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Zeta functions of Dirac and Laplace-type operators over finite cylinders

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Abstract

In this paper, a complete description of the zeta functions and corresponding zeta determinants for Dirac and Laplace-type operators over finite cylinders using the contour integration method, for example described in [K. Kirsten, Spectral Functions in Mathematics and Physics, Chapman & Hall/CRC Press, Boca Raton, 2001] is given. Different boundary conditions, local and non-local ones, are considered. The method is shown to be very powerful in that it is easily adapted to each situation and in that answers are very elegantly obtained.

1. Introduction

Many branches of mathematics and physics are characterized by frequent appearance of spectral functions [47]. These functions are associated with suitable sequences of numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, which, for most applications, are eigenvalues of Laplace-type operators. In different ways, certain properties of physical systems or Riemannian manifolds are then encoded in this spectrum and relevant information can be found by suitably organizing the spectrum in form of adequate functions.

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Probably the most important spectral function is the so-called zeta function, which is directly related to topics such as analytic torsion [63], the heat kernel [42,68], Casimir energy [12,34,56], and effective actions [13,16,37]. For the purpose of describing these relationships a little further let us assume we have given a discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$. Then the zeta function is defined in generalization to the Riemann zeta function as

$$\zeta(s) = \sum_i \lambda_i^{-s},$$

where $\Re s$ of the complex parameter s is assumed sufficiently large such as to make this sum convergent [70]. It can be shown that a meromorphic continuation of $\zeta(s)$ to the whole complex plane exists with isolated poles at known locations [33,66]. The relation to the topics mentioned and described in the following shows that the zeta function is indeed a very intelligent and relevant organization of the spectrum.

(1) The residues of the zeta function modulo a multiplicative constant equal the heat kernel coefficients in the small-time asymptotic behavior of the heat kernel $\sum_i e^{-\lambda_i t}$ [66].
(2) The derivative of the zeta function at $s = 0$ describes the analytical torsion of a manifold. This definition was first introduced by the mathematicians Ray and Singer [63], when they tried to give a definition of the Reidemeister-Franz torsion [41], a combinatorial topological invariant of a manifold, in analytic terms. That the two definitions in fact agree for compact manifolds without boundary was independently proven by Cheeger [17] and Müller [57]. In physics the use of zeta functions, in particular of $\zeta'(0)$, took its origin in ambiguities of dimensional regularization when applied to quantum field theory in curved spacetime [27,45]. More generally, functional determinants provide the so-called one-loop approximation to quantum field theories in the path integral formulation [19,38,65].
(3) The properties of $\zeta(s)$ at $s = -1/2$ contain information about the Casimir effect. The residue is closely related to the renormalization of a quantum field theory [18,71]. The finite part is needed for a full understanding of this effect [34,35,47].

These mentioned connections make it very desirable to have effective analytical tools available for the complete analysis of zeta functions. The obvious problem is that an explicit knowledge of the eigenvalues λ_k, which can serve as a starting point for any calculation, is in general only guaranteed for highly symmetric regions like the torus, the sphere, or regions bounded by parallel planes. For these manifolds detailed calculations have been performed, for a summary see for example [16,34,35]. However, for cases where the spectrum is not known explicitly only a few general methods for the analysis of (properties of) zeta functions are available.

For example for one-dimensional situations the determinants of differential and difference operators have been related to boundary values of solutions of the operators [14,31,39,40,50,51]. When the operator is a conformally covariant differential operator exact results may sometimes be obtained by transforming to a simpler operator where the answer is known [8,24,25,30]. The same kind of philosophy, in this context, applies to results from analytical surgery where results for determinants of Laplace-type operators on different suitable manifolds are related to each other [15,52,58,60,61].

Another class of situations where an explicit analysis can be performed is the class where although eigenvalues are *not* known explicitly, an implicit eigenvalue equation for them is known. The starting point of this approach makes use of a suitable contour integral representation of the zeta function involving this implicit eigenvalue equation. The approach has been developed in [9–11] in the context of calculations on the ball and generalized cone of arbitrary dimensions, see [47] for a review. Recently, results already available in one dimension have been rederived and generalized in this formalism in a much simplified way in [48,49]. It is the aim of the present article to apply the contour integration technique to the geometry of finite cylinders. As an introduction to the method, we derive well-known properties of the Riemann zeta function in Section 2. The basic ideas of the approach are described here and applied later on in all following sections. In these sections, we analyze zeta functions associated with Laplacians, respectively, Dirac operators over finite cylinders. Different boundary conditions are considered and exact expressions for the zeta functions are found and used to find the determinants. Appendix A gives two integrals needed in the main body of the text.

