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On the noncommutative residue and the heat trace expansion on conic manifolds

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Abstract. Given a cone pseudodifferential operator P we give a full asymptotic expansion as $t \to 0^+$ of the trace $\operatorname{Tr} Pe^{-tA}$, where A is an elliptic cone differential operator for which the resolvent exists on a suitable region of the complex plane. Our expansion contains $\log t$ and *new* $(\log t)^2$ terms whose coefficients are given explicitly by means of residue traces. Cone operators are contained in some natural algebras of pseudodifferential operators on which unique trace functionals can be defined. As a consequence of our explicit heat trace expansion, we recover all these trace functionals.

1. Introduction

On a smooth compact manifold M without boundary we may consider the quotient algebra $\mathcal{A} = \Psi^{\mathbb{Z}}_{c\ell}(M)/\Psi^{-\infty}(M)$, where $\Psi^{\mathbb{Z}}_{c\ell}(M)$ is the algebra of classical pseudodifferential operators of integral order and $\Psi^{-\infty}(M)$ is its ideal of smoothing elements. Operators of order less than $-\dim M$ are of trace class, but the L^2-trace cannot be extended to the whole algebra $\mathcal{A}$. However, M. Wodzicki [21] and V. Guillemin [8] independently introduced a new functional $\operatorname{Res} : \mathcal{A} \to \mathbb{C}$ which vanishes on commutators, so defines a trace on $\mathcal{A}$. This trace functional is called the *noncommutative residue* or *Wodzicki residue*, and is unique in the sense that any other trace on $\mathcal{A}$ is a constant multiple of Res. A detailed survey about this trace can be found for instance in [9].

The noncommutative residue is closely related to the zeta function of operators and to the generalized heat trace asymptotics. In fact, $\operatorname{Res}(P)$ can be defined (up to a constant) as the residue at $z = 0$ of $\operatorname{Tr} PA^{-z}$, for some fixed invertible pseudodifferential operator A; the complex power being defined as by Seeley [20]. Alternatively, $\operatorname{Res}(P)$ can be defined as the coefficient of $\log t$ in the asymptotic expansion as $t \to 0^+$ of $\operatorname{Tr} Pe^{-tA}$. Using these different but equivalent definitions, the noncommutative residue has been extended to various algebras of pseudodifferential operators on manifolds with and without boundaries (cf. [3], [6], [11]) including manifolds with singularities, cf. [17], [18]. The purpose of this paper is

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to obtain a generalized heat trace expansion for cone operators, and use it to recover the corresponding noncommutative residue(s) that can be associated to them.

Let M be a compact manifold with boundary and let x denote a fixed boundary defining function. Let $\Psi_b^m(M)$ be the space of b-pseudodifferential operators over M, cf. Section 2. On manifolds with conic singularities, spaces of the form $x^{-p}\Psi_b^m(M)$, $p \in \mathbb{R}$, arise as natural spaces of pseudodifferential operators containing the cone differential operators.

As originally done by R. Melrose and V. Nistor in [17] for cusp operators, it is helpful to consider the following algebra

$$x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M) = \bigcup_{p\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} x^{-p}\Psi_b^m(M).$$

In this algebra there is an ideal of smoothing operators

$$\mathcal{I} := x^{\infty}\Psi_b^{-\infty}(M) = \bigcap_{p\in\mathbb{Z}} x^{-p}\Psi_b^{-\infty}(M),$$

and we can consider the following quotient algebras:

$$\mathcal{I}_\sigma := x^{\infty}\Psi_b^{\mathbb{Z}}(M)/\mathcal{I}, \quad \mathcal{I}_\partial := x^{-\mathbb{Z}}\Psi_b^{-\infty}(M)/\mathcal{I},$$

$$\mathcal{A}_\sigma := x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/x^{-\mathbb{Z}}\Psi_b^{-\infty}(M),$$
$$\mathcal{A}_\partial := x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/x^{\infty}\Psi_b^{\mathbb{Z}}(M),$$

$$\mathcal{A}_{\partial,\sigma} := x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/\{x^{-\mathbb{Z}}\Psi_b^{-\infty}(M) + x^{\infty}\Psi_b^{\mathbb{Z}}(M)\}.$$

These quotients 'separate' the filtration given by the order of the operators from the filtration given by the power of x. It turns out that on these algebras there are three 'unique' trace functionals $\mathrm{Tr}_{\partial,\sigma}$ on $\mathcal{A}_\sigma$, $\mathcal{A}_\partial$ and $\mathcal{A}_{\partial,\sigma}$, Tr_∂ on $\mathcal{I}_\partial$, and Tr_σ on $\mathcal{I}_\sigma$. They are conic versions of the functionals studied in [17]. $\mathrm{Tr}_{\partial,\sigma}$ and Tr_σ were also considered in [18]. Their definitions and basic properties are given in Section 2.

In [5, 4], the first author defines natural conditions (called parameter-ellipticity) on a cone differential operator $A \in x^{-m}\mathrm{Diff}_b^m(M)$ which ensure that the heat operator e^{-tA} exists. Moreover, a full asymptotic expansion of $\mathrm{Tr}\, e^{-tA}$ was obtained. Later in [12], this expansion was generalized to $\mathrm{Tr}\, Pe^{-tA}$ where P is a cone differential operator. In the present work, the techniques of [12] are expanded to analyze the generalized heat trace expansion when P is pseudodifferential.

Let $P \in x^{-p}\Psi_b^{m'}(M)$, $p, m' \in \mathbb{R}$, be a pseudodifferential cone operator. Assume that $p < m$ so that Pe^{-tA} is of trace class (on appropriate weighted Sobolev spaces) for all $t > 0$. In Section 5, we obtain an asymptotic expansion of the form

$$\mathrm{Tr}\, Pe^{-tA} \sim_{t\to 0^+} \sum_{k=0}^{\infty} c_k\, t^{\xi_k} + \sum_{k=0}^{\infty} c_k'(\log t)\, t^{\eta_k} + \sum_{k=0}^{\infty} c_k''(\log t)^2\, t^{\omega_k}$$

and give explicit formulas for all the $\log t$ coefficients c_k' and c_k'' in terms of the trace functionals mentioned above, see Theorems 5.1, 5.2 and their corollaries. In particular, the coefficient of $\log t$ is

$$-\frac{1}{m}\operatorname{Tr}_\sigma(P) - \frac{1}{m}\operatorname{Tr}_\partial(P) - \frac{1}{m^2}\operatorname{Tr}_{\partial,\sigma}(P),$$

and $\operatorname{Tr}_{\partial,\sigma}(P) = -m^2 \times$ the coefficient of $(\log t)^2$.

Our results rely on the parametrix construction within a parameter-dependent calculus which is presented in Section 3, and on basic properties of the Laplace and Mellin transforms discussed in Section 4. Let us finally mention that the study of heat trace asymptotics on manifolds with conic singularities was initiated by J. Cheeger [1, 2] and further developed by many other authors, a list of references can be found in [4], [12].

2. Operator algebras and trace functionals

Recall that an n-dimensional manifold with corners M is a manifold with atlas given by local models of the form $[0,\infty)^k \times \mathbb{R}^{n-k}$, where k can run anywhere between 0 and n. (This definition is suitable for our purposes, but is a bit more general than the standard definition, cf. [15].) For any $\alpha \in \mathbb{R}$, the b-alpha density bundle Ω_b^α is the line bundle on M with local basis of the form $|(dx/x)dy|^\alpha$ on a patch $[0,\infty)_x^k \times \mathbb{R}_y^{n-k}$. We either use $\alpha = 1/2$ or $\alpha = 1$; in the latter case, we will write Ω_b for Ω_b^1.

We now review the definition of b-pseudodifferential operators. The standard reference is Melrose's book [15]. Let M be an n-dimensional compact manifold with connected boundary $Y = \partial M$. Recall that M_b^2 is the manifold with corners that has an atlas consisting of the usual coordinate patches on $M^2 \setminus Y^2$ together with polar coordinate patches over Y^2 in M^2. The typical picture of M_b^2 is shown in Figure 2.1. The boundary hypersurfaces *lb*, *rb*, and *ff* stand for "left boundary", "right boundary", and "front face".

