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Geometric BVPs, Hardy spaces, and the Cauchy integral and transform on regions with corners

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Abstract

In this paper we give a new perspective on the Cauchy integral and transform and Hardy spaces for Dirac-type operators on manifolds with corners of codimension two. Instead of considering Banach or Hilbert spaces, we use polyhomogeneous functions on a geometrically "blown-up" version of the manifold called the total boundary blow-up introduced by Mazzeo and Melrose [R.R. Mazzeo, R.B. Melrose, Analytic surgery and the eta invariant, Geom. Funct. Anal. 5 (1) (1995) 14–75]. These polyhomogeneous functions are smooth everywhere on the original manifold except at the corners where they have a "Taylor series" (with possible log terms) in polar coordinates. The main application of our analysis is a complete Fredholm theory for boundary value problems of Dirac operators on manifolds with corners of codimension two.

MSC: 58J28; 58J52

Keywords: Boundary value problems; Calderón projector; Cauchy integral; Cauchy transform; Dirac-type operators; Fredholm theory; Manifolds with corners

1. Introduction

Boundary value problems for Dirac operators on manifolds with smooth boundary have a complete systematic Fredholm theory [15,20,107] developed by Calderón and Seeley. Now take a Dirac operator on a smooth manifold (e.g. $\mathbb{R}^n$) and restrict the Dirac operator to a manifold with corners inside the manifold (e.g. a polyhedral region in $\mathbb{R}^n$; cf. Section 2.1). For the restricted Dirac operator there is no complete Fredholm theory, rather each situation is handled via ad hoc

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methods (cf. [30,41,58,72,80,84,97,101,102] and many more references in the sequel). One goal of this paper is to develop a complete Fredholm theory (see Theorem 2.5) on polyhomogeneous functions when the corners are of codimension two. This Fredholm theory is developed through the Cauchy integral and transform.

The Cauchy integral (= the Poisson operator) and the Cauchy transform (= the Calderón projector) have impacted many areas of science such as analytic function theory [56], physics [60], Fourier analysis [64], several complex variables [94], representation theory [40] and of course boundary value problems [12] to name a few areas. These singular integral operators are defined on function spaces associated to boundaries of manifolds and it is well known that when the boundary is smooth, these operators preserve smooth functions; however, when the boundary is not smooth, smooth functions are not preserved. For this reason, when the boundary is not smooth, the space of smooth functions is always completed to Banach spaces (Besov, Hölder, Sobolev, etc.) where many tools are available to facilitate the analysis of the Cauchy integral and transform [17,40,81,82]. In this paper we propose a geometric alternative (Theorems 2.1 and 2.2): We analyze the Cauchy integral and transform not on functions on the original manifold but on "smooth" (polyhomogeneous) functions on a geometrically "blown-up" version of the manifold called the total boundary blow-up introduced by Mazzeo and Melrose [71]. As remarked above, we apply our analysis of the Cauchy integral and transform to develop a complete Fredholm theory for boundary value problems of Dirac operators on manifolds with corners of codimension two (Theorem 2.5)—we determine exactly when a boundary value problem is "elliptic," that is, Fredholm. To precisely state our results, we recall the prerequisite material as it was developed, much of which we later generalize to the context in which we work.

1.1. Complex analysis and the Cauchy integral

We begin our introduction with a review of some well-known properties of the Cauchy integral (see [110] for more history). Let $X \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be a smooth two-dimensional compact manifold with boundary in the complex plane where we orient the boundary in the usual counterclockwise manner. Then by Cauchy's integral formula [12,27] we know that if f is a holomorphic function on X, then

$$f(x) = \frac{i}{2\pi} \int\limits_{\partial X} \frac{f(y)}{x-y}\, dy \quad \text{for all } x \in \mathring{X} \text{ (the interior of } X\text{)}.$$

The right-hand side of this equality defines a continuous map

$$\mathcal{K} : C^\infty(\partial X) \to C^\infty(X)$$

called the *Cauchy integral* or *Poisson operator*

$$(\mathcal{K}\varphi)(x) := \frac{i}{2\pi} \int\limits_{\partial X} \frac{\varphi(y)}{x-y}\, dy \quad \text{for all } \varphi \in C^\infty(\partial X),$$

where $(\mathcal{K}\varphi)(x)$ is defined on ∂X by continuity from the interior of X. For generalization later on, we rewrite $\mathcal{K}$ in the following way

$$\mathcal{K}\varphi = \frac{i}{2\pi}\int_{\partial X}\frac{\varphi(y)}{x-y}\,dy = \frac{i}{2\pi}\int_{\partial X}\frac{\varphi(y)}{x-y}\frac{dy}{ds}\,ds(y)$$
$$= \frac{i}{2\pi}\int_{\partial X}\frac{\varphi(y)}{x-y}T(y)\,ds(y) = \frac{1}{2\pi}\int_{\partial X}\frac{\varphi(y)}{x-y}N(y)\,ds(y), \tag{1.1}$$

where s denotes the arclength parameter, T is the unit tangent vector to ∂X and $N = iT$ is the *inward* unit normal. By Cauchy's integral formula we know that if f is holomorphic on X, then $f = \mathcal{K}(f|_{\partial X})$, which says that $\mathcal{K}$ reproduces holomorphic functions from their boundary values. Thus, $\mathcal{K}$ can be called a "reproducing kernel." From this it follows that

$$\operatorname{ran}\mathcal{K} = \ker\big(\bar\partial : C^\infty(X) \to C^\infty(X)\big) = \text{holomorphic functions on } X.$$

The "local implies global" property of $\mathcal{K}$ (local properties of f on ∂X determines f on all of X) is a precursor to many of the topics covered in global analysis.

There is another way to write $\mathcal{K}$ that will be important in the sequel. Consider the *Cauchy–Riemann operator*

$$\bar\partial := \partial_{x_1} + i\partial_{x_2} : C^\infty\big(\mathbb{R}^2\big) \to C^\infty\big(\mathbb{R}^2\big). \tag{1.2}$$

Although this operator is not invertible, it does have an inverse on compactly supported functions:

$$\bar\partial^{-1}u := \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{u(y)}{x-y}\,dy \quad \text{for all } u \in C_c^\infty\big(\mathbb{R}^2\big).$$

Then with this definition, we have $\bar\partial \circ \bar\partial^{-1} = \bar\partial^{-1} \circ \bar\partial = \mathrm{Id}$ on $C_c^\infty(\mathbb{R}^2)$. Then the last expression in (1.1) shows that

$$\mathcal{K}\varphi = \bar\partial^{-1}(\delta_{\partial X}N\varphi) \quad \text{for all } \varphi \in C^\infty(\partial X), \tag{1.3}$$

where $\delta_{\partial X}$ is the delta function concentrated on ∂X and where the image of $\bar\partial^{-1}$ is initially restricted to $\mathring{X}$ and then extended to ∂X by continuity. This formula can be used to generalize the Cauchy integral operator to any differential operator with an inverse on compactly supported functions.

An operator closely related to the Cauchy integral operator is the *Cauchy transform* or *Calderón projector*

$$\mathcal{C} : C^\infty(\partial X) \to C^\infty(\partial X)$$

defined as

$$\mathcal{C}\varphi := (\mathcal{K}\varphi)|_{\partial X} \quad \text{for all } \varphi \in C^\infty(\partial X).$$

If we define the *Hardy* or *Cauchy-data space* of $\bar\partial$ over X by

$$\mathcal{H}(\bar\partial) := \big\{\phi|_{\partial X} \;\big|\; \phi \in C^\infty(X),\ \bar\partial\phi = 0\big\},$$

then it is well known (and easy to prove) that $\mathcal{C}$ has the following properties: $\mathcal{C}^2 = \mathcal{C}$, $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}(\bar{\partial})$, and $\operatorname{ran}\mathcal{C} = \mathcal{H}(\bar{\partial})$. Here are the main properties of the Cauchy integral operator and Cauchy transform:

$$\mathcal{K}: C^\infty(\partial X) \to C^\infty(X), \qquad \operatorname{ran}\mathcal{K} = \ker\bigl(\bar{\partial}: C^\infty(X) \to C^\infty(X)\bigr)$$

and

$$\mathcal{C}: C^\infty(\partial X) \to C^\infty(\partial X), \qquad \mathcal{C}^2 = \mathcal{C}, \qquad \mathcal{C} = \mathrm{Id} \quad \text{on } \mathcal{H}(\bar{\partial}), \qquad \operatorname{ran}\mathcal{C} = \mathcal{H}(\bar{\partial}).$$

1.2. Dirac-type operators

Let (M, g) be a smooth Riemannian manifold (not necessarily compact) without boundary and let E, F be vector bundles over M. A *Dirac-type operator*

$$\mathcal{D}: C^\infty(M, E) \to C^\infty(M, F)$$

is an elliptic first-order differential operator mapping sections of E to sections of F such that $\sigma(\mathcal{D}^*\mathcal{D}) = g$ where σ denotes principal symbol. Some of the main examples include the original (Riemannian version of the) Dirac operator of P.A.M. Dirac [34], the Hodge–Dirac operator on differential forms [31], the Darboux–Dirac operator on anti-holomorphic differential forms [52], and of course the spin and spin$^{\mathbb{C}}$ Dirac operators [8,46]; see Section 3.1 for more examples. An example that is a "hot" topic nowadays is Clifford analysis, which we shall cover in Section 3.2.

For a Dirac-type operator $\mathcal{D}$ on a (not necessarily compact) smooth Riemannian manifold M, motivated by the Cauchy–Riemann operator described earlier we shall assume that $\mathcal{D}$ has an inverse $\mathcal{D}^{-1}$ on compactly supported sections. Thus, our working assumptions are

$$\text{(I)} \quad \mathcal{D}: C^\infty(M, E) \to C^\infty(M, F) \quad \text{is of Dirac-type} \tag{1.4}$$

and there is an operator

$$\begin{aligned} \text{(II)} \quad & \mathcal{D}^{-1}: C_c^\infty(M, F) \to C^\infty(M, E) \quad \text{with} \\ & \mathcal{D} \circ \mathcal{D}^{-1} = \mathrm{Id} \quad \text{on } C_c^\infty(M, F) \quad \text{and} \quad \mathcal{D}^{-1} \circ \mathcal{D} = \mathrm{Id} \quad \text{on } C_c^\infty(M, E). \end{aligned} \tag{1.5}$$

In practice, we shall be working on a neighborhood of a compact manifold $X \subseteq M$ with boundary (or even corners) inside of M; then an *arbitrary* Dirac-type operator can be made invertible on a neighborhood of X via the invertible double construction of Wojciechowski [15,114]. Thus, the assumption (1.5) is not restrictive.

Let X be a compact manifold with smooth boundary in M with $\dim X = \dim M$. In analogy with (1.3), we define the *Cauchy integral* or *Poisson operator* by

$$\mathcal{K}\varphi = \mathcal{D}^{-1}(\delta_{\partial X} G\varphi) \quad \text{for all } \varphi \in C^\infty(\partial X, E), \tag{1.6}$$

where $\delta_{\partial X}$ is the delta function of the boundary, $G: E \to F$ is the principal symbol of $\mathcal{D}$ evaluated on the inward pointing unit normal vector field to ∂X, and where the image of $\mathcal{D}^{-1}$ is initially restricted to $\mathring{X}$ and then extended to ∂X by continuity.

In fact, a deep result of Calderón and Seeley [20,107] states that

$$\mathcal{K}: C^\infty(\partial X, E) \to C^\infty(X, E) \quad \text{and} \quad \operatorname{ran}\mathcal{K} = \ker\bigl(\mathcal{D}: C^\infty(X, E) \to C^\infty(X, F)\bigr). \tag{1.7}$$

The *Cauchy transform* or *Calderón projector* is the operator

$$\mathcal{C}: C^\infty(\partial X, E) \to C^\infty(\partial X, E)$$

defined by

$$\mathcal{C}\varphi := (\mathcal{K}\varphi)|_{\partial X} \quad \text{for all } \varphi \in C^\infty(\partial X, E). \tag{1.8}$$

If we define the *Hardy* or *Cauchy-data space* of $\mathcal{D}$ over X by

$$\mathcal{H}(\mathcal{D}) := \bigl\{\phi|_{\partial X} \bigm| \phi \in C^\infty(X, E),\ \mathcal{D}\phi = 0\bigr\},$$

then by works of Calderón and Seeley [20,107], the following properties hold:

$$\begin{gathered} \mathcal{C}: C^\infty(\partial X, E) \to C^\infty(\partial X, E), \\ \mathcal{C}^2 = \mathcal{C}, \qquad \mathcal{C} = \mathrm{Id} \quad \text{on } \mathcal{H}(\mathcal{D}), \qquad \operatorname{ran}\mathcal{C} = \mathcal{H}(\mathcal{D}). \end{gathered} \tag{1.9}$$

There is a complete Fredholm theory for boundary value problems involving the Calderón projector: If $\mathcal{P}$ is any pseudodifferential projection on $C^\infty(\partial X, E)$, define

$$\mathcal{D}_{\mathcal{P}} := \mathcal{D}: \operatorname{dom}(\mathcal{D}_{\mathcal{P}}) \to C^\infty(X, F),$$

where

$$\operatorname{dom}(\mathcal{D}_{\mathcal{P}}) := \bigl\{\phi \in C^\infty(X, E) \bigm| \mathcal{P}(\phi|_{\partial X}) = 0\bigr\},$$

then the following elegant Fredholm criterion holds [15]:

$$\begin{gathered} \mathcal{D}_{\mathcal{P}} \text{ is Fredholm if and only if } \mathcal{P}\mathcal{C}: \mathcal{H}(\mathcal{D}) \to \operatorname{ran}\mathcal{P} \text{ is Fredholm, in which case} \\ \operatorname{ind}\mathcal{D}_{\mathcal{P}} = \operatorname{ind}\bigl(\mathcal{P}\mathcal{C}: \mathcal{H}(\mathcal{D}) \to \operatorname{ran}\mathcal{P}\bigr). \end{gathered} \tag{1.10}$$

2. Goal of this paper and statement of main results

The goal of this paper is to extend the results (1.7), (1.9), and (1.10) for Dirac-type operators on manifolds with boundary to the case when X has corners.

2.1. Manifolds with corners

Specifically, we are interested in the case of a *manifold with corners of codimension two* $X \subseteq M$. This means that if $n = \dim M$, then X is locally diffeomorphic to either $\mathbb{R}^n$ (the interior points), $[0, 1) \times \mathbb{R}^{n-1}$ (near the boundary but away from the corners), or $[0, 1)^2 \times \mathbb{R}^{n-2}$ (near the corners). See Fig. 1 for a couple of examples. It turns out that the results (1.7), (1.9), and (1.10) are *not true* when the smoothness of X is relaxed. Indeed, define $C^\infty(X) := C^\infty(M)|_X$ and $C^\infty(\partial X) := C^\infty(M)|_{\partial X}$, and consider, for example, the case of $X \subseteq \mathbb{R}^2$ (a "tear drop" of angle $0 < \theta_0 < 2\pi$) given in Fig. 2.

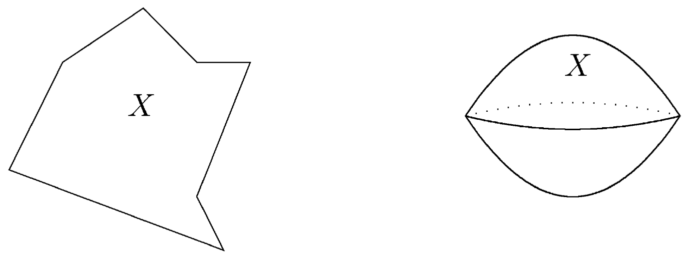

Fig. 1. Examples of manifolds with corners of codimension two; the left-hand manifold in $\mathbb{R}^2$ and the right-hand manifold in $\mathbb{R}^3$. The right manifold is a solid "flattened sphere" in $\mathbb{R}^3$. Around the edge, this manifold is diffeomorphic to $[0,1)^2 \times \mathbb{S}^1$.

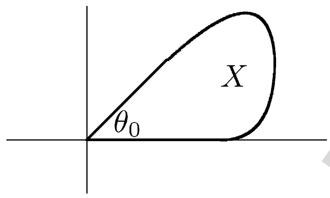

Fig. 2. A manifold with a corner in the plane.

Example 1. If $(\mathcal{K}\varphi)(x) := \frac{i}{2\pi}\int_{\partial X} \frac{\varphi(y)}{x-y}\,dy$, then $\mathcal{K} : C^\infty(\partial X) \to C^\infty(X)$ if and only if $\theta_0 = \pi$; that is, if and only if ∂X is smooth.

Sufficiency is just Calderón–Seeley and to prove necessity, consider $\varphi(x_1, x_2) = x_1$, which is a perfectly infinitely differentiable function. Then modulo a smooth function, for $x \in \mathring{X}$ (the interior of X) in terms of polar coordinates we can express $(\mathcal{K}\varphi)(x)$ as an integral over the rays $y = \rho$, $y = \rho e^{i\theta_0}$ for $0 \leqslant \rho \leqslant 1$:

$$\int_{\partial X} \frac{y_1}{x-y}\,dy \equiv \int_0^1 \frac{\rho}{x-\rho}\,d\rho - \int_0^1 \frac{\rho\cos\theta_0}{x-\rho e^{i\theta_0}} e^{i\theta_0}\,d\rho.$$

The integrals on the right can be evaluated explicitly using the partial fractions formula $\frac{a}{x-a} = \frac{x}{x-a} - 1$, and we find that modulo a smooth function at $x = 0$,

$$\mathcal{K}\varphi = \int_{\partial X} \frac{y_1}{x-y}\,dy \equiv \left(1 - \cos\theta_0 e^{-i\theta_0}\right) x \log(-x), \tag{2.1}$$

where $\log(-x)$ is the principal logarithm defined on X; note that if $x \in \mathring{X}$, then $-x$ has argument in $(-\pi, \pi)$. In particular, in general we have a logarithmic singularity at $x = 0$. Of course, the only angle for which this singularity vanishes is $\theta_0 = \pi$; in this case, the sector is not really a sector, it is flat at the origin and hence ∂X is actually smooth.

This simple example shows that in the general case, in order for the results of Calderón and Seeley (1.7), (1.9) to be valid, one needs to use spaces which are strictly larger than C^∞; in other words, C^∞ is *too small*. Common choices of function spaces include the familiar Banach spaces: L^p spaces, Hölder spaces, Besov spaces, and Sobolev spaces, where the formulas (1.7), (1.9) do hold in appropriate spaces. We remark that the proofs of boundedness are nontrivial because in the presence of corners, the operators $\mathcal{K}$ and $\mathcal{C}$ are *not* pseudodifferential but at best are singular integral operators [111] (when the boundary is C^1 some C^∞ techniques can still be employed [38,96], but when the boundary is Lipschitz as in our case, the C^∞ techniques breakdown). The fact that the standard Cauchy transform in the plane is bounded on $L^p(\partial X)$ was proved by Coifman, McIntosh, and Meyer [26] (cf. [25]) generalizing a result of Calderón [21]. This was later generalized by Murray [87] and then by McIntosh [74] to the Clifford–Dirac case in $\mathbb{R}^{n+1}$, which has many applications to the growing field of Clifford analysis; see for example [14,54, 61–63]. Finally, the Clifford–Dirac case was later generalized to the full Dirac operator case on a Riemannian manifold by Mitrea [69,83,84]. See also [1–6,9,49,65,112] and the books [17,41, 55,81,82,93] for related results and more references. The methods used to prove boundedness on these Banach spaces work for Lipschitz domains and use, amongst other methods, singular harmonic measures (e.g. the Carleson measure), elliptic estimates, the Calderón rotation method (or commutator method), variational principles, a priori estimates, Clifford or Haar wavelets systems, and Littlewood–Paley theory.

2.2. A geometric "smooth function" approach via blow-up

The goal of this paper is to develop a "smooth" theory for the Poisson operator and Calderón projector on a manifold with corners. There are two obstructions to this goal: (1) We have to define what "smooth" means because we already know that (1.7), (1.9) fail if "smooth" is meant in the usual sense; (2) None of the techniques mentioned earlier (singular harmonic measures, etc.) that work for Banach spaces can be used for "smooth" functions so the standard approaches cannot be applied.

The idea of "smoothness" comes from a closer analysis of Example 1, in particular, the formula (2.1). Observe that if we introduce polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ in (2.1), then modulo a smooth function, for $\varphi(x_1, x_2) = x_1$, we can write

$$\mathcal{K}\varphi = \int_{\partial X} \frac{\varphi(y)}{x-y}\,dy \equiv f(\theta) r\log(r) + r g(\theta), \tag{2.2}$$

where $f, g \in C^\infty([0, \theta_0])$ (there are formulas for f, g but these are not important). Geometrically, the introduction of polar coordinates radially "blows-up" the corner as seen in Fig. 3; see Melrose [78,79] for the general notion. The manifold obtained by using polar coordinates near the original corner instead of the original rectangular coordinates is called the *total boundary blow-up of* X, denoted by X_{tb}, and was introduced by Mazzeo and Melrose in [71].

In view of (2.2) we shall define "smoothness" as functions that are smooth on the interior of X, smooth up to the boundary hypersurfaces of X away from the corners, and which near the corners look like (2.2) in polar coordinates. To make this precise, for a general manifold X with corners of codimension two, we first form the new manifold X_{tb} by introducing polar coordinates near the corners of X; see Fig. 4 for a couple of examples. Thus, if $Y \subseteq X$ is a corner, then writing

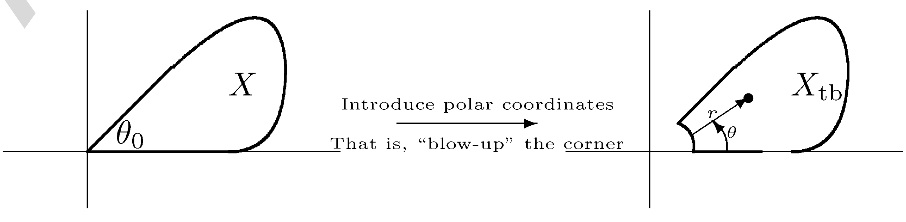

Fig. 3. "Blowing-up" the corner (introducing polar coordinates) forms the manifold X_{tb}.

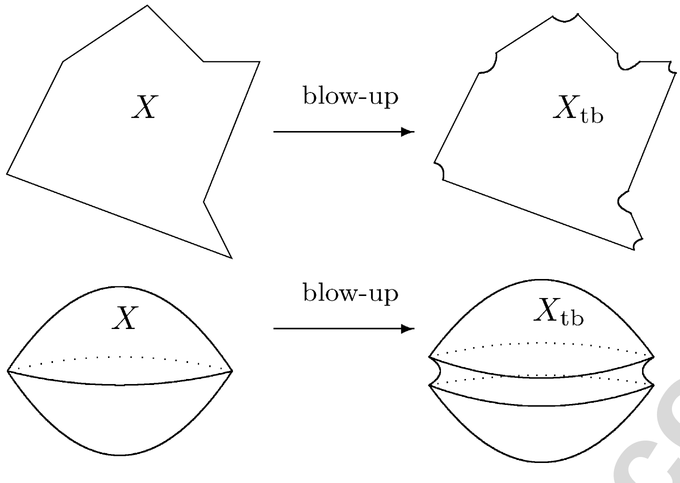

Fig. 4. Two examples (a polygonal region in the plane and a solid squashed sphere) showing X_{tb}, the total boundary blow-up, which is obtained by blowing up the corners.

$$X \cong [0,1)_{x_1} \times [0,1)_{x_2} \times Y \tag{2.3}$$

near Y, introducing polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, we have

$$X_{\mathrm{tb}} \cong [0,\varepsilon)_r \times \left[0, \frac{\pi}{2}\right]_\theta \times Y, \quad \varepsilon > 0. \tag{2.4}$$

(Note that $\cong$ in (2.3) means "diffeomorphic to" *not* "isometric to"; in other words, the hypersurfaces intersecting that Y do not have to intersect at 90°, they can intersect at any angle, but any angle in $\mathbb{R}^2$ is diffeomorphic to the standard upper right quadrant.)

For each corner Y in X, let $\overline{\mathcal{I}}_Y \subseteq \mathbb{C} \times \mathbb{N}_0$ be a discrete subset with $\mathbb{N}_0 = \{0,1,2,\ldots\}$; see Sections 4 and 5.3 and the definitions (5.9), (4.16) for thorough discussions of the properties of such a set, called an index set (the bar over $\overline{\mathcal{I}}_Y$ means that this index set is "completed" in a certain sense). Let $\overline{\mathcal{I}} = \{\overline{\mathcal{I}}_Y \mid Y \text{ is a corner of } X\}$. We define the *polyhomogeneous space* of functions $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}})$ as the set of all functions ϕ that are C^∞ on the interior of X, C^∞ up to the interior of every boundary hypersurface of X, and if $Y \subseteq X$ is a corner of X, then in polar coordinates, (r,θ,y) near the corner as in (2.4), we have

$$\phi \sim \sum_{(\alpha,k)\in\overline{\mathcal{I}}_Y} r^\alpha (\log r)^k \psi_{\alpha,k}(\theta,y), \tag{2.5}$$

where $\psi_{\alpha,k}(\theta,y) \in C^\infty([0,\frac{\pi}{2}] \times Y)$. The meaning of "$\sim$" basically means that for any $N \in \mathbb{N}$,

$$\phi = \sum_{(\alpha,k)\in\overline{\mathcal{I}}_Y,\, \Re\alpha \leqslant N} r^\alpha (\log r)^k \psi_{\alpha,k}(\theta,y) + r^N \phi_N(r,\theta,y), \tag{2.6}$$

where $\phi_N(r,\theta,y)$ is continuous in $r \in [0,\varepsilon)$ and smooth in $(\theta,y) \in [0,\frac{\pi}{2}] \times Y$; see Section 4 for a precise definition. We define $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X) := (\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}))|_{\partial X}$, where $|_{\partial X}$ means restriction to the interior of each boundary hypersurface of X; this makes perfect sense since elements in $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}})$ are C^∞ (in particular, continuous) up to the interior each boundary hypersurface of X. The spaces $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}})$ and $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X)$ have natural topologies explained easiest by convergence properties; for example, if $\{\phi_j\}$ is a sequence in $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}})$, then $\phi_j \to \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}})$ means that $\phi_j \to \phi$ in the C^∞ topology away from the blown-up corners and near the blown-up corners, the coefficients of ϕ_j in the expansion (2.6) (for any N) converge to the corresponding coefficients of ϕ; see Section 5.3 for the precise topology on $\mathcal{A}^{\overline{\mathcal{I}}}$.