By starting of with the Riemann zeta function we want to emphasize that in a certain sense the mathematical prerequisites required to read a significant part of this paper is that of an undergraduate student; in fact, at a certain level, perhaps the two most sophisticated mathematical formulas that we will need for this paper are the formulas

$$\frac{\pi}{\sin \pi s} = \Gamma(s)\Gamma(1-s) \quad \text{and} \quad \int_0^\infty \frac{x^{-2s+2a-1}}{(1+x^2)^w}\,\mathrm{d}x = \frac{\Gamma(a-s)\Gamma(s-a+w)}{2\Gamma(w)}$$

the so-called Reflection Formula and the Beta function [1, Chapter 6]. The only "advanced fact," we will need is that if Δ_Y is a Laplace-type operator over a compact $(n-1)$-dimensional manifold Y, then the zeta function $\zeta_{\Delta_Y}(s)$ has a meromorphic extension to $\mathbb{C}$ with isolated simple poles at (see [42, p. 112])

$$s = \frac{n-1-k}{2} \quad \text{for } k = 0,1,2,\ldots \quad \text{and} \quad \frac{n-1-k}{2} \notin \{0\} \cup -\mathbb{N}. \tag{1.1}$$

Note that if D_Y is a Dirac-type operator over Y (so that D_Y^2 is of Laplace-type), then the zeta function $\zeta_{D_Y^2}(s)$ has the meromorphic structure described in (1.1). The level of prerequisites illustrates the power and elegance of the contour integration method.

2. The Riemann zeta function

To illustrate the basic idea of the contour integration method, we shall analyze the Riemann zeta function $\zeta_{\mathcal{R}}(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. For reasons that will be evident in a moment, we shall instead look at $\zeta_{\mathcal{R}}(2s) = \sum_{n=1}^\infty \frac{1}{n^{2s}}$. The starting point of the contour integration method is to determine an equation $F(\lambda) = 0$ whose roots squared are exactly n^2 for $n = 1,2,\ldots$ This is easy

$$\lambda^2 = n^2 \quad \text{with } n \in \mathbb{N} \quad \Longleftrightarrow \quad F(\lambda) := \frac{\sin(\pi\lambda)}{\lambda} = 0.$$

The next step in the contour integration method is to use the Argument Principle (or Cauchy's formula) [20, p. 123] from elementary complex analysis to write

$$\zeta_{\mathcal{R}}(2s) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-2s} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log F(\lambda)\,\mathrm{d}\lambda, \tag{2.1}$$

where γ is the imaginary axis as shown in Fig. 1. (Here, we used that each zero occurs with multiplicity one and we use the standard branch for λ^{-2s}.) Of course, if we so desired we can choose γ to be any contour surrounding the positive integers as long as it does not pass through $(-\infty,0)\times\{0\}$, which is the branch cut of λ^{-2s}. However, choosing γ as the imaginary axis, we can put the integral in (2.1) into a nice form. To do so, we use that

$$\mathrm{i}^{-2s} = (\mathrm{e}^{\mathrm{i}\pi/2})^{-2s} = \mathrm{e}^{-\mathrm{i}\pi s} \quad \text{and} \quad (-\mathrm{i})^{-2s} = (\mathrm{e}^{-\mathrm{i}\pi/2})^{-2s} = \mathrm{e}^{\mathrm{i}\pi s}$$

and write the contour integral as follows:

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\gamma \lambda^{-2s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\log F(\lambda)\,\mathrm{d}\lambda &= \frac{1}{2\pi\mathrm{i}}\left\{-\int_0^\infty (\mathrm{i}x)^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x\right.\\ &\quad\left.+\int_0^\infty(-\mathrm{i}x)^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(-\mathrm{i}x)\,\mathrm{d}x\right\}\\ &= \frac{1}{2\pi\mathrm{i}}\left(-\mathrm{e}^{-\mathrm{i}\pi s}+\mathrm{e}^{\mathrm{i}\pi s}\right)\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x,\end{aligned}$$

where we used that $F(\lambda)$ is an *even* function. Thus, we have

$$\zeta_{\mathcal{R}}(2s) = \frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x \tag{2.2}$$

which is an integral representation valid for $1/2 < \Re s < 1$, the limitations coming from the behavior of the integrand as x approaches 0 and ∞. To analyze the integral, observe that

$$F(\mathrm{i}x) = \frac{\sin(\pi\mathrm{i}x)}{\mathrm{i}x} = \frac{\mathrm{e}^{\pi x}}{2x}\left(1-\mathrm{e}^{-2\pi x}\right).$$

We now plug this into the above integral; however, because $(1-\mathrm{e}^{-2\pi x})$ vanishes at $x=0$, we separate out the small x behavior:

$$\zeta_{\mathcal{R}}(2s) = \frac{\sin\pi s}{\pi}\int_0^1 x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x + \frac{\sin\pi s}{\pi}\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x.$$

For $\Re(s) > 1/2$ we compute the large x integral:

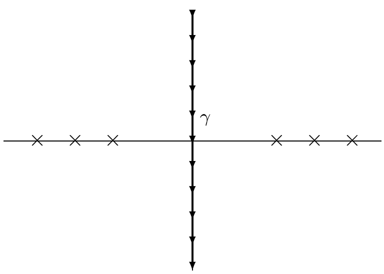

Fig. 1. The contour γ for the zeta function. The $\times$'s represent the zeros of $F(\lambda)$.

$$\begin{aligned}\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x &= \int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{\pi x}}{2x}\left(1-\mathrm{e}^{-2\pi x}\right)\right)\mathrm{d}x = \pi\int_1^\infty x^{-2s}\,\mathrm{d}x \\ &\quad -\int_1^\infty x^{-2s-1}\mathrm{d}x + \int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2\pi x}\right)\mathrm{d}x \\ &= \frac{\pi}{2s-1}-\frac{1}{2s}+\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2\pi x}\right)\mathrm{d}x\end{aligned}$$

which provides the analytical continuation to all values of s. Multiplying this formula by $\frac{\sin(\pi s)}{\pi}$, for $\Re(s) < 1$ we obtain

$$\begin{aligned}\zeta_{\mathcal{R}}(2s) = \frac{\sin\pi s}{\pi}\int_0^1 x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x + \frac{\sin\pi s}{2s-1}-\frac{\sin\pi s}{2\pi s}+\frac{\sin\pi s}{\pi}\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x} \\ \times\log\left(1-\mathrm{e}^{-2x\pi}\right)\mathrm{d}x.\end{aligned} \tag{2.3}$$

From this formula we can *easily* prove the following well-known Lemma.