For symmetry reasons, it is common practice to have b-pseudodifferential operators act on b-half densities rather than on functions. Given $m \in \mathbb{R}$, the space $\Psi_b^m(M, \Omega_b^{\frac{1}{2}})$ consists of operators A on $C^\infty(M, \Omega_b^{\frac{1}{2}})$ that have a Schwartz kernel K_A satisfying the following two conditions:

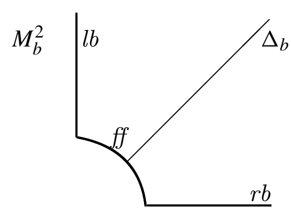

Fig. 2.1. The manifold M_b^2 is M^2 "blown-up" at the corner Y^2. The submanifold Δ_b is the the diagonal in M^2 lifted to M_b^2; that is, written in the polar coordinates of M_b^2.

1. given $\varphi \in C_c^\infty(M_b^2 \setminus \Delta_b)$, we can write φK_A in the form $k \cdot \nu$, where k is a smooth function on M_b^2 vanishing to infinite order at *lb* and *rb*, and where $\nu \in C^\infty(M_b^2, \Omega_b^{\frac{1}{2}})$,
2. given a coordinate patch of M_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}_\eta^n$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_A = \int e^{i\xi\cdot\eta} a(y,\xi) đ\xi \cdot \nu, \tag{2.1}$$

where $\nu \in C^\infty(M_b^2, \Omega_b^{\frac{1}{2}})$, and where $a(y,\xi)$ is a classical pseudodifferential symbol of order m.

Henceforth, we fix a boundary defining function x for Y. Let $p, m \in \mathbb{R}$. Given $A \in x^{-p}\Psi_b^m(M, \Omega_b^{\frac{1}{2}})$, we define its (Wodzicki) residue density $\omega(A)$ as follows, cf. [22]. Write the kernel of A locally as in (2.1) above. Observe that since $\Delta_b \cong M$, the coordinate patch $\mathcal{U}$ can be considered a coordinate patch on M. We define $\omega(A)$ locally on the patch $\mathcal{U}$ by

$$\omega_y(A) := \int_{|\xi|=1} a_{-n}(y,\xi) đ\xi \cdot \nu|_{\Delta_b}, \tag{2.2}$$

where $a_{-n}(y,\xi)$ is the homogeneous term of degree $-n$ in the symbol expansion of $a(y,\xi)$. Note that if $m \notin \mathbb{Z}$, then $\omega(A) = 0$ since $a(y,\xi)$ has no homogeneous component of degree $-n$. Also note that $\nu|_{\Delta_b} \in C^\infty(M, \Omega_b)$ as one can check using local coordinates. The local expression (2.2) turns out to be independent of coordinates; a nice proof of this fact can be found in [3]. Thus, the local expressions (2.2) actually define a global density $\omega(A) \in x^{-p}C^\infty(M, \Omega_b)$.

We now review various trace functional introduced by Melrose and Nistor [17, Sec. 5], and in a slightly different setting by Schrohe [18]. We remark that Melrose and Nistor studied cusp operators, although by continuity many of their results apply to b-operators. In what follows, we will freely use their results presented from the perspective we need them.

The functional $\mathrm{Tr}_{\partial,\sigma}$. Given $P \in x^{-p}\Psi_b^m(M, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$ and $p \in \mathbb{N}_0$, we define

$$\mathrm{Tr}_{\partial,\sigma}(P) := \frac{1}{p!}\int_Y \partial_x^p\{x^p\,\omega(P)\}|_{x=0}. \tag{2.3}$$

The expression $\partial_x^p\{x^p\,\omega(P)\}|_{x=0}$ is defined as follows. Since $\omega(P)$ belongs to $x^{-p}C^\infty(M, \Omega_b)$, in a collar $M \cong [0,1)_x \times Y$ near Y, we can write $x^p\,\omega(P) = f(x)\frac{dx}{x}$, where $f(x)$ is a smooth family of densities on Y. We then define $\partial_x^p\{x^p\,\omega(P)\}|_{x=0} := \partial_x^p f(0)$. It is not obvious that $\mathrm{Tr}_{\partial,\sigma}(P)$ is defined independent of the boundary defining function x. We will prove that it is independently defined following Lemma 2.1 below.

A more transparent way to view $\mathrm{Tr}_{\partial,\sigma}(P)$ is as follows. Expand $\omega(P)$ near the boundary Y in a Taylor series:

$$\omega(P) \sim \sum_{j=-p}^{\infty} x^j\, \omega(P_j) \frac{dx}{x}.$$

Then as $\frac{1}{p!}\partial_x^p \{x^p\, \omega(P)\}|_{x=0} = \omega(P_0)$, we have

$$\mathrm{Tr}_{\partial,\sigma}(P) = \int_Y \omega(P_0). \tag{2.4}$$

Thus, $\mathrm{Tr}_{\partial,\sigma}$ is a natural way to restrict the Wodzicki residue trace to the boundary. If $p \notin \mathbb{N}_0$, then $\mathrm{Tr}_{\partial,\sigma}(P)$ is defined to be zero.

Lemma 2.1. *Let $u \in x^{-p}C^\infty(M, \Omega_b)$, $p \in \mathbb{R}$. Then the function $\mathbb{C} \ni z \mapsto \int_M x^z u$ is well-defined for $\Re z > p$ and it extends to be a meromorphic function on $\mathbb{C}$ with only simple poles at $z = p,\ p-1,\ p-2, \ldots$. In particular, it has a pole at $z = 0$ if and only if $p \in \mathbb{N}_0$ in which case, its residue is given by $\frac{1}{p!}\int_Y \partial_x^p \{x^p u\}|_{x=0}$. The regular value of the function $\mathbb{C} \ni z \mapsto \int_M x^z u$ is called the b-integral of u and is denoted by ${}^b\!\!\int_M u$.*

Proof. In a neighborhood $M \cong [0,1)_x \times Y$ of Y, we can write $u = x^{-p} v(x) \frac{dx}{x}$ where $v(x)$ is a smooth family of densities on Y. Expanding $v(x)$ in a Taylor series at $x = 0$ gives the expansion $u \sim \sum_{j=0}^\infty x^{-p+j} u_j \frac{dx}{x}$. Then our lemma follows since $\int_0^1 x^{z-p+j} \frac{dx}{x} = \frac{1}{z-p+j}$. In particular, if $p \in \mathbb{N}_0$, we get the expansion $u \sim \sum_{j=-p}^\infty x^j u_{p+j} \frac{dx}{x}$, where

$$u_{p+j} = \frac{1}{(p+j)!}\partial_x^{p+j} v(0) = \frac{1}{(p+j)!}\partial_x^{p+j}\{x^p u\}|_{x=0}$$

for $j = -p, -p+1, \ldots$ □

Lemma 2.2. *The functional $\mathrm{Tr}_{\partial,\sigma}$ is defined independent of the choice of boundary defining function x.*

Proof. Suppose that x' is another boundary defining function. Then $x' = ax$ where $0 < a \in C^\infty(M)$. Denote by $\mathrm{Tr}_{\partial,\sigma,x}$ and $\mathrm{Tr}_{\partial,\sigma,x'}$ the functional $\mathrm{Tr}_{\partial,\sigma}$ defined using the boundary defining functions x and x', respectively. Then by Lemma 2.1,

$$\mathrm{Tr}_{\partial,\sigma,x}(P) - \mathrm{Tr}_{\partial,\sigma,x'}(P) = \mathrm{Res}_1 \int_M x^z P(z),$$

where Res_1 denotes the residue at $z = 0$, and where $P(z) = (1 - a^z)\,\omega(P)$. Since $P(0) = 0$, by Lemma 2.1 it follows that $\int_M x^z P(z)$ is regular at $z = 0$. Thus, $\mathrm{Tr}_{\partial,\sigma,x}(P) = \mathrm{Tr}_{\partial,\sigma,x'}(P)$. □