Such "polyhomogeneous functions" or "functions of asymptotic type" have occurred in a variety of contexts involving manifolds with singularities; see for example, Callias [22], Kondrat'ev [57], Maz'ja and Plamenevskiĭ [70], Mazzeo [72,73], Melrose and Mendoza [80], Rempel and Schulze [92], Schulze [100–102], and references therein. Recall our assumptions:

$$\text{(I)}\quad \mathcal{D}: C^\infty(M,E) \to C^\infty(M,F) \quad \text{is of Dirac-type}$$

over a smooth Riemannian manifold M and (which can always be achieved on a neighborhood of any compact manifold with corners in M) there is an operator (II)

$$\begin{aligned}\text{(II)}\quad & \mathcal{D}^{-1}: C_c^\infty(M,F) \to C^\infty(M,E) \quad \text{with}\\ & \mathcal{D}\circ\mathcal{D}^{-1} = \mathrm{Id} \quad \text{on } C_c^\infty(M,F) \quad \text{and} \quad \mathcal{D}^{-1}\circ\mathcal{D} = \mathrm{Id} \quad \text{on } C_c^\infty(M,E).\end{aligned}$$

Writing the Dirac operator in polar coordinates near any corner of the manifold with corners X one can show that

$$\mathcal{D}: \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F).$$

As a side note, in Theorem 5.4 we prove that this map is *surjective* for any $\overline{\mathcal{I}} > -1$, where $\overline{\mathcal{I}} > -1$ just means that all the powers α in (2.4) are strictly larger than -1. In the following theorem we extend the Calderón–Seeley result (1.7) to corners.

Theorem 2.1. *Let X be a compact manifold with corners of codimension two in M with* $\dim X = \dim M$ *and fix any index family $\overline{\mathcal{I}} > -1$. Then the Poisson operator defined by* (1.6) *extends to a continuous linear map*

$$\mathcal{K}: \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$$

such that

$$\operatorname{ran}\mathcal{K} = \ker\bigl(\mathcal{D}: \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)\bigr).$$

Furthermore, $\mathcal{K}$ reproduces harmonic functions as seen in Theorem 2.4 *below.*

The condition $\overline{\mathcal{I}} > -1$ guarantees that if $\varphi \in \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$, then $\varphi \in L^1(\partial X, E)$, so $\mathcal{K}\varphi$ is at least defined as an L^1 function [69,83,84]; the point of the theorem is that $\mathcal{K}\varphi$ has the same structure as φ itself. Define the Cauchy–Hardy space by

$$\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) := \bigl\{\phi|_{\partial X} \bigm| \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E),\ \mathcal{D}\phi = 0\bigr\},$$

where $\phi|_{\partial X}$ really means to restrict ϕ to the interior of each boundary hypersurface of X. We now extend the Calderón–Seeley result (1.9) to corners.

Theorem 2.2. *Let X be a compact manifold with corners of codimension two in M with* $\dim X = \dim M$ *and fix any index family $\overline{\mathcal{I}} > -1$. Then the Calderón projector given by* (1.8) *defines a continuous linear projector*

$$\mathcal{C}: \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$$

whose range is exactly the Cauchy–Hardy space $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$. In other words,

$$\mathcal{C}^2 = \mathcal{C}, \qquad \mathcal{C} = \mathrm{Id} \quad \text{on } \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}), \quad \text{and} \quad \operatorname{ran} \mathcal{C} = \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}).$$

For the specific example $M = \mathbb{R}^{n+1}$, $E = F = C\ell(\mathbb{R}^n)$, the Clifford algebra, and $\mathcal{D}$ the Clifford–Dirac operator considered in Section 3.2, the scalar part of

$$\mathcal{K}: \mathcal{A}^{\overline{\mathcal{I}}}\big(\partial X, C\ell\big(\mathbb{R}^n\big)\big) \to \mathcal{A}^{\overline{\mathcal{I}}}\big(X_{\mathrm{tb}}, C\ell\big(\mathbb{R}^n\big)\big)$$

is exactly the *double-layer potential operator* for the Laplacian on X_{tb} and the scalar part of

$$2\mathcal{C} - \mathrm{Id}: \mathcal{A}^{\overline{\mathcal{I}}}\big(\partial X, C\ell\big(\mathbb{R}^n\big)\big) \to \mathcal{A}^{\overline{\mathcal{I}}}\big(\partial X, C\ell\big(\mathbb{R}^n\big)\big)$$

is exactly the *singular double-layer potential* for the Laplacian on X_{tb}; see Eq. (3.4) (cf. [26,82]). Therefore, Theorem 2.2 immediately implies the following.

Corollary 2.3. *Let $X \subseteq \mathbb{R}^{n+1}$ be a compact manifold with corners of codimension two with* $\dim X = n + 1$ *and fix any index family $\overline{\mathcal{I}} > -1$. Then the double-layer potential defines a continuous linear map from $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, C\ell(\mathbb{R}^n))$ to $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, C\ell(\mathbb{R}^n))$ and the singular double-layer potential operator defines a continuous linear map on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, C\ell(\mathbb{R}^n))$.*

In a future paper we hope to use this corollary to prove existence of solutions on *polyhomogeneous spaces* to the Dirichlet and Neumann problems of the Laplacian on a manifold with corners of codimension two. This of course has already been done on Banach spaces, see for example the work of Dahlberg [29], Jerison and Kenig [50], Verchota [113]. See [75,82], for proofs of the existence of solutions to Laplace's equation on Banach spaces using Clifford analysis techniques.

Before discussing Fredholm properties of Dirac operators, we generalize Cauchy's theorem to polyhomogeneous spaces on manifolds with corners. Let M_{tb} denote the manifold obtained by blowing up the corners of X in M, then blowing up the hypersurfaces of X in M as seen in Fig. 5. In the following theorem, $\mathcal{K}$ denotes the "full" Poisson operator, which means we initially restrict the image of $\mathcal{D}^{-1}$ in Eq. (1.6) to $M \setminus \partial X$.

Theorem 2.4. *For $\overline{\mathcal{I}} > -1$, the Poisson operator defines a continuous linear map*

$$\mathcal{K}: \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(M_{\mathrm{tb}}, E).$$

Moreover, if $\mathcal{D}\phi = 0$, then Cauchy's formula holds

$$(\mathcal{K}\varphi)(x) = \begin{cases} \phi(x), & x \in X_{\mathrm{tb}}, \\ 0, & x \in M_{\mathrm{tb}} \setminus X_{\mathrm{tb}}. \end{cases}$$

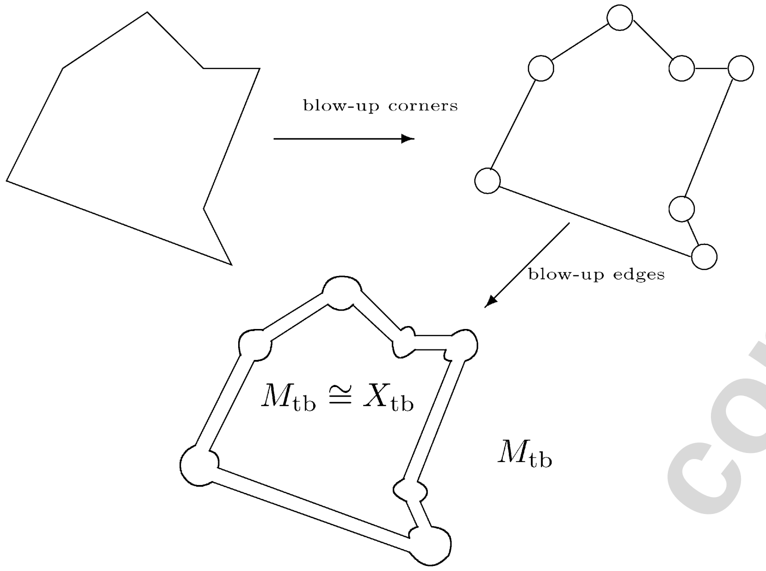

Fig. 5. Blowing up the corners then blowing up the edges forms M_{tb}. Here, M_{tb} is given by the $\mathbb{R}^2\setminus$ the "hallways ∪ bubbles" and consists of two connected components. Note that the bounded component is diffeomorphic to X_{tb}.

Note that in Theorem 2.1, we were really talking about the restriction of the "full" Poisson operator to the subset $X_{\mathrm{tb}} \subseteq M_{\mathrm{tb}}$.

2.3. A complete Fredholm theory

It is well known (see the discussion by Mitrea [84, p. 209]) that there does not exist a general notion of *regular elliptic problem* for a Dirac operator on a manifold with corners but instead each specific problem must be dealt with via ad hoc methods; cf. also the books [30,41,58,101,102]. However, using the results in this paper, we can give a complete and general notion of regular elliptic BVP on polyhomogeneous spaces.

Let $\mathcal{P}$ be a (not necessarily continuous) projection on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$; that is, $\mathcal{P}$ is a linear map on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$ with $\mathcal{P}^2 = \mathcal{P}$. Consider the operator

$$\mathcal{D}_{\mathcal{P}} := \mathcal{D} : \mathrm{dom}(\mathcal{D}_{\mathcal{P}}) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F), \tag{2.7}$$

where

$$\mathrm{dom}(\mathcal{D}_{\mathcal{P}}) := \big\{\phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \,\big|\, \mathcal{P}(\phi|_{\partial X}) = 0\big\}.$$

In the following theorem we give a *complete* characterization of boundary value problems (2.7) that are Fredholm.

Theorem 2.5 *(Complete Fredholm theory). For $\overline{\mathcal{I}} > -1$ and an arbitrary projection $\mathcal{P}$ on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$, the operator*

$$\mathcal{D}_{\mathcal{P}} : \mathrm{dom}(\mathcal{D}_{\mathcal{P}}) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)$$

is Fredholm; that is, has a finite-dimensional kernel and cokernel, if and only if the operator

$$\mathcal{PC}: \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) \to \operatorname{ran} \mathcal{P}$$

is Fredholm, in which case

$$\operatorname{ind} \mathcal{D}_{\mathcal{P}} = \operatorname{ind}\big(\mathcal{PC}: \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) \to \operatorname{ran} \mathcal{P}\big).$$

In fact, $\mathcal{D}_{\mathcal{P}}$ and $\mathcal{PC}$ have isomorphic kernels and cokernels.

Remark 2.6. Therefore, for a projection $\mathcal{P}$ on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$ defining a boundary value problem $\mathcal{D}_{\mathcal{P}}$ for the Dirac operator $\mathcal{D}$, it makes sense to *define* this boundary value problem to be *elliptic* if and only if $\mathcal{PC}: \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) \to \operatorname{ran} \mathcal{P}$ is Fredholm.

Finally, we remark that in a future paper, we hope to discuss the pseudodifferential nature of the Cauchy integral and transform as an element of a Boutet de Monvel [16] type calculus. See [42,51,53,59,98,103,104] for more on the Boutet de Monvel calculus on manifolds with boundary and other singular manifolds. In our case, the Boutet de Monvel calculus is related to pseudodifferential calculi on manifolds with edges; see the works of Schulze and collaborators [59,89,97,99,106], especially *without* the transmission condition [105]; more specifically, the calculus is defined on the union of the hypersurfaces of a manifold with corners with special compatibility conditions between the hypersurfaces.

The rest of this paper is structured as follows. We begin in Section 3 by discussing and providing examples of Dirac-type operators. Next, in Section 4 we give the necessary background on manifolds with corners, blow-ups, and polyhomogeneous spaces. We also study the mapping properties of various transforms (Fourier, Laplace, etc.) on polyhomogeneous spaces. The main results in this paper rely heavily on the mapping properties of pseudodifferential operators on polyhomogeneous spaces. We study such mapping properties in Sections 5 and 6. In Section 7 we apply the results of the previous sections to prove our main results on the Cauchy integral and transform. Finally, in Section 8 we prove Theorem 2.5 on Fredholm boundary value problems for Dirac operators on polyhomogeneous spaces.

3. Dirac operators, Clifford algebras, and Clifford analysis

This section serves as background to Dirac operators. We especially focus on the Clifford–Dirac operator since this operator demands little prerequisites to define (unlike, for example, the spin Dirac operator [8,46]) and this operator plays a large rôle in the increasingly important and growing field of Clifford analysis [95].

3.1. Dirac operators

For vector bundles E and F over a Riemannian manifold M, a first-order differential operator

$$\mathcal{D}: C^{\infty}(M, E) \to C^{\infty}(M, F)$$

is said to be of *Dirac-type* if $\mathcal{D}$ is elliptic and the principal symbol of $\mathcal{D}^*\mathcal{D}$ is the metric. Here are some common Dirac-type operators.

(1) *The Cauchy–Riemann operator.* The most elementary Dirac operator is the Cauchy–Riemann operator $\overline{\partial} = \partial_{x_1} + i\,\partial_{x_2}$ already discussed in the introduction.
(2) *The Hodge–Dirac operator [31].* If d^* is the formal adjoint of the exterior derivative $d : C^\infty(M, \Lambda) \to C^\infty(M, \Lambda)$ acting on smooth forms on M, then $\mathcal{D} := d + d^*$ is a Dirac-type operator.
(3) *The $\overline{\partial}$-operator.* If M is a complex manifold and $\overline{\partial}^*$ is the formal adjoint of the Cauchy–Riemann operator $\overline{\partial} : C^\infty(M, \Lambda^{0,k}) \to C^\infty(M, \Lambda^{0,k+1})$ acting on smooth $0, k$ forms on M, then $\mathcal{D} := \overline{\partial} + \overline{\partial}^*$ is a Dirac-type operator. This operator plays a key rôle in the Riemann–Roch–Hirzebruch theorem [13].
(4) *The quaternionic Dirac operator.* Let $\mathbb{H}$ denote the quaternions; thus, if i, j, k denote the imaginary units of $\mathbb{H}$, then $i^2 = j^2 = k^2 = ijk = -1$, $ij = k$, $jk = i$, $ki = j$, and multiplication is anticommutative. The operator, studied by Fueter [39],

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} : C^\infty\big(\mathbb{R}^4, \mathbb{H}\big) \to C^\infty\big(\mathbb{R}^4, \mathbb{H}\big)$$

is a Dirac-type operator. Here, the inner product on $\mathbb{H}$ is $a\overline{a}$ with $\overline{a}$ denoting the conjugate of a. Then

$$\mathcal{D}^* = -\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3},$$

so $\mathcal{D}^*\mathcal{D} = -\sum_{\ell=0}^{3} \partial_{x_\ell}^2 = \Delta$ and therefore $\mathcal{D}$ is indeed a Dirac-type operator.
(5) *The octonionic Dirac operator.* Here, $\mathbb{O}$, the octonions or Cayley algebra, is the only nonassociative division algebra with real scalars and is showing much promise as a framework for studying the fundamental particles [10,35,36,44,91]. An element of this algebra is a sum

$$\xi = a_0 + a_1 i + a_2 j + a_3 k + a_4 k\ell + a_5 j\ell + a_6 i\ell + a_7 \ell,$$

where each basis element by definition squares to -1. Therefore, $\mathbb{O}$ has seven imaginary units. Multiplication of these imaginary units is governed by Fig. 6. Here, each straight line and the middle circle (joining three imaginary units) is to be thought of as a circle with the orientation of the line governed by the arrow on the line. Thus, the octonionic imaginary units are grouped into seven quaternionic subalgebras given by the lines and the middle circle. For example, the bottom line represents the multiplication $(i\ell)k = j\ell$, $k(j\ell) = i\ell$,

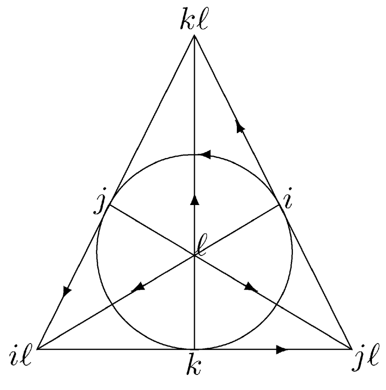

Fig. 6. A pictorial of the octonions adapted from Dray and Manogue [36].

and $(j\ell)(i\ell) = k$ and their anticommutators. Conjugation of ξ, $\bar{\xi}$, is the element defined by switching the sign of each imaginary unit. The norm of an octonian ξ is defined by

$$|\xi|^2 := \xi\bar{\xi} = a_0^2 + a_1^2 + \cdots + a_7^2.$$

One of the several types of Dirac operators associated to the octonions is the operator

$$\mathcal{D} : C^\infty(\mathbb{R}^8, \mathbb{O}) \to C^\infty(\mathbb{R}^8, \mathbb{O})$$

defined by the formula

$$\mathcal{D} := \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + \cdots + \ell\frac{\partial}{\partial x_7},$$

which is easily checked to be of Dirac-type.

3.2. The Clifford–Dirac operator

The Clifford algebra on Euclidean space and its corresponding Dirac operator are higher-dimensional analogs of the complex numbers and the Cauchy–Riemann operator; see [33] for a detailed account. For $n \geqslant 1$, recall that $C\ell(\mathbb{R}^n)$, introduced in 1878 by Clifford [23], is the $\mathbb{R}$-algebra generated by $n+1$ independent vectors $1, e_1, \ldots, e_n$ with $1 \in \mathbb{R}$ the multiplicative identity, governed by the rules

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad \text{for all } i, j = 1, \ldots, n. \tag{3.1}$$

If $e_0 := 1$, then we can consider $\mathbb{R}^{n+1} \subseteq C\ell(\mathbb{R}^n)$ as the span of $\{e_0, e_1, \ldots, e_n\}$, and if we put $e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$ for any increasing list $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, then e_0 together with all the e_I's form a basis of $C\ell(\mathbb{R}^n)$. Thus, any element $a \in C\ell(\mathbb{R}^n)$ can be written in the form

$$a = \sum_{|I| \geqslant 0} a_I e_I, \tag{3.2}$$

where $|I| = i_1 + \cdots + i_k$. In particular, $C\ell(\mathbb{R}^n)$ is 2^n-dimensional. Note that if $n = 1$, then the identity (3.1) is just $e_1^2 = -1$. Hence, identifying e_1 with the complex number i, $C\ell(\mathbb{R}^1)$ is simply the complex numbers $\mathbb{C}$. Similarly, $C\ell(\mathbb{R}^2) \equiv \mathbb{H}$.

We define the Dirac operator (cf. [18,86])

$$\mathcal{D} : C^\infty(\mathbb{R}^{n+1}, C\ell(\mathbb{R}^n)) \to C^\infty(\mathbb{R}^{n+1}, C\ell(\mathbb{R}^n))$$

by the formula

$$\mathcal{D} := \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}.$$

Here, $C^\infty(\mathbb{R}^{n+1}, C\ell(\mathbb{R}^n))$ is the set of all smooth Clifford algebra-valued functions $\phi : \mathbb{R}^{n+1} \to C\ell(\mathbb{R}^n)$, where smooth means that we have $\phi(x) = \sum_{|I| \geqslant 0} \phi_I(x) e_I$ with $\phi_I : \mathbb{R}^{n+1} \to \mathbb{R}$ smooth for each I. Then

$$\mathcal{D}\phi = \sum_{j=0}^{n} \sum_{|I| \geqslant 0} \frac{\partial \phi_I}{\partial x_j} e_j e_I.$$

There is another Dirac operator defined by $\mathcal{D}'\phi = \sum_{j=0}^{n} \sum_{|I| \geqslant 0} \frac{\partial \phi_I}{\partial x_j} e_I e_j$ (which is usually denoted by $\phi\mathcal{D}$ with ϕ on the left of $\mathcal{D}$), but we focus on $\mathcal{D}$ although everything stated below for $\mathcal{D}$ has an analogous statement for $\mathcal{D}'$.

If $n = 1$, and we identify e_1 with i, then $\mathcal{D} = \bar{\partial} = \partial_{x_0} + i\partial_{x_1}$, hence for general n, the operator $\mathcal{D}$ is a higher-dimensional analog of the Cauchy–Riemann operator. A function ϕ is (*left-*)*monogenic* if $\mathcal{D}\phi = 0$ and the field of *Clifford analysis* is the study of monogenic functions. Thus, Clifford analysis is a higher-dimensional analog of complex analysis. In fact, all the familiar theorems from complex analysis: Cauchy's theorem, Morera's theorem, Liouville's theorem, Weierstrass' theorem on uniform convergence, the maximum modulus principle, the mean-value theorem, and so forth, have analogs in Clifford analysis [17,19,40,90]. Much of modern-day Clifford analysis was developed, amongst others, by Delanghe [32], Eichhorn [37], Iftimie [48], and Hestenes [45]; see Ryan [95] for a recent survey, and Clifford analysis has impacted many areas of mathematics; see for example [115] for an application of Clifford analysis to analyze the full water wave equation.

To define the formal adjoint $\mathcal{D}^*$ we need an inner product on $C\ell(\mathbb{R}^n)$. Given $a \in C\ell(\mathbb{R}^n)$ written as in (3.2) we define the scalar part of a as $\mathrm{Sc}(a) := a_0$, the coefficient of e_0 in (3.2) and we define the Clifford conjugate of a as the element

$$\bar{a} := \sum_{|I| \geqslant 0} a_I \bar{e}_I, \quad \text{where } \bar{e}_I := (-1)^k e_{i_k} e_{i_{k-1}} \cdots e_2 e_1,\ \bar{e}_0 = e_0.$$

Note that $e_I \bar{e}_I = 1$ and it is easy to check that for any $a, b \in C\ell(\mathbb{R}^n)$, we have $\overline{ab} = \bar{b}\bar{a}$. We define an inner product on $C\ell(\mathbb{R}^n)$ by

$$\langle a, b \rangle := \mathrm{Sc}(a\bar{b}) = \sum_{|I| \geqslant 0} a_I b_I, \quad \text{for all } a, b \in C\ell(\mathbb{R}^n).$$

Then, for $\phi, \psi \in C_c^\infty(\mathbb{R}^{n+1}, C\ell(\mathbb{R}^n))$, we define

$$\langle \phi, \psi \rangle := \int \langle \phi(x), \psi(x) \rangle \, dx = \int \mathrm{Sc}\big(\phi(x)\overline{\psi}(x)\big)\, dx = \mathrm{Sc}\bigg(\int \phi(x)\overline{\psi}(x)\, dx\bigg).$$

A straightforward computation shows that

$$\mathcal{D}^* = -\overline{\mathcal{D}} = -\sum_{j=0}^{n} \bar{e}_j \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_0} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}.$$

Of course, this formula generalizes the identity $(\partial_{x_0} + i\partial_{x_1})^* = -\partial_{x_0} + i\partial_{x_1}$ for the Cauchy–Riemann operator. One then easily finds that

$$\mathcal{D}^*\mathcal{D} = \mathcal{D}\mathcal{D}^* = -\sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2} = \Delta.$$

Hence, $\mathcal{D}$ is a Dirac-type operator.

We now define the Cauchy integral and transforms in this setting. First, it is well known that the fundamental solution of the Laplacian Δ is the operator

$$E\phi = \int E(x-y)\phi(y)\,dy, \qquad E(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, & \text{if } n = 1, \\ -\frac{1}{\sigma_n}\frac{1}{|x|^{n-1}}, & \text{if } n > 1, \end{cases}$$

where $\sigma_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ is the area of $\mathbb{S}^n$, and which satisfies $E\Delta = \Delta E = \mathrm{Id}$ on Schwartz functions. It is then straightforward to check that if

$$e(x) := \mathcal{D}^* E(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}}$$

with $\overline{x} = x_0 - e_1x_1 - \cdots - e_nx_n$, then on Schwartz functions, we have

$$\mathcal{D}^{-1}\phi = \int e(x-y)\phi(y)\,dy = \frac{1}{\sigma_n}\int_{\mathbb{R}^{n+1}} \frac{\overline{x-y}}{|x-y|^{n+1}}\phi(y)\,dy, \tag{3.3}$$

in the sense that $\mathcal{D}\circ\mathcal{D}^{-1}\phi = \mathcal{D}^{-1}\circ\mathcal{D}\phi = \phi$ for all Schwartz functions ϕ. Note that when $n = 1$, this formulas reduces to

$$\mathcal{D}^{-1}\phi = \frac{1}{2\pi}\int \frac{1}{x-y}\phi(y)\,dy_0\,dy_1.$$

In terms of the complex number $w = y_0 + iy_1$, we have $dw \wedge \overline{dw} = -2i\,dy_0 \wedge dy_1$, therefore

$$\overline{\partial}^{-1}\phi(z) = \frac{1}{4\pi i}\int \frac{\phi(w)}{w-z}\,dw \wedge \overline{dw},$$

the well-known formula from elementary complex analysis. Note: Usually there is a factor of $1/2$ in the definition of $\overline{\partial}$, namely, $\overline{\partial}$ is usually defined as $\frac{1}{2}(\partial_{x_0} + i\partial_{x_1})$, in which case the factor in front of the above integral is $\frac{1}{2\pi i}$ and not $\frac{1}{4\pi i}$.

In view of (3.3) and the definition (1.6), the Cauchy integral or Poisson operator in the Clifford analysis context is the operator

$$\mathcal{K}\varphi = \frac{1}{\sigma_n}\int_{\partial X} \frac{\overline{x-y}}{|x-y|^{n+1}} G(y)\varphi(y)\,dS(y), \tag{3.4}$$

where G is Clifford multiplication by the inward pointing normal vector to ∂X and S is surface measure.