Lemma 2.1. *The Riemann zeta function $\zeta_{\mathcal{R}}(s)$ has a meromorphic extension to the whole complex plane with only a single pole at $s = 1$. This pole is simple and*

$$\begin{aligned}&\mathrm{Res}_{s=1}\zeta_{\mathcal{R}}(s) = 1, \qquad \zeta_{\mathcal{R}}(-2k) = 0, \quad k = 1,2,3,\ldots, \\ &\zeta_{\mathcal{R}}(0) = -\frac{1}{2}, \quad \text{and} \quad \zeta_{\mathcal{R}}'(0) = -\frac{1}{2}\log(2\pi).\end{aligned}$$

Proof. The function $\frac{\mathrm{d}}{\mathrm{d}x}\log(1-\mathrm{e}^{-2x\pi}) = 2\pi\mathrm{e}^{-2x\pi}(1-\mathrm{e}^{-2x\pi})^{-1}$ is exponentially decreasing as $x\to\infty$, which implies that the integral $\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2x\pi}\right)\mathrm{d}x$ is *entire*. For the first integral in (2.3), notice that $\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x) = \mathrm{i}F'(\mathrm{i}x)/F(\mathrm{i}x)$ is an odd function of x (since $F(\mathrm{i}x)$ is even in x) and analytic at $x = 0$, therefore $\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x) = \sum_{n=1}^\infty a_n x^{2n-1}$ for some constants a_n. Since for $\Re s < 1$ we have

$$\int_0^1 x^{-2s+2n-1}\mathrm{d}x = \frac{1}{-2s+2n}$$

the meromorphic continuation of the integral $\int_0^1 x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x$ has poles only at $s\in\mathbb{N}$. Since $\sin(\pi s)$ vanishes at the integers, it follows that the function $\frac{\sin\pi s}{\pi}\int_0^1 x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x$ has a meromorphic extension (from $\Re s < 1$), which is entire and vanishes when $s = 0,-1,-2,\ldots$ Using the facts we just learned about the integrals appearing in

$$\begin{aligned}\zeta_{\mathcal{R}}(2s) = &\frac{\sin\pi s}{\pi}\int_0^1 x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x + \frac{\sin\pi s}{2s-1}-\frac{\sin\pi s}{2\pi s} \\ &+\frac{\sin\pi s}{\pi}\int_1^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2x\pi}\right)\mathrm{d}x\end{aligned}$$

the fact that $\zeta_{\mathcal{R}}(s)$ has a meromorphic extension to the whole complex plane with only a single pole at $s = 1$, and the properties $\mathrm{Res}_{s=1}\zeta_{\mathcal{R}}(s) = 1$, $\zeta_{\mathcal{R}}(-2k) = 0, \quad k = 1,2,3,\ldots$, and $\zeta_{\mathcal{R}}(0) = -\frac{1}{2}$, then follow easily.

To compute the derivative $\zeta_{\mathcal{R}}'(0)$, we take $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}$ of (2.3) and use that $F(\mathrm{i}) = \frac{\mathrm{e}^\pi}{2}(1-\mathrm{e}^{-2\pi})$ and $F(0) = \pi$, to get

$$
\begin{aligned}
2\zeta'_{\mathcal{R}}(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\log F(\mathrm{i}x)\,\mathrm{d}x - \pi + \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2\pi x}\right)\mathrm{d}x \\
&= \log F(\mathrm{i}) - \log F(0) - \pi - \log\left(1-\mathrm{e}^{-2\pi}\right) \\
&= \log\left(\frac{\mathrm{e}^{\pi}}{2}\left(1-\mathrm{e}^{-2\pi}\right)\right) - \log\pi - \pi - \log\left(1-\mathrm{e}^{-2\pi}\right) = -\log(2\pi).
\end{aligned}
$$

This completes our proof. □

Before moving on to zeta functions of Laplace-type operators, recall that the Hurwitz zeta function $\zeta_{\mathcal{H}}(s,a)$ is defined by (see [72])

$$\zeta_{\mathcal{H}}(s,a) = \sum_{k=0}^{\infty}(k+a)^{-s} \quad \text{for } 0 < a < 1. \tag{2.4}$$

Imitating the proof of Lemma 2.1, one can derive the following theorem easily.