We now consider in what sense the functional $\mathrm{Tr}_{\partial,\sigma}$ defines a trace. To do so, we need to introduce some operator algebras. The main algebra is

$$x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}}) = \bigcup_{p\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} x^{-p}\Psi_b^m(M,\Omega_b^{\frac{1}{2}}).$$

Consider now the following quotient algebras:

$$\begin{aligned}\mathcal{A}_\sigma &:= x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}})/x^{-\mathbb{Z}}\Psi_b^{-\infty}(M,\Omega_b^{\frac{1}{2}}),\\ \mathcal{A}_\partial &:= x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}})/x^{\infty}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}}),\end{aligned}$$

$$\mathcal{A}_{\partial,\sigma} := x^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}})/\{x^{-\mathbb{Z}}\Psi_b^{-\infty}(M,\Omega_b^{\frac{1}{2}}) + x^{\infty}\Psi_b^{\mathbb{Z}}(M,\Omega_b^{\frac{1}{2}})\}.$$

Melrose and Nistor [17] prove that a functional analogous to $\mathrm{Tr}_{\partial,\sigma}$ defines a unique trace functional on quotient algebras of cusp operators. The corresponding statement in the cone setting is the following.

Theorem 2.3. *The functional* $\mathrm{Tr}_{\partial,\sigma}$ *defines a trace on the algebras* $\mathcal{A}_\sigma$, $\mathcal{A}_\partial$, *and* $\mathcal{A}_{\partial,\sigma}$, *and is the unique such trace in the sense that any other trace on any of these algebras is a constant multiple of* $\mathrm{Tr}_{\partial,\sigma}$.

The functional Tr_∂. Fix a holomorphic family $Q(z)\in x^{\alpha z}\Psi_b^{\beta z}(M,\Omega_b^{\frac{1}{2}})$, $\alpha,\beta\in\mathbb{R}$, of operators such that $Q(0)=\mathrm{Id}$. Let $P\in x^{-p}\Psi_b^m(M,\Omega_b^{\frac{1}{2}})$, $p,m\in\mathbb{R}$, and denote by $(PQ(z))|_\Delta$ the restriction of the Schwartz kernel of $PQ(z)$ to the diagonal Δ in M^2. Then by [17, Lem. 4], $(PQ(z))|_\Delta$ defines a meromorphic function on $\mathbb{C}$, taking values in $x^{\alpha z-p}C^\infty(M,\Omega_b)$ (using that $\Delta\cong M$), with possible simple poles at those $z\in\mathbb{C}$ with $\beta z=-n-m+k$ for $k=0,1,\ldots$.
In particular,

$$\mathrm{Res}_0(PQ(z))|_\Delta \in x^{-p}C^\infty(M,\Omega_b)$$

is well defined, where Res_0 means “the regular value at $z=0$”. We define

$$\mathrm{Tr}_\partial(P) := \frac{1}{p!}\int_Y \partial_x^p\{x^p\,\mathrm{Res}_0(PQ(z))|_\Delta\}|_{x=0}. \tag{2.5}$$

If $p\notin\mathbb{N}_0$, then $\mathrm{Tr}_\partial(P)$ is defined to be zero. The functional Tr_∂ was first introduced in [17]. The same argument found in Lemma 2.2 shows that Tr_∂ is defined independent of the boundary defining function chosen. (Unfortunately, Tr_∂ does depend on the regularizing operator $Q(z)$; however, its dependence on $Q(z)$ can be explicitly determined, see [17, Lem. 11].)

For P of sufficiently large negative order, $\mathrm{Tr}_\partial(P)$ has a natural interpretation. Indeed, if $m<-n$, then observe that $(PQ(z))|_\Delta$ is regular at $z=0$ with value $K_P|_\Delta\in x^{-p}C^\infty(M,\Omega_b)$. Assume that $p\in\mathbb{N}_0$ and expand $K_P|_\Delta$ in Taylor series at $x = 0$: $K_P|_\Delta \sim \sum_{j=-p}^{\infty} x^j K_{P,j}|_\Delta \frac{dx}{x}$. Then as $\partial_x^p \{x^p K_P|_\Delta\}|_{x=0} = K_{P,0}|_\Delta$, we have

$$\mathrm{Tr}_\partial(P) = \int_Y K_{P,0}|_\Delta.$$

Thus, for P of sufficiently negative order, $\mathrm{Tr}_\partial(P)$ is a type of L^2-trace of P restricted to the boundary of M.

Consider now the algebra

$$\mathcal{I}_\partial := x^{-\mathbb{Z}}\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}})/x^{\infty}\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}}).$$

The next theorem follows from analogous results proved in [17].

Theorem 2.4. *The functional* Tr_∂ *defines a trace on* $\mathcal{I}_\partial$*, and it is unique in the sense that any other trace on* $\mathcal{I}_\partial$ *is a constant multiple of* Tr_∂.

The trace functional $\mathbf{Tr}_\sigma$. Given $P \in x^{-p}\Psi_b^m(M, \Omega_b^{\frac{1}{2}})$, $p, m \in \mathbb{R}$, we define

$$\mathrm{Tr}_\sigma(P) := {}^b\!\!\int_M \omega(P), \tag{2.6}$$

where ${}^b\!\!\int$ is the b-integral introduced in Lemma 2.1. This functional was first introduced in [18] when P vanishes at the boundary (in which case, ${}^b\!\!\int_M \omega(P) = \int_M \omega(P)$ is just the usual integral of $\omega(P)$ over M). In the generality presented in (2.6), Tr_σ was first studied in [17]. Tr_σ is the natural generalization of the Wodzicki residue trace to cone operators.

To see how the functional Tr_σ depends on the boundary defining function x, we first prove the following lemma.

Lemma 2.5. *Let $x' = ax$ be another boundary defining function for Y where $0 < a \in C^\infty(M)$ and let $u \in x^{-p}C^\infty(M, \Omega_b)$, $p \in \mathbb{R}$. Denote by ${}^{b,x}\!\!\int_M u$ and ${}^{b,x'}\!\!\int_M u$ the b-integral of u as defined by using the boundary defining function x and x', respectively. Then,*

$$ {}^{b,x}\!\!\int_M u = {}^{b,x'}\!\!\int_M u + \frac{1}{p!}\int_Y \partial_x^p\{x^p \log a\, u\}|_{x=0}. \tag{2.7}$$

If $p \notin \mathbb{N}_0$, then the last term is understood to be equal to zero.

Proof. We can write $(x')^z u - x^z u = z x^z a(z) u$, where $a(z) = (a^z - 1)/z$. As $a(0) = \log a$, we have

$$ {}^{b,x'}\!\!\int_M u \;-\; {}^{b,x}\!\!\int_M u = \mathrm{Res}_0\, z \int_M x^z a(z) u = \mathrm{Res}_1 \int_M x^z \log a\, u,$$

where Res_1 denotes the residue at $z = 0$. The identity (2.7) now follows from Lemma 2.1. □

Lemma 2.6. *Given another boundary defining function $x' = ax$ where $0 < a \in C^\infty(M)$, the trace functionals Tr_σ defined with respect to x' and x, respectively, are related by*

$$\mathrm{Tr}_{\sigma,x'}(P) - \mathrm{Tr}_{\sigma,x}(P) = \mathrm{Tr}_{\partial,\sigma}(\log a\, P).$$

Proof. By Lemma 2.5, we have

$$\begin{aligned}\mathrm{Tr}_{\sigma,x'}(P) - \mathrm{Tr}_{\sigma,x}(P) &= {}^{b,x}\!\!\int_M \omega(P) - {}^{b,x'}\!\!\int_M \omega(P) \\ &= \frac{1}{p!}\int_Y \partial_x^p \{x^p\, \omega(\log a\, P)\}|_{x=0} \\ &= \mathrm{Tr}_{\partial,\sigma}(\log a\, P).\end{aligned}$$

□

The final algebra we consider is

$$\mathcal{I}_\sigma := x^\infty \Psi_b^{\mathbb{Z}}(M, \Omega_b^{\frac{1}{2}})/x^\infty \Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}}).$$

The next theorem follows from analogous results proved in [17].