4. Polyhomogeneous expansions and asymptotics of h-transforms

In this section we first review some ideas concerning analysis on manifolds with corners; standard references include [77,78], and [73]. The main results in this section occur in Sections 4.2 and 4.3 where we describe in great detail the properties of "h-transforms" (generalizations of Laplace and Fourier transforms) on polyhomogeneous functions. In particular, Theorems 4.4 and 4.8 will be used quite a bit in the sequel.

4.1. Analysis on manifolds with corners

We begin with a definition of a manifold with corners. An n-dimensional *manifold with corners* Z is a Hausdorff paracompact topological space with C^∞ structure given by local coordinate patches of the form

$$[0,1)^k \times (-1,1)^{n-k}, \tag{4.1}$$

where k can be any integer between 0 and n depending on where the patch is located in Z. A *codimension d face* of Z is the closure of a connected component of points in Z which are origins of charts of the form (4.1) with $k = d$. The set $M_d(Z)$ denotes the set of all codimension d faces of Z. The *codimension* of Z is the largest k that can occur in a local chart (4.1). For example, a manifold with boundary is just a codimension one manifold with corners.

Remark 4.1. In the terminology of Melrose [76], we should technically call Z a *tied manifold*. A manifold with corners in the sense of Melrose [78] requires that each *boundary hypersurface* H, or codimension one face, be embedded in the sense that it has a globally defined boundary defining function; a nonnegative function in $C^\infty(Z)$ that vanishes only on H where it has a nonzero differential. An example of a manifold with corners in the sense of this paper that is not a manifold with corners in the sense of [78] is the tear drop shown in Fig. 2 of Section 2.

4.1.1. Asymptotic expansions

We now discuss asymptotic expansions. Let $\mathcal{U} = [0,1)^k_x \times (-1,1)^{n-k}_y$. Then for $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, the space of symbols $S^a(\mathcal{U})$ consists of smooth functions $u \in C^\infty(\mathring{\mathcal{U}})$ that can be expressed in the form

$$u(x,y) = x_1^{-a_1} \cdots x_k^{-a_k} v(x,y), \tag{4.2}$$

where all "b-derivatives" [78] (that is, partials of the form $(x\partial_x)^\alpha \partial_y^\beta v(x,y)$) of $v(x,y)$ are locally bounded functions on $\mathcal{U}$; that is, bounded on compact subsets of $\mathcal{U}$. If u has compact support, then for any $N = 0, 1, 2, \ldots$ we define

$$\|u\|_{S^a,N} := \sum_{|\beta|+|\gamma| \leqslant N} \left\| (x\partial_x)^\beta \partial_y^\gamma v(x,y) \right\|_\infty. \tag{4.3}$$

By taking a compact exhaustion of $\mathcal{U}$ we can make $S^a(\mathcal{U})$ into a Fréchet space using the norms (4.3) on compact subsets of $\mathcal{U}$; however, we will not need to do this although we shall use (4.3) later to define semi-norms on polyhomogeneous spaces.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. A ($C^\infty$) *index set* $\mathcal{I}$ is a discrete subset of $\mathbb{C} \times \mathbb{N}_0$ such that

- $(\alpha, k) \in \mathcal{I} \Rightarrow (\alpha, \ell) \in \mathcal{I}$ for all $0 \leqslant \ell \leqslant k$;
- given any $N \in \mathbb{R}$, the set $\{(\alpha, k) \in \mathcal{I} \mid \Re\alpha \leqslant N\}$ is finite (where $\Re$ denotes the real part);
- $(\alpha, k) \in \mathcal{I} \Rightarrow (\alpha + j, k) \in \mathcal{I}$ for all $j \in \mathbb{N}_0$.

If $\beta \in \mathbb{R}$, to say that $\mathcal{I} > \beta$ we mean that for any $(\alpha, k) \in \mathcal{I}$, we have $\Re\alpha > \beta$.

Given an index set $\mathcal{I}$, a function $u \in S^a(\mathcal{U})$ is said to have an *asymptotic expansion* at $x_1 = 0$ with index set $\mathcal{I}$ if, for each $N > 0$, we can write

$$u(x, y) = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha\leqslant N} x_1^\alpha (\log x_1)^k u_{(\alpha,k)}(x', y) + x_1^N u_N(x, y), \tag{4.4}$$

where if $a = (a_1, a')$, then $u_N(x, y) \in S^{(0,a')}(\mathcal{U})$, $u_{(\alpha,k)}(x', y) \in S^{a'}(\mathcal{U}')$ with $x = (x_1, x')$ and $\mathcal{U}' = [0, 1)^{k-1}_{x'} \times (-1, 1)^{n-k}_y$. It is common to write

$$u(x, y) \sim \sum_{(\alpha,k)\in\mathcal{I}} x_1^\alpha (\log x_1)^k u_{(\alpha,k)}(x', y)$$

to express the asymptotic expansion. For example, note that if $\mathcal{I} = \emptyset$, then the expansion property (4.4) holds if and only if for all $N > 0$, we can write $u = x_1^N u_N(x, y)$ with $u_N(x, y) \in S^a(\mathcal{U})$. This is equivalent to the statement that u vanishes with all derivatives at $x_1 = 0$. For another example, taking $\mathcal{I} = \mathbb{N}_0$, we see from (4.4) that for any $N > 0$, we have

$$u(x, y) = \sum_{0\leqslant j\leqslant N} x_1^j u_j(x', y) + x_1^N u_N(x, y) \tag{4.5}$$

with $u_N(x, y) \in S^a(\mathcal{U})$ and $u_j(x', y) \in S^{a'}(\mathcal{U}')$. This just means that $u(x, y)$ is C^∞ in x_1 even down to $x_1 = 0$ and this expansion is nothing more than the Taylor expansion of $u(x, y)$ taken at $x_1 = 0$. Of course, asymptotic expansions at any other boundary $x_i = 0$ for $i = 2, \dots, k$ are defined similarly.

On a manifold with corners Z we can define symbol spaces and asymptotic expansions by reducing to the local cases described above. Henceforth assume that Z has finitely many hypersurfaces $H_1, \dots, H_m$. For $a = (a_1, \dots, a_m) \in \mathbb{R}^m$, a function $u \in C^\infty(\mathring{Z})$ is said to be in the symbol space $S^a(Z)$ if given any coordinate patch $\mathcal{U} = [0, 1)^k_x \times (-1, 1)^{n-k}_y$ and compactly supported function $\varphi \in C^\infty_c(\mathcal{U})$, we have $\varphi u \in S^{a_\mathcal{U}}(\mathcal{U})$ where $a_\mathcal{U} = (a_{i_1}, \dots, a_{i_k})$ with $\{x_1 = 0\} = H_{i_1} \cap \mathcal{U}, \dots, \{x_k = 0\} = H_{i_k} \cap \mathcal{U}$. Note that $S^{a_\mathcal{U}}(\mathcal{U})$ has already been defined above.

A function $u \in S^a(Z)$ is said to have an asymptotic expansion at a hypersurface H with index set $\mathcal{I}$, if for any patch $\mathcal{U} = [0, 1)_{x_1} \times \mathcal{U}'$ on Z as in (4.1) with $H \cap \mathcal{U} = \{x_1 = 0\}$, and for any function $\varphi \in C^\infty_c(\mathcal{U})$, the function φu has an asymptotic expansion at $x_1 = 0$ with index set $\mathcal{I}$ in the sense described in (4.4). (One can show that the notion of symbol space and asymptotic expansion is independent of the choice of local coordinates (4.1); see [67,77].)

Let $\mathcal{H}$ be a collection of hypersurfaces of Z and let $\mathcal{I}$ be a collection of index sets $\mathcal{I} = \{\mathcal{I}_H \mid H \in \mathcal{H}\}$ associated to the collection of hypersurfaces $\mathcal{H}$. If $\beta \in \mathbb{R}$, to say that $\mathcal{I} > \beta$ we mean that for any $\mathcal{I}_H \in \mathcal{I}$, we have $\mathcal{I}_H > \beta$ and if $\beta \in \mathbb{C}$, we define $\mathcal{I} + \beta = \{\mathcal{I}_H + \beta\}$ where $\mathcal{I}_H + \beta = \{(\alpha + \beta, k) \mid (\alpha, k) \in \mathcal{I}_H\}$. We denote by $\mathcal{A}^\mathcal{I}(Z)$ the space of functions u such that (i) $u \in S^a(Z)$ for some $a \in \mathbb{R}^m$ (depending on $\mathcal{I}$ and not on u), (ii) for each $H \in \mathcal{H}$ the function u has an expansion at H with index set $\mathcal{I}_H$, and (iii) if $H \notin \mathcal{H}$, then u has an expansion at H

with index set $\mathcal{I}_H := \mathbb{N}_0$. By the example given around (4.5), for $H \notin \mathcal{H}$, u is smooth up to H and note that the values of the a_i's in $S^a(Z)$ are related to the leading terms in the expansion (4.4). The letter "$\mathcal{A}$" stands for "asymptotics." If Z has exactly one hypersurface, then an index family $\mathcal{I}$ consists of just one set $\mathcal{I} = \{\mathcal{J}\}$ for an index set $\mathcal{J}$, in which case we denote $\mathcal{A}^{\mathcal{I}}(Z)$ by $\mathcal{A}^{\mathcal{J}}(Z)$.

The space $\mathcal{A}^{\mathcal{I}}(Z)$ has a Fréchet space structure described as follows. Let $u \in \mathcal{A}^{\mathcal{I}}(Z)$. Then $u \in S^a(Z)$ for some $a \in \mathbb{R}^m$ depending on $\mathcal{I}$. Let $H \in M_1(Z)$ and let $\mathcal{U} = [0,1)^k_x \times (-1,1)^{n-k}_y$ be a coordinate patch as in (4.1) with $H \cap \mathcal{U} = \{x_1 = 0\}$. Then for any $\varphi \in C_c^\infty(\mathcal{U})$ and $N \in \mathbb{N}_0$, we can write $(\varphi u)(x,y)$ as in (4.4) where $\mathcal{I}_H = \mathbb{N}_0$ if $H \notin \mathcal{H}$:

$$\varphi(x,y)u(x,y) = \sum_{(\alpha,k)\in\mathcal{I}_H,\, \Re\alpha \leqslant N} x_1^\alpha (\log x_1)^k u_{(\alpha,k)}(x',y) + x_1^N u_N(x,y), \tag{4.6}$$

where if $\varphi u \in S^{a_{\mathcal{U}}}(\mathcal{U})$ (using the notation in the definition of $S^a(Z)$ above) and $a_{\mathcal{U}} = (a_{i_1}, a')$, then $u_N(x,y) \in S^{(0,a')}(\mathcal{U})$, $u_{(\alpha,k)}(x',y) \in S^{a'}(\mathcal{U}')$ with $x = (x_1, x')$ and $\mathcal{U}' = [0,1)^{k-1}_{x'} \times (-1,1)^{n-k}_y$. We define

$$\|\varphi u\|_{H,N} := \|u_N\|_{S^{(0,a')},N} + \sum_{(\alpha,k)\in\mathcal{I}_H,\, \Re\alpha\leqslant N} \|u_{(\alpha,k)}\|_{S^{a'},N},$$

where the norms $\|\ \|_{S^{(0,a')},N}$ and $\|\ \|_{S^{a'},N}$ are defined in (4.3). Let $\{(\mathcal{U}_j, \varphi_j)\}$ be a partition of unity of Z where the $\mathcal{U}_j$'s are charts of the form (4.1) and $\varphi_j \in C_c^\infty(\mathcal{U}_j)$. For each $H \in M_1(Z)$, let J_H be the set of j's such that $\mathcal{U}_j \cap H \neq \emptyset$ and let J be those j's such that $\mathcal{U}_j$ is in the interior of Z. Let $\|\ \|_{j,C^N}$ be a C^N norm over $\mathcal{U}_j$ for $j \in J$. For each $N \in \mathbb{N}_0$ we define

$$\|u\|_N := \sum_{H\in M_1(Z)} \sum_{j\in J_H,\, j\leqslant N} \|\varphi_j u\|_{H,N} + \sum_{j\in J,\, j\leqslant N} \|\varphi_j u\|_{j,C^N}. \tag{4.7}$$

With these norms, $\mathcal{A}^{\mathcal{I}}(Z)$ becomes a Fréchet space. One can check that in this topology, a sequence $\{u_i\}$, with $u_i \in \mathcal{A}^{\mathcal{I}}(Z)$, converges to $u \in \mathcal{A}^{\mathcal{I}}(Z)$ if and only if on compact subsets of the interior of Z, $u_i \to u$ in the C^∞-topology, and in any expansion of the sort (4.6), the coefficients in the expansion of φu_i converge to the corresponding coefficients of φu; that is, $u_{i,(\alpha,k)}(x',y) \to u_{(\alpha,k)}(x',y)$ in $S^{a'}(\mathcal{U}')$ and $u_{iN}(x,y) \to u_N(x,y)$ in $S^{(0,a')}(\mathcal{U})$.

4.1.2. Blow-ups and compactifications

Let Z be a manifold with corners and let $Y \subseteq Z$ be an embedded submanifold of codimension ℓ, which by definition means that near each point of Y there is a coordinate patch $\mathcal{U} = [0,1)^k \times (-1,1)^{n-k}$ of the form (4.1) on Z such that $Y \cap \mathcal{U}$ is equal to the zero set of exactly ℓ of the coordinates $x_1, \dots, x_n$. The blow-up $[Z;Y]$ of Z along Y is then a new manifold with corners that has an atlas consisting of the usual coordinate patches on $Z \setminus Y$ together with polar coordinate patches over Y in Z. For instance, if

$$\mathcal{U} = [0,1)^p \times (-1,1)^q \times [0,1)^{k-p} \times (-1,1)^{n-k-q}$$

with

$$Y \cap \mathcal{U} = \{0\} \times \{0\} \times [0,1)^{k-p} \times (-1,1)^{n-k-q},$$

then local coordinates for $[Z; Y]$ are given by the original coordinates on the factor $[0,1)_y^{k-p} \times (-1,1)_{y'}^{n-k-q}$ together with polar coordinates on $[0,1)_x^p \times (-1,1)_{x'}^q$:

$$r = \left|(x,x')\right| \geqslant 0, \qquad \omega = \frac{(x,x')}{|(x,x')|} \in \mathbb{S}^{p+q-1}.$$

The *blow-down map* $\beta : [Z; Y] \to Z$ is the map that takes a point in polar coordinates back to the original coordinates: $(r, \omega, y, y') \mapsto (x, x', y, y')$ where $(x, x') = r\omega$. For example, if $Z = \mathbb{R}^n$ and $Y = \{0\}$, then

$$\left[\mathbb{R}^n; \{0\}\right] = [0,\infty) \times \mathbb{S}^{n-1}$$

is just the standard polar coordinates $r = |x|$ and $\omega = x/|x|$. The blow-down map in this case is just the map $(r, \omega) \mapsto x = r\omega$.

Back to our general discussion involving $[Z; Y]$, if $X \subseteq Z$ is a closed subset of Z, then the *lift* of X into $[Z; Y]$, $\beta^* X \subseteq [Z; Y]$, is defined under the following conditions:

- if $Z \subseteq Y$, then we define $\beta^* X := \beta^{-1}(X)$;
- if $X = \overline{X \setminus Y}$, then we define $\beta^* X := \overline{\beta^{-1}(Y \setminus X)}$.

If X satisfies either of these two conditions, and if in addition, $\beta^* X$ is an embedded submanifold of $[Z; Y]$, then $[Z; Y]$ blown-up along $\beta^* X$ is defined, and we denote it by $[Z; Y; X] \equiv [[Z; Y]; \beta^* X]$. A family $\mathcal{Y} = \{Y_1, \dots, Y_N\}$ of embedded submanifolds of Z is said to intersect *normally* if the conormal bundles of $Y_1, \dots, Y_N$ are independent at intersections; for example, this trivially holds when $Y_1, \dots, Y_N$ are pairwise disjoint. For any normal family, the iterated blow-up

$$[Z; Y_{i_1}; \cdots; Y_{i_N}] = \left[\left[\cdots\left[[Z; Y_{i_1}]; Y_{i_2}\right]\cdots; Y_{i_N}\right]\right] \tag{4.8}$$

is defined independent of the ordering [76]. This manifold is denoted by $[Z; \mathcal{Y}]$. For example, if Z is a manifold with corners of codimension two, then the submanifolds in $\mathcal{Y} = M_2(Z)$ are pairwise disjoint, so $Z_{\mathrm{tb}} := [Z; M_2(Z)]$ is defined.

We now describe compactifications. We denote by $\overline{[0,\infty)}$ the *compactification* of the interval $[0,\infty)$, which is obtained from $[0,\infty)$ by adding a point $0'$ at infinity and using the function ρ defined by $\rho = 0$ at $0'$ and $\rho := \frac{1}{x}$ for $x \in (0,\infty)$ as the coordinate function near $0'$. The manifold $\overline{[0,\infty)}$ has two boundary components, 0 and $0'$. The compactification of $\mathbb{R}^n$, denoted by $\overline{\mathbb{R}^n}$, is the disjoint union of $\mathbb{R}^n$ with the sphere $\mathbb{S}^{n-1}$, which forms the "boundary at infinity." $\overline{\mathbb{R}^n}$ has the usual structure on the $\mathbb{R}^n$ portion of $\overline{\mathbb{R}^n}$ and outside of the point $0 \in \mathbb{R}^n$, where we identify $\mathbb{R}^n \setminus 0 \equiv (0,\infty)_r \times \mathbb{S}^{n-1}$ using polar coordinates, we identify $\overline{\mathbb{R}^n}$ with $\overline{[0,\infty)_r} \times \mathbb{S}^{n-1} \setminus (\{0\} \times \mathbb{S}^{n-1})$. Thus, outside of the origin, $(\rho, \omega) \in [0,\infty) \times \mathbb{S}^{n-1}$ with $\rho = \frac{1}{r}$ define coordinates on $\overline{\mathbb{R}^n}$.

4.2. h-Transform expansions at ∞

Let $h(\xi) \in C^\infty(\mathbb{R}^n)$ be a smooth bounded function. Then for $r > 0$ and $\phi(\xi)$ a locally integrable function on $\mathbb{R}^n$, we shall analyze the following integral "transform"

$$\mathcal{T}_h(\phi, r) := \int_{\mathbb{R}^n} h(r\xi)\phi(\xi)\, d\xi, \tag{4.9}$$

when this integral can be given a meaning; see [66] for related results. Specifically, we shall assume that the function $h(\xi)$ is one of the following types:

Example 1: $h(\xi) = e^{-|\xi|}$. More generally we can consider $h(\xi) = e^{-t|\xi|}$ for $t \in (0, \infty)$. The h-transform (4.9) then represents a type of "Laplace transform."

Example 2: $h(\xi) = e^{i\gamma\cdot\xi}$ where $\gamma \in \mathbb{S}^{n-1}$. More generally we can consider $h(\xi) = e^{i\gamma\cdot\xi}$ for $\gamma \in \mathbb{R}^n$ nonzero. In this case, the integral (4.9) must be interpreted as an oscillatory integral as e.g. in Hörmander's book [47, Section 7.8].

Example 3: We can combine Examples 1 and 2 with

$$h(\xi) := e^{i\gamma\cdot\xi - t|\xi|}$$

where $(\gamma, t) \in \mathbb{R}^n \times [0, \infty)$ with $|\gamma|^2 + t^2 > 0$. When $\gamma = 0$ we have Example 1 and when $t = 0$ we have Example 2.

In this section we shall study the asymptotic expansions of $\mathcal{T}_h(\phi, r)$ as $r \to \infty$ for $\phi \in \mathcal{A}^{\mathcal{I}}([\mathbb{R}^n; \{0\}])$ compactly supported. Here, we recall from Section 4.1 that $[\mathbb{R}^n; \{0\}] = [0, \infty)_r \times \mathbb{S}^{n-1}$ is the manifold $\mathbb{R}^n$ blown-up at $\{0\}$. In Section 4.3 we consider the behavior of $\mathcal{T}_h(\phi, r)$ as $r \to 0$. To study $r \to \infty$, we introduce the variable $\rho = 1/r$ and study

$$\mathcal{T}_h\left(\phi, \frac{1}{\rho}\right) = \int h\left(\frac{\xi}{\rho}\right)\phi(\xi)\, d\xi$$

as $\rho \to 0$. We begin with the following lemma. In this section, our standing assumption is that $h(\xi)$ is a function described in Examples 1–3 above.

Lemma 4.2. *Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) \equiv 1$ near 0 and $\chi(\xi) \equiv 0$ outside a neighborhood of 0. Then given any function $\phi(\xi)$ of the form $\phi(\xi) = |\xi|^\alpha (\log|\xi|)^k a(\xi/|\xi|)$ where $a(\omega) \in C^\infty(\mathbb{S}^{n-1})$, $\Re\alpha > -n$ and $k \in \mathbb{N}_0$, there are constants $a_j \in \mathbb{C}$, such that*

$$\mathcal{T}_h\left(\chi\phi, \frac{1}{\rho}\right) = \int h\left(\frac{\xi}{\rho}\right)\chi(\xi)\phi(\xi)\, d\xi \equiv \rho^{\alpha+n} \sum_{j=0}^{k} a_j (\log\rho)^j, \tag{4.10}$$

modulo a smooth function on $[0, \infty)_\rho$ that vanishes to infinite order at $\rho = 0$.

Proof. Scaling $\xi \mapsto \rho\xi$, one can see that for some constants $c_{k,j}$, we have

$$\phi(\rho\xi) = \rho^\alpha |\xi|^\alpha a\big(\xi/|\xi|\big) \sum_{j=0}^{k} c_{k,j} (\log\rho)^{k-j} \big(\log|\xi|\big)^j.$$

Therefore, changing variables $\xi \mapsto \rho\xi$, we see that

$$\mathcal{T}_h\left(\chi\phi, \frac{1}{\rho}\right) = \int h\left(\frac{\xi}{\rho}\right)\chi(\xi)\phi(\xi)\,d\xi$$
$$= \rho^{\alpha+n}\sum_{j=0}^{k} c_{k,j}(\log\rho)^{k-j}\int h(\xi)\chi(\rho\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{j}a\big(\xi/|\xi|\big)\,d\xi. \quad (4.11)$$

Let us fix j and consider the function

$$f_j(\rho) := \int h(\xi)\chi(\rho\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{j}a\big(\xi/|\xi|\big)\,d\xi.$$

Note that $f_j(\rho)$ is smooth for $\rho \in (0,\infty)$; this is obvious if $t > 0$ in $h(\xi) = e^{i\gamma\cdot\xi - t|\xi|}$ and if $t = 0$, this smoothness follows from the oscillatory integral definition of the right-hand side of $f_j(\rho)$. We claim that $f_j'(\rho)$ is smooth on $[0,\infty)$ and vanishes to infinite order at $\rho = 0$. To see this, observe that since $\rho\partial_\rho\chi(\rho\xi) = (\xi\cdot\partial_\xi\chi)(\rho\xi)$, we have

$$\rho\partial_\rho f_j(\rho) = \int h(\xi)(\xi\cdot\partial_\xi\chi)(\rho\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{j}a\big(\xi/|\xi|\big)\,d\xi.$$

Changing variables $\xi \mapsto \xi/\rho$ and expanding just as we did to get (4.11), we can write

$$\rho\partial_\rho f_j(\rho) = \rho^{-\alpha-n}\sum_{i=0}^{j} c_{j,i}(-\log\rho)^{j-i}\int h\left(\frac{\xi}{\rho}\right)(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{i}a\big(\xi/|\xi|\big)\,d\xi.$$

Since $\chi(\xi) \equiv 1$ near 0 and $\chi(\xi) \equiv 0$ outside a neighborhood of 0, the function

$$(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{i}a\big(\xi/|\xi|\big)$$

is a smooth compactly supported function on $\mathbb{R}^n$ vanishing near the origin. It follows that for the examples of h we are considering that the function

$$\int h\left(\frac{\xi}{\rho}\right)(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{\alpha}\big(\log|\xi|\big)^{i}a\big(\xi/|\xi|\big)\,d\xi$$

is a smooth function of $\rho \in [0,\infty)$ vanishing to infinite order (that is, with all derivatives) at $\rho = 0$—for $h(\xi) = e^{i\gamma\cdot\xi - t|\xi|}$ with $t > 0$ this statement is obvious and in the Fourier transform example $h(\xi) = e^{i\gamma\cdot\xi}$, this statement is just the well-known fact that the Fourier transform of a Schwartz function is again a Schwartz function. In particular, $\rho\partial_\rho f_j(\rho)$ is smooth and vanishes to infinite order at $\rho = 0$. Therefore $\partial_\rho f_j(\rho)$ is smooth on $[0,\infty)$ and vanishes to infinite order at $\rho = 0$, and so by integration, we see that

$$f_j(\rho) = f_j(0) + g_j(\rho),$$

where $g_j(\rho)$ is smooth on $[0,\infty)$ and vanishes to infinite order at $\rho = 0$. Substituting this into (4.11) we see that

$$\mathcal{T}_h\left(\chi\phi, \frac{1}{\rho}\right) \equiv \rho^{\alpha+n} \sum_{j=0}^{k} c_{k,j} (\log\rho)^{k-j} f_j(0)$$

modulo a function vanishing to infinite order at $\rho = 0$. This concludes the proof. □

Next, we need the following lemma.

Lemma 4.3. *If $a(r,\omega) \in r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$, with $N \geqslant 0$, is compactly supported, then the function*

$$A(\rho) := \int h\left(\frac{\xi}{\rho}\right) a(\xi)\, d\xi,$$

where $a(\xi) := a(|\xi|, \xi/|\xi|)$, defines an element of $\rho^{N+n-1} S^0([0,\infty)_\rho)$.