Lemma 2.2. *The Hurwitz zeta function $\zeta_{\mathcal{H}}(s,a)$ has a meromorphic extension to the whole complex plane with only a single pole at $s = 1$. This pole is simple with residue equal to* 1, *and*

$$\zeta_{\mathcal{H}}(0,a) = \frac{1}{2} - a \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\zeta_{\mathcal{H}}(s,a) = \log(\Gamma(a)) - \frac{1}{2}\log(2\pi).$$

3. Laplace-type operators over finite cylinders

We continue by studying the zeta-function for Laplace-type operators

$$\Delta = -\partial_u^2 + \Delta_Y : H^2([0,R]\times Y,E) \to L^2([0,R]\times Y,E),$$

where Y is a compact manifold without boundary, E is a Hermitian vector bundle over $[0,R]\times Y$ and Δ_Y is a Laplace-type operator acting on sections of $E_0 := E|_Y$. We can also take Y with boundary, in which case we impose local boundary conditions so that Δ_Y has a well-behaved discrete spectrum. Throughout this section,

$$0 \leqslant \mu_1^2 \leqslant \mu_2^2 \leqslant \mu_3^2 \leqslant \cdots$$

denote the eigenvalues of Δ_Y, each repeated according to its multiplicity. In what follows, similar remarks as in Section 2 regarding the existence of different representations of the zeta function are appropriate. However, these are always straightforward and we will not emphasize all the time in which region expressions are well-defined.

3.1. Dirichlet conditions

As an introduction to the contour integral method for Laplace-type operators over finite cylinders, we begin with the "simplest" of all boundary conditions, the Dirichlet condition; we denote the resulting operator by $\Delta_{\mathcal{D}}$. Thus, we consider

$$\Delta_{\mathcal{D}} : \left\{\phi \in H^2([0,R]\times Y,E) | \phi(0) = 0 = \phi(R)\right\} \to L^2([0,R]\times Y,E).$$

Recall from Section 2 concerning the Riemann zeta function that the starting point of our method is determining an implicit eigenvalue equation. Let us fix an eigenvalue λ^2 and work over the λ^2-eigenspace of $\Delta_{\mathcal{D}}$. Then, for φ taking values in the μ_k^2-eigenspace of Δ_Y, we have

$$\Delta_{\mathcal{D}}\varphi = \lambda^2\varphi \Longleftrightarrow \left(-\partial_u^2 + \mu_k^2\right)\varphi = \lambda^2\varphi \Longleftrightarrow \varphi'' = (\mu_k^2 - \lambda^2)\varphi.$$

Therefore,

$$\varphi = a\cosh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) + b\sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right). \tag{3.1}$$

Imposing Dirichlet boundary conditions, we see that

$$\varphi(0) = 0 \Rightarrow \varphi = b\sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right)$$

and then,

$$\varphi(R) = 0 \Rightarrow \sinh\left(\sqrt{\mu_k^2 - \lambda^2}R\right) = 0.$$

Since, $\lambda = \mu_k$ leads to the trivial solution, we can see that over the μ_k^2-eigenspace of Δ_Y

$$\lambda^2 \text{ is an eigenvalue of } \Delta_{\mathcal{D}} \Longleftrightarrow F_k(\lambda) := \frac{\sinh\left(\sqrt{\mu_k^2 - \lambda^2}R\right)}{\sqrt{\mu_k^2 - \lambda^2}} = 0.$$

Notice that $F_k(\lambda)$ is an entire even function of λ that is nonzero at $\lambda = 0$. In conclusion, $F_k(\lambda)$ is an entire even function of λ whose zeros squared are exactly the (nonzero) eigenvalues of $-\partial_u^2 + \mu_k^2$ with Dirichlet conditions. We now form the zeta function of $\Delta_{\mathcal{D}}$ via contour integrals. Of course, we can easily see that

$$\lambda^2 = \frac{\pi^2 n^2}{R^2} + \mu_k^2, \qquad n \in \mathbb{N},$$

so we could just write down the zeta function of $\Delta_{\mathcal{D}}$ in terms of Epstein-type zeta functions and apply the Poisson summation formula [52,58]. But we shall form the zeta function using the contour integral method since it generalizes to cases when the eigenvalues are not explicit. Thus, following the Riemann zeta function example, using the Argument Principle (or Cauchy's formula) we write

$$\zeta_{\Delta_{\mathcal{D}}}(s) = \sum_{k=1}^{\infty} \frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x.$$

In the case when $\mu_k = 0$, we know that eigenvalues are of the form $\frac{\pi^2 n^2}{R^2}$, so

$$\begin{aligned}\sum_{\mu_k=0} \frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x &= \sum_{\mu_k=0} \zeta_k(s) = h_Y \sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2/R^2)^s} \\ &= h_Y \frac{R^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = h_Y \frac{R^{2s}}{\pi^{2s}} \zeta_{\mathcal{R}}(2s),\end{aligned} \tag{3.2}$$

where $h_Y := \dim\ker \Delta_Y$ and $\zeta_{\mathcal{R}}(s)$ is the Riemann zeta function studied in Section 2. Let us consider the case when $\mu_k \neq 0$. In this case, by the definition of $F_k(\lambda)$, we have

$$F_k(\mathrm{i}x) = \frac{\sinh\left(\sqrt{\mu_k^2 + x^2}R\right)}{\sqrt{\mu_k^2 + x^2}} = \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2\sqrt{\mu_k^2 + x^2}}\left(1 - \mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)$$

therefore

$$\begin{aligned}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x &= \int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2\sqrt{\mu_k^2+x^2}}\left(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\right)\mathrm{d}x\\ &= \int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x\\ &\quad+\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.\end{aligned}$$

