A duality for the algebra of conditional logic

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REFERENCES

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Three-valued Condiditional Logic

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

	'
T	F
F	T
U	U

Weak Regular Extension

$\underset{w}{\wedge}$	T	F	U
T	T	F	U
F	F	F	U
U	U	U	U

$\overset{w}{\vee}$	T	F	U
T	T	T	U
F	T	F	U
U	U	U	U

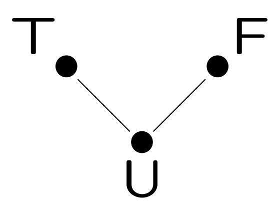
	'
T	F
F	T
U	U

Natural Duality
C as a Structured Topological Space

- $C = \{\mathtt{T},\mathtt{F},\mathtt{U}\}$

- Discrete topology

- *Definedness* partial order



- Partial operations

Dual to C-algebra A is $P = \mathrm{Hom}_{\mathcal{C}}(A, C)$.

Proposition 1. *Let A be a finite C-algebra, $e : A \hookrightarrow C^n$ an embeddding and $\varphi \in \mathrm{Hom}(A, C)$.*
Let $\mathbf{n} = \{1, \ldots, n\}$.

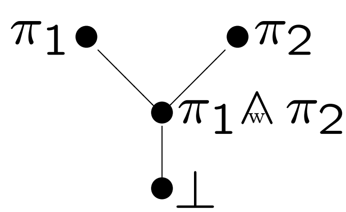
1. *If $\varphi(A) = \{\mathtt{T},\mathtt{F}\}$, then $\varphi = \pi_j \circ e$ for some $j \in \mathbf{n}$.*

2. *If $\varphi(A) = \{\mathtt{T},\mathtt{F},\mathtt{U}\}$, then there is $J \subseteq \mathbf{n}$ such that*

$$\varphi = \left(\bigwedge_{j \in J}^{\mathrm{w}} \pi_j \right) \circ e$$

Example 2. $A_7 = \{UU, UT, UF, TU, FU, TT, FF\}$
is a subalgebra of C^2.

$$P = \mathrm{Hom}_{\mathcal{C}}(A_7, C) = \{\perp, \pi_1 \mathbin{\wedge_{\mathrm{w}}} \pi_2, \pi_1, \pi_2\}$$



In poset $P = \mathrm{Hom}_{\mathcal{C}}(A, C)$

"bounded above" implies "greatest lower bound".

Proposition 3. *Let A be a C-algebra and $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Hom}_{\mathcal{C}}(A, C)$. If $\varphi_1, \varphi_2 \leq \varphi_3$, then $\varphi_1 \wedge_{\mathrm{w}} \varphi_2 \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ and $\varphi_1 \wedge_{\mathrm{w}} \varphi_2 = \mathrm{glb}(\varphi_1, \varphi_2)$.*

Stone space topology on $P = \mathrm{Hom}_{\mathcal{C}}(A, C)$

subspace topology of the product topology on C^A.

Duality

$\mathcal{C}$: category of C-algebras

$\mathcal{P}$: category of posets with bottom element satisfying the condition of Proposition 3.

For $P \in \mathcal{P}$ the dual C algebra is

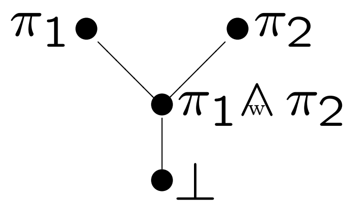
$A = \mathrm{Hom}_{\mathcal{P}}(P, C)$.

It is a subalgebra of C^P.

For finite P we have:
Theorem 4. *Let $P \in \mathcal{P}$ be finite. Then, there is a finite C-algebra A such that $P \approx \mathrm{Hom}_{\mathcal{C}}(A, C)$.*

Theorem 5. *Let A be a C-algebra, and $P = \mathrm{Hom}_{\mathcal{C}}(A, C)$. Then A embeds as a subalgebra of $\mathrm{Hom}_{\mathcal{P}}(P, C)$.*

Example 6. *For $A_7 = \{UU, UT, UF, TU, FU, TT, FF\}$ and $P = \mathrm{Hom}_{\mathcal{C}}(A_7, C)$*



$\mathrm{Hom}_{\mathcal{P}}(P, C)$ *is an 11-element algebra which contains a copy of A_7.*

$$\{UUUU, UUUT, UUUF, UUTU, UUFU, UTTT, UFFF, \\ UUTT, UUFF, UUTF, UUFT\},$$

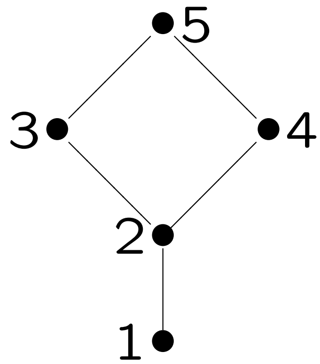
Dual equivalence obtained by adding structure on the posets side.

Additional structure includes a *partial binary operation*

domain is the subalgebra A_7 of C^2,

operation given by the *weak and*.

Example 7. *Let P be the poset given by*



Then

$$
\begin{aligned}
A &= \mathsf{Hom}_{\mathcal{P}}(P, C) \\
&= \{UUUUU, UUUUT, UUUUF, UUUTT, UUUFF, \\
&\quad UUTUT, UUFUF, UTTTT, UFFFF\}
\end{aligned}
$$

Note that $UUTTT$ and $UUFFF$ are poset morphisms, but they do not preserve $2 = \mathsf{glb}(3, 4)$, hence do not belong to A.

We have

$$\mathsf{Hom}_{\mathcal{C}}(A, C) = \{\pi_1 = \bot, \pi_2 = \pi_3 \wedge \pi_4, \pi_3, \pi_4, \pi_5\} \approx P.$$