Theorem 2.7. *The functional Tr_σ defines a trace on $\mathcal{I}_\sigma$, and it is unique in the sense that any other trace on $\mathcal{I}_\sigma$ is a constant multiple of Tr_σ.*

3. Parameter-dependent calculus

Let $E \to M$ be a smooth vector bundle. The space $x^{-m}\,\mathrm{Diff}_b^m(M, E)$, $m > 0$, is the space of differential operators that near the boundary take the form

$$A = x^{-m} \sum_{k+|\alpha|\le m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

with coefficients smooth up to $x = 0$. To such an A we associate the operator

$$A_0 = r^{-m} \sum_{k+|\alpha|\le m} a_{k\alpha}(0, y)(rD_r)^k D_y^\alpha$$

by freezing the coefficients at $r = 0$. Recall that $D = \frac{1}{i}\partial$.

In order to define the Sobolev spaces on which these operators act continuously, we fix a boundary defining function x, a b-density $d\mu$, and a smooth hermitian metric on $E \to M$. Then, $L_b^2(M, E)$ is the Hilbert space of sections in E that are square integrable with respect to $d\mu$. Recall that, on a collar neighborhood of $\partial M = \{0\} \times Y$, $d\mu$ can be written as $d\mu = \frac{dx}{x} \otimes dy$, where dy is a smooth density on Y. For any $m \in \mathbb{N}$, the space $H_b^m(M, E)$ consists of those elements $u \in L_b^2(M, E)$ such that $Lu \in L_b^2(M, E)$ for every operator $L \in \mathrm{Diff}_b^m(M, E)$. For an arbitrary $s \in \mathbb{R}$, the space $H_b^s(M, E)$ can be defined by duality and interpolation.

On the manifold $Y^\wedge := \mathbb{R}_+ \times Y$ we also consider the spaces $\mathcal{K}^{s,\alpha}(Y^\wedge, E)$, $s, \alpha \in \mathbb{R}$, defined as follows. Let $\omega \in C_c^\infty(\overline{\mathbb{R}}_+)$ with $\omega(r) = 1$ near $r = 0$. Then $\mathcal{K}^{s,\alpha}(Y^\wedge, E)$ consists of distributions u such that $\omega u \in r^\alpha H_b^s(Y^\wedge, E)$ and such that given any coordinate patch $\mathcal{U}$ on Y diffeomorphic to an open subset of $\mathbb{S}^{n-1}$ and function $\varphi \in C_c^\infty(\mathcal{U})$, we have $(1-\omega)\varphi\, u \in H^s(\mathbb{R}^n, E)$ where $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ is identified with $\mathbb{R}^n \setminus \{0\}$ via polar coordinates.

Definition 3.1. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M, \Omega_b^{\frac{1}{2}})$ and let Λ be a sector in $\mathbb{C}$ containing the origin. The operator family $A - \lambda$ is said to be parameter-elliptic with respect to $\alpha \in \mathbb{R}$ on Λ if and only if*

1. *$\sigma_{\psi,b}^m(A)(\xi) - \lambda$ is invertible for all $\xi \neq 0$ and $\lambda \in \Lambda$,*
2. *$\operatorname{spec}_b(A) \cap \{z \in \mathbb{C} \mid \Im z = -\alpha\} = \varnothing$,*
3. *$A_0 - \lambda : \mathcal{K}^{s,\alpha}(Y^\wedge, \Omega_b^{\frac{1}{2}}) \to \mathcal{K}^{s-m,\alpha-m}(Y^\wedge, \Omega_b^{\frac{1}{2}})$ is invertible for every $\lambda \in \Lambda$ sufficiently large, and for some $s \in \mathbb{R}$.*

Here $\sigma_{\psi,b}^m(A)(\xi)$ is the principal b-symbol of $x^m A$ and $\operatorname{spec}_b(A)$ denotes its boundary spectrum, cf. [4, Sec. 3].

This concept of ellipticity is similar to the one introduced by Schulze [19] for the analysis on manifolds with edges. The following proposition is proved in [4, Th. 3.2].

Proposition 3.2. *If $A - \lambda$ is parameter-elliptic with respect to α on Λ, then*

$$A - \lambda : x^\alpha H_b^s(M, \Omega_b^{\frac{1}{2}}) \to x^{\alpha-m} H_b^{s-m}(M, \Omega_b^{\frac{1}{2}})$$

is invertible for every $\lambda \in \Lambda$ sufficiently large, and all $s \in \mathbb{R}$.

The parameter-dependent ellipticity of $A - \lambda$ turns out to be a necessary condition for the previous proposition to hold with a uniformly bounded inverse. Thus any positive selfadjoint cone operator on $x^{\alpha-m} H_b^{s-m}(M, \Omega_b^{\frac{1}{2}})$ with domain $x^\alpha H_b^s(M, \Omega_b^{\frac{1}{2}})$ is indeed parameter-elliptic with respect to α.

Example 3.3. Let x be a fixed boundary defining function, and let g be a Riemannian metric on M which, near the boundary, coincides with the cone metric $dx^2 + x^2 g_Y$, where g_Y is a metric on Y. The corresponding measure is of the form $x^n d\mu$ for a b-measure $d\mu$. Let Δ_g be the Laplace-Beltrami operator associated to the metric g. This operator is by definition symmetric on $L^2(M, x^n d\mu) = x^{-n/2} L_b^2(M)$. Therefore, the operator $-x^{n/2-1}\Delta_g\, x^{-n/2+1}$ is symmetric on $x^{-1}L_b^2(M)$ which implies the symmetry of

$$A = -x^{n/2-1}\Delta_g\, x^{-n/2+1} + x^{-2}a^2$$

for every real number a. Now, for a function $u \in C_c^\infty(M)$ supported near the boundary, we have

$$\Delta_g u = -x^{-2}\Big((xD_x)^2 u - i(n-2)(xD_x)u - \Delta_Y u\Big),$$

where Δ_Y is the Laplacian associated to g_Y on Y. Then,

$$-x^{n/2-1}\Delta_g\, x^{-n/2+1}u = x^{-2}\left((xD_x)^2u - \Delta_Y u + \tfrac{(n-2)^2}{4}u\right)$$

and so $Au = x^{-2}\left((xD_x)^2u + L_{Y,a}u\right)$, where $L_{Y,a} = -\Delta_Y + \frac{(n-2)^2}{4} + a^2$. For $a > 1$ the boundary spectrum of x^2A does not intersect the critical strip $\{\sigma \in \mathbb{C} \mid |\Im\sigma| < 1\}$ so that A with domain $xH_b^2(M)$ is selfadjoint on $x^{-1}L_b^2(M)$. In particular, $A - \lambda$ is parameter-elliptic with respect to $\alpha = 1$ on any sector $\Lambda \subset \mathbb{C}$ contained in the resolvent set of A.

We next analyze the resolvent $(A-\lambda)^{-1}$ within a suitable parameter-dependent pseudodifferential calculus. We begin by describing the corresponding parameter-dependent symbols as defined in [14]. Related symbol classes can be found in [7].