By Lemma A.1 in Appendix A, we know that

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x = R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}$$

and

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left((\mu_k^2+x^2)^{1/2}\right)\mathrm{d}x = \frac{1}{2}\mu_k^{-2s}.$$

We therefore have

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x = R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}-\frac{1}{2}\mu_k^{-2s}.$$

Thus,

$$\begin{aligned}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x &= R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}-\frac{1}{2}\mu_k^{-2s}\\ &\quad+\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.\end{aligned}$$

In conclusion, we have proved the following theorem:

Theorem 3.1. *We have*

$$\begin{aligned}\zeta_{\Delta_{\mathcal{D}}}(s) &= h_Y\frac{R^{2s}}{\pi^{2s}}\zeta_{\mathcal{R}}(2s)+R\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}\zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)-\frac{1}{2}\zeta_{\Delta_Y}(s)\\ &\quad+\sum_{\mu_k>0}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.\end{aligned}$$

In particular, $\zeta_{\Delta_{\mathcal{D}}}(s)$ has a meromorphic extension to the whole complex plane except for simple poles at $s=\frac{n-k}{2}$ with $k = 0,1,2,\ldots$ and $\frac{n-k}{2}\notin\{0\}\cup-\mathbb{N}$.

Proof. Since, $\frac{\mathrm{d}}{\mathrm{d}x}\log(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}})$ is decreasing exponentially in both μ_k and x, the series $\sum_{\mu_k>0}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log(1-\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}})\,\mathrm{d}x$ is entire. Our statement about the meromorphic properties of $\zeta_{\Delta_{\mathcal{D}}}(s)$ now follows from those of $\zeta_{\mathcal{R}}(2s)$ (from Lemma 2.1) and $\zeta_{\Delta_Y}(s)$ (from (1.1)). $\square$

Now using the facts from Lemma 2.1:

$$\zeta_{\mathcal{R}}(0)=-\tfrac{1}{2}\quad\text{and}\quad\zeta'_{\mathcal{R}}(0)=-\tfrac{1}{2}\log(2\pi)$$

so that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} h_Y \frac{R^{2s}}{\pi^{2s}} \zeta_{\mathcal{R}}(2s) = -h_Y \log R + h_Y \log \pi - h_Y \log(2\pi) = -h_Y \log R - h_Y \log 2$$
$$= -h_Y \log(2R) \tag{3.3}$$

we can *easily* and almost *automatically* compute the log-zeta determinant:

$$\begin{aligned}\zeta'_{\Delta_{\mathcal{D}}}(0) &= -h_Y \log(2R) + \frac{R}{2\sqrt{\pi}} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}\left(s - \frac{1}{2}\right) \right) - \frac{1}{2} \zeta'_{\Delta_Y}(0) \\ &\quad + \sum_{\mu_k > 0} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1 - \mathrm{e}^{-2R\sqrt{\mu_k^2 + x^2}}\right) \mathrm{d}x \\ &= -h_Y \log(2R) + \frac{R}{2\sqrt{\pi}} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}\left(s - \frac{1}{2}\right) \right) - \frac{1}{2} \zeta'_{\Delta_Y}(0) \\ &\quad - \sum_{\mu_k > 0} \log\left(1 - \mathrm{e}^{-2R\mu_k}\right).\end{aligned}$$

Here, we notice that $\sum_{\mu_k > 0} \log(1 - \mathrm{e}^{-2R\mu_k})$ converges absolutely as $\log(1+z) = \mathcal{O}(z)$ for $|z|$ small. We have now proved the following:

Theorem 3.2. *We have*

$$\det{}_\zeta \Delta_{\mathcal{D}} = \frac{(2R)^{h_Y}}{\sqrt{\det \Delta_Y}} \mathrm{e}^{CR} \cdot \prod_{\mu_k > 0} \left(1 - \mathrm{e}^{-2R\mu_k}\right),$$

where $C = -\frac{1}{2\sqrt{\pi}} \frac{d}{ds}\Big|_{s=0} \left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}\left(s - \frac{1}{2}\right) \right)$.

This result for the determinant agrees with the result presented in [52,58]. We remark that the constant C in [52] was accidentally written incorrectly as $C = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \left(\Gamma(s)^{-1} \zeta_{D_Y^2}(s - 1/2) \right)$. Fortunately, [52] was interested in ratios of determinants, so the main result is correct because the (incorrect) constants cancelled in the end.