Proof. Recalling that $h(\xi) = e^{i\gamma\cdot\xi - t|\xi|}$, we need to show that

$$\rho^{-N-n+1} A(\rho) = \rho^{-N-n+1} \int e^{i\frac{\gamma}{\rho}\cdot\xi} e^{-t\frac{|\xi|}{\rho}} a(\xi)\, d\xi \in S^0\big([0,\infty)_\rho\big).$$

Note that $A(\rho)$ certainly has this property if $t > 0$. Indeed, making the change of variables $\xi \mapsto \rho\xi$ in the formula for $A(\rho)$ we obtain

$$\rho^{-N-n+1} A(\rho) = \rho^{-N+1} \int e^{i\gamma\cdot\xi} e^{-t|\xi|} a(\rho\xi)\, d\xi.$$

Using that $a(r,\omega) \in r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$ is compactly supported in r, it is easy to check that $\rho^{-N-n+1} A(\rho) \in S^0([0,\infty))$ for $t > 0$. Therefore we might as well assume from the very beginning that $t = 0$ and $|\gamma| > 0$. Summarizing, all we have to do is show that

$$\rho^{-N-n+1} A(\rho) = \rho^{-N-n+1} \int e^{i\frac{\gamma}{\rho}\cdot\xi} a(\xi)\, d\xi \in S^0\big([0,\infty)\big).$$

Before proving this, we note that using the definition of $r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$, one can check that for any β, there is a constant C such that

$$\big|\partial_\xi^\beta a(\xi)\big| \leqslant C|\xi|^{N-|\beta|}, \quad \text{for } |\xi| \leqslant 1. \tag{4.12}$$

Observe that if we define $L_\gamma := -\frac{\gamma}{|\gamma|^2} \cdot D_\xi = -\frac{1}{i}\sum_{k=1}^n \frac{\gamma_k}{|\gamma|^2}\partial_{\xi_k}$, then

$$L_\gamma e^{i\frac{\gamma}{\rho}\cdot\xi} = -\rho^{-1} e^{i\frac{\gamma}{\rho}\cdot\xi}.$$

Hence, using the estimates (4.12) to justify integrating by parts, we have

$$\rho^{-N-n+1}A(\rho)=(-1)^{N+n-1}\int\big(L_\gamma^{N+n-1}e^{i\frac{\gamma}{\rho}\cdot\xi}\big)a(\xi)\,d\xi=\int e^{i\frac{\gamma}{\rho}\cdot\xi}L_\gamma^{N+n-1}a(\xi)\,d\xi$$
$$=\int e^{i\frac{\gamma}{\rho}\cdot\xi}a_0(\xi)\,d\xi,$$

where $a_0(\xi):=L_\gamma^{N+n-1}a(\xi)$. By definition of our symbol spaces, it follows that in polar coordinates, $a_0(r,\omega)\in S^{n-1}([0,\infty)_r\times\mathbb{S}^{n-1}_\omega)$ and of course is still compactly supported in r. Thus, we are reduced to proving the statement: if $a(r,\omega)\in S^{n-1}_c([0,\infty)_r\times\mathbb{S}^{n-1}_\omega)$, where the subscript c denotes compact support, then

$$A(\rho):=\int e^{i\frac{\gamma}{\rho}\cdot\xi}a(\xi)\,d\xi$$

is an element of $S^0([0,\infty))$. To prove this, first of all observe that $a(\xi)$ is integrable because $a(\xi)=\mathcal{O}(|\xi|^{-n+1})$ near $\xi=0$, so A is bounded. We claim that

$$\rho\partial_\rho A(\rho)=\int e^{i\frac{\gamma}{\rho}\cdot\xi}a_1(\xi)\,d\xi,\quad\text{where } a_1\in S^{n-1}_c\big([0,\infty)_r\times\mathbb{S}^{n-1}_\omega\big).\tag{4.13}$$

Once we prove this statement, given any $k\in\mathbb{N}$, induction proves that

$$(\rho\partial_\rho)^kA(\rho)=\int e^{i\frac{\gamma}{\rho}\cdot\xi}a_k(\xi)\,d\xi,\quad\text{where } a_k\in S^{n-1}_c\big([0,\infty)_r\times\mathbb{S}^{n-1}_\omega\big).$$

Since a_k is integrable, $(\rho\partial_\rho)^kA(\rho)$ is bounded. Thus, $A(\rho)\in S^0([0,\infty))$. So, it remains to prove (4.13). To do so, we use the estimates (4.12) to justify integrating by parts and find that

$$\rho\partial_\rho A(\rho)=-\rho^{-1}\int e^{i\frac{\gamma}{\rho}\cdot\xi}(i\gamma\cdot\xi)a(\xi)\,d\xi$$
$$=\int\big(L_\gamma e^{i\frac{\gamma}{\rho}\cdot\xi}\big)(i\gamma\cdot\xi)a(\xi)\,d\xi$$
$$=-\int e^{i\frac{\gamma}{\rho}\cdot\xi}L_\gamma\big[(i\gamma\cdot\xi)a(\xi)\big]\,d\xi.$$

Note that $L_\gamma[(i\gamma\cdot\xi)a(\xi)]\in S^{n-1}_c([0,\infty)_r\times\mathbb{S}^{n-1}_\omega)$. Thus, (4.13) holds. $\square$

We can now prove the following theorem.

Theorem 4.4. *If $\mathcal{I}>-n$, then*

$$\phi\in\mathcal{A}^{\mathcal{I}}_c\big([\mathbb{R}^n;\{0\}]\big)\quad\Longrightarrow\quad\mathcal{T}_h(\phi)\in\mathcal{A}^{\mathcal{I}+n}\big(\overline{[0,\infty)}\big),$$

where the subscript c denotes compact support and where on the right, $\mathcal{I}+n$ is associated to the boundary at infinity $0'$ of the compactification $\overline{[0,\infty)}$. (Thus, $\mathcal{T}_h(\phi)$ is smooth at $0\in\overline{[0,\infty)}$.)

Proof. If $\mathcal{I} > -n$ and $\phi \in \mathcal{A}_c^{\mathcal{I}}([\mathbb{R}^n; \{0\}])$, then using the formula (4.9), it is easily seen that $\mathcal{T}_h(\phi, r)$ is smooth at $r = 0$ (recalling that $\phi(\xi)$ has compact support). Thus, we just need to show that an expansion of the sort

$$\phi(r, \omega) \sim \sum_{(\alpha,k)\in\mathcal{I}} r^\alpha (\log r)^k a_{\alpha,k}(\omega) \quad \text{as } r \to 0,$$

implies an expansion of the sort

$$\mathcal{T}_h\left(\phi, \frac{1}{\rho}\right) \sim \sum_{(\alpha,k)\in\mathcal{I}} b_{\alpha,k} \rho^{\alpha+n} (\log \rho)^k \quad \text{as } \rho \to 0.$$

Let $\phi_{\alpha,k} = r^\alpha (\log r)^k a_{\alpha,k}(\omega)$ and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 1$ near 0 and $\chi(\xi) \equiv 0$ outside a neighborhood of 0. Then fixing any large $N \gg 0$, consider the finite expansion

$$\phi(r\omega) = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N} \chi(r\omega)\phi_{\alpha,k}(r\omega) + f_N(r, \omega),$$

where $f_N(r, \omega) \in r^N S^0([0, \infty)_r \times \mathbb{S}_\omega^{n-1})$ and vanishes for r large. Thus,

$$\mathcal{T}_h\left(\phi, \frac{1}{\rho}\right) = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N} \mathcal{T}_h\left(\chi\phi_{\alpha,k}, \frac{1}{\rho}\right) + \mathcal{T}_h\left(f_N, \frac{1}{\rho}\right).$$

By Lemma 4.2 we know that $\mathcal{T}_h(\chi\phi_{\alpha,k}, \frac{1}{\rho})$ is of the form

$$\mathcal{T}_h\left(\chi\phi_{\alpha,k}, \frac{1}{\rho}\right) \equiv \rho^{\alpha+n} \sum_{\ell=0}^{k} b_{\alpha,k,\ell} (\log \rho)^\ell$$

modulo a smooth function on $[0, \infty)_\rho$ that vanishes to infinite order at $\rho = 0$. Because $f_N(r, \omega) \in r^N S^0([0, \infty)_r \times \mathbb{S}_\omega^{n-1})$ and is compactly supported in r, by Lemma 4.3 we know that $\mathcal{T}_h(f_N, \frac{1}{\rho}) \in \rho^{N+n-1} S^0([0, \infty)_\rho)$. Since $N \gg 0$ was completely arbitrary, this completes our proof. □

Remark 4.5. By the proof of this theorem and the lemmas it used, one can check that if $\phi_t \in \mathcal{A}^{\mathcal{I}}([\mathbb{R}^n; \{0\}])$ depends smoothly on a parameter t and has support in a fixed compact set for all parameters t, then $\mathcal{T}_h(\phi_t) \in \mathcal{A}^{\mathcal{I}+n}(\overline{[0, \infty)})$ also depends smoothly on t. Here, "smoothly" has a well-defined meaning since recall that $\mathcal{A}^{\mathcal{I}}([\mathbb{R}^n; \{0\}])$ and $\mathcal{A}^{\mathcal{I}+n}(\overline{[0, \infty)})$ have natural Fréchet topologies, and differentiation for any Fréchet space-valued function is a well-defined notion.

4.3. h-Transform expansions at 0

We now give a "converse" to Theorem 4.4. We first prove the following lemma, which is akin to Lemma 4.2. The proof is similar, but there is a big twist pointed out near the middle of the proof.

Lemma 4.6. *Let $\phi = \chi(\xi)a_{\beta,k}(\xi/|\xi|)|\xi|^{-\beta-n}(\log|\xi|)^k$ where $\beta \in \mathbb{C}$ and $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is zero for $|\xi| \leqslant 1$ and identically 1 for $|\xi| \geqslant 2$. Then the h-transform*

$$\mathcal{T}_h(\phi, r) = \int h(r\xi)\phi(\xi)\,d\xi$$

has the following expansion as $r \to 0$

$$\mathcal{T}_h(\phi, r) \sim r^\beta \sum_{j=0}^{k} a_j(\log r)^j + r^\beta \sum_{j=0}^{k} b_j(\log r)^{j+1} + \sum_{\ell=0}^{\infty}\sum_{j=0}^{k} c_{\ell,j} r^\ell (\log r)^j,$$

where $b_j = 0$ if $\beta \notin \mathbb{N}_0$.

Proof. Observe that

$$\phi(\xi/r) = \chi(\xi/r)r^{\beta+n}|\xi|^{-\beta-n}a_{\beta,k}\big(\xi/|\xi|\big)\sum_{j=0}^{k} d_{k,j}(\log r)^{k-j}\big(\log|\xi|\big)^j,$$

for some constants $d_{k,j}$, therefore changing variables $\xi \mapsto \xi/r$, we see that

$$\begin{aligned}\mathcal{T}_h(\phi, r) &= r^{-n}\int h(\xi)\phi(\xi/r)\,d\xi \\ &= r^\beta \sum_{j=0}^{k} d_{k,j}\big(\log r\big)^{k-j}\int h(\xi)\chi(\xi/r)|\xi|^{-\beta-n}\big(\log|\xi|\big)^j a_{\beta,k}\big(\xi/|\xi|\big)\,d\xi. \qquad (4.14)\end{aligned}$$

Let us fix j and consider the function

$$f_j(r) := \int h(\xi)\chi(\xi/r)|\xi|^{-\beta-n}\big(\log|\xi|\big)^j a_{\beta,k}\big(\xi/|\xi|\big)\,d\xi;$$

note that $f_j(r)$ is smooth for $r \in (0,\infty)$. We now expand $f_j(r)$ by expanding $f_j'(r)$. To do so, observe that since $r\partial_r(\chi(\xi/r)) = -(\xi\cdot\partial_\xi\chi)(\xi/r)$, we have

$$r\partial_r f_j(r) = -\int h(\xi)(\xi\cdot\partial_\xi\chi)(\xi/r)|\xi|^{-\beta-n}\big(\log|\xi|\big)^j a_{\beta,k}\big(\xi/|\xi|\big)\,d\xi.$$

Changing variables $\xi \mapsto r\xi$ and expanding, we can write

$$r\partial_r f_j(r) = r^{-\beta}\sum_{i=0}^{j} d_{j,i}(-\log r)^{j-i}\int h(r\xi)(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{-\beta-n}\big(\log|\xi|\big)^i a_{\beta,k}\big(\xi/|\xi|\big)\,d\xi.$$

Observe that the function $(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{-\beta-n}(\log|\xi|)^i a_{\beta,k}(\xi/|\xi|)$ is a smooth compactly supported function on $\mathbb{R}^n$ (vanishing near the origin). It follows that the function

$$\int h(r\xi)(\xi\cdot\partial_\xi\chi)(\xi)|\xi|^{-\beta-n}\big(\log|\xi|\big)^i a_{\beta,k}\big(\xi/|\xi|\big)\,d\xi$$

is a smooth function of $r \in [0, \infty)$. Hence, after expanding this integral at $r = 0$, we obtain

$$r\partial_r f_j(r) \sim \sum_{\ell=0}^{\infty} \sum_{i=0}^{j} r^{-\beta+\ell} d_{j,i,\ell} (\log r)^{j-i} \quad \text{as } r \to 0$$

for some constants $d_{j,i,\ell}$, or after division by r, we have

$$\partial_r f_j(r) \sim \sum_{\ell=0}^{\infty} \sum_{i=0}^{j} d_{j,i,\ell} r^{-\beta+\ell-1} (\log r)^{j-i} \quad \text{as } r \to 0. \tag{4.15}$$

In the proof of Lemma 4.2, at this point the proof was basically finished because the function $\partial_\rho f_j(\rho)$ in that proof was identically zero in Taylor series at $\rho = 0$; in our present situation, $\partial_r f_j(r)$ has an expansion at $r = 0$, which needs to be integrated to find $f_j(r)$. To this end, observe that an integration by parts argument shows that for any complex number $\tau \neq -1$, we have

$$\int t^\tau (\log t)^k \, dt = \frac{t^{\tau+1}}{\tau+1} (\log t)^k - \frac{k}{\tau+1} \int t^\tau (\log t)^{k-1} \, dt.$$

Using this recurrence formula, an induction argument shows that modulo an integration constant,

$$\int t^\tau (\log t)^k \, dt = \sum_{j=0}^{k} C_{\tau,j} t^{\tau+1} (\log t)^j, \quad \tau \neq -1,$$

for some constants $C_{\tau,j}$; when $\tau = -1$, we have

$$\int t^\tau (\log t)^k \, dt = \frac{1}{k+1} (\log t)^{k+1}, \quad \tau = -1.$$

Now integrating both sides of (4.15) and using the formulas for $\int t^\tau (\log t)^k \, dt$, we get a formula of the type

$$f_j(r) \sim D_j + \sum_{i=0}^{j} C_{ji} (\log r)^{j-i+1} + \sum_{\ell=0}^{\infty} \sum_{i=0}^{j} D_{j,i,\ell} r^{-\beta+\ell} (\log r)^{j-i} \quad \text{as } r \to 0,$$

for some constants D_j, C_{ji}, $D_{j,i,\ell}$ and where $C_{ji} = 0$ if $\beta \notin \mathbb{N}_0$. Now substituting this expression for $f_j(r)$ into (4.14) we obtain, as $r \to 0$,

$$\begin{aligned} \mathcal{T}_h(\phi, r) &= r^\beta \sum_{j=0}^{k} d_{k,j} (\log r)^{k-j} f_j(r) \\ &\sim r^\beta \sum_{j=0}^{k} d_{k,j} D_j (\log r)^{k-j} + r^\beta \sum_{j=0}^{k} \sum_{i=0}^{j} d_{k,j} C_{ji} (\log r)^{k-i+1} \end{aligned}$$

$$+\sum_{\ell=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} D_{j,i,\ell} d_{k,j} r^{\ell} (\log r)^{k-i}.$$

This expansion completes our proof. □

Next, we need the following technical result.

Lemma 4.7. *Let $a_1, a_2, a_3, \ldots \in \mathcal{A}^{\mathcal{I}}([0,\infty))$ and suppose that $f : (0,\infty) \to \mathbb{C}$ has the property that for all $N \in \mathbb{N}$ sufficiently large, we can write*

$$f = \sum_{i=1}^{j} a_i + f_j,$$

where $f_j \in C^{\ell_j}([0,\infty))$ with $\ell_j \to \infty$ as $j \to \infty$. Then $f \in \mathcal{A}^{\mathcal{I}\cup\mathbb{N}_0}([0,\infty))$.

Proof. For fixed $N_0 \in \mathbb{N}$, we shall prove that

$$f = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N_0} r^{\alpha} (\log r)^k b_{\alpha,k} + \sum_{k=0}^{N_0-1} c_k r^k + r^{N_0} g_{N_0},$$

where $g_{N_0} \in S^0([0,\infty))$; this proves our result. To prove this, choose $N \in \mathbb{N}$ and then choose j such that $\ell_j > N_0 + N$. Then by assumption we can write $f = \sum_{i=1}^{j} a_i + f_j$, where $f_j \in C^{\ell_j}([0,\infty)) \subseteq C^{N_0+N}([0,\infty))$. In particular, expanding f_j in Taylor series up to order r^{N_0} we obtain

$$f_j = \sum_{k=0}^{N_0-1} c_k r^k + r^{N_0} f_{N_0}(r),$$

where $f_{N_0}(r) \in C^N([0,\infty))$. Expanding each a_i up to a term that is $\mathcal{O}(r^{N_0})$ we obtain

$$a_i = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N_0} r^{\alpha} (\log r)^k a_{i,\alpha,k} + r^{N_0} a_{i N_0}(r),$$

where $a_{i N_0}(r) \in S^0([0,\infty))$. Hence,

$$\begin{aligned} f &= \sum_{i=1}^{j}\Bigg(\sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N_0} r^{\alpha} (\log r)^k a_{i,\alpha,k} + r^{N_0} a_{i N_0}(r)\Bigg) + \sum_{k=0}^{N_0-1} c_k r^k + r^{N_0} f_{N_0}(r) \\ &= \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N_0} r^{\alpha} (\log r)^k b_{\alpha,k} + \sum_{k=0}^{N_0-1} c_k r^k + r^{N_0} g_{N_0}(r), \end{aligned}$$

where $b_{\alpha,k} = \sum_{i=1}^{j} a_{i,\alpha,k}$ and $g_{N_0}(r) = \sum_{i=1}^{j} a_{i N_0}(r) + f_{N_0}(r)$. Since $f_{N_0}(r) \in C^N([0,\infty))$ it follows that $(r\partial_r)^k g_{N_0}(r)$ is locally bounded on $[0,\infty)$ for all $k \leqslant N$ and $g_{N_0}(r)$ is in C^N

for r away from zero. Of course, since N is completely arbitrary, it follows that $g_{N_0}(r)$ is in $S^0([0,\infty))$. This completes our proof. $\square$

We now prove a converse statement to Theorem 4.4. To state this result, given an index set $\mathcal{I}$ we define the index set $\widehat{\mathcal{I}}$ as the index set defined as follows:

$$\boxed{(\alpha,k)\in\mathcal{I} \quad\Longrightarrow\quad \begin{cases}(\alpha,k)\in\widehat{\mathcal{I}},\\ (\alpha,k+1)\in\widehat{\mathcal{I}} \quad \text{if } \alpha\in\mathbb{N}_0,\\ (j,k)\in\widehat{\mathcal{I}} \quad \text{for all } j\in\mathbb{N}_0.\end{cases}} \tag{4.16}$$

We box the formula because it plays such an important rôle in this paper. The following theorem gives a partial "converse" to Theorem 4.4.

Theorem 4.8. *We have*

$$\phi\in\mathcal{A}^{\mathcal{I}+n}\big(\overline{\mathbb{R}^n}\big) \quad\Longrightarrow\quad \mathcal{T}_h(\phi)\in\mathcal{A}^{\widehat{\mathcal{I}}}\big([0,\infty)\big).$$

Proof. More explicitly, we need to show that an expansion of the sort

$$\phi\left(\frac{1}{\rho},\omega\right)\sim\sum_{(\beta,k)\in\mathcal{I}}\rho^{\beta+n}(\log\rho)^k a_{\beta,k}(\omega) \quad \text{as } \rho\to 0,$$

implies an expansion of the sort

$$\mathcal{T}_h(\phi,r)\sim\sum_{(\beta,k)\in\widehat{\mathcal{I}}} b_{\beta,k}r^\beta(\log r)^k \quad \text{as } r\to 0.$$

If $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near 0 and $\chi(\xi)\equiv 1$ outside a neighborhood of 0, then

$$\mathcal{T}_h(\phi)=\mathcal{T}_h(\chi\phi)+\mathcal{T}_h\big((1-\chi)\phi\big).$$

Since a smooth compactly supported function gives rise to an h-transform that is smooth at $r=0$, the term $\mathcal{T}_h((1-\chi)\phi)$ is smooth at $r=0$, so we may focus on $\mathcal{T}_h(\chi\phi)$. Put

$$\phi_{\beta,k}(\xi):=\chi\left(\frac{\omega}{\rho}\right)\rho^{\beta+n}(\log\rho)^k a_{\beta,k}(\omega)=\chi(\xi)|\xi|^{-\beta-n}\big(-\log|\xi|\big)^k a_{\beta,k}\big(\xi/|\xi|\big),$$

where $\rho=1/|\xi|$ and $\omega=\xi/|\xi|$. Then fixing any large $N\gg 0$, consider the finite expansion

$$\chi\phi=\sum_{(\beta,k)\in\mathcal{I},\,\Re\beta\leqslant N}\phi_{\beta,k}(\xi)+f_N(\rho,\omega),$$

where $f_N(\rho,\omega)\in\rho^N S^0([0,\infty)_\rho\times\mathbb{S}^{n-1}_\omega)$ and vanishes for ρ large (that is, near the origin in $\mathbb{R}^n$). In particular,

$$\mathcal{T}_h(\chi\phi,r)=\sum_{(\beta,k)\in\mathcal{I},\,\Re\beta\leqslant N}\mathcal{T}_h(\phi_{\beta,k},r)+\mathcal{T}_h(f_N,r).$$

By Lemma 4.6 we know that as $r \to 0$ we can write

$$\mathcal{T}_h(\phi_{\beta,k}, r) \sim r^\beta \sum_{j=0}^{k} a_j (\log r)^j + r^\beta \sum_{j=0}^{k} b_j (\log r)^{j+1} + \sum_{\ell=0}^{\infty} \sum_{j=0}^{k} c_{\ell,j} r^\ell (\log r)^j,$$

where $b_j = 0$ if $\beta \notin \mathbb{N}_0$, which implies that $\phi_{\beta,k} \in \mathcal{A}^{\widehat{\mathcal{I}}}([0,\infty))$ by definition of $\widehat{\mathcal{I}}$ in (4.16). Because $f_N(\rho, \omega) \in \rho^N S^0([0,\infty)_\rho \times \mathbb{S}^{n-1}_\omega) = \mathcal{O}(|\xi|^{-N})$ and is compactly supported in ρ, it follows that

$$\mathcal{T}_h(f_N, r) = \int h(r\xi) f_N(\xi)\, d\xi \in C^{N-n-1}\big([0,\infty)\big).$$

Since $N - n - 1 \to \infty$ as $N \to \infty$, by Lemma 4.7 it follows that $\mathcal{T}_h(\phi) \in \mathcal{A}^{\widehat{\mathcal{I}} \cup \mathbb{N}_0}([0,\infty))$. Finally, since $\mathbb{N}_0 \subseteq \widehat{\mathcal{I}}$, we have $\mathcal{A}^{\widehat{\mathcal{I}} \cup \mathbb{N}_0}([0,\infty)) = \mathcal{A}^{\widehat{\mathcal{I}}}([0,\infty))$ and our proof is complete. □

Remark 4.9. Remark 4.5 holds for Theorem 4.8 as well: If $\phi_t \in \mathcal{A}^{\mathcal{I}+n}(\overline{\mathbb{R}^n})$ depends smoothly on a parameter t, then $\mathcal{T}_h(\phi_t) \in \mathcal{A}^{\widehat{\mathcal{I}}}([0,\infty))$ also depends smoothly on t.

5. PDOs on blown-up spaces I

Suppose we have a (classical) pseudodifferential operator on a smooth manifold without boundary. If we blow-up codimension two submanifolds inside M, what are the mapping properties (if any) of the pseudodifferential operator on polyhomogeneous functions on the resulting blown-up manifold? The goal of this section is to answer this question (see Theorem 5.2) and then use the answer to derive properties of our Dirac operator (see Theorems 5.3 and 5.4).

5.1. Symbols and polyhomogeneous functions

Let M be a smooth manifold without boundary and let $N \subseteq M$ be a connected codimension two embedded submanifold of M; thus, centered at each point in N there is a coordinate patch $\mathcal{U}$ of M such that

$$\mathcal{U} = \mathcal{V} \times \mathcal{W}, \tag{5.1}$$

where $\mathcal{V} \subseteq \mathbb{R}^2$ is open containing $(0,0)$ and $\mathcal{W} \subseteq \mathbb{R}^{n-2}$ is open such that $\mathcal{U} \cap N = \{(0,0)\} \times \mathcal{W}$. Assume that $\phi \in \mathcal{A}^{\mathcal{I}}([M;N])$ is supported in the coordinate patch (5.1). Recall from (4.4) this means that with $(u,v) = r\omega$ written in polar coordinates, where (u,v) are the rectangular coordinates on $\mathcal{V}$, we have, for any $N \gg 0$,

$$\phi(r\omega, y) = \sum_{(\alpha,k)\in\mathcal{I},\, \Re\alpha \leqslant N} r^\alpha (\log r)^k a_{\alpha,k}(\omega, y) + r^N f_N(r, \omega, y), \tag{5.2}$$

where the coefficients $a_{\alpha,k}(\omega, y)$ are smooth functions of ω and y and are compactly supported in y, and $f_N(r,\omega,y)$ is bounded with all b-derivatives in $r \in [0,\infty)$ and derivatives in ω and y.