For $m, p \in \mathbb{R}$ and $d > 0$ we define $S^{m,p,d}(\mathbb{R}^n;\Lambda)$ as the space of functions $a \in C^\infty(\mathbb{R}^n \times \Lambda)$ such that

$$|\partial_\xi^\alpha \partial_\lambda^\beta a(\xi,\lambda)| \le C_{\alpha\beta}(1+|\xi|)^{m-p-|\alpha|}(1+|\xi|+|\lambda|^{1/d})^{p-d|\beta|}.$$

The space $S_r^{m,p,d}(\mathbb{R}^n;\Lambda)$, $p/d \in \mathbb{Z}$, consists of elements $a \in S^{m,p,d}(\mathbb{R}^n;\Lambda)$ such that if we set

$$\tilde{a}(\xi,z) := z^{p/d}a(\xi,1/z),$$

then $\tilde{a}(\xi,z)$ is smooth at $z=0$, and

$$|\partial_\xi^\alpha \partial_z^\beta \tilde{a}(\xi,z)| \le C_{\alpha\beta}(1+|\xi|)^{m-p-|\alpha|+d|\beta|}(1+|z||\xi|^d)^{p/d-|\beta|}$$

uniformly for $|z| \le 1$. Further let $S_{r,c\ell}^{m,p,d}(\mathbb{R}^n;\Lambda)$ be the space of elements $a \in S_r^{m,p,d}(\mathbb{R}^n;\Lambda)$ that, for every $N \in \mathbb{N}$, admit a decomposition

$$a(\xi,\lambda) = \sum_{j=0}^{N-1}\chi(\xi)a_{m-j}(\xi,\lambda) + r_N(\xi,\lambda), \tag{3.1}$$

where $r_N \in S_r^{m-N,p,d}(\mathbb{R}^n;\Lambda)$, $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) = 0$ for $|\xi| \le \frac{1}{2}$ and $\chi(\xi) = 1$ for $|\xi| \ge 1$, and where each $a_{m-j}(\xi,\lambda)$ is anisotropic homogeneous of degree $m-j$, i.e.,

$$a_{m-j}(\delta\xi,\delta^d\lambda) = \delta^{m-j}a_{m-j}(\xi,\lambda) \quad \text{for } \delta > 0,$$

and $z^{p/d}a_{m-j}(\xi,1/z)$ is smooth at $z=0$. Finally, $a \in S_{r,c\ell}^{m,p,d}(\mathbb{R}^n;\Lambda)$ is said to be *holomorphically tempered* if it is holomorphic in a neighborhood of Λ, and there exists an $\varepsilon > 0$ such that each $a_{m-j}(\xi,\lambda)$ is holomorphic in the region

$$\left\{(\xi,\lambda) \in (\mathbb{R}^n\setminus\{0\})\times\mathbb{C} \mid \lambda \in \Lambda \text{ or } |\lambda| \le \varepsilon|\xi|^d \text{ or } \tfrac{1}{\varepsilon}|\xi|^d \le |\lambda|\right\}.$$

We now define our corresponding spaces of parameter-dependent cone operators, cf. [12, 13]. Let ρ denote a boundary defining function for the front face *ff* of M_b^2. Our first space $\Psi_{c,\Lambda}^{m,p,d}(M, \Omega_b^{\frac{1}{2}})$ is defined as the space of those operator families $A(\lambda) \in C^\infty(\Lambda, \Psi_b^m(M, \Omega_b^{\frac{1}{2}}))$ whose Schwartz kernels $K_{A(\lambda)}$ satisfy the following two conditions:

1. given $\varphi \in C_c^\infty(M_b^2 \setminus \Delta_b)$, then $\varphi K_{A(\lambda)} = R(\rho^d \lambda)$, where $R(\lambda)$ is smooth in λ taking values in $\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}})$ and satisfies the estimates
$$|\partial_\lambda^\alpha R(\lambda)| \le C_\alpha (1 + |\lambda|)^{p/d - |\alpha|}, \text{ for all } \alpha \in \mathbb{N}_0, \ \lambda \in \Lambda,$$
2. given a coordinate patch on M_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}_\eta^n$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, then
$$\varphi K_{A(\lambda)} = \int e^{i\xi\cdot\eta} a(y, \xi, \rho^d \lambda) đ\xi \cdot \nu,$$
where $\nu \in C^\infty(M_b^2, \Omega_b^{\frac{1}{2}})$, and where
$$y \mapsto a(y, \xi, \lambda) \in C^\infty(\mathcal{U}, S_{r,c\ell}^{m,p,d}(\mathbb{R}^n; \Lambda)).$$

Furthermore, $A(\lambda)$ is holomorphically tempered if it is holomorphic in a neighborhood of Λ and its corresponding local symbols are holomorphically tempered.

We sketch the definition of our next parameter-dependent space; see [12] for the precise definition. Assume that a branch of log is defined and fixed on Λ, and let
$$\Lambda^{1/d} := \{\lambda^{1/d} \mid \lambda \in \Lambda \setminus 0\} \cup \{0\}.$$
Let $\mathcal{T}$ denote the manifold $M_b^2 \times \Lambda^{1/d}$ blown-up at the submanifold $\mathit{ff} \times \{\lambda^{1/d} = \infty\}$. The boundary hypersurface of $\mathcal{T}$ originating from $M_b^2 \times \{\lambda^{1/d} = \infty\}$ is called the "boundary at ∞". We define $\Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(M, \Omega_b^{\frac{1}{2}})$ as the class of operators whose Schwartz kernels can be written in the form $k \cdot \nu$, where $\nu \in C^\infty(M_b^2, \Omega_b^{\frac{1}{2}})$, and where k is a function on $\mathcal{T}$ which vanishes to infinite order at the boundary at ∞, and is of asymptotic type $\mathcal{E}$ at the rest of the faces of $\mathcal{T}$. Here, k is of asymptotic type $\mathcal{E}$ means, roughly speaking, that if H is a boundary hypersurface of $\mathcal{T}$ (except the boundary at ∞), then $\mathcal{E}$ associates to H a set of numbers E_H such that k can be expanded at H in powers and powers of the logarithm of the corresponding boundary defining function; the powers that may occur in the expansion are given exactly by the set E_H.

Our final space of operators is $\Psi_\Lambda^{-\infty,\mathcal{F}}(M, \Omega_b^{\frac{1}{2}})$, where $\mathcal{F} = (F_1, F_2)$ is an index family for M^2. An element $S(\lambda) \in \Psi_\Lambda^{-\infty,\mathcal{F}}(M, \Omega_b^{\frac{1}{2}})$ is a family of operators whose Schwartz kernels $K_{S(\lambda)}$ can be written in the form $k \cdot \nu$, where $\nu \in C^\infty(M^2, \Omega_b^{\frac{1}{2}})$, and where k is a function on $M^2 \times \Lambda$ that vanishes to infinite order at $\lambda = \infty$, and is of asymptotic type $\mathcal{F}$ on M^2.

The next theorem follows from corresponding results in [13] and [14] .

Theorem 3.4. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M, \Omega_b^{\frac{1}{2}})$ be such that $A-\lambda$ is parameter-elliptic with respect to $\alpha \in \mathbb{R}$ on a sector $\Lambda \subset \mathbb{C}$. Let $P \in x^{-p}\Psi_b^{m'}(M, \Omega_b^{\frac{1}{2}})$, $p, m' \in \mathbb{R}$. Then we get a decomposition*

$$P(A-\lambda)^{-1} = Q(\lambda) + R(\lambda) + S(\lambda),$$

where $Q(\lambda) \in x^{m-p}\Psi_{c,\Lambda}^{m'-m,-m,m}(M, \Omega_b^{\frac{1}{2}})$, $R(\lambda) \in x^{-p}\Psi_{c,\Lambda}^{-\infty,m,\mathcal{G}(\alpha)}(M, \Omega_b^{\frac{1}{2}})$, and $S(\lambda) \in x^{-p}\Psi_{\Lambda}^{-\infty,\mathcal{F}(\alpha)}(M, \Omega_b^{\frac{1}{2}})$ for some index families $\mathcal{G}(\alpha)$ and $\mathcal{F}(\alpha)$. $Q(\lambda)$ being holomorphically tempered.

Remark 3.5. The index families $\mathcal{G}(\alpha)$ and $\mathcal{F}(\alpha)$ are given explicitly in terms of the boundary spectrum $\operatorname{spec}_b(A)$. For their precise definition we refer to [12, Section 3.2]. They will play no role in the trace expansions that we are looking at, so we do not need their explicit descriptions here.