3.2. Neumann conditions

We now consider the zeta function and zeta determinant for the Neumann Laplacian $\Delta_{\mathcal{N}}$:

$$\Delta_{\mathcal{N}} : \left\{\phi \in H^2([0,R] \times Y, E) | \phi'(0) = 0 = \phi'(R)\right\} \to L^2([0,R] \times Y, E).$$

We start off by finding the eigenvalues of the Neumann Laplacian. To this end, we use formula (3.1) over the μ_k^2-eigenspace of Δ_Y:

$$\varphi = a \cosh\left(\sqrt{\mu_k^2 - \lambda^2} u\right) + b \sinh\left(\sqrt{\mu_k^2 - \lambda^2} u\right).$$

Imposing Neumann boundary conditions, we see that

$$\varphi'(0) = 0 \Rightarrow \varphi = a \cosh\left(\sqrt{\mu_k^2 - \lambda^2} u\right)$$

and for $\lambda^2 \neq \mu_k^2$,

$$\varphi'(R) = 0 \Rightarrow \sinh\left(\sqrt{\mu_k^2 - \lambda^2} R\right) = 0.$$

Therefore, we can see that over the μ_k^2-eigenspace of Δ_Y

$$\lambda^2 \neq \mu_k^2 \text{ is an eigenvalue of } \Delta_{\mathcal{N}} \Longleftrightarrow F_k(\lambda) := \frac{\sinh\left(\sqrt{\mu_k^2 - \lambda^2}R\right)}{\sqrt{\mu_k^2 - \lambda^2}} = 0.$$

For $\lambda^2 = \mu_k^2$, $\varphi =$ *constant* is a *nontrivial* solution. Hence, $\zeta_{\Delta_{\mathcal{N}}}(s) - \zeta_{\Delta_{\mathcal{D}}}(s) = \zeta_{\Delta_Y}(s)$ and the result for the zeta function and the determinant are easily obtained from Theorems 3.1 and 3.2, respectively.

3.3. Mixed Dirichlet Neumann conditions

We now consider the zeta function and zeta determinant for the Laplacian $\Delta_{\mathcal{M}}$ where we put mixed Dirichlet and Neumann conditions:

$$\Delta_{\mathcal{M}} : \left\{\phi \in H^2([0,R] \times Y, E) | \phi(0) = 0, \phi'(R) = 0\right\} \to L^2([0,R] \times Y, E);$$

of course, a similar result holds if we consider $\phi'(0) = 0$ and $\phi(R) = 0$. We again use the formula (3.1) over the μ_k^2-eigenspace of Δ_Y:

$$\varphi = a\cosh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) + b\sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right).$$

Imposing Dirichlet boundary condition at $u = 0$, we see that

$$\varphi(0) = 0 \Rightarrow \varphi = b\sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right)$$

and then,

$$\varphi'(R) = 0 \Rightarrow \cosh\left(\sqrt{\mu_k^2 - \lambda^2}R\right) = 0.$$

Therefore, we can see that over the μ_k^2-eigenspace of Δ_Y

$$\lambda^2 \neq 0 \text{ is an eigenvalue of } \Delta_{\mathcal{M}} \Longleftrightarrow F_k(\lambda) := \cosh\left(\sqrt{\mu_k^2 - \lambda^2}R\right) = 0.$$

Then, as in the previous cases, we have

$$\zeta_{\Delta_{\mathcal{M}}}(s) = \sum_{k=1}^{\infty} \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x.$$

First, when $\mu_k = 0$, we know that eigenvalues are of the form $\frac{\pi^2(n+1/2)^2}{R^2}$, $n \in \mathbb{N}$ so

$$\begin{aligned}\sum_{\mu_k=0} \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x &= \sum_{\mu_k=0} \zeta_k(s) = h_Y \frac{R^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty} \frac{1}{(n+1/2)^{2s}} \\ &= h_Y \frac{R^{2s}}{\pi^{2s}} \zeta_{\mathcal{H}}\left(2s, \frac{1}{2}\right), \qquad (3.4)\end{aligned}$$

where $h_Y := \dim\ker \Delta_Y$ and $\zeta_{\mathcal{H}}(s,a)$ is the Hurwitz zeta function defined in (2.4) with the properties in Lemma 2.2. When $\mu_k \neq 0$, by definition

$$F_k(\mathrm{i}x) = \cosh\left(\sqrt{\mu_k^2 + x^2}R\right) = \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\left(1 + \mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right).$$

Hence, proceeding as in Section 3.1, we obtain

$$\frac{\sin \pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x = R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)} + \frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.$$

In conclusion, we have proved the following theorem:

Theorem 3.3. *We have*

$$\zeta_{\Delta_{\mathcal{M}}}(s) = h_Y\frac{R^{2s}}{\pi^{2s}}\zeta_{\mathcal{H}}\left(2s,\frac{1}{2}\right) + R\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}\zeta_{\Delta_Y}\left(s-\frac{1}{2}\right) + \sum_{\mu_k>0}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.$$

In particular, $\zeta_{\Delta_{\mathcal{M}}}(s)$ has a meromorphic extension to the whole complex plane except for simple poles at $s = \frac{n-k}{2}$ with $k = 0, 1, 2, \ldots$ and $\frac{n-k}{2} \notin \{0\} \cup -\mathbb{N}$.