Note that the expansion (5.2) and the fact that ϕ is compactly supported in y are equivalent to the statement

$$y \mapsto \phi(u, v, y) \in C_c^\infty\big(\mathcal{W}; \mathcal{A}^{\mathcal{I}}\big([\mathcal{U}; \{0\}]\big)\big),$$

the compactly supported functions in $\mathcal{W}$ with values in the space $\mathcal{A}^{\mathcal{I}}([\mathcal{U}; \{0\}])$. Here we recall that $\mathcal{A}^{\mathcal{I}}([\mathcal{U}; \{0\}])$ has a natural Fréchet topology so the notion $\phi \in C_c^\infty(\mathcal{W}; \mathcal{A}^{\mathcal{I}}([\mathcal{U}; \{0\}]))$ is well defined. Below we shall use similar Fréchet space-valued notation so we do not have to write out long expansions of the sort (5.2). For example, $\mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^m}))$, which appears in Lemma 5.1 below, denotes the space of all $\mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^m})$-valued Schwartz functions on $\mathbb{R}^n$. For any $\alpha \in \mathbb{C}$, we define

$$S_{c\ell}^{-\alpha}\big(\mathbb{R}^p\big) := \mathcal{A}^{\alpha + \mathbb{N}_0}\big(\overline{\mathbb{R}^p}\big),$$

where $\alpha + \mathbb{N}_0 = \{\alpha + k \mid k \in \mathbb{N}_0\}$. It is straightforward to check that $a(\eta) \in S_{c\ell}^{-\alpha}(\mathbb{R}^p)$ just means that $a(\eta)$ is a classical symbol of order α in the usual sense: For any N we can write

$$a(\eta) = \sum_{j=0}^{N} \chi(\eta) a_j\left(\frac{\eta}{|\eta|}\right)|\eta|^{-\alpha-j} + \big(1 + |\eta|^2\big)^{-\frac{\alpha+N+1}{2}} b_N(\eta), \tag{5.3}$$

where $\chi(\eta) = 0$ near $\eta = 0$ and $\chi(\eta) = 1$ for $\eta \geqslant 1$, $a_j \in C^\infty(\mathbb{S}^{p-1})$, and for any γ the function $|\eta|^{|\gamma|}\partial_\eta^\gamma b_N(\eta)$ is bounded.

The following lemma will be an important ingredient in the theorems in the subsequent sections.

Lemma 5.1. *For any $\alpha \in \mathbb{C}$ and index set $\mathcal{I}$, if $a(\xi, \tau) \in S_{c\ell}^{-\alpha}(\mathbb{R}^m \times \mathbb{R}^n)$ and $\tau \mapsto \varphi(\xi, \tau) \in \mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^m}))$, then*

$$\tau \mapsto a(\xi, \tau)\varphi(\xi, \tau) \in \mathscr{S}\big(\mathbb{R}^n; \mathcal{A}^{\alpha+\mathcal{I}}\big(\overline{\mathbb{R}^m}\big)\big).$$

Proof. Since $\varphi(\xi, \tau) \in \mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^m}))$, for any $N \gg 0$ we can write

$$\varphi(\xi, \tau) = \sum_{(\beta,k)\in\mathcal{I},\, \Re\beta \leqslant N} \chi(\xi)|\xi|^{-\beta}\big(\log|\xi|\big)^k \varphi_{\beta,k}(\tau) + \big(1 + |\xi|^2\big)^{-\frac{N+1}{2}} R_N(\xi, \tau),$$

where $\chi(\xi) = 0$ near $\xi = 0$ and $\chi(\xi) = 1$ for $|\xi| \geqslant 1$, $\varphi_{\beta,k}(\tau) \in \mathscr{S}(\mathbb{R}^n)$, and for any multi-indices μ, ν, γ,

$$\xi^\mu \tau^\gamma \partial_\xi^\mu \partial_\tau^\nu R_N(\xi, \tau) \quad \text{is bounded.} \tag{5.4}$$

Now let $\varphi(\tau) \in \mathscr{S}(\mathbb{R}^n)$ and *suppose* we can show that for any $N \gg 0$, we can write

$$a(\xi, \tau)\varphi(\tau) = \sum_{j=0}^{N} \chi(\xi)|\xi|^{-\alpha-j}\psi_j(\tau) + \big(1 + |\xi|^2\big)^{-\frac{\alpha+N+1}{2}} S_N(\xi, \tau), \tag{5.5}$$

where $\psi_j(\tau) \in \mathscr{S}(\mathbb{R}^n)$, and $S_N(\xi,\tau)$ has the same boundedness property (5.4) as $R_N(\xi,\tau)$ above. Then writing

$$a(\xi,\tau)\varphi(\xi,\tau) = \sum_{(\beta,k)\in\mathcal{I},\, \Re\beta\leqslant N} \chi(\xi)|\xi|^{-\beta}\big(\log|\xi|\big)^k a(\xi,\tau)\varphi_{\beta,k}(\tau) + \big(1+|\xi|^2\big)^{-\frac{N+1}{2}} a(\xi,\tau)R_N(\xi,\tau),$$

and applying the expansion (5.5) to $a(\xi,\tau)\varphi_{\beta,k}(\tau)$ for each β, k, it follows that

$$\tau \mapsto a(\xi,\tau)\varphi(\xi,\tau) \in \mathscr{S}\big(\mathbb{R}^n_\tau; \mathcal{A}^{\alpha+\mathcal{I}}\big(\overline{\mathbb{R}^m_\xi}\big)\big),$$

which is exactly the statement of this lemma. Thus, it suffices to prove (5.5).

To prove (5.5) we first use the expansion (5.3) of elements in the symbol space $S^{-\alpha}_{c\ell}(\mathbb{R}^m\times\mathbb{R}^n)$: For any N we can write

$$a(\xi,\tau) = \sum_{j=0}^{N} \chi(\xi,\tau)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j} + \big(1+|\xi|^2+|\tau|^2\big)^{-\frac{\alpha+N+1}{2}} b_N(\xi,\tau),$$

where $\langle\xi,\tau\rangle = (|\xi|^2+|\tau|^2)^{1/2}$, $\chi(\xi,\tau)=0$ near $(\xi,\tau)=0$ and $\chi(\xi,\tau)=1$ for $\langle\xi,\tau\rangle \geqslant 1$, $a_j(\theta)\in C^\infty(\mathbb{S}^{m+n-1})$, and for any γ,

$$\langle\xi,\tau\rangle^{|\gamma|}\partial^\gamma_{(\xi,\tau)} b_N(\xi,\tau) \quad \text{is bounded.}$$

Thus,

$$\begin{aligned} a(\xi,\tau)\varphi(\tau) &= \sum_{j=0}^{N} \chi(\xi,\tau)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau) + \big(1+|\xi|^2+|\tau|^2\big)^{-\frac{\alpha+N+1}{2}} b_N(\xi,\tau)\varphi(\tau) \\ &= \sum_{j=0}^{N} \chi(\xi,\tau)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau) \\ &\quad + \big(1+|\xi|^2\big)^{-\frac{\alpha+N+1}{2}}\left[\left(1+\frac{|\tau|^2}{1+|\xi|^2}\right)^{-\frac{\alpha+N+1}{2}} b_N(\xi,\tau)\varphi(\tau)\right]. \end{aligned}$$

The remainder term in brackets is easily shown to satisfy the boundedness property (5.4). Therefore, to prove (5.5) it suffices to prove that for each j,

$$\chi(\xi,\tau)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau)$$

has an expansion of the form (5.5). To see this, first write

$$\begin{aligned} &\chi(\xi,\tau)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau) \\ &\quad = \chi(\xi)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau) + \big(\chi(\xi,\tau)-\chi(\xi)\big)a_j(\theta)\langle\xi,\tau\rangle^{-\alpha-j}\varphi(\tau). \end{aligned}$$

Notice that for $|\xi| \geqslant 1$, we have $\chi(\xi,\tau) - \chi(\xi) = 1 - 1 = 0$, so we can drop the last term and focus on the first term $\chi(\xi) a_j(\theta) \langle \xi,\tau\rangle^{-\alpha-j} \varphi(\tau)$.

In summary, we have reduced our theorem to deriving an expansion of the sort (5.5) for the special case

$$a(\xi,\tau) = \chi(\xi) b(\theta) \langle \xi,\tau\rangle^{-\alpha} \quad \text{and} \quad \varphi(\tau) \in \mathscr{S}\big(\mathbb{R}^n\big),$$

where $b(\theta) \in C^\infty(\mathbb{S}^{m+n-1})$. By choosing a partition of unity of the sphere $\mathbb{S}^{m+n-1}$, we may further consider two cases for any fixed $c > 0$:

Case 1. $a(\xi,\tau)$ is supported in an angle of the form $|\xi| \leqslant c|\tau|$.
Case 2. $a(\xi,\tau)$ is supported in an angle of the form $|\tau| \leqslant c|\xi|$.

In Case 1, for any $N \gg 0$, we can write

$$a(\xi,\tau)\varphi(\tau) = \big(1 + |\xi|^2\big)^{-N/2} \cdot \left(\frac{1+|\xi|^2}{1+|\tau|^2}\right)^{N/2} a(\xi,\tau)\varphi_N(\tau),$$

where $\varphi_N(\tau) = (1+|\tau|^2)^{N/2}\varphi(\tau)$ is still a Schwartz function and the function $(\frac{1+|\xi|^2}{1+|\tau|^2})^{N/2}$ is bounded. Therefore, if $a(\xi,\tau)$ is supported in an angle $|\xi| \leqslant c|\tau|$, then because of the decaying term $(1+|\xi|^2)^{-N/2}$ and the arbitrariness of N it is clear that such a term will contribute to the residual term in (5.5). Therefore, we may consider Case 2 when $a(\xi,\tau)$ is supported in an angle of the form $|\tau| \leqslant c|\xi|$. In this case, we may use projective coordinates

$$\theta = (\theta_1,\theta_2) = \left(\frac{\xi}{|\xi|}, \frac{\tau}{|\xi|}\right),$$

which are coordinates on the sphere $\mathbb{S}^{m+n-1}$ in the region where $|\tau| \leqslant c|\xi|$ (in fact, even in the larger region where $|\xi| > 0$). In this case, we have

$$a(\xi,\tau)\varphi(\tau) = \chi(\xi)\langle \xi,\tau\rangle^{-\alpha}\varphi(\tau) b\left(\frac{\xi}{|\xi|}, \frac{\tau}{|\xi|}\right).$$

Expanding $b(\theta_1,\theta_2)$ in a partial Taylor expansion at $\theta_2 = 0$, we get

$$b\left(\frac{\xi}{|\xi|}, \frac{\tau}{|\xi|}\right) = \sum_{|\gamma|=0}^{N} b_\gamma\left(\frac{\xi}{|\xi|}\right)\tau^\gamma |\xi|^{-|\gamma|} + \sum_{|\gamma|=N+1} \tau^\gamma |\xi|^{-N-1} b_\gamma(\theta_1,\theta_2), \tag{5.6}$$

where the b_γ's are smooth. By the binomial theorem we have

$$(1+x)^{-\alpha/2} = \sum_{j=0}^{N} c_j x^j + x^{N+1} f_N(x),$$

where $f_N \in C^\infty([0,\infty))$. Using this expansion we have

$$\langle \xi, \tau \rangle^{-\alpha} = \big(|\xi|^2 + |\tau|^2\big)^{-\alpha/2} = |\xi|^{-\alpha}\left(1 + \frac{|\tau|^2}{|\xi|^2}\right)^{-\alpha/2}$$

$$= \sum_{j=0}^{N} c_j |\xi|^{-\alpha-2j}|\tau|^{2j} + |\xi|^{-\alpha-2N-2}|\tau|^{2N+2} f_N\left(\frac{|\tau|^2}{|\xi|^2}\right). \tag{5.7}$$

Finally, multiplying the expansions (5.6) and (5.7) and using that $\tau^\gamma |\tau|^{2j}\varphi(\tau)$ is Schwartz for any nonnegative powers γ and j, we see that

$$a(\xi, \tau)\varphi(\tau) = \chi(\xi)\langle \xi, \tau\rangle^{-\alpha}\varphi(\tau) \cdot b\left(\frac{\xi}{|\xi|}, \frac{\tau}{|\xi|}\right)$$

has the desired expansion (5.5). □

5.2. Inverse on the total space

Let

$$\mathcal{D} : C^\infty(M, E) \to C^\infty(M, F)$$

be a Dirac-type operator and consider the coordinate patch $\mathcal{U} = \mathcal{V} \times \mathcal{W}$ in (5.1) considered earlier. If (u, v) denote the coordinates on $\mathcal{V}$, then over $\mathcal{U}$ we have

$$\mathcal{D} = G_1 \partial_u + G_2 \partial_v + B,$$

where $G_1, G_2 : E \to F$ are bundle maps over $\mathcal{U}$ and $B = B_{(u,v)}$ is a first-order differential operator on $\mathcal{W}$ depending on (u, v). Choosing polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$ in the $\mathcal{V}$ factor, one can check that

$$\mathcal{D} = (\cos\theta G_1 + \sin\theta G_2)\partial_r + (\cos\theta G_2 - \sin\theta G_1)\frac{1}{r}\partial_\theta + B.$$

From this formula and the expansion (5.2) for a typical element of $\mathcal{A}^{\mathcal{I}}([M; N])$, one can show that the Dirac-type operator has the mapping property

$$\mathcal{D} : \mathcal{A}^{\mathcal{I}}\big([M; N], E\big) \to \mathcal{A}^{\mathcal{I}-1}\big([M; N], F\big),$$

for an arbitrary index family $\mathcal{I}$. More generally, Theorem 5.2 below gives the mapping properties for *any* pseudodifferential operator on such polyhomogeneous spaces. If $\mathcal{I} = \{\mathcal{I}_H\}$ is an index family associated to some boundary hypersurfaces of a manifold with corners, then we define $\widehat{\mathcal{I}} := \{\widehat{\mathcal{I}}_H\}$ where $\widehat{\mathcal{I}}_H$ is defined in (4.16).

Theorem 5.2. *Any classical pseudodifferential operator $A \in \Psi^m(M, E, F)$ of order $m \in \mathbb{C}$ defines a continuous linear map*

$$A : \mathcal{A}_c^{\mathcal{I}}\big([M; N], E\big) \to \mathcal{A}^{\widehat{\mathcal{I}-m}}\big([M; N], F\big),$$

for any index family $\mathcal{I} > -2$ and where the subscript c on the left denotes compactly supported sections.

Proof. In this proof we drop the bundles for notational simplicity. Since the Schwartz kernel of A is smoothing off the diagonal, we may assume that A is supported in a single coordinate patch of M, and in particular, in a coordinate patch near N

$$\mathcal{U} = \mathcal{V} \times \mathcal{W},$$

where $\mathcal{V} \subseteq \mathbb{R}^2$ is open containing $(0,0)$ and $\mathcal{W} \subseteq \mathbb{R}^{n-2}$ is open such that $\mathcal{U} \cap N = \{(0,0)\} \times \mathcal{W}$. If (u, v) denotes the coordinates on $\mathcal{V}$, then over the coordinate patch $\mathcal{U}$, modulo a smoothing operator we have

$$A\phi = \int_{\mathbb{R}^n} e^{iu\xi + iv\eta + iy\cdot\tau} a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau)\, đ\xi\, đ\tau\, đ\eta,$$

where $a(u, v, y, \xi, \eta, \tau)$ is the complete symbol of A, which is a classical (polyhomogeneous) symbol of order m in (ξ, η, τ). Assume that $\phi \in \mathcal{A}^{\mathcal{I}}([M; N])$ is supported in the coordinate patch $\mathcal{U} = \mathcal{V} \times \mathcal{W}$. This means that

$$y \mapsto \phi(u, v, y) \in C_c^\infty\big(\mathcal{W}; \mathcal{A}^{\mathcal{I}}\big([\mathcal{U}; \{0\}]\big)\big),$$

the compactly supported functions in $\mathcal{W}$ with values in the space $\mathcal{A}^{\mathcal{I}}([\mathcal{U}; \{0\}])$. Taking the Fourier transform in y, we see that

$$\tau \mapsto \int_{\mathbb{R}^{n-2}} e^{-iy\cdot\tau}\phi(u, v, y)\, dy \in \mathscr{S}\big(\mathbb{R}^{n-2}; \mathcal{A}^{\mathcal{I}}\big([\mathcal{U}; \{0\}]\big)\big), \tag{5.8}$$

the space of Schwartz functions with values in $\mathcal{A}^{\mathcal{I}}([\mathcal{U}; \{0\}])$. Now using Theorem 4.4 it follows that taking the Fourier transform of (5.8) with respect to (u, v), we have

$$\tau \mapsto \hat{\phi}(\xi, \eta, \tau) \in \mathscr{S}\big(\mathbb{R}^{n-2}; \mathcal{A}^{\mathcal{I}+2}\big(\overline{\mathbb{R}^2}\big)\big).$$

Since $a(u, v, y, \xi, \eta, \tau)$ is a classical symbol of order m, by Lemma 5.1 we see that for fixed u, v, y,

$$\tau \mapsto a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau) \in \mathscr{S}\big(\mathbb{R}^{n-2}; \mathcal{A}^{\mathcal{I}-m+2}\big(\overline{\mathbb{R}^2}\big)\big).$$

Of course, this is smooth and compactly supported in $(u, v, y) \in \mathcal{U}$, so we actually have

$$\big((u, v, y), \tau\big) \mapsto a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau) \in C^\infty\big(\mathcal{U} \times \mathbb{R}^{n-2}; \mathcal{A}^{\mathcal{I}-m+2}\big(\overline{\mathbb{R}^2}\big)\big),$$

and is Schwartz in τ. Integrating out the τ variable by taking the inverse Fourier transform in τ, we get

$$(u, v, y) \mapsto \int_{\mathbb{R}^n} e^{iy\cdot\tau} a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau)\, d\tau \in C_c^\infty\big(\mathcal{U}; \mathcal{A}^{\mathcal{I}-m+2}\big(\overline{\mathbb{R}^2}\big)\big).$$

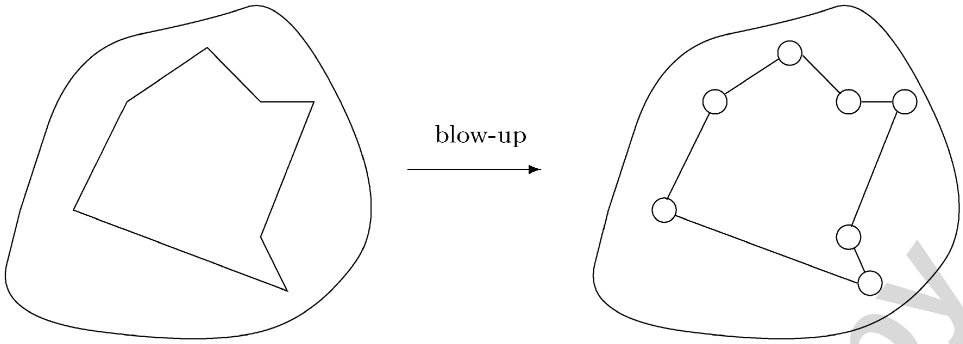

Fig. 7. Blowing up the corners $M_2(X)$ of X in the extended manifold M forms $[M; M_2(X)]$.

Finally, taking the inverse transform in (ξ, η) and using Theorem 4.8, we get

$$\int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau)\, đ\xi\, đ\tau\, đ\eta \in C_c^\infty\big(\mathcal{W}; \mathcal{A}^{\widehat{\mathcal{I}-m}}\big([\mathcal{U}; \{0\}]\big)\big),$$

which is exactly what we wanted to show. $\square$

Let $X \subseteq M$ be a compact manifold with corners of codimension two with $\dim X = \dim M$. Recall from Section 4.1 that $M_d(X)$ denotes the set of codimension d faces of X. Since X is of codimension two, $M_2(X)$ consists of finitely many pairwise disjoint smooth compact connected manifolds without boundary, the dimension of each equal to $\dim M - 2$. In particular, we can define the blown-up space (see Fig. 7)

$$\big[M; M_2(X)\big]$$

via (4.8) by simply blowing-up (or taking polar coordinates around) each element of $M_2(X)$. We henceforth assume that $\mathcal{D}^{-1}$ exists on compactly supported functions—the assumption (1.5). Then we have the following result.

Theorem 5.3. *We have*

$$\mathcal{D}^{-1} : \mathcal{A}_c^{\mathcal{I}}\big(\big[M; M_2(X)\big], F\big) \to \mathcal{A}^{\widehat{\mathcal{I}+1}}\big(\big[M; M_2(X)\big], E\big)$$

for any index family $\mathcal{I} > -2$.

Proof. We have

$$\big[M; M_2(X)\big] = [M; Y_1; Y_2; \cdots; Y_\ell],$$

where $Y_1, \dots, Y_\ell$ are the connected codimension two components of X. Since $\mathcal{D}^{-1}$ is pseudodifferential (not necessarily properly supported but this makes no difference), its Schwartz kernel is smoothing off the diagonal in $M \times M$, so it suffices to assume that X has exactly one codimension two face N and we just have to show that

$$\mathcal{D}^{-1} : \mathcal{A}_c^{\mathcal{I}}\big([M; N], F\big) \to \mathcal{A}^{\widehat{\mathcal{I}+1}}\big([M; N], E\big)$$

for any index set $\mathcal{I} > -2$. But this follows from the previous theorem and the fact that $\mathcal{D}^{-1} \in \Psi^{-1}(M, F, E)$. $\square$

5.3. Surjectivity of $\mathcal{D}$

Given any index family $\mathcal{I} = \{\mathcal{I}_Y \mid Y \in M_2(X)\}$, we can consider $\mathcal{I}$ as an index family on X_{tb} where we associate $\mathcal{I}_Y$ to the (unique) boundary face in X_{tb} obtained by the blow-up of Y in forming $X_{\mathrm{tb}} := [X; M_2(X)]$. Let $\overline{\mathcal{I}}$ denote the collection of all index families $\mathcal{F} = \{\mathcal{F}_Y \mid Y \in M_2(X)\}$ such that

$$\boxed{\mathcal{F} \in \overline{\mathcal{I}} \quad \Longleftrightarrow \quad \mathcal{F} \text{ is an index family and } \mathcal{F}_Y \subseteq \mathcal{I}_Y \cup (\mathbb{N}_0 \times \mathbb{N}_0) \quad \forall Y \in M_2(X).} \tag{5.9}$$

Then we define $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$ as the set of all sections of E such that

$$\phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \quad \Longleftrightarrow \quad \phi \in \mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}}, E), \quad \text{where } \mathcal{F} \in \overline{\mathcal{I}}.$$

Thus, an element of $\mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}}, E)$ basically has the same expansion as an element of $\mathcal{A}^{\mathcal{I}}(X_{\mathrm{tb}}, E)$ except, because of the union with $\mathbb{N}_0 \times \mathbb{N}_0$, one can add smooth expansions multiplied by logarithms. An important property of $\overline{\mathcal{I}}$ is that if $\mathcal{F} \in \overline{\mathcal{I}}$, then $\widehat{\mathcal{F}} \in \overline{\mathcal{I}}$, which follows from the definition (4.16) of $\widehat{\mathcal{F}}$.

We can put a natural complete metric topology on $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$ such that if $\{\phi_j\}$ is a sequence in $\mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$, then $\phi_j \to \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$ if and only if for j sufficiently large, $\phi_j, \phi \in \mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}}, E)$ for some fixed index set $\mathcal{F} \in \overline{\mathcal{I}}$ and $\phi_j \to \phi$ in $\mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}}, E)$.

Theorem 5.3 will be used to prove the following result.

Theorem 5.4. *For any index family $\mathcal{I} > -2$ associated to $M_2(X)$, the Dirac operator*

$$\mathcal{D} : \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)$$

is surjective.

Proof. Let $\psi \in \mathcal{A}^{\mathcal{F}-1}(X_{\mathrm{tb}}, F)$ where $\mathcal{F} \in \overline{\mathcal{I}}$. We claim that we can extend ψ to an element $\tilde{\psi} \in \mathcal{A}_c^{\mathcal{F}-1}([M; M_2(X)], F)$. Indeed, observe that if $Y \in M_2(X)$ and $M \cong (-1, 1)^2 \times Y$ and $X \cong [0, 1)^2 \times Y$ near $Y \subseteq X$, then writing these coordinates in polar form we have

$$\bigl[M; M_2(X)\bigr] \cong [0, \varepsilon)_r \times \mathbb{S}^1 \times Y, \qquad X_{\mathrm{tb}} \cong [0, \varepsilon)_r \times \left[0, \frac{\pi}{2}\right] \times Y, \quad \varepsilon > 0.$$

Now the extension claim is clear since a function $\psi(r, \theta, y)$ in the coordinates $[0, \varepsilon)_r \times [0, \frac{\pi}{2}] \times Y$ can always be extended to a function $\tilde{\psi}(r, \theta, y)$ on $[0, \varepsilon)_r \times \mathbb{S}^1 \times Y$. Note that $\tilde{\psi}$ is not unique, but this non-uniqueness is irrelevant in what follows. By Theorem 5.3, we know that

$$\tilde{\phi} := \mathcal{D}^{-1} \tilde{\psi} \in \mathcal{A}^{\widehat{\mathcal{F}}}\bigl(\bigl[M; M_2(X)\bigr], E\bigr).$$

Restricting to $X_{\mathrm{tb}} \subseteq [M; M_2(X)]$, we get an element

$$\phi := \tilde{\phi}|_{X_{\mathrm{tb}}} \in \mathcal{A}^{\widehat{\mathcal{F}}}(X_{\mathrm{tb}}, E) \quad \Longrightarrow \quad \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E).$$

This section satisfies

$$\mathcal{D}\phi = \mathcal{D}(\tilde{\phi}|_{X_{\mathrm{tb}}}) = \big(\mathcal{D}\mathcal{D}^{-1}\tilde{\psi}\big)\big|_{X_{\mathrm{tb}}} = \tilde{\psi}|_{X_{\mathrm{tb}}} = \psi.$$

Therefore, $\mathcal{D} : \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)$ is surjective as stated. □

6. PDOs on blown-up spaces II

In Theorem 5.3 we proved that $\mathcal{D}^{-1}$ maps polyhomogeneous spaces on X_{tb} to polyhomogeneous spaces on X_{tb} (really $[M; M_2(X)]$ but it is useful to think of this mapping property in terms of $X_{\mathrm{tb}} \subseteq [M; M_2(X)]$). The goal of this section is to prove Theorem 6.3, which implies that $\mathcal{D}^{-1}$ maps polyhomogeneous spaces from the *boundary* ∂X to polyhomogeneous spaces on the total space X_{tb}. This fact will be used in Section 7 to analyze the Cauchy integral and transform.