4. Analysis of the Laplace and Mellin transforms

In order to give a precise representation of the coefficients in the asymptotic expansion of $\operatorname{Tr} Pe^{-tA}$ we first want to analyze the Laplace transform of a holomorphically tempered family $Q(\lambda) \in x^{m-p}\Psi_{c,\Lambda}^{m'-m,-m,m}(M; \Omega_b^{\frac{1}{2}})$, $m', p \in \mathbb{R}$, $m > \max(0, p)$. Here, we assume that Λ is a sector of the form

$$\{\lambda \in \mathbb{C} \mid \varepsilon \le \arg\lambda \le 2\pi - \varepsilon\} \text{ for some } 0 < \varepsilon < \pi/2. \tag{4.1}$$

Since $Q(\lambda)$ is holomorphically tempered, there exists $\delta > 0$ such that $Q(\lambda)$ is holomorphic on and to the left of the contour

$$\Upsilon := \{\lambda \in \mathbb{C} \mid \lambda \in \partial\Lambda \text{ for } |\lambda| \ge \delta, \text{ or } |\lambda| = \delta \text{ for } 2\pi - \varepsilon \le \arg\lambda \le \varepsilon\}.$$

For $t > 0$ define

$$\mathcal{Q}(t) = \frac{i}{2\pi}\int_\Upsilon e^{-t\lambda} Q(\lambda)d\lambda. \tag{4.2}$$

Then $\mathcal{Q}(t)$ is an operator of trace class on $x^{\alpha-m}H_b^s(M, \Omega_b^{\frac{1}{2}})$ for every $s \in \mathbb{R}$. This relies on the fact that for each $t > 0$, the kernel of $\mathcal{Q}(t)$ is a smooth function on M^2 vanishing to infinite order at ∂M^2, cf. [14].

Let $M_b^2 \cong [0,1)_x \times Y \times \mathbb{R}_\eta^n$ near the face *ff* with $\Delta_b \cong [0,1)_x \times Y \times \{0\}$, see Figure 2.1. For simplicity, in the analysis that follows we assume that the kernel of $Q(\lambda)$ is supported on this coordinate patch. Furthermore, since the variables on Y enter in the analysis more or less as parameters (cf. [14]), we also omit the variables on Y. Thus, we can write

$$K_{Q(\lambda)} = \int e^{i\xi\cdot\eta} q(x, \xi, x^m\lambda) đ\xi \;\left|\tfrac{dx}{x} d\eta\right|^{1/2}, \tag{4.3}$$

where $q(x, \xi, \lambda) \in S^{m'-m,-m,m}_{r,c\ell}(\mathbb{R}^n; \Lambda)$ and $q(x, \xi, \lambda) = O(x^{m-p})$. Thus,

$$K_{\mathcal{Q}(t)} = \int e^{i\xi\cdot\eta}\mathcal{L}_c(q)(t, x, \xi)đ\xi \ \left|\tfrac{dx}{x}d\eta\right|^{1/2},$$

where

$$\mathcal{L}_c(q)(t, x, \xi) = \frac{i}{2\pi}\int_{\Upsilon} e^{-t\lambda}q(x, \xi, x^m\lambda)d\lambda.$$

Finally,

$$\operatorname{Tr}\mathcal{Q}(t) = \iint \mathcal{L}_c(q)(t, x, \xi)đ\xi \ \tfrac{dx}{x}. \tag{4.4}$$

Motivated by the relation between the heat trace and the zeta function via the Mellin transform, we define

$$B(z) := \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\operatorname{Tr}\mathcal{Q}(t)dt. \tag{4.5}$$

Thus, by means of the inverse Mellin transform we get

$$\operatorname{Tr}\mathcal{Q}(t) = \frac{1}{2\pi i}\int_{\Re z=\delta} t^{-z}B(z)\Gamma(z)dz \tag{4.6}$$

for any sufficiently large $\delta \in \mathbb{R}$. The following proposition (which is just an application of Cauchy's theorem) suggests that the asymptotic expansion of $\operatorname{Tr}\mathcal{Q}(t)$ as $t \to 0^+$ is determined by the poles of $B(z)\Gamma(z)$.

Proposition 4.1. *Suppose that $\psi(z)$ is meromorphic with a single pole of order $\ell + 1$ at $z = w$. Let*

$$u(t) = \frac{1}{2\pi i}\int_\gamma t^{-z}\psi(z)dz,$$

where γ is a simple closed curve around w. Then, we can write

$$u(t) = \sum_{k=0}^{\ell}\frac{(-1)^k}{k!}t^{-w}(\log t)^k r_{k+1},$$

where $r_1, r_2, \ldots$ are the coefficients of the Laurent expansion of ψ.

Now, our goal is to determine all the poles of $B(z)\Gamma(z)$, then push the contour $\Re z = \delta$ in (4.6) down to $\Re z = -\infty$. Every time we pass a pole of $B(z)\Gamma(z)$ we pick up a contour integral of $B(z)\Gamma(z)$ around that pole, which by Proposition 4.1, contributes powers and powers of the logarithm of t to the expansion of $\operatorname{Tr}\mathcal{Q}(t)$.

We first determine the poles of $B(z)$. To do so, we want to write it in terms of $q(x, \xi, \lambda)$. By (4.4) and (4.5), and the fact that

$$\lambda^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-t\lambda}dt,$$

we have

$$\begin{aligned}
B(z) &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \operatorname{Tr} \mathcal{Q}(t)dt \\
&= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \iint \mathcal{L}_c(q)(t,x,\xi) đ\xi \, \tfrac{dx}{x} dt \\
&= \iint \left(\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \mathcal{L}_c(q)(t,x,\xi)dt \right) đ\xi \, \tfrac{dx}{x} \\
&= \iint \left(\frac{i}{2\pi} \int_\Upsilon \lambda^{-z} q(x,\xi,x^m\lambda)d\lambda \right) đ\xi \, \tfrac{dx}{x} \\
&= \iint \hat{q}(x,\xi,z) đ\xi \, \tfrac{dx}{x},
\end{aligned}$$

where

$$\begin{aligned}
\hat{q}(x,\xi,z) &:= \frac{i}{2\pi} \int_\Upsilon \lambda^{-z} q(x,\xi,x^m\lambda)d\lambda \\
&= x^{mz-m} \frac{i}{2\pi} \int_\Upsilon \lambda^{-z} q(x,\xi,\lambda)d\lambda \quad (\lambda \to x^{-m}\lambda).
\end{aligned}$$

According to (4.3), $\hat{q}(x,\xi,z)$ is a local symbol of the operator

$$\mathcal{M}(Q)(z) := \frac{i}{2\pi} \int_\Upsilon \lambda^{-z} Q(\lambda)d\lambda.$$

By the definition of the space $S^{m'-m,-m,m}_{r,c\ell}(\mathbb{R}^n;\Lambda)$, cf. (3.1), a straight forward computation shows that $\hat{q}(x,\xi,z)$ is a classical symbol of order $-mz+m'$. Now let us write

$$B(z) = \int_M \mathcal{M}(Q)(z)|_\Delta \quad \text{with} \quad \mathcal{M}(Q)(z)|_\Delta = \int_{\mathbb{R}^n} \hat{q}(x,\xi,z)đ\xi \, \tfrac{dx}{x}. \tag{4.7}$$

As in [14], $\mathcal{M}(Q)(z)|_\Delta$ extends to be a meromorphic function on $\mathbb{C}$ having simple poles at $z=-z_k$, $z_k = \frac{k-m'-n}{m}$, $k \in \mathbb{N}_0$, with residues

$$\operatorname{Res}_1 \mathcal{M}(Q)(-z_k)|_\Delta = \frac{1}{m}\, \omega(\mathcal{M}(Q)(-z_k)),$$

where $\omega(\mathcal{M}(Q)(-z_k))$ is the corresponding Wodzicki residue density, i.e.,

$$\omega(\mathcal{M}(Q)(-z_k)) = \int_{|\xi|=1} \hat{q}_{-n}(x,\xi,-z_k)đ\xi \, \tfrac{dx}{x},$$

where $\hat{q}_{-n}(x,\xi,-z_k)$ denotes the homogeneous component of order $-n$ in the expansion of $\hat{q}(x,\xi,-z_k)$. Note that $\hat{q}(x,\xi,-z_k)$ is a classical symbol of order $k-n \in \mathbb{Z}$ since $\hat{q}(x,\xi,z)$ is classical of order $-mz+m'$.