Proof. According to Lemma 2.2, the pole structure of $\zeta_{\mathcal{H}}(s, 1/2)$ is the same as the pole structure of $\zeta_{\mathcal{R}}(s)$. We can also prove this directly:

$$\zeta_{\mathcal{H}}\left(s,\frac{1}{2}\right) = \sum_{k=0}^\infty\left(k+\frac{1}{2}\right)^{-s} = 2^s\sum_{k=0}^\infty(2k+1)^{-s} = 2^s\left(\zeta_{\mathcal{R}}(s) - \sum_{k=1}^\infty(2k)^{-s}\right)$$
$$= 2^s(1-2^{-s})\zeta_{\mathcal{R}}(s) = (2^s-1)\zeta_{\mathcal{R}}(s)$$

which implies our claim. Then this theorem just follows from the previous computation and Theorem 3.1. $\square$

Using

$$\zeta_{\mathcal{H}}\left(0,\frac{1}{2}\right) = 0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\zeta_{\mathcal{H}}\left(s,\frac{1}{2}\right) = -\frac{1}{2}\log 2$$

from Lemma 2.2, so that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} h_Y\frac{R^{2s}}{\pi^{2s}}\zeta_{\mathcal{H}}\left(2s,\frac{1}{2}\right) = -h_Y\log 2 \tag{3.5}$$

now we can *easily* and almost *automatically* compute the log-zeta determinant:

$$\zeta'_{\Delta_{\mathcal{M}}}(0) = -h_Y\log 2 + \frac{R}{2\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)\right) + \sum_{\mu_k>0}\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x$$
$$= -h_Y\log 2 + \frac{R}{2\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)\right) - \sum_{\mu_k>0}\log\left(1+\mathrm{e}^{-2R\mu_k}\right).$$

Here, we notice that $\sum_{\mu_k>0}\log(1+\mathrm{e}^{-2R\mu_k})$ converges absolutely. We have now proved the following

Theorem 3.4. *We have*

$$\det{}_\zeta\Delta_{\mathcal{M}} = 2^{h_Y}\mathrm{e}^{CR}\cdot\prod_{\mu_k>0}\left(1+\mathrm{e}^{-2R\mu_k}\right),$$

where $C = -\frac{1}{2\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)\right)$.

This result for the determinant agrees with the result presented in [55].

3.4. Periodic boundary conditions

We now consider the zeta function and zeta determinant for the Laplacian $\Delta_{\mathcal{P}}$ where we put periodic boundary conditions:

$$\Delta_{\mathcal{P}}:\left\{\phi\in H^2([0,R]\times Y,E)|\phi(R)=\phi(0),\phi'(R)=\phi'(0)\right\}\to L^2([0,R]\times Y,E).$$

We start off by finding the eigenvalues of $\Delta_{\mathcal{P}}$. To this end, we use the formula (3.1) over the μ_k^2-eigenspace of Δ_Y:

$$\varphi = a\cosh\left(\sqrt{\mu_k^2-\lambda^2}u\right)+b\sinh\left(\sqrt{\mu_k^2-\lambda^2}u\right).$$

Imposing the condition $\phi(R)=\phi(0)$, we see that

$$a\left(\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1\right)+b\sinh\left(\sqrt{\mu_k^2-\lambda^2}R\right)=0$$

and from the condition $\phi'(R)=\phi'(0)$ we get

$$a\sinh\left(\sqrt{\mu_k^2-\lambda^2}R\right)+b\left(\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1\right)=0.$$

For both of a and b not to be zero, we must have

$$\det\begin{pmatrix}\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1 & \sinh\left(\sqrt{\mu_k^2-\lambda^2}R\right)\\ \sinh\left(\sqrt{\mu_k^2-\lambda^2}R\right) & \cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1\end{pmatrix}=0$$

that is,

$$\left(\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1\right)^2-\sinh^2\left(\sqrt{\mu_k^2-\lambda^2}R\right)=0$$

or multiplying out and using that $\cosh^2 z-\sinh^2 z=1$, we obtain

$$2-2\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)=0.$$

Thus, over the μ_k^2-eigenspace of Δ_Y

$$\lambda^2 \text{ is an eigenvalue of } \Delta_{\mathcal{P}} \iff F_k(\lambda):=\cosh\left(\sqrt{\mu_k^2-\lambda^2}R\right)-1=0,\quad \mu_k\neq 0$$

and

$$\lambda^2 \neq 0 \text{ is an eigenvalue of } \Delta_{\mathcal{P}} \Longleftrightarrow F_k(\lambda) := \frac{1-\cos(\lambda R)}{\lambda^2} = 0, \quad \mu_k = 0.$$

To evaluate the zeta function of $\Delta_{\mathcal{P}}$, we write

$$\zeta_{\Delta_{\mathcal{P}}}(s) = \sum_{k=1}^{\infty} \zeta_k(s), \qquad \zeta_k(s) := \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x.$$

We shall study each $\zeta_k(s)$. Consider first the case when $\mu_k = 0$. In this case, we know that nonzero eigenvalues are of the form $\frac{(2\pi)^2 n^2}{R^2}$ where $n \in \mathbb{Z} - \{0\}$, so

$$\sum_{\mu_k=0} \zeta_k(s) = 2h_Y \frac{R^{2s}}{(2\pi)^{2s}} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2h_Y \frac{R^{2s}}{(2\pi)^{2s}} \zeta_{\mathcal{R}}(2s). \tag{3.6}$$

In the case when $\mu_k \neq 0$, the function $F_k(\mathrm{i}x)$ can easily be found:

$$\begin{aligned} F_k(\mathrm{i}x) &= \left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2} + \frac{\mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}}{2} \right) - 1 \\ &= \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2} \left(1 + \mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}} - 2\mathrm{e}^{-R\sqrt{\mu_k^2+x^2}} \right) \\ &= \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2} \left(1 - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}} \right)^2. \end{aligned}$$

Substituting this formula for $F_k(\mathrm{i}x)$ into $\frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x$, we see that

$$\zeta_k(s) = \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log \left(\mathrm{e}^{R\sqrt{\mu_k^2+x^2}} \right) \mathrm{d}x + \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log \left(1 - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}} \right)^2 \mathrm{d}x.$$

By Lemma A.1, the first integral is equal to

$$\frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log \left(\mathrm{e}^{R\sqrt{\mu_k^2+x^2}} \right) \mathrm{d}x = R\mu_k^{-2s+1} \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}$$

therefore,

$$\zeta_k(s) = R\mu_k^{-2s+1} \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)} + \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log \left(1 - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}} \right)^2 \mathrm{d}x.$$

Summing over all k, we have proved the following theorem. (The meromorphy statements follow just as in Theorem 3.1.)