6.1. Preliminary lemmas

We need a couple lemmas. Here is the first.

Lemma 6.1. *Let $\mathcal{U}$ be an open subset of $[0, \infty)_v \times \mathbb{R}^m_y$, for any index set $\mathcal{I}$ let*

$$a(v, y, \xi, \tau) \in C^\infty\big(\mathcal{U}_{(v,y)}; \mathscr{S}\big(\mathbb{R}^n_\tau; \mathcal{A}^{\mathcal{I}}\big(\overline{\mathbb{R}^p_\xi}\big)\big)\big),$$

let $q(\xi, \tau)$ be a positive-definite quadratic form and define $\langle 1, \xi, \tau\rangle^2 := 1 + q(\xi, \tau)$, and finally, define

$$f(v, y, \xi) := \int_{\mathbb{R}^n} e^{-v\langle 1,\xi,\tau\rangle + iy\cdot\tau} a(v, y, \xi, \tau)\, d\tau.$$

Then we can write

$$f(v, y, \xi) = e^{-v\langle 1,\xi,0\rangle} g(v, y, \xi),$$

where $g(v, y, \xi) \in C^\infty(\mathcal{U}_{(v,y)}; \mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^p_\xi}))$.

Proof. For simplicity assume that $a(v, y, \xi, \tau)$ has support for ξ near $\partial\overline{\mathbb{R}^p}$; the case when a has compact support in $\xi \in \mathring{\overline{\mathbb{R}^p}} \equiv \mathbb{R}^p$ is much simpler. Then the statement that $a(v, y, \xi, \tau) \in C^\infty(\mathcal{U}_{(v,y)}; \mathscr{S}(\mathbb{R}^n_\tau; \mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^p_\xi})))$ just means that in terms of the variables $\rho = 1/|\xi|$ and $\omega = \xi/|\xi|$ near $\partial\overline{\mathbb{R}^p}$, for any $N \gg 0$ we have

$$a\left(v,y,\frac{\omega}{\rho},\tau\right)=\sum_{(\alpha,k)\in\mathcal{I},\,\Re\alpha\leqslant N}\rho^{\alpha}(\log\rho)^{k}\chi(\rho)\varphi_{\alpha,k}(v,y,\omega,\tau)+\rho^{N}R_N(v,y,\rho,\omega,\tau),\tag{6.1}$$

where $\chi(\rho)$ is compactly supported and equals 1 for ρ near 0, $\varphi_{\alpha,k}(v,y,\omega,\tau)\in C^{\infty}(\mathcal{U}_{(v,y)}\times\mathbb{S}^{p-1}_{\omega};\mathscr{S}(\mathbb{R}^n_{\tau}))$, and for any multi-indices $\ell,\beta,\mu,\nu,\gamma$,

$$\rho^{\ell}\tau^{\gamma}\partial^{\beta}_{(v,y)}\partial^{\ell}_{\rho}\partial^{\mu}_{\omega}\partial^{\nu}_{\tau}R_N(v,y,\rho,\omega,\tau)\quad\text{is bounded.}\tag{6.2}$$

Since $\langle 1,\xi,\tau\rangle=(1+q(\xi,\tau))^{1/2}$ and $\langle 1,\xi,0\rangle=(1+q(\xi,0))^{1/2}$, we have

$$\langle 1,\xi,\tau\rangle-\langle 1,\xi,0\rangle=\frac{q(\xi,\tau)-q(\xi,0)}{\langle 1,\xi,\tau\rangle+\langle 1,\xi,0\rangle},$$

so

$$g(v,y,\xi)=e^{v\langle 1,\xi,0\rangle}f(v,y,\xi)=\int_{\mathbb{R}^n}e^{-v(\langle 1,\xi,\tau\rangle-\langle 1,\xi,0\rangle)+iy\cdot\tau}a(v,y,\xi,\tau)\,d\tau$$

$$=\int_{\mathbb{R}^n}e^{-v\frac{q(\xi,\tau)-q(\xi,0)}{\langle 1,\xi,\tau\rangle+\langle 1,\xi,0\rangle}+iy\cdot\tau}a(v,y,\xi,\tau)\,d\tau.$$

Replacing ξ with $\xi=\omega/\rho$, and for any real a, putting $\langle a,\xi,\tau\rangle^2:=a^2+q(\xi,\tau)$, and then using that

$$\langle 1,\omega/\rho,\tau\rangle^2=1+q(\omega/\rho,\tau)=\rho^{-2}\big(\rho^2+q(\omega,\rho\tau)\big)\quad\Longrightarrow\quad\langle 1,\omega/\rho,\tau\rangle=\rho^{-1}\langle\rho,\omega,\rho\tau\rangle,$$

we obtain

$$g\left(v,y,\frac{\omega}{\rho}\right)=\int_{\mathbb{R}^n}e^{-vh(\rho,\omega,\tau)+iy\cdot\tau}a\left(v,y,\frac{\omega}{\rho},\tau\right)d\tau,$$

where

$$h(\rho,\omega,\tau):=\frac{1}{\langle\rho,\omega,\rho\tau\rangle+\langle\rho,\omega,0\rangle}\cdot\frac{q(\omega,\rho\tau)-q(\omega,0)}{\rho}.$$

Therefore, using (6.1) we conclude that for any N,

$$g\left(v,y,\frac{\omega}{\rho}\right)=\sum_{(\alpha,k)\in\mathcal{I},\,\Re\alpha\leqslant N}\rho^{\alpha}(\log\rho)^{k}\chi(\rho)\int_{\mathbb{R}^n}e^{-vh(\rho,\omega,\tau)+iy\cdot\tau}\varphi_{\alpha,k}(v,y,\omega,\tau)\,d\tau$$

$$+\rho^{\alpha+N+1}\int_{\mathbb{R}^n}e^{-vh(\rho,\omega,\tau)+iy\cdot\tau}R_N(v,y,\rho,\omega,\tau)\,d\tau.\tag{6.3}$$

Noting that $q(\omega,\rho\tau)-q(\omega,0)$ vanishes at $\rho=0$ so that $h(\rho,\omega,\tau)$ is infinitely differentiable at $\rho=0$, and recalling that $\varphi_{\alpha,k}(v,y,\omega,\tau)$ and $R_N(v,y,\rho,\omega,\tau)$ are Schwartz in τ, it follows that

$$\int_{\mathbb{R}^n} e^{-vh(\rho,\omega,\tau)+iy\cdot\tau}\varphi_{\alpha,k}(v,y,\omega,\tau)\,d\tau \tag{6.4}$$

is infinitely differentiable at $\rho=0$ (and C^∞ in v, y, ω), and also

$$\int_{\mathbb{R}^n} e^{-vh(\rho,\omega,\tau)+iy\cdot\tau}R_N(v,y,\rho,\omega,\tau)\,d\tau$$

has the same properties as R_N in (6.2) except of course without the τ derivatives. Expanding the integral (6.4) in Taylor series at $\rho=0$ up to the ρ^N term and replacing the expansion into (6.3), we conclude that $g(v,y,\xi)\in C^\infty(\mathcal{U}_{(v,y)};\mathcal{A}^{\mathcal{I}}(\overline{\mathbb{R}^p_\xi}))$, which completes our proof. □

We need one more lemma.

Lemma 6.2. *For any index set $\mathcal{I}>-1$, let $\psi(t,\tau)\in\mathscr{S}(\mathbb{R}^n_\tau;\mathcal{A}^{\mathcal{I}}([0,\infty)_t))$ be compactly supported in t, let $\langle 1,\eta,\tau\rangle^2:=1+q(\eta,\tau)$ with $q(\eta,\tau)$ a positive-definite quadratic form on $\mathbb{R}^m\times\mathbb{R}^n$ and define*

$$f(\eta,\tau):=\int_0^\infty e^{-t\langle 1,\eta,\tau\rangle}\psi(t,\tau)\,dt.$$

Then $f(\eta,\tau)\in\mathscr{S}(\mathbb{R}^n_\tau;\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}^m_\eta}))$.

Proof. Throughout this proof, we put $\langle a,\eta,\tau\rangle^2:=a^2+q(\eta,\tau)$ and $\langle a,\eta\rangle:=\langle a,\eta,0\rangle$. Observe that

$$\langle 1,\eta,\tau\rangle=\langle 1,\eta,0\rangle+\big(\langle 1,\eta,\tau\rangle-\langle 1,\eta,0\rangle\big)=\langle 1,\eta\rangle+\left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}\right),$$

so we can rewrite $f(\eta,\tau)$ as

$$f(\eta,\tau)=\int_0^\infty e^{-t\langle 1,\eta,\tau\rangle}\psi(t,\tau)\,dt=\int_0^\infty e^{-t\langle 1,\eta\rangle}e^{-t\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}}\psi(t,\tau)\,dt. \tag{6.5}$$

Fix $N\gg 0$. Then we can write

$$\begin{aligned} e^{-t\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}} &= \sum_{j=0}^N c_j t^j\left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}\right)^j \\ &\quad + t^{N+1}\left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}\right)^{N+1}E_N\left(t\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle+\langle 1,\eta\rangle}\right), \end{aligned}$$

where $c_j = (-1)^{j-1}/j!$ and where $E_N(x) = \frac{(-1)^{N+1}}{(N+1)!}\int_0^1 e^{-ux}\,du$. Replacing this expression into (6.5) we obtain

$$f(\eta,\tau) = \sum_{j=0}^{N} c_j \int_0^\infty e^{-t\langle 1,\eta\rangle} t^j \left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)^j \psi(t,\tau)\,dt$$
$$+ \left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)^{N+1} \int_0^\infty e^{-t\langle 1,\eta\rangle} t^{N+1} E_N\left(t\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)\psi(t,\tau)\,dt. \tag{6.6}$$

Making the change of variables $t \mapsto t/\langle 1,\eta\rangle$ in the last integral, we see that

$$\int_0^\infty e^{-t\langle 1,\eta\rangle} t^{N+1} E_N\left(t\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)\psi(t,\tau)\,dt$$
$$= \langle 1,\eta\rangle^{-N-2} \int_0^\infty e^{-t} t^{N+1} E_N\left(\frac{t}{\langle 1,\eta\rangle}\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)\psi\left(\frac{t}{\langle 1,\eta\rangle},\tau\right)dt.$$

Recalling that $\psi(t,\tau)$ is Schwartz in τ, so that powers of τ multiplied with $\psi(t,\tau)$ have no effect on the Schwartz property in τ, it can be checked that the last term in (6.6) decays like $\langle 1,\eta\rangle^{-N-1}$ and is Schwartz in τ, even with all derivatives in η,τ. Therefore, our lemma follows once we prove the following claim:

$$\int_0^\infty e^{-t\langle 1,\eta\rangle} t^j \left(\frac{q(\eta,\tau)-q(\eta,0)}{\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle}\right)^j \psi(t,\tau)\,dt \in \mathscr{S}\left(\mathbb{R}^n_\tau; \mathcal{A}^{\mathcal{I}+j+1}\left(\overline{\mathbb{R}^m_\eta}\right)\right).$$

To prove this, observe that $q(\eta,\tau)-q(\eta,0)$ has at most a linear factor of η so that $(q(\eta,\tau) - q(\eta,0))^j$ is at most of order $|\eta|^j$ in η, and observe that $|\eta|^j\mathcal{A}^{\mathcal{J}}(\overline{\mathbb{R}^m}) \subseteq \mathcal{A}^{\mathcal{J}-j}(\overline{\mathbb{R}^m})$ for any index set $\mathcal{J}$. Also observe that $t^j\mathcal{A}^{\mathcal{I}}([0,\infty)) \subseteq \mathcal{A}^{\mathcal{I}+j}([0,\infty))$. With these observations in mind, it suffices to prove that for fixed j,

$$f_j(\eta,\tau) := \int_0^\infty e^{-t\langle 1,\eta\rangle} \frac{1}{(\langle 1,\eta,\tau\rangle + \langle 1,\eta\rangle)^j}\psi(t,\tau)\,dt \in \mathscr{S}\left(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}+2j+1}\left(\overline{\mathbb{R}^m}\right)\right),$$

where $\psi(t,\tau) \in \mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}+j}([0,\infty)))$ is compactly supported in t. Note that

$$\langle 1,\eta\rangle = q^{\frac{1}{2}}(\eta,0) + \left(\left(1+q(\eta,0)\right)^{\frac{1}{2}} - q^{\frac{1}{2}}(\eta,0)\right) = \langle 0,\eta\rangle + \left(\frac{1}{\langle 1,\eta\rangle + \langle 0,\eta\rangle}\right),$$

so if we put $\rho = 1/|\eta|$ and $\omega = \eta/|\eta|$, then after some simplification,

$$\langle 1,\omega/\rho\rangle = \rho^{-1}\langle 0,\omega\rangle + \left(\frac{\rho}{\langle \rho,\omega\rangle + \langle 0,\omega\rangle}\right).$$

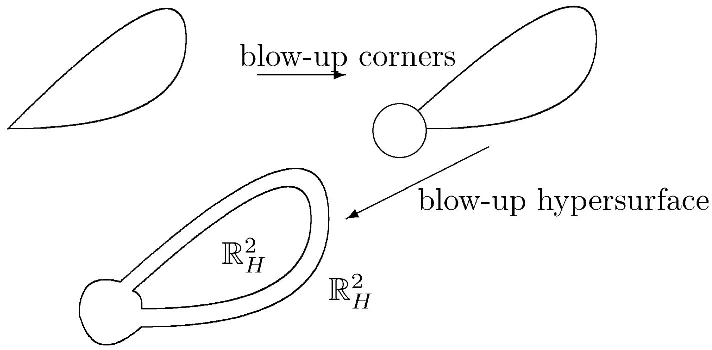

Fig. 8. A tied submanifold $H \subseteq \mathbb{R}^2$ and the blown-up manifold $\mathbb{R}^2_H$. Here, $\mathbb{R}^2_H = \mathbb{R}^2 \setminus$ the ("narrow hallway" $\cup$ "bubble") and consists of two connected components.

Hence,

$$f_j\left(\frac{\omega}{\rho}, \tau\right) = \rho^j \int_0^\infty e^{-t\langle 0,\omega\rangle/\rho} e^{-\frac{t\rho}{\langle\rho,\omega\rangle + \langle 0,\omega\rangle}} \frac{1}{(\langle\rho,\omega,\rho\tau\rangle + \langle\rho,\omega\rangle)^j} \psi(t,\tau)\, dt.$$

The function

$$e^{-\frac{t\rho}{\langle\rho,\omega\rangle + \langle 0,\omega\rangle}} \frac{1}{(\langle\rho,\omega,\rho\tau\rangle + \langle\rho,\omega\rangle)^j}$$

is infinitely differentiable at $\rho = 0$, so expanding this function in Taylor series at $\rho = 0$, we get an expansion in ρ with coefficients in terms of powers of t and powers of τ. The powers of τ multiplied with $\psi(t,\tau)$ have no effect on the Schwartz property in τ, and $t^k \mathcal{A}^{\mathcal{J}}([0,\infty)) \subseteq \mathcal{A}^{\mathcal{J}+k}([0,\infty))$ for any index set $\mathcal{J}$, so to prove that $f_j(\eta,\tau) \in \mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}+2j+1}(\overline{\mathbb{R}^m}))$ it suffices to prove that

$$\rho^j \int_0^\infty e^{-t\langle 0,\omega\rangle/\rho} \psi(t,\tau)\, dt \in \mathscr{S}\big(\mathbb{R}^n_\tau; \mathcal{A}^{\mathcal{I}+2j+k+1}\big(\overline{\mathbb{R}^m_\eta}\big)\big),$$

where $\psi(t,\tau) \in \mathscr{S}(\mathbb{R}^n; \mathcal{A}^{\mathcal{I}+j+k}([0,\infty)))$ is compactly supported in t. However, this fact follows from Theorem 4.4 and the fact that $\rho^j \mathcal{A}^{\mathcal{J}}(\overline{\mathbb{R}^m}) \subseteq \mathcal{A}^{\mathcal{J}+j}(\overline{\mathbb{R}^m})$ for any index set $\mathcal{J}$. $\square$

6.2. *Inverse from the boundary to the interior*

A *tied submanifold with boundary of codimension one in M* is just a codimension one face of a manifold with corners of codimension two $X \subseteq M$ with $\dim X = \dim M$. For example, the upper left pictures in Fig. 8 (the boundary of a "tear drop") and Fig. 9 (a curved segment) are examples of tied submanifolds with boundary of codimension one in $\mathbb{R}^2$. The word "tied" has to do with Fig. 8, where we see that the boundary is "tied" to itself [76]. We denote by $M_1(H) := \{Y \in M_2(X) \mid Y \subseteq H\}$. For example, $M_1(H)$ consists of just the vertex in the upper left picture in Fig. 8 and the two end points of the curved segment in Fig. 9. We define

$$M_H := \big[M; M_1(H); H\big].$$

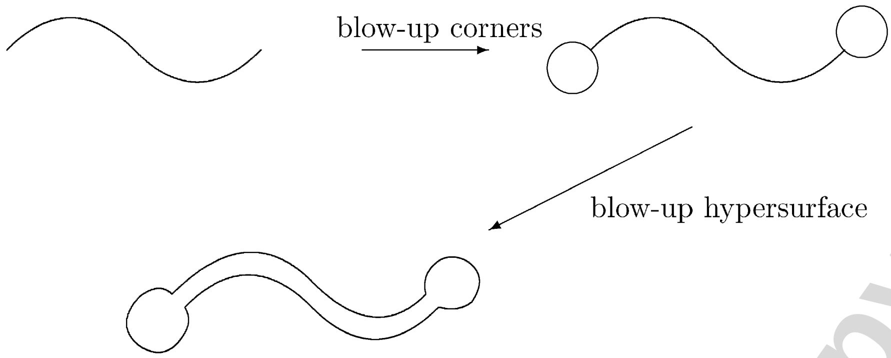

Fig. 9. A tied submanifold $H \subseteq \mathbb{R}^2$ and the blown-up manifold $\mathbb{R}^2_H$. Here, $\mathbb{R}^2_H = \mathbb{R}^2 \setminus$ the "curved dumbbell."

See Figs. 8 and 9 for examples. The proof of the following theorem is necessarily long and detailed and probably should be omitted at a first reading.

Theorem 6.3. *If $H \subseteq M$ is a compact tied submanifold with boundary of codimension one in M, then for any index family $\mathcal{I} > -1$ associated to $M_1(H)$, we have a continuous linear map*

$$\phi \mapsto \mathcal{D}^{-1}\delta_H G_H \phi : \mathcal{A}^{\mathcal{I}}(H, E) \to \mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E),$$

where $\widehat{\mathcal{I}}$ is defined in (4.16) *and is associated to the faces in M_H obtained from the blow-ups of the boundary components of H, δ_H is the delta function concentrated on H, and $G_H : E \to F$ is the principal symbol of $\mathcal{D}$ evaluated on the inward pointing unit normal vector field to H.*

Proof. Given $\phi \in \mathcal{A}^{\mathcal{I}}(H, E)$ we need to prove that $\mathcal{D}^{-1}\delta_H G_H \phi \in \mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E)$. As with the proof of Theorem 5.3, since the Schwartz kernel of $\mathcal{D}^{-1}$ is smoothing off the diagonal, it suffices to work in a single coordinate patch. In a coordinate patch near the interior of H (that is, away from $M_1(H)$) we can consider H as part of a smooth boundary of a manifold with boundary in M, so we can prove this result using techniques from the smooth boundary case as in [15,19, 24,43,107] and other papers. The new phenomenon is near a corner, say Y. If H happens to be tied at Y as in the far left picture in Fig. 10, then by choosing a partition of unity of H we can assume that $\phi \in \mathcal{A}^{\mathcal{I}}(H, E)$ is supported on only one of the ends entering Y; for example, in the far left picture in Fig. 10 we may assume that ϕ is supported on the horizontal portion of H. Thus, for the rest of this proof, we may assume that H takes the form in Fig. 11. Now consider a coordinate patch on M near a corner $Y \subseteq H$:

$$\mathcal{U} = \mathcal{V}_{(u,v)} \times \mathcal{W}_y,$$

where the subscripts denote the notation for coordinates, $\mathcal{V} \subseteq \mathbb{R}^2$ is open containing $(0, 0)$ and $\mathcal{W} \subseteq \mathbb{R}^{n-2}$ is open such that $\mathcal{U} \cap Y = \{(0, 0)\} \times \mathcal{W}$. We assume that in this coordinate patch, $\mathcal{U} \cap H$ is the horizontal line $\{u \geqslant 0, v = 0\}$ such as seen in Fig. 11, properties of which we shall be more specific later in our proof.

For a compactly supported function ϕ over $\mathcal{U}$, modulo a smoothing operator we have

$$\mathcal{D}^{-1}\phi = \int_{\mathbb{R}^n} e^{iu\xi + iv\eta + iy\cdot\tau} a(u, v, y, \xi, \eta, \tau)\hat{\phi}(\xi, \eta, \tau)\, đ\xi\, đ\tau\, đ\eta,$$

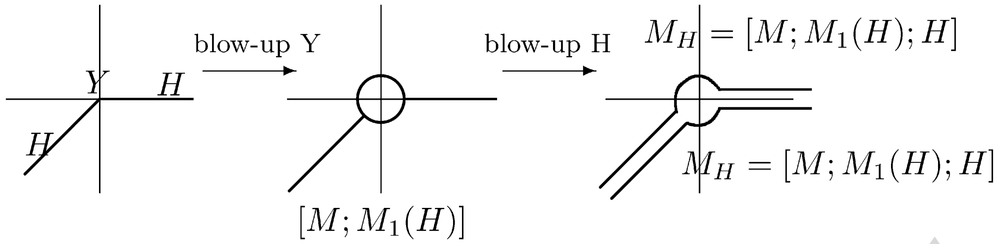

Fig. 10. If H is tied, then M_H is as shown in the figure.

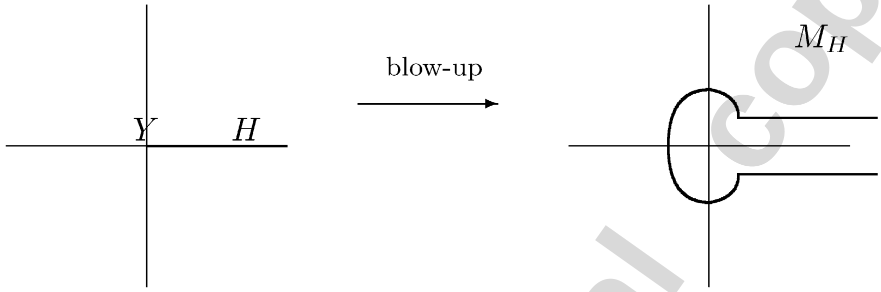

Fig. 11. Blowing up the manifold M at Y, then at H.

where $a(u, v, y, \xi, \eta, \tau)$ is the complete symbol of $\mathcal{D}^{-1}$, which is a classical symbol of order -1. Assume that $\phi \in \mathcal{A}^{\mathcal{I}}(H, E)$, where $\mathcal{I} > -1$, is supported in $\mathcal{U} = \mathcal{V} \times \mathcal{W}$. Thus, we may write

$$\phi = \kappa(u)\varphi(u, y),$$

where $\kappa(u)$ is the Heaviside function and $\varphi(u, y) \in C_c^\infty(\mathcal{W}, \mathcal{A}^{\mathcal{I}}([\mathbb{R}; \{0\}]))$. Therefore,

$$\delta_H G_H \phi = G_H \kappa(u)\delta(v)\varphi(u, y),$$

where $\delta(v)$ is the delta function. Hence,

$$\mathcal{D}^{-1}\delta_H G_H \varphi = \int_{\mathbb{R}^n} e^{iu\xi + iv\eta + iy\cdot\tau} a(u, v, y, \xi, \eta, \tau) G_H \widehat{(\kappa\varphi)}(\xi, \tau)\, đ\xi\, đ\eta\, đ\tau. \tag{6.7}$$

We need to prove that this function defines an element of $\mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E)$, which in local polar coordinates (r, θ, y) is just $C^\infty([0, 2\pi]_\theta \times \mathcal{W}; \mathcal{A}^{\widehat{\mathcal{I}}}(\overline{[0, \infty)_r}))$ where we drop the vector bundle E from the notation just for simplicity. To go about doing this, note that since $a(u, v, y, \xi, \eta, \tau)$ is classical of order -1 we have an expansion

$$a(u, v, y, \xi, \eta, \tau) \sim \sum_{j=1}^{\infty} \chi(\xi, \eta, \tau) a_j(u, v, y, \xi, \eta, \tau),$$

where $\chi(\xi, \eta, \tau)$ vanishes for $|(\xi, \eta, \tau)|$ near 0 and equal to 1 for $|(\xi, \eta, \tau)| \geqslant 1$ and a_j is homogeneous of degree $-j$ in (ξ, η, τ). Recall that $\sim$ means that for any N we have

$$a(u, v, y, \xi, \eta, \tau) - \sum_{j=1}^{N} \chi(\xi, \eta, \tau) a_j(u, v, y, \xi, \eta, \tau) = r_N(u, v, y, \xi, \eta, \tau) \in S_{c\ell}^{-1-N},$$

a symbol of order $-1-N$. Replacing this expansion into (6.7) we obtain

$$\mathcal{D}^{-1}\delta_H G_H \phi = \sum_{j=1}^{N} \phi_j + f_N,$$

where

$$\phi_j := \int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} \chi(\xi,\eta,\tau) a_j(u,v,y,\xi,\eta,\tau) G_H\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\eta\, đ\tau$$

and

$$f_N := \int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} r_N(u,v,y,\xi,\eta,\tau) G_H\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\eta\, đ\tau.$$

Since $(\kappa\varphi)$ is L^1 we know that $\widehat{(\kappa\varphi)}(\xi,\tau)$ is bounded and hence

$$r_N(u,v,y,\xi,\eta,\tau) G_H\widehat{(\kappa\varphi)}(\xi,\tau) = \mathcal{O}\big((1+|\xi|+|\eta|+|\tau|)^{-N-1}\big).$$

It follows that taking $N \geqslant n$ we have

$$f_N = \int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} r_N(u,v,y,\xi,\eta,\tau) G_H\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\eta\, đ\tau \in C^{N-n}(\mathcal{U},E).$$

Since $N-n\to\infty$ as $N\to\infty$, by (a similar version of) Lemma 4.7 all we have to do is prove that for each j, the function

$$\phi_j := \int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} \chi(\xi,\eta,\tau) a_j(u,v,y,\xi,\eta,\tau) G_H\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\eta\, đ\tau \tag{6.8}$$

defines an element of $\mathcal{A}^{\widehat{\mathcal{I}}}(M_H,E)$, which in local polar coordinates (r,θ,y) is just $C^\infty([0,2\pi]_\theta \times \mathcal{W}; \mathcal{A}^{\widehat{\mathcal{I}}}(\overline{[0,\infty)}_r))$ where we henceforth drop vector bundles from the notation for simplicity. In order to do this we need to consider two cases shown in Fig. 12: When (u,v) is in part of a sector away from the negative real axis and when (u,v) is in a sector containing the negative real axis. As seen below, the proof of the second case is different from the first case; the subtle reason is that if H is continued to the negative real axis, then H actually has no boundary at Y.