Now Lemma 2.1 implies that if $u \in x^{-p}C_c^\infty(\overline{\mathbb{R}}_+)$, then $\int_0^\infty x^{mz}u(x)\frac{dx}{x}$ has poles at $z = -\frac{j-p}{m}$, $j \in \mathbb{N}_0$, with corresponding residues $\frac{1}{m}\frac{1}{j!}\partial_x^j(x^p u)|_{x=0}$. Thus, as $\mathcal{M}(Q)(z)|_\Delta \in x^{mz-p}C_c^\infty(\overline{\mathbb{R}}_+,\Omega_b)$, the family $B(z)$ has the poles of $\mathcal{M}(Q)(z)|_\Delta$

and additional poles at $z = -\frac{j-p}{m}$. On the other hand, $\Gamma(z)$ has simple poles at $z = -\ell$, $\ell \in \mathbb{N}_0$, with residues given by $\frac{(-1)^\ell}{\ell!}$. Altogether we get that $B(z)\Gamma(z)$ may have poles at

$$\begin{aligned} z &= -\ell && \text{for } \ell = 0, 1, 2, \ldots \\ z &= -\frac{j-p}{m} && \text{for } j = 0, 1, 2, \ldots \\ z &= -\frac{k-m'-n}{m} && \text{for } k = 0, 1, 2, \ldots \end{aligned} \tag{4.8}$$

These are simple, double or triple order poles depending on j, k and ℓ. In any case, only double and triple order poles may produce log terms in the asymptotic expansion of Tr $\mathcal{Q}(t)$, cf. Proposition 4.1. In summary, the poles of $B(z)\Gamma(z)$ arise because of the poles of $\Gamma(z)$, integrating $\mathcal{M}(Q)(z)|_\Delta$ in x, and from the poles of $\mathcal{M}(Q)(z)|_\Delta$ itself. Using this information, we now write down all the possible combinations for the higher order poles of $B(z)\Gamma(z)$.

Second order poles of B(z)Γ(z). Suppose that this function has a pole at $z = \zeta$. According to (4.8) there are three cases where ζ may be a double pole.

Case 1: $\zeta = -\frac{j-p}{m} = -\frac{k-m'-n}{m} \notin -\mathbb{N}_0$. Then $j = p - \zeta m$, and the second order residue at $\zeta = -z_k$ is

$$\begin{aligned} r_2 &= \Gamma(\zeta)\frac{1}{mj!}\partial_x^j\left\{x^{p-\zeta m}\frac{1}{m}\,\omega(\mathcal{M}(Q)(\zeta))\right\}\Big|_{x=0} \\ &= \frac{\Gamma(-z_k)}{m^2(p+z_k m)!}\partial_x^{p+z_k m}\left\{x^{p+z_k m}\,\omega(\mathcal{M}(Q)(-z_k))\right\}\Big|_{x=0}. \end{aligned}$$

Case 2: $\zeta = -\ell = -\frac{k-m'-n}{m} \in -\mathbb{N}_0$ and $\zeta \neq -\frac{j-p}{m}$ for any j. Then

$$r_2 = \frac{(-1)^\ell}{m\ell!}\,{}^b\!\!\int \omega(\mathcal{M}(Q)(-\ell)).$$

Case 3: $\zeta = -\ell = -\frac{j-p}{m}$ and $\zeta \neq -\frac{k-m'-n}{m}$ for any k. In this case,

$$\begin{aligned} r_2 &= \frac{(-1)^\ell}{\ell!}\frac{1}{mj!}\partial_x^j\left\{x^j\ \mathrm{Res}_0\,\mathcal{M}(Q)(-\ell)|_\Delta\right\}\Big|_{x=0} \\ &= \frac{(-1)^\ell}{m\ell!(p+\ell m)!}\partial_x^{p+\ell m}\left\{x^{p+\ell m}\ \mathrm{Res}_0\,\mathcal{M}(Q)(-\ell)|_\Delta\right\}\Big|_{x=0}, \end{aligned}$$

where $\mathrm{Res}_0\,\mathcal{M}(Q)(-\ell)|_\Delta$ means the regular value of the kernel of $\mathcal{M}(Q)(z)$ restricted to the diagonal Δ in M^2, evaluated at $z = -\ell$.

Third order poles of B(z)Γ(z). They may only occur when in (4.8)

$$\ell = \frac{j-p}{m} = \frac{k-m'-n}{m} \quad \text{for some } j, k, \ell \in \mathbb{N}_0.$$

In this case, the third order residue is given by

$$r_3 = \frac{(-1)^\ell}{\ell!}\frac{1}{mj!}\partial_x^j\Big\{x^j\frac{1}{m}\,\omega(\mathcal{M}(Q)(-\ell))\Big\}\Big|_{x=0}$$
$$= \frac{(-1)^\ell}{m^2\ell!(p+\ell m)!}\partial_x^{p+\ell m}\Big\{x^{p+\ell m}\,\omega(\mathcal{M}(Q)(-\ell))\Big\}\Big|_{x=0}.$$

The second order residue can be written as a sum of the following three expressions:

$$r_{2,1} = \frac{\Gamma_0(-\ell)}{m^2(p+\ell m)!}\partial_x^{p+\ell m}\Big\{x^{p+\ell m}\,\omega(\mathcal{M}(Q)(-\ell))\Big\}\Big|_{x=0},$$

where $\Gamma_0(-\ell)$ denotes the regular value of $\Gamma(-\ell)$,

$$r_{2,2} = \frac{(-1)^\ell}{m\ell!}\,{}^b\!\!\int\,\omega(\mathcal{M}(Q)(-\ell)),$$
$$r_{2,3} = \frac{(-1)^\ell}{m\ell!(p+\ell m)!}\partial_x^{p+\ell m}\Big\{x^{p+\ell m}\,\mathrm{Res}_0\,\mathcal{M}(Q)(-\ell)|_\Delta\Big\}\Big|_{x=0}.$$

As a consequence of Proposition 4.1 and the previous discussion, we obtain the following theorem.

Theorem 4.2. *The trace of $\mathcal{Q}(t)$ admits an asymptotic expansion*

$$\mathrm{Tr}\,\mathcal{Q}(t) \sim_{t\to 0^+} \sum_{k=0}^\infty a_k\,t^{(k-p)/m} + \sum_{k=0}^\infty \big\{b_k + \beta_k\log t\big\}\,t^k$$
$$+ \sum_{k=0}^\infty \big\{c_k + \gamma_k\log t + \delta_k(\log t)^2\big\}\,t^{z_k},$$

where $z_k = \frac{k-m'-n}{m}$ with $n = \dim M$. Moreover,

$$\beta_k = -\frac{(-1)^k}{mk!}\,{}^b\!\!\int\,\omega(\mathcal{M}(Q)(-k))$$
$$-\frac{(-1)^k}{mk!(p+km)!}\int_Y \partial_x^{p+km}\Big\{x^{p+km}\,\mathrm{Res}_0\,\mathcal{M}(Q)(-k)|_\Delta\Big\}\Big|_{x=0},$$
$$\gamma_k = -\frac{\Gamma_0(-z_k)}{m^2(p+z_km)!}\int_Y \partial_x^{p+z_km}\Big\{x^{p+z_km}\,\omega(\mathcal{M}(Q)(-z_k))\Big\}\Big|_{x=0},$$
$$\delta_k = -\frac{(-1)^{z_k}}{m^2z_k!(p+z_km)!}\int_Y \partial_x^{p+z_km}\Big\{x^{p+z_km}\,\omega(\mathcal{M}(Q)(-z_k))\Big\}\Big|_{x=0}.$$

If in any of the factorials $(p+km)!$, $(p+z_km)!$, or $(z_k)!$ the number is not in $\mathbb{N}_0$, we define the corresponding coefficient to be 0.