Theorem 3.5. *We have*

$$\begin{aligned} \zeta_{\Delta_{\mathcal{P}}}(s) &= 2h_Y \frac{R^{2s}}{(2\pi)^{2s}} \zeta_{\mathcal{R}}(2s) + R \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)} \zeta_{\Delta_Y}\left(s - \frac{1}{2} \right) \\ &\quad + \sum_{\mu_k>0} \frac{\sin \pi s}{\pi} \int_0^{\infty} x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log \left(1 - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}} \right)^2 \mathrm{d}x. \end{aligned}$$

In particular, $\zeta_{\Delta_{\mathcal{P}}}(s)$ has a meromorphic extension to the whole complex plane except for simple poles at $s = \frac{n-k}{2}$ with $k = 0, 1, 2, \ldots$ and $\frac{n-k}{2} \notin \{0\} \cup -\mathbb{N}$.

A similar computation as we did in (3.3) shows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} 2h_Y \frac{R^{2s}}{(2\pi)^{2s}} \zeta_{\mathcal{R}}(2s) = -2h_Y \log R$$

therefore

$$\begin{aligned}\zeta'_{\Delta_{\mathcal{P}}}(0) &= -2h_Y \log R + \frac{R}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)\right) + \sum_{\mu_k>0} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1-\mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}\right)^2 \mathrm{d}x \\ &= -2h_Y \log R - CR - \sum_{\mu_k>0} \log\left(1-\mathrm{e}^{-2R\mu_k}\right)^2,\end{aligned}$$

where $C = -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}(s-\frac{1}{2}))$. Thus, we have

Theorem 3.6. *We have*

$$\det{}_\zeta \Delta_{\mathcal{P}} = R^{2h_Y} \mathrm{e}^{CR} \prod_{\mu_k>0} \left(1-\mathrm{e}^{-2R\mu_k}\right)^2,$$

where $C = -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{\Delta_Y}\left(s-\frac{1}{2}\right)\right)$.

Note that periodic boundary conditions are related to finite temperature quantum field theory of a scalar field. Therefore results like Theorems 3.5 and 3.6 can be found in that context [28,46,59].

4. Dirac-type operators

4.1. Dirac-type operators over finite cylinders

We now consider a Dirac-type operator

$$\eth : H^1([0,R] \times Y, S) \to L^2([0,R] \times Y, S),$$

where S is a Clifford bundle over $[0,R]\times Y$. We assume that $\eth$ is of product form

$$\eth = G(\partial_u + D_Y), \tag{4.1}$$

where G is a bundle automorphism of $S_0 := S|_Y$ and D_Y is a Dirac operator acting on $C^\infty(Y,S_0)$ such that $G^2 = -\mathrm{Id}$ and $GD_Y = -D_Y G$. Since $G^2 = -\mathrm{Id}$, we have $S = S^+ \oplus S^-$, where $S^\pm$ are the $\pm i$ eigensubbundles of G. Now the finite cylinder $[0,R]\times Y$ has boundaries, so we have to impose boundary conditions. For concreteness, we shall consider separated boundary conditions; that is, boundary conditions separately imposed at $u = 0$ and $u = R$. Moreover, breaking up

$$L^2([0,R] \times Y, S) = L^2([0,R], V) \oplus L^2([0,R], V^\perp),$$

where $V := \ker D_Y$ and $V^\perp$ is the orthogonal complement of V within $L^2(Y, S_0)$, the operator $\eth$ breaks up as

$$\eth = G\partial_u \oplus G(\partial_u + D_Y) \quad \text{over } H^1([0,R], V) \oplus H^1([0,R], V^\perp)$$

and we shall impose separated boundary conditions individually on $H^1([0,R], V)$ and $H^1([0,R], V^\perp)$. On $H^1([0,R], V)$, we completely characterize *all* self-adjoint extensions of $\eth = G\partial_u$ with separated boundary conditions. Put $V^+ := \ker(D_Y) \cap C^\infty(Y, S^+)$ and

$V^{-}:=\ker(D_Y)\cap C^{\infty}(Y,S^{-})$ and let Π_0 denote the orthogonal projection onto the zero eigenspace of D_Y.

Proposition 4.1. *$G\partial_u$ has self-adjoint extensions with separated boundary conditions if and only if* $\dim V^{+}=\dim V^{-}$, *in which case, self-adjoint extensions are in one-to-one correspondence with involutions σ_1,σ_2 over V that anticommute with G, and the boundary conditions are given by*

$$\frac{1+\sigma_1}{2}\Pi_0 