Case 1. In this case, we may assume initially that $|v|>0$. We can write (6.8) as

$$\begin{aligned}\phi_j &= \int_{\mathbb{R}^n} e^{iu\xi+iv\eta+iy\cdot\tau} \chi(\xi,\eta,\tau) a_j(u,v,y,\xi,\eta,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\eta\, đ\tau \\ &= \int_{\mathbb{R}^{n-1}} e^{iu\xi+iy\cdot\tau} b_j(u,v,y,\xi,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi\, đ\tau, \end{aligned} \tag{6.9}$$

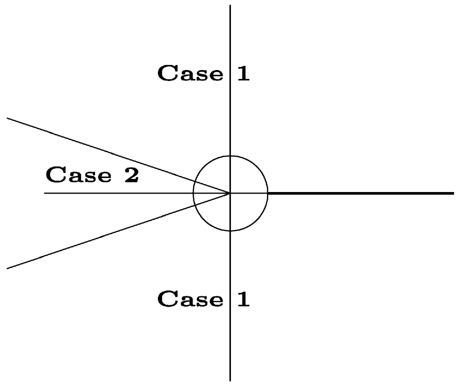

Fig. 12. Cases 1 and 2 are when (u, v) is away and near the negative real axis.

where

$$b_j(u, v, y, \xi, \tau) := \int_{\mathbb{R}} e^{iv\eta} \chi(\xi, \eta, \tau) a_j(u, v, y, \xi, \tau, \eta) G_H \, đ\eta; \tag{6.10}$$

by Dirichlet's test this integral exists (as an improper Riemann integral) because the symbol $a_j(u, v, y, \xi, \tau, \eta)$ decays like $|\eta|^{-j}$ with $j \geqslant 1$ and $e^{iv\eta}$ oscillates. In order to get a nice formula for $a_j(u, v, y, \xi, \tau, \eta)$ that will make the properties of b_j transparent, we now choose coordinates so that the metric takes the form

$$g = dv^2 + h(v),$$

where $h(v)$ is a metric in the u, y coordinates; this can be done by choosing a normal vector to $\{v = 0\}$ and using geodesic flow in the direction of the normal to define the coordinate v. This implies, in particular, that the metric looks like

$$g\big((\xi, \eta, \tau), (\xi, \eta, \tau)\big) = \eta^2 + \lambda(\xi, \tau), \tag{6.11}$$

where $\lambda(\xi, \tau) = \lambda(u, v, y, \xi, \tau)$ is a homogeneous polynomial of degree two in (ξ, τ), and positive for $(\xi, \tau) \neq 0$. This implies that

$$\mathcal{D} = G_1 \partial_u + G_2 \partial_v + B, \quad \text{where } G_2 = G_H,$$

and the principal symbol of $a(u, v, y, \xi, \eta, \tau)$ is equal to

$$a_1(u, v, y, \xi, \eta, \tau) = \sigma_1(\mathcal{D})^{-1} = \frac{\sigma_1(\mathcal{D}^*)}{\sigma_2(\mathcal{D}^*\mathcal{D})} = \frac{(-i\eta + \sigma(u, v, y, \xi, \tau))G_2^{-1}}{\eta^2 + \lambda(\xi, \tau)}, \tag{6.12}$$

where σ is the principal symbol of $(G_1 \partial_u + B)^* G_2$. Moreover, from the explicit local parametrix construction of $\mathcal{D}$ [42,88,108,109] we know that $a(u, v, y, \xi, \eta, \tau)$ has a rational expansion in ξ, τ, η in the sense that

$$a(u, v, y, \xi, \eta, \tau) \sim \sum_{j=1}^{\infty} \chi(\xi, \eta, \tau) \frac{p_j(u, v, y, \xi, \eta, \tau)}{(\eta^2 + \lambda(\xi, \tau))^j}, \tag{6.13}$$

with $p_j(u,v,y,\xi,\tau,\eta)$ a polynomial of degree j in (ξ,τ,η). The proof of the rational expansion (6.13) is by induction using that $a_1(u,v,y,\xi,\eta,\tau)$ has this property by (6.12). Thus, we may assume that

$$a_j(u,v,y,\xi,\eta,\tau)=\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta^2+\lambda(\xi,\tau))^j}. \tag{6.14}$$

Consider now, for each j, the integral

$$\int_{\mathbb{R}} e^{iv\eta}a_j(u,v,y,\xi,\tau,\eta)\,đ\eta=\int_{\mathbb{R}} e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta^2+\lambda(\xi,\tau))^j}\,đ\eta.$$

Assume for a moment that $v<0$ so we are in the bottom part of the sector labelled Case 1 in Fig. 12. Then factoring $\eta^2+\lambda(\xi,\tau)=(\eta+i\lambda^{1/2}(\xi,\tau))(\eta-i\lambda^{1/2}(\xi,\tau))$ with $\lambda^{1/2}(\xi,\tau)=\sqrt{\lambda(\xi,\tau)}$ and using Cauchy's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}} e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta^2+\lambda(\xi,\tau))^j}\,đ\eta&=\int_{\mathbb{R}} e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta+i\lambda^{\frac12}(\xi,\tau))^j(\eta-i\lambda^{\frac12}(\xi,\tau))^j}\,đ\eta\\&=\frac{-i}{(j-1)!}\left(\frac{\partial}{\partial\eta}\right)^{j-1}\bigg|_{\eta=-i\lambda^{\frac12}(\xi,\tau)}\left(e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta-i\lambda^{\frac12}(\xi,\tau))^j}\right).\end{aligned} \tag{6.15}$$

Here, recalling that $v<0$, we shifted the line $\mathbb{R}=\{\Im\xi=0\}$ down to $\{\Im\xi=-\infty\}$ where the integral vanishes, and we picked up a pole at $\xi=-i\lambda^{1/2}(\tau,\eta)$. Applying the product rule to the last term in (6.15) we obtain

$$\int_{\mathbb{R}} e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta^2+\lambda(\xi,\tau))^j}\,đ\eta=e^{v\lambda^{\frac12}(\xi,\tau)}q_j^-(u,v,y,\xi,\tau),\quad\text{for } v<0,$$

where

$$q_j^-(u,v,y,\xi,\tau)=\sum_{k=0}^{j-1}v^kq_{jk}^-(u,v,y,\xi,\tau)$$

with $q_{jk}^-(u,v,y,\xi,\tau)$ a homogeneous function in (ξ,τ) (not rational) of degree $1-j+k$. An identical computation, but this time shifting the contour $\mathbb{R}=\{\Im\xi=0\}$ up to $\{\Im\xi=+\infty\}$, we obtain

$$\int_{\mathbb{R}} e^{iv\eta}\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta^2+|(\xi,\tau)|^2)^j}\,đ\eta=e^{-v\lambda^{\frac12}(\xi,\tau)}q_j^+(u,v,y,\xi,\tau),\quad\text{for } v>0,$$

where

$$q_j^+(u,v,y,\xi,\tau)=\sum_{k=0}^{j-1}v^kq_{jk}^+(u,v,y,\xi,\tau)$$

with $q_{jk}^{+}(u,v,y,\xi,\tau)$ a homogeneous function in (ξ,τ) of degree $1-j+k$. In view of (6.10) it follows that

$$b_j(u,v,y,\xi,\tau)=e^{\mp v\langle 1,\xi,\tau\rangle}q_j^{\pm}(u,v,y,\xi,\tau),\quad \text{for } v=\pm|v|,$$

where

$$q_j^{\pm}(u,v,y,\xi,\tau)\in S_{c\ell}^{0},$$

a symbol of order 0, where $\langle 1,\xi,\tau\rangle^2 := 1+\lambda(\xi,\tau)$, and where "for $v=\pm|v|$" is shorthand for "for $v>0$ and $v<0$, respectively." Note that although the expression for $b_j(u,v,y,\xi,\tau)$ depends initially on $v>0$ and $v<0$, each of the functions $q^{+}(u,v,y,\xi,\tau)$ and $q^{-}(u,v,y,\xi,\tau)$ extends canonically to be smooth for all $(u,v,y)\in\mathcal{U}$. Now by (6.9), we have

$$\phi_j=\int\limits_{\mathbb{R}^{n-1}} e^{iu\xi\mp v\langle 1,\xi,\tau\rangle+iy\cdot\tau}q_j^{\pm}(u,v,y,\xi,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\,đ\xi\,đ\tau,\quad \text{for } v=\pm|v|. \tag{6.16}$$

Noting that $\kappa(u)\varphi(u,y)\in C_c^{\infty}(\mathcal{W},\mathcal{A}^{\mathcal{I}}([0,\infty)))\subseteq C_c^{\infty}(\mathcal{W},\mathcal{A}^{\mathcal{I}}([\mathbb{R};\{0\}]))$ (it happens to vanish on the negative real axis too) it follows that

$$\tau\mapsto\int\limits_{\mathbb{R}^{n-2}} e^{-iy\cdot\tau}\kappa(u)\varphi(u,y)\,dy\in\mathscr{S}\big(\mathbb{R}_\tau^{n-2};\mathcal{A}^{\mathcal{I}}\big([\mathbb{R};\{0\}]\big)\big), \tag{6.17}$$

the space of Schwartz functions with values in $\mathcal{A}^{\mathcal{I}}([\mathbb{R};\{0\}])$. Now using Theorem 4.4, taking the Fourier transform in u we have

$$\tau\mapsto\widehat{(\kappa\varphi)}(\xi,\tau)\in\mathscr{S}\big(\mathbb{R}_\tau^{n-2};\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}})\big).$$

Since $q^{\pm}(u,v,y,\xi,\tau)$ is a classical symbol of order 0, by Lemma 5.1, we see that for fixed u,v,y,

$$\tau\mapsto q_j^{\pm}(u,v,y,\xi,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\in\mathscr{S}\big(\mathbb{R}_\tau^{n-2};\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}})\big).$$

Of course, this is smooth in $(u,v,y)\in\mathcal{U}$, so we actually have

$$\big((u,v,y),\tau\big)\mapsto q_j^{\pm}(u,v,y,\xi,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\in C^{\infty}\big(\mathcal{U}_{(u,v,y)}\times\mathbb{R}_\tau^{n-2};\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}})\big),$$

and is Schwartz in τ. Now employing Lemma 6.1, we can write

$$\int\limits_{\mathbb{R}^{n-2}} e^{\mp v\langle 1,\xi,\tau\rangle+iy\cdot\tau}q_j^{\pm}(u,v,y,\xi,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\,đ\tau=e^{\mp v\langle 1,\xi\rangle}s_j^{\pm}(u,v,y,\xi),$$

for $v=\pm|v|$, where $\langle 1,\xi\rangle^2 := 1+\lambda(\xi,0)$ and

$$(u,v,y)\mapsto s_j^{\pm}(u,v,y,\xi)\in C^{\infty}\big(\mathcal{U};\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}})\big).$$

In conclusion, we have

$$\phi_j = \int_{\mathbb{R}} e^{iu\xi \mp v\langle 1,\xi\rangle} s_j^{\pm}(u, v, y, \xi)\, đ\xi, \quad \text{for } v = \pm|v|.$$

Modulo a smooth function we can write this as

$$\phi_j \equiv \int_{\mathbb{R}} e^{iu\xi \mp v|\xi|} s_j^{\pm}(u, v, y, \xi)\, đ\xi, \quad \text{for } v = \pm|v|,$$

where we used that $\lambda(\xi, 0) = |\xi|^2$, the Riemannian length of ξ, by the formula (6.11). If $(u, v) = r\omega$, where $\omega = (\omega_1, \omega_2) \in A$ with $A \subseteq \mathbb{S}^1$ is a fixed arch in either the upper or lower Case 1 regions in Fig. 12, then we can write this as

$$\phi_j \equiv \int_{\mathbb{R}} e^{r(i\omega_1\xi \mp \omega_2|\xi|)} s_j^{\pm}(r\omega, y, \xi)\, đ\xi, \quad \text{for } \omega_2 = \pm|\omega_2|.$$

This fits into the h-transform situation of Theorem 4.8 with $h(\xi) = e^{i\omega_1\xi \mp \omega_2|\xi|}$ (see Example 3 of Section 4.2). Hence, by Theorem 4.8 and Remark 4.9 we know that

$$\phi_j(r\omega, y) \in C^\infty\big(A_\theta \times \mathcal{W}; \mathcal{A}^{\widehat{\mathcal{I}}}\big(\overline{[0,\infty)_r}\big)\big).$$

This implies that $\phi_j \in \mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E)$ as required and completes Case 1. We remark that $\int_{\mathbb{R}} e^{r(i\omega_1\xi - \omega_2|\xi|)} s_j^{+}(r\omega, y, \xi)\, đ\xi$ and $\int_{\mathbb{R}} e^{r(i\omega_1\xi + \omega_2|\xi|)} s_j^{-}(r\omega, y, \xi)\, đ\xi$ define smooth functions on $[0, \pi]$ and $[\pi, 2\pi]$, respectively. However, this does not imply that $\phi_j \in C^\infty([0, 2\pi]_\theta \times \mathcal{W}; \mathcal{A}^{\widehat{\mathcal{I}}}(\overline{[0,\infty)_r}))$ because it is not obvious that these integrals agree when $\theta = \pi$! (In fact, at 0 and 2π, these integrals are different!) This is why we need a second case to deal with angles near π.

Case 2. In this case, we may assume that $u < 0$ and we are working near $v = 0$. As with Case 1, the starting point is again (6.8):

$$\phi_j = \int_{\mathbb{R}^n} e^{iu\xi + iv\eta + iy\cdot\tau} \chi(\xi, \eta, \tau) a_j(u, v, y, \xi, \eta, \tau) \widehat{(\kappa\varphi)}(\xi, \tau)\, đ\xi\, đ\eta\, đ\tau.$$

The fact that $u < 0$ plays an important part here because in this case, we write

$$\phi_j = \int_{\mathbb{R}^{n-1}} e^{iv\eta + iy\cdot\tau} c_j(u, v, y, \eta, \tau)\, đ\eta\, đ\tau, \tag{6.18}$$

where

$$c_j(u, v, y, \eta, \tau) := \int_{\mathbb{R}} e^{iu\xi} \chi(\xi, \eta, \tau) a_j(u, v, y, \xi, \eta, \tau) \widehat{(\kappa\varphi)}(\xi, \tau)\, đ\xi. \tag{6.19}$$

Thus, in this case, the roles of ξ and η are switched!

We now proceed as we did in Case 1 but with appropriate switching of the roles of ξ and η taking care to note the special twists that occur because of the switching. First, we now choose coordinates so that the metric takes the form

$$g = du^2 + \tilde{h}(u),$$

where $\tilde{h}(u)$ is a metric in the v, y coordinates; this can be done by choosing a vector field locally near Y that is tangent and pointing inward to H and using geodesic flow in the direction of the vector field to define the coordinate u. This implies, in particular, that the metric looks like

$$g\big((\xi,\eta,\tau),(\xi,\eta,\tau)\big) = \xi^2 + \mu(\eta,\tau), \tag{6.20}$$

where $\mu(\eta,\tau) = \mu(u,v,y,\eta,\tau)$ is a homogeneous polynomial of degree two in (η,τ) and positive for $(\eta,\tau) \neq 0$. In this case, we can take the expansion in (6.13) to look like (cf. the related series (6.14))

$$a(u,v,y,\xi,\eta,\tau) \sim \sum_{j=1}^{\infty} \chi(\xi,\eta,\tau) \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2 + \mu(\eta,\tau))^j}$$

with $q_j(u,v,y,\xi,\tau,\eta)$ a polynomial of degree j in (ξ,η,τ). Thus, for Case 2 we shall (and may) assume that

$$a_j(u,v,y,\xi,\eta,\tau) = \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2 + \mu(\eta,\tau))^j}.$$

In order to study the integral (6.19), we need to analyze each integral

$$\int_{\mathbb{R}} e^{iu\xi} a_j(u,v,y,\xi,\tau,\eta) \widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi = \int_{\mathbb{R}} e^{iu\xi} \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2 + \mu(\eta,\tau))^j} \widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi.$$

Writing $\xi^2 + \mu(\eta,\tau) = (\xi + i\mu^{1/2}(\eta,\tau))(\xi - i\mu^{1/2}(\eta,\tau))$ and using Cauchy's theorem exactly as we did in (6.15), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} e^{iu\xi} \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2 + \mu(\eta,\tau))^j} \widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi \\
&\quad = \int_{\mathbb{R}} e^{iu\xi} \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi + i\mu^{\frac{1}{2}}(\eta,\tau))^j (\xi - i\mu^{\frac{1}{2}}(\eta,\tau))^j} \widehat{(\kappa\varphi)}(\xi,\tau)\, đ\xi \\
&\quad = \frac{-i}{(j-1)!} \left(\frac{\partial}{\partial\xi}\right)^{j-1} \bigg|_{\xi = -i\mu^{\frac{1}{2}}(\eta,\tau)} \left(e^{iu\xi} \frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi - i\mu^{\frac{1}{2}}(\eta,\tau))^j} \widehat{(\kappa\varphi)}(\xi,\tau) \right).
\end{aligned} \tag{6.21}$$

Here, recalling that $u<0$, we shifted the line $\mathbb{R}=\{\Im\xi=0\}$ down to $\{\Im\xi=-\infty\}$ and we picked up a pole at $\xi=-i\mu^{1/2}(\eta,\tau)$. Moreover, it is important to note that $\kappa(t)\varphi(t,y)$ is supported in $t\geqslant 0$, so that

$$\widehat{(\kappa\varphi)}(\xi,\tau)=\int_0^\infty\int_{\mathbb{R}^{n-2}} e^{-it\xi-iy\cdot\tau}\varphi(t,y)\,dy\,dt,$$

and therefore if $\xi=z_1+iz_2\in\mathbb{C}$ with $z_1,z_2\in\mathbb{R}$ and $z_2<0$, then

$$\widehat{(\kappa\varphi)}(\xi,\tau)=\int_0^\infty\int_{\mathbb{R}^{n-2}} e^{-itz_1+tz_2-iy\cdot\tau}\varphi(t,y)\,dy\,dt$$

is bounded as $z_2\to-\infty$. Therefore the integral remainder in (6.21) does indeed vanish as we take the contour $\mathbb{R}=\{\Im\xi=0\}$ down to $\{\Im\xi=-\infty\}$. Observe that

$$\begin{aligned}\widehat{(\kappa\varphi)}\bigl(-i\mu^{\frac{1}{2}}(\eta,\tau),\tau\bigr)&=\int_0^\infty\int_{\mathbb{R}^{n-2}} e^{-t\mu^{\frac{1}{2}}(\eta,\tau)-iy\cdot\tau}\varphi(t,y)\,dy\,dt\\&=\int_0^\infty e^{-t\mu^{\frac{1}{2}}(\eta,\tau)}\psi(t,\tau)\,dt,\end{aligned}$$

where

$$\psi(t,\tau)=\int_{\mathbb{R}^{n-2}} e^{-iy\cdot\tau}\varphi(t,y)\,dy\in\mathscr{S}\bigl(\mathbb{R}^{n-2};\mathcal{A}^{\mathcal{I}}\bigl([0,\infty)\bigr)\bigr)\subseteq\mathscr{S}\bigl(\mathbb{R}^{n-2};\mathcal{A}^{\mathcal{I}}\bigl([\mathbb{R};\{0\}]\bigr)\bigr)$$

and vanishes for $t<0$. We are now in the situation of Lemma 6.2, which implies that away from $(\eta,\tau)=(0,0)$, we have

$$\widehat{(\kappa\varphi)}\bigl(-i\mu^{\frac{1}{2}}(\eta,\tau),\tau\bigr)=\int_0^\infty e^{-t\mu^{\frac{1}{2}}(\eta,\tau)}\psi(t,\tau)\,dt\in\mathscr{S}\bigl(\mathbb{R}^{n-2}_\tau;\mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}_\eta})\bigr).$$

Since for any k,

$$\partial_\xi^k\widehat{(\kappa\varphi)}(\xi,\tau)=(-i)^k\int_0^\infty\int_{\mathbb{R}^{n-2}} e^{-it\xi-iy\cdot\tau}t^k\varphi(t,y)\,dy\,dt,$$

and $t^k\mathcal{A}^{\mathcal{I}}([\mathbb{R};\{0\}])\subseteq\mathcal{A}^{\mathcal{I}+k}([\mathbb{R};\{0\}])$, we have

$$\bigl(\partial_\xi^k\widehat{(\kappa\varphi)}\bigr)\bigl(-i\mu^{\frac{1}{2}}(\eta,\tau),\tau\bigr)\in\mathscr{S}\bigl(\mathbb{R}^{n-2}_\tau;\mathcal{A}^{\mathcal{I}+k+1}(\overline{\mathbb{R}_\eta})\bigr).$$

Hence, by Lemma 5.1 and using the product rule noting that $\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta - i\mu^{1/2}(\xi,\tau))^j}$ is a symbol of order zero, we see that for arbitrary k,

$$\left(\frac{\partial}{\partial \xi}\right)^k\bigg|_{\xi=-i\mu^{\frac{1}{2}}(\eta,\tau)} \left(\frac{p_j(u,v,y,\xi,\eta,\tau)}{(\eta - i\mu^{\frac{1}{2}}(\xi,\tau))^j}\widehat{(\kappa\varphi)}(\xi,\tau)\right)$$
$$\in C^\infty\big(\mathcal{U}_{(u,v,y)}; \mathscr{S}\big(\mathbb{R}^{n-2}_\tau; \mathcal{A}^{\mathcal{I}+k+1}(\overline{\mathbb{R}_\eta})\big)\big)$$
$$\subseteq C^\infty\big(\mathcal{U}_{(u,v,y)}; \mathscr{S}\big(\mathbb{R}^{n-2}_\tau; \mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}_\eta})\big)\big).$$

Now using the product rule in (6.21) we have

$$\int_{\mathbb{R}} e^{iu\xi}\frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2+\mu(\eta,\tau))^j}\widehat{(\kappa\varphi)}(\xi,\tau)\,đ\xi = e^{u\mu^{\frac{1}{2}}(\eta,\tau)}s_j(u,v,y,\eta,\tau), \quad \text{for } u<0,$$

where

$$s_j(u,v,y,\eta,\tau) \in C^\infty\big(\mathcal{U}_{(u,v,y)}; \mathscr{S}\big(\mathbb{R}^{n-2}_\tau; \mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}_\eta})\big)\big).$$

Thus, in view of (6.19), it follows that

$$\begin{aligned} c_j(u,v,y,\eta,\tau) &:= \int_{\mathbb{R}} e^{iu\xi}\chi(\xi,\eta,\tau)a_j(u,v,y,\xi,\eta,\tau)\widehat{(\kappa\varphi)}(\xi,\tau)\,đ\xi \\ &= \int_{\mathbb{R}} e^{iu\xi}\chi(\xi,\eta,\tau)\frac{q_j(u,v,y,\xi,\eta,\tau)}{(\xi^2+\mu(\eta,\tau))^j}\widehat{(\kappa\varphi)}(\xi,\tau)\,đ\xi \\ &= e^{u\langle 1,\eta,\tau\rangle}q_j(u,v,y,\eta,\tau), \end{aligned}$$

where $u<0$, $\langle 1,\eta,\tau\rangle^2 := 1+\mu(\eta,\tau)$, and where

$$q_j(u,v,y,\eta,\tau) \in C^\infty\big(\mathcal{U}_{(u,v,y)}; \mathscr{S}\big(\mathbb{R}^{n-2}_\tau; \mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}_\eta})\big)\big).$$

Hence, by (6.18), we have

$$\phi_j = \int_{\mathbb{R}^{n-1}} e^{u\langle 1,\eta,\tau\rangle + iv\eta + iy\cdot\tau}q_j(u,v,y,\eta,\tau)\,đ\eta\,đ\tau.$$

Now employing Lemma 6.1, we can write

$$\int_{\mathbb{R}^{n-2}} e^{u\langle 1,\eta,\tau\rangle + iy\cdot\tau}q_j(u,v,y,\eta,\tau)\,đ\tau = e^{u\langle 1,\eta\rangle}\tilde{q}_j(u,v,y,\eta),$$

where $\langle 1,\eta\rangle^2 := 1+\mu(\eta,0) = 1+|\eta|^2$ (with $|\eta|^2$ denoting the Riemannian squared length of η by (6.20)), and

$$(u,v,y) \mapsto \tilde{q}_j(u,v,y,\eta) \in C^\infty\big(\mathcal{U}; \mathcal{A}^{\mathcal{I}+1}(\overline{\mathbb{R}}_\eta)\big).$$

In conclusion, we have

$$\phi_j = \int_{\mathbb{R}} e^{u\langle 1,\eta\rangle + iv\eta} \tilde{q}_j(u, v, y, \eta)\, đ\eta, \quad \text{for } u < 0.$$

Up to a smooth function we can write this as

$$\phi_j \equiv \int_{\mathbb{R}} e^{u|\eta| + iv\eta} \tilde{q}_j(u, v, y, \eta)\, đ\eta, \quad \text{for } u < 0.$$

We are now in the exact same situation as we were at the end of Case 1 (but now we have η instead of ξ and now $u < 0$ instead of having v's of various signs in that case). Therefore, by the same argument based on Theorem 4.8, $\phi_j \in \mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E)$ as required.