5. Heat trace expansion

Let Λ be a sector of the form (4.1). Let P and A be as in Theorem 3.4. Thus,

$$P(A-\lambda)^{-1} = Q(\lambda) + R(\lambda) + S(\lambda), \tag{5.1}$$

where the family $Q(\lambda) \in x^{m-p}\Psi^{m'-m,-m,m}_{c,\Lambda}(M, \Omega_b^{\frac{1}{2}})$ is holomorphically tempered, $R(\lambda) \in x^{-p}\Psi^{-\infty,\mathcal{G}(\alpha)}_{c,\Lambda}(M, \Omega_b^{\frac{1}{2}})$, and $S(\lambda) \in x^{-p}\Psi^{-\infty,\mathcal{F}(\alpha)}_{\Lambda}(M, \Omega_b^{\frac{1}{2}})$ for some index families $\mathcal{G}(\alpha)$ and $\mathcal{F}(\alpha)$. Define $\mathcal{R}(t)$ and $\mathcal{S}(t)$ in the same way as $\mathcal{Q}(t)$ was defined in (4.2). Then as shown in [14], $\mathcal{R}(t)$ and $\mathcal{S}(t)$ are also operators of trace class and we get a decomposition

$$\operatorname{Tr}(Pe^{-tA}) = \operatorname{Tr}\mathcal{Q}(t) + \operatorname{Tr}\mathcal{R}(t) + \operatorname{Tr}\mathcal{S}(t), \quad t > 0.$$

By the same theorems of loc. cit. the trace of $\mathcal{R}(t)$ admits an expansion

$$\operatorname{Tr}\mathcal{R}(t) \sim \sum_{k=0}^{\infty} r_k\, t^{(k-p)/m} \qquad \text{as } t \to 0^+,$$

and $\operatorname{Tr}\mathcal{S}(t)$ vanishes to infinite order at $t = 0$. On the other hand, the trace of $\mathcal{Q}(t)$ admits the asymptotic expansion given in Theorem 4.2. Thus, $\operatorname{Tr}(Pe^{-tA})$ has the same expansion as $\operatorname{Tr}\mathcal{Q}(t)$. To provide a nicer expression for the second term appearing in the formula for β_k in Theorem 4.2, we proceed as follows. Denote by $Q_0(\lambda)$ the function $Q(\lambda)$ in (5.1) for $P = \mathrm{Id}$. Then, since $Q_0(\lambda)$ is equal to $(A-\lambda)^{-1}$ modulo $\Psi_b^{-\infty}$, formally speaking,

$$\mathcal{M}(Q_0)(z) = \frac{i}{2\pi}\int_{\Upsilon} \lambda^{-z} Q_0(\lambda)d\lambda \sim \frac{i}{2\pi}\int_{\Upsilon} \lambda^{-z}(A-\lambda)^{-1}d\lambda = A^{-z}.$$

Thus, although the complex power A^z does not exist in general, we can still associate a useful meaning to it:

$$A^z := \mathcal{M}(Q_0)(-z). \tag{5.2}$$

Moreover, the symbolic properties of $Q_0(\lambda)$ imply that $\mathcal{M}(Q_0)(0) = \mathrm{Id}$ and that, modulo $\Psi_b^{-\infty}$, $A^k\,\mathcal{M}(Q_0)(-z+k) = \mathcal{M}(Q_0)(-z)$ for any $k \in \mathbb{N}_0$.

Thus, we have proved:

Theorem 5.1. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M, \Omega_b^{\frac{1}{2}})$ be such that $A - \lambda$ is parameter-elliptic with respect to some $\alpha \in \mathbb{R}$ on a sector $\Lambda \subset \mathbb{C}$ of the form* (4.1). *Let $P \in x^{-p}\Psi_b^{m'}(M, \Omega_b^{\frac{1}{2}})$, $p, m' \in \mathbb{R}$, and assume that $m > p$. Then*

$$\begin{aligned}\operatorname{Tr} Pe^{-tA} \sim_{t\to 0^+} &\sum_{k=0}^{\infty} a_k\, t^{(k-p)/m} + \sum_{k=0}^{\infty} \left\{b_k + \beta_k \log t\right\} t^k \\ &+ \sum_{k=0}^{\infty} \left\{c_k + \gamma_k \log t + \delta_k (\log t)^2\right\} t^{z_k},\end{aligned}$$

where $z_k = \frac{k-m'-n}{m}$ *with* $n = \dim M$. *Moreover, the coefficients* β_k, γ_k *and* δ_k *are given explicitly by*

$$\begin{aligned}\beta_k &= -\frac{(-1)^k}{mk!} {}^b\!\!\int \omega(PA^k) \\ &\quad - \frac{(-1)^k}{mk!(p+km)!} \int_Y \partial_x^{p+km} \Big\{ x^{p+km} \operatorname{Res}_0(PA^k A^z)|\Delta|_{z=0} \Big\} \Big|_{x=0}, \\ \gamma_k &= -\frac{\Gamma_0(-z_k)}{m^2(p+z_k m)!} \int_Y \partial_x^{p+z_k m} \Big\{ x^{p+z_k m}\, \omega(PA^{z_k}) \Big\} \Big|_{x=0}, \\ \delta_k &= -\frac{(-1)^{z_k}}{m^2(z_k)!(p+z_k m)!} \int_Y \partial_x^{p+z_k m} \Big\{ x^{p+z_k m}\, \omega(PA^{z_k}) \Big\} \Big|_{x=0}.\end{aligned}$$

Again, if in any of the factorials the number is not in $\mathbb{N}_0$, *we define the corresponding coefficient to be* 0. *The meaning of the powers* A^z *and* A^{z_k} *is given in* (5.2).

Trace functionals revisited. By means of the generalized heat trace expansion obtained above, we can recover the unique trace functionals on the algebras $\mathcal{I}_\sigma$, $\mathcal{I}_\partial$, $\mathcal{A}_\sigma$, $\mathcal{A}_\partial$ and $\mathcal{A}_{\sigma,\partial}$ from Section 2.

In the following results, the functional $\operatorname{Tr}_\partial$ is defined using the holomorphic family A^z given in (5.2).

Theorem 5.2. *In the expansion of* $\operatorname{Tr} Pe^{-tA}$ *given in Theorem 5.1, the coefficients* β_k, γ_k *and* δ_k *can be written as*

$$\begin{aligned}\beta_k &= -\frac{(-1)^k}{mk!}\big(\operatorname{Tr}_\sigma(PA^k) + \operatorname{Tr}_\partial(PA^k)\big), \\ \gamma_k &= -\frac{\Gamma_0(-z_k)}{m^2} \operatorname{Tr}_{\partial,\sigma}(PA^{z_k}), \\ \delta_k &= -\frac{(-1)^{z_k}}{m^2(z_k)!} \operatorname{Tr}_{\partial,\sigma}(PA^{z_k}).\end{aligned}$$

In particular, the coefficient of $\log t$ *is*

$$-\frac{1}{m}\operatorname{Tr}_\sigma(P) - \frac{1}{m}\operatorname{Tr}_\partial(P) - \frac{1}{m^2}\operatorname{Tr}_{\partial,\sigma}(P),$$

and

$$\operatorname{Tr}_{\partial,\sigma}(P) = -m^2 \ \times \ \textit{the coefficient of } (\log t)^2.$$

Corollary 5.3. *Suppose that* $P \in x^{-p}\Psi_b^{m'}(M, \Omega_b^{\frac{1}{2}})$ *with* $p < 0$. *Then there are no* $(\log t)^2$ *terms in the expansion of* $\operatorname{Tr} Pe^{-tA}$, *and*

$$\operatorname{Tr}_\sigma(P) = -m \ \times \ \textit{the coefficient of } \log t.$$

Corollary 5.4. *Suppose that* $P \in x^{-p}\Psi_b^{m'}(M, \Omega_b^{\frac{1}{2}})$ *with* $m' < -n$. *Then there are no* $(\log t)^2$ *terms in the expansion of* $\operatorname{Tr} Pe^{-tA}$, *and*

$$\operatorname{Tr}_\partial(P) = -m \ \times \ \textit{the coefficient of } \log t.$$

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