. } b = x^{-1}ax$$

$\sim$ and $\sim_G$ are equivalence relations. The equivalence classes are the conjugacy classes.

S-conjugated

$$a \sim_{pS} b \quad \text{iff } \exists x, y \in S, \text{ s.t. } a = xy, b = yx$$

Denote by $\sim_S$ the transitive closure of $\sim_{pS}$.

As a binary operation

$$a * x := x^{-1}ax. \tag{1}$$

Proposition 1. *The binary operation of group conjugation satisfies the following laws:*

(L1) $(x * y) * z = (x * z) * (y * z)$ *(right self-distributivity)*

(L2) $x * x = x$ *(idempotency)*

(L3) $x * z = y * z \ \Rightarrow\ x = y$ *(right cancellation)*

(L4) $x * y = x \ \Rightarrow\ y * x = y$ *(symmetry)*

Conjugation without inverses

$$b = a * x \quad \text{if and only if} \quad xb = ax \tag{2}$$

Conjugation in semigroups

Proposition 2. *Let S be a left cancellative semigroup. The partial operation of conjugation, as defined in* (2) *satisfies* (L1,L2,L4)*.*
If S is right cancellative, then it satisfies (L3)*.*

In the identities (L1–L2) this means that when both sides are defined, they are equal. For the quasi-identities (L3–L4), the meaning is that whenever the terms in the antecedent equation are defined and equal, the terms in the consequent equation are also defined and equal.

Left cancellation is used only to have the operation well-defined, but not in the proof. Without left-cancellation the result holds for the binary relation ρ_x

$$a\rho_x b \quad \text{if and only if} \quad xb = ax \tag{3}$$

Inverse semigroups

Regular semigroup: inverses exist
$(\forall x \in X)(\exists y \in S)$ s.t. $xyx = x$ and $yxy = y$.

Inverse semigroup: inverses exist and are unique.
i-conjugation:

$$a * x := x^{-1}ax \tag{4}$$

i-conjugation fails (L1–L4) in general. (*symmetric inverse semigroup*)

Proposition 3. *S an inverse semigroup, $x \in S$.*

1. *xx^{-1} and $x^{-1}x$ are idempotents, the left and right identities for x.*
2. *Every idempotent in S is its own inverse.*
3. *Idempotents commute.*
4. *The product of idempotents is idempotent.*
5. *Left and right identities are all the idempotents of S.*
6. *The set of idempotents form an inverse subsemigroup.*

Proposition 4. *Let S be an inverse semigroup.*

1. *$a \leq b$ defined by $ab^{-1} = aa^{-1}$, is a compatible partial order on S.*

2. *$a \in S$, $e \in E$, $f = aa^{-1}$ the left identity of a.*
   $ea \leq a$, with equality if and only if $f \leq e$.

S acts by i-conjugation on E.

$e \in E$ is **stable** under $a \in S$ if $\quad a^{-1}ea = e$
it is **weakly-stable** if $\quad a^{-1}ea \leq e$
Notice that $a^{-1}ea \leq a^{-1}a$.

Proposition 5. *Let $a \in S$ and $e \in E$. Consider*

(i) e is stable under i-conjugation by a,

(ii) e and a commute,

(iii) e is weakly-stable under i-conjugation by a,

then $(i) \Rightarrow (ii) \Rightarrow (iii)$. Strict implications.

Proposition 6. *Let S be an inverse semigroup, $a, x \in S$.*

1. *The i-conjugate of a by x, $a * x = x^{-1}ax$ is a conjugate of a by x, in the sense of (2), if and only if the left identity of x is weakly-stable under i-conjugation by a^{-1}.*

2. *If a has a conjugate by x, in the sense of (2), then $a * x$ is the smallest such conjugate.*

Theorem 7. *Let S be an inverse semigroup.*
For $a, x \in S$, the conjugation of a by x defined as the smallest b satisfying (2), if it exists, is a well-defined partial operation.
It is equal to (4) when defined.
It satisfies the laws (L1,L2,L4).

Conjualgebras

A *(partial) conjualgebra* consists of a set A with a (partial) binary operation $*$ that satisfies the Laws (L1–L4).
For $x \in A$, the *centralizer* of x is the set

$$C_x := \{a \in A | a * x = a\}$$

Proposition 8. *Let $(A, *)$ be a conjualgebra, and $x \in A$.*

1. *The map*

$$\begin{array}{rccc} \rho_x : & A & \longrightarrow & A \\ & a & \longmapsto & a * x \end{array}$$

   is an injective homomorphism of A.

2. *The centralizer C_x is a subalgebra of A, i.e. the subalgebra $[\rho_x = 1_A]$.*

3. *$A * x$, the image of ρ_x, is a subalgebra of A.*

4. *$C_x \leq A * x \leq A$.*

Theorem 9. *Let A be a conjualgebra, and let* $\mathrm{IEnd}(A)$ *be the semigroup of injective endomorphisms of A.*

1. $\mathrm{IEnd}(A)$ *is a subsemigroup of the symmetric inverse semigroup $\mathcal{J}_A$, but not an inverse subsemigroup in general.*

2. *The map*

$$\begin{array}{rccc} \rho : & A & \longrightarrow & \mathrm{IEnd}(A) \\ & x & \longmapsto & \rho_x \end{array}$$

has the property that ρ_{x*y} is a conjugate of ρ_x by ρ_y. When considered inside $\mathcal{J}_A$, it satisfies the inequality

$$\rho_{x*y} \geq \rho_x * \rho_y.$$