Finally, we remark that all the lemmas used in this proof depend continuously on their function data (as can be seen by examining the proofs). Therefore, the map

$$\mathcal{A}^{\mathcal{I}}(H, E) \ni \phi \mapsto \mathcal{D}^{-1}\delta_H G_H \phi \in \mathcal{A}^{\widehat{\mathcal{I}}}(M_H, E)$$

depends continuously on ϕ. $\square$

7. The Cauchy integral and transform on manifolds with corners

Using Theorem 6.3 we can now define the Cauchy integral and transform; see Lemma 7.1. The main results of this section include the Borel–Pompeiu formula (a generalization of Cauchy's integral formula) in Theorem 7.2, which implies Theorems 2.1 and 2.4 on the Cauchy integral (= Poisson operator). We also prove Theorem 2.2 on the Cauchy transform (= Calderón projector).

7.1. Definition of the Cauchy integral and transform

Let $X \subseteq M$ be a compact manifold corners of codimension two with $\dim X = \dim M$.

The *Cauchy integral* or *Poisson operator* is defined by

$$\mathcal{K} := \sum_{H \in M_1(X)} \mathcal{D}^{-1}\delta_H G_H,$$

where δ_H is the delta function concentrated on H and $G_H : E \to F$ is the principal symbol of $\mathcal{D}$ evaluated on the inward pointing unit normal vector field to H. Let $\mathcal{I} > -1$ be an index family associated to $M_2(X)$. Then applying Theorem 6.3 to each of $\mathcal{D}^{-1}\delta_H G_H$ it follows that

$$\mathcal{K} : \mathcal{A}^{\mathcal{I}}(\partial X, E) \to \mathcal{A}^{\widehat{\mathcal{I}}}(M_{\mathrm{tb}}, E),$$

where $\mathcal{A}^{\mathcal{I}}(\partial X, E) := (\mathcal{A}^{\mathcal{I}}(X_{\mathrm{tb}}, E))|_{\partial X}$ with $|_{\partial X}$ meaning restriction to the interior of each boundary hypersurface of X, where (see Fig. 5 in Section 2)

$$M_{\mathrm{tb}} := \big[M; M_2(X); M_1(X)\big],$$

and $\widehat{\mathcal{I}}$ is the index set defined in (4.16) associated to the faces coming from the blow-ups of $M_2(X)$. If $\varphi \in \mathcal{A}^{\mathcal{I}}(\partial X, E)$, then $\mathcal{K}\varphi \in \mathcal{A}^{\widehat{\mathcal{I}}}(M_{\mathrm{tb}}, E)$, so restriction to $X_{\mathrm{tb}} \subseteq M_{\mathrm{tb}}$ we get $(\mathcal{K}\varphi)|_{X_{\mathrm{tb}}} \in \mathcal{A}^{\widehat{\mathcal{I}}}(X_{\mathrm{tb}}, E)$ and on the interior of X_{tb}, we have

$$\mathcal{D}(\mathcal{K}\varphi) = \sum_{H \in M_1(X)} \mathcal{D}\mathcal{D}^{-1}\delta_H G_H \varphi = \sum_{H \in M_1(X)} \delta_H G_H \varphi = 0.$$

Therefore, restriction to $\mathring{X}_{\mathrm{tb}} \subseteq M_{\mathrm{tb}}$, we have $\mathcal{K}\varphi \in \ker(\mathcal{D} \text{ on } \mathcal{A}^{\widehat{\mathcal{I}}}(X_{\mathrm{tb}}, E))$. Hence, if for any index set $\mathcal{J}$ associated to $M_2(X)$, we define

$$\gamma^+_{\partial X} : \mathcal{A}^{\mathcal{J}}(M_{\mathrm{tb}}, E) \to \mathcal{A}^{\mathcal{J}}(\partial X, E)$$

as the restriction map $\gamma^+_{\partial X} := |_{\partial X}$, from the interior of $X_{\mathrm{tb}} \subset M_{\mathrm{tb}}$ to the hypersurface components of ∂X minus the corners of X, then $\gamma^+_{\partial X}\mathcal{K}\varphi \in \mathcal{H}_{\widehat{\mathcal{I}}}(\mathcal{D})$ by definition of the Cauchy–Hardy space

$$\mathcal{H}_{\widehat{\mathcal{I}}}(\mathcal{D}) := \bigl\{\phi|_{\partial X} \bigm| \phi \in \mathcal{A}^{\widehat{\mathcal{I}}}(X_{\mathrm{tb}}, E),\ \mathcal{D}\phi = 0\bigr\},$$

where $\phi|_{\partial X} := \gamma^+_{\partial X}\phi$. Therefore, if we define the *Cauchy transform* or *Calderón projector*

$$\mathcal{C} : \mathcal{A}^{\mathcal{I}}(\partial X, E) \to \mathcal{A}^{\widehat{\mathcal{I}}}(\partial X, E)$$

by

$$\mathcal{C}\varphi := \gamma^+_{\partial X}\mathcal{K}\varphi \quad \text{for all } \varphi \in \mathcal{A}^{\mathcal{I}}(\partial X, E),$$

then $\operatorname{ran}\mathcal{C} \subseteq \mathcal{H}_{\widehat{\mathcal{I}}}(\mathcal{D})$. We summarize our findings in the following lemma.

Lemma 7.1. *For any index family $\mathcal{I} > -1$ associated to $M_2(X)$, the Cauchy integral defines a continuous linear map*

$$\mathcal{K} := \sum_{H \in M_1(X)} \mathcal{D}^{-1}\delta_H G_H : \mathcal{A}^{\mathcal{I}}(\partial X, E) \to \mathcal{A}^{\widehat{\mathcal{I}}}(M_{\mathrm{tb}}, E)$$

with

$$\operatorname{ran}\bigl(\gamma_{X_{\mathrm{tb}}}\mathcal{K}\bigr) \subseteq \ker\bigl(\mathcal{D} : \mathcal{A}^{\widehat{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\widehat{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)\bigr),$$

where $\gamma_{X_{\mathrm{tb}}}$ denotes restriction from M_{tb} to X_{tb}. The Cauchy transform defines a continuous linear map

$$\mathcal{C} := \gamma^+_{\partial X}\mathcal{K} : \mathcal{A}^{\mathcal{I}}(\partial X, E) \to \mathcal{A}^{\widehat{\mathcal{I}}}(\partial X, E) \quad \textit{with } \operatorname{ran}\mathcal{C} \subseteq \mathcal{H}_{\widehat{\mathcal{I}}}(\mathcal{D}).$$

Having defined the Cauchy integral and Calderón–Cauchy projector, we now investigate their main properties.

7.2. Properties of the Cauchy integral (Poisson operator)

The first formula in the following theorem is the so-called the Borel–Pompeiu theorem; see [85] for a historical review. In the following theorem the function χ on M_{tb} is defined by

$$\chi := \begin{cases} 1 & \text{on } X_{\mathrm{tb}}, \\ 0 & \text{on } M_{\mathrm{tb}} \setminus X_{\mathrm{tb}}. \end{cases}$$

The following theorem, in particular, proves Theorem 2.4.

Theorem 7.2 *(Borel–Pompeiu's formula). For any index family $\mathcal{I} > -1$ associated to $M_2(X)$, if $\phi \in \mathcal{A}^{\mathcal{I}}(X_{\mathrm{tb}}, E)$ and $\varphi := \phi|_{\partial X} \in \mathcal{A}^{\mathcal{I}}(\partial X, E)$, then*

$$(\mathcal{K}\varphi)(x) + \big(\mathcal{D}^{-1}\chi\mathcal{D}\phi\big)(x) = \begin{cases} \phi(x), & x \in X_{\mathrm{tb}}, \\ 0, & x \in M_{\mathrm{tb}} \setminus X_{\mathrm{tb}}. \end{cases}$$

In particular, if $\mathcal{D}\phi = 0$, then "Cauchy's formula" holds:

$$(\mathcal{K}\varphi)(x) = \begin{cases} \phi(x), & x \in X_{\mathrm{tb}}, \\ 0, & x \in M_{\mathrm{tb}} \setminus X_{\mathrm{tb}}. \end{cases}$$

Proof. Extend ϕ to M_{tb} via $\tilde{\phi} := \chi\phi$. In order to apply $\mathcal{D}$ to $\tilde{\phi}$, we work locally (at least for the moment) so consider near a corner Y as shown in Fig. 13, and by choosing a partition of unity of X_{tb} if necessary and by reflecting if necessary we assume that ϕ is supported near Y and near $\theta = 0$ in Fig. 13.

Specify the coordinates (u, v, y) near Y in M with $H = \{u \geqslant 0, v = 0\}$ such that the metric takes the form

$$g = dv^2 + h(v),$$

where $h(v)$ is a metric in the u, y coordinates and in these coordinates write

$$\mathcal{D} = G_1\partial_u + G_2\partial_v + B.$$

Introducing polar coordinates (r, θ), this becomes

$$\mathcal{D} = (G_1\cos\theta + G_2\sin\theta)\partial_r + \frac{1}{r}(G_2\cos\theta - G_1\sin\theta)\partial_\theta + B,$$

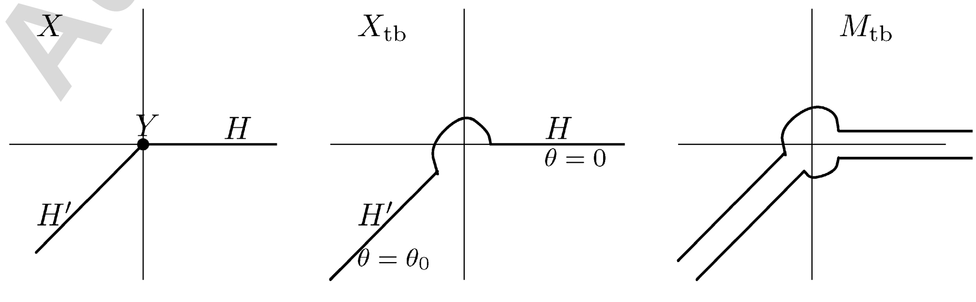

Fig. 13. The manifolds X_{tb} and M_{tb} near the blow-ups.

and

$$\tilde{\phi} = \kappa(\theta)\phi(r,\theta),$$

where κ is the Heaviside function on $\mathbb{R}$. Applying $\mathcal{D}$ to $\tilde{\phi}$ and using that the derivative of the Heaviside function is the delta distribution, it follows that

$$\mathcal{D}\tilde{\phi} = \frac{1}{r} G_2 \delta(\theta)\phi(r,0) + \kappa(\theta)\mathcal{D}\phi(r,\theta).$$

Since Clifford multiplication by the inward pointing normal on H is exactly G_2 and since $\phi(r,0) = \varphi(r)$, as a distribution acting on $C_c^\infty(M,F)$ we have

$$\begin{aligned}\left(\frac{1}{r} G_2 \delta(\theta)\phi(r,0)\right)(\psi) &= \int_M \frac{1}{r}\bigl\langle \psi(r,\theta), G_H \delta(\theta)\varphi(r)\bigr\rangle r\,dr\,d\theta\,dy \\ &= \int_M \bigl\langle \psi(r,0), G_H\varphi(r)\bigr\rangle\,dr\,dy = (G_H\delta_H\varphi)(\psi).\end{aligned}$$

Hence,

$$\mathcal{D}\tilde{\phi} = \delta_H \otimes G_H\varphi + \chi\mathcal{D}\phi.$$

In conclusion, summing up our local results, we have

$$\mathcal{D}\tilde{\phi} = \sum_{H\in M_1(X)} \delta_H G_H \varphi + \chi\mathcal{D}\phi.$$

Applying $\mathcal{D}^{-1}$ to both sides we get

$$\tilde{\phi} = \sum_{H\in M_1(X)} \mathcal{D}^{-1}\delta_H G_H\varphi + \mathcal{D}^{-1}\chi\mathcal{D}\phi =: \mathcal{K}\varphi + \mathcal{D}^{-1}\chi\mathcal{D}\phi.$$

This proves our result. □

7.3. Proof of Theorem 2.1 on the Poisson operator

Recall from (5.9) that for any index set $\mathcal{I}$ associated to $M_2(X)$, we have

$$\phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}},E) \quad\Longleftrightarrow\quad \phi \in \mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}},E),$$

where $\mathcal{F} \in \overline{\mathcal{I}}$ means that $\mathcal{F}$ is an index family associated to $M_2(X)$ and for each $Y \in M_2(X)$, $\mathcal{F}_Y \subseteq \mathcal{I}_Y \cup (\mathbb{N}_0 \times \mathbb{N}_0)$. As already mentioned, an element of $\mathcal{A}^{\mathcal{F}}(X_{\mathrm{tb}},E)$ basically has the same expansions as an element of $\mathcal{A}^{\mathcal{I}}(X_{\mathrm{tb}},E)$ except one can add smooth expansions multiplied by logarithms. One of the key properties of $\overline{\mathcal{I}}$ is that if $\mathcal{F} \in \overline{\mathcal{I}}$, then $\widehat{\mathcal{F}} \in \overline{\mathcal{I}}$.

To prove Theorem 2.1, we shall prove that for any index family $\mathcal{I} > -1$ associated to $M_2(X)$,

$$\gamma_{X_{\mathrm{tb}}}\mathcal{K} : \mathcal{A}^{\overline{\mathcal{I}}}(\partial X,E) \to \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}},E), \tag{7.1}$$

where $\gamma_{X_{\mathrm{tb}}}$ denotes restriction from M_{tb} to X_{tb} and

$$\operatorname{ran}(\gamma_{X_{\mathrm{tb}}}\mathcal{K}) = \ker\bigl(\mathcal{D}: \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)\bigr). \tag{7.2}$$

In fact, by Lemma 7.1 we know that

$$\gamma_{X_{\mathrm{tb}}}\mathcal{K}: \mathcal{A}^{\mathcal{F}}(\partial X, E) \to \mathcal{A}^{\widehat{\mathcal{F}}}(X_{\mathrm{tb}}, E)$$

with

$$\operatorname{ran}\bigl(\gamma_{X_{\mathrm{tb}}}\mathcal{K} \text{ on } \mathcal{A}^{\mathcal{F}}(\partial X, E)\bigr) \subseteq \ker\bigl(\mathcal{D} \text{ on } \mathcal{A}^{\widehat{\mathcal{F}}}(X_{\mathrm{tb}}, E)\bigr).$$

Since $\mathcal{F} \in \overline{\mathcal{I}} \Rightarrow \widehat{\mathcal{F}} \in \overline{\mathcal{I}}$, (7.1) follows and (7.2) will follow if we can show the inclusion

$$\ker\bigl(\mathcal{D}: \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F)\bigr) \subseteq \operatorname{ran}\bigl(\gamma_{X_{\mathrm{tb}}}\mathcal{K} \text{ on } \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)\bigr).$$

However, this inclusion follows immediately from Cauchy's formula in Theorem 7.2.

7.4. Proof of Theorem 2.2 on the Calderón projector

We need to prove that for any index family $\mathcal{I} > -1$ associated to $M_2(X)$, the Cauchy transform $\mathcal{C}$ defines a continuous linear map

$$\mathcal{C}: \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$$

such that

(i) $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) := \{\phi|_{\partial X} \mid \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E),\ \mathcal{D}\phi = 0\}$;
(ii) $\mathcal{C}^2 = \mathcal{C}$ on $\mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$; that is, $\mathcal{C}$ is a projection;
(iii) $\operatorname{ran}\mathcal{C} = \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$.

Step I. We first show that $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$. Let $\varphi = \phi|_{\partial X}$ where $\phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E)$ and $\mathcal{D}\phi = 0$; we need to show that $\mathcal{C}\varphi = \varphi$. But, by Borel–Pompeiu's formula, we have

$$(\mathcal{K}\varphi)(x) = \begin{cases} \phi(x), & x \in X_{\mathrm{tb}}, \\ 0, & x \in M_{\mathrm{tb}} \setminus X_{\mathrm{tb}}. \end{cases}$$

Hence,

$$\mathcal{C}\varphi := \gamma^{+}_{\partial X}(\mathcal{K}\varphi) = \gamma^{+}_{\partial X}(\phi) = \varphi.$$

Hence, $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$.

Step II. From Lemma 7.1 we know that $\operatorname{ran}\mathcal{C} \subseteq \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$. Therefore, since $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$ by Step I, we have

$$\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) = \mathrm{Id}\bigl(\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})\bigr) = \mathcal{C}\bigl(\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})\bigr) \subseteq \operatorname{ran}\mathcal{C}.$$

Therefore $\operatorname{ran}\mathcal{C} = \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$.

Step III. Finally, we show that $\mathcal{C}^2 = \mathcal{C}$. Let $\varphi \in \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$. Then by Lemma 7.1, we know that $\mathcal{C}\varphi \in \mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$. Hence, as $\mathcal{C} = \mathrm{Id}$ on $\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D})$ by Step I, we see that

$$\mathcal{C}^2\varphi = \mathcal{C}(\mathcal{C}\varphi) = \mathrm{Id}(\mathcal{C}\varphi) = \mathcal{C}\varphi.$$

Therefore $\mathcal{C}^2 = \mathcal{C}$ and our proof is complete.

8. Fredholm properties on polyhomogeneous spaces

In this section we prove Theorem 2.5 characterizing Fredholm boundary value problems for Dirac operators on manifolds with corners of codimension two.

8.1. BVPs for linear maps between vector spaces

The proof of Theorem 2.5 is based on Theorem 8.1 below concerning "abstract" BVPs. This theorem was proved with J. Park in [68], but in order to keep this article self-contained, we give an abbreviated proof of the theorem whose details can be filled in or looked up in [68]. Let V_0, V_1, V_2 be vector spaces (finite- or infinite-dimensional) and let

$$A : V_1 \to V_2, \qquad \gamma : V_1 \to V_0$$

be linear surjective maps. Suppose that there is a projection $\mathcal{C} : V_0 \to V_0$ whose image is the "Hardy space" of A

$$\mathcal{H}(A) := \gamma \ker A = \{\gamma\phi \mid \phi \in V_1,\ A\phi = 0\} \subseteq V_0.$$

Also suppose that the "unique continuation property" holds:

$$\psi \in \operatorname{ran}\mathcal{C} \iff \exists!\phi \in V_1, \quad A\phi = 0 \quad \text{and} \quad \gamma\phi = \psi.$$

The novelty of the following "abstract" boundary value problem for linear maps is that it makes no mention of topology (Hilbert space, Fréchet space, etc.) so it can be applied to a wide range of situations. (In [68] the following result was stated for topological vector spaces but the proof is purely linear algebraic and makes no mention of topology.)

Theorem 8.1. *(See [68].) For an arbitrary projection $\mathcal{P} : V_0 \to V_0$, the operator*

$$A_{\mathcal{P}} : \mathrm{dom}(A_{\mathcal{P}}) \to V_2,$$

where

$$\mathrm{dom}(A_{\mathcal{P}}) := \{\phi \in V_1 \mid \mathcal{P}(\gamma\phi) = 0\} \subseteq V_1,$$

and the operator

$$\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P}$$

have isomorphic kernels and cokernels

$$\ker A_{\mathcal{P}} \cong \ker(\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P}), \qquad \frac{V_2}{\operatorname{ran} A_{\mathcal{P}}} \cong \frac{\operatorname{ran}\mathcal{P}}{\operatorname{ran}(\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P})}.$$

In particular, $A_{\mathcal{P}}$ is Fredholm; *that is, it has a finite-dimensional kernel and cokernel, if and only if $\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P}$ is Fredholm, in which case*

$$\operatorname{ind} A_{\mathcal{P}} = \operatorname{ind}(\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P}).$$

Proof. We need to prove that $\ker A_{\mathcal{P}} \cong \ker \mathcal{P}\mathcal{C}$ and $\operatorname{coker} A_{\mathcal{P}} \cong \operatorname{coker} \mathcal{P}\mathcal{C}$. That $\ker A_{\mathcal{P}} \cong \ker \mathcal{P}\mathcal{C}$ is easy: By "unique continuation," it follows that

$$\begin{aligned} \psi \in \operatorname{ran}\mathcal{C}, \quad \mathcal{P}\psi = 0 \quad &\Longleftrightarrow \quad \exists! \phi, \quad A\phi = 0 \quad \text{and} \quad \gamma\phi = \psi \quad \text{and} \quad \mathcal{P}\psi = 0 \\ &\Longleftrightarrow \quad \exists! \phi, \quad A\phi = 0 \quad \text{and} \quad \mathcal{P}\gamma\phi = 0 \quad \text{and} \quad \gamma\phi = \psi \\ &\Longleftrightarrow \quad \exists! \phi \in \ker A_{\mathcal{P}} \quad \text{with } \gamma\phi = \psi. \end{aligned}$$

Therefore the map

$$\ker A_{\mathcal{P}} \ni \phi \mapsto \gamma\phi \in \ker \mathcal{P}\mathcal{C}$$

is an isomorphism.

To prove that $\operatorname{coker} A_{\mathcal{P}} \cong \operatorname{coker} \mathcal{P}\mathcal{C}$, we define a map

$$f : V_0 \to V_2 / \operatorname{ran} A_{\mathcal{P}} =: \operatorname{coker} A_{\mathcal{P}}$$

as follows. Let $\psi \in V_0$. Then there is a $\phi \in V_1$ such that $\gamma\phi = \psi$. We define

$$f(\psi) := [A\phi] \in V_2 / \operatorname{ran} A_{\mathcal{P}},$$

where [ ] denotes equivalence class. It is easy to check that f is well defined independent of the choice of $\phi \in V_1$ with $\gamma\phi = \psi$, and, since $A : V_1 \to V_2$ is surjective, f is also surjective. It is also easy to check that $f : \ker \mathcal{P} \to \operatorname{ran} A_{\mathcal{P}}$, therefore f descends to a (still surjective) map on the quotient:

$$\tilde{f} : V_0 / \ker \mathcal{P} \to V_2 / \operatorname{ran} A_{\mathcal{P}}.$$

Since $\mathcal{P}$ is a projection, we have a canonical isomorphism $\operatorname{ran}\mathcal{P} \cong V_0 / \ker \mathcal{P}$, therefore we obtain a surjective map $\tilde{f} : \operatorname{ran}\mathcal{P} \to V_2 / \operatorname{ran} A_{\mathcal{P}}$. Finally, one can show that $\ker \tilde{f} = \operatorname{ran}(\mathcal{P}\mathcal{C} : \operatorname{ran}\mathcal{C} \to \operatorname{ran}\mathcal{P})$. It follows that $\tilde{f}$ descends to an isomorphism of vector spaces

$$\operatorname{coker} \mathcal{P}\mathcal{C} := \operatorname{ran}\mathcal{P} / \operatorname{ran}\mathcal{P}\mathcal{C} \xrightarrow{\cong} V_2 / \operatorname{ran} A_{\mathcal{P}} =: \operatorname{coker} A_{\mathcal{P}},$$

which completes our proof. $\square$

8.2. The Fredholm theorem

To prove Theorem 2.5, let $\mathcal{I} > -1$ be any index family associated to $M_2(X)$ and consider

$$\mathcal{D} : \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}-1}(X_{\mathrm{tb}}, F), \qquad \gamma^+_{\partial X} : \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E),$$

where $\gamma^+_{\partial X} := |_{\partial X}$ means restriction to (the hypersurface components of) ∂X, which is obviously surjective. Also, $\mathcal{D}$ is surjective by Theorem 5.4 and by Theorem 2.2 we know that

$$\mathcal{C} : \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E) \to \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$$

has image equal to the Cauchy data space of $\mathcal{D}$:

$$\mathcal{H}_{\overline{\mathcal{I}}}(\mathcal{D}) := \gamma^+_{\partial X} \ker_{\overline{\mathcal{I}}} \mathcal{D} = \left\{ \gamma^+_{\partial X}\phi \;\middle|\; \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E),\ \mathcal{D}\phi = 0 \right\} \subseteq \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$$

and

$$\psi \in \operatorname{ran} \mathcal{C} \quad \Longleftrightarrow \quad \exists! \phi \in \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E), \quad \mathcal{D}\phi = 0 \quad \text{and} \quad \gamma^+_{\partial X}\phi = \psi.$$

Note that the uniqueness statement is just the celebrated unique continuation principle for Dirac operators; see for example [7,11,28]. Therefore, with

$$A = \mathcal{D}, \quad V_1 = \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, E), \quad V_2 = \mathcal{A}^{\overline{\mathcal{I}}}(X_{\mathrm{tb}}, F), \quad \gamma = \gamma^+_{\partial X}, \quad V_0 = \mathcal{A}^{\overline{\mathcal{I}}}(\partial X, E)$$

we have satisfied all the conditions of Theorem 8.1. Applying Theorem 8.1 to this situation gives Theorem 2.5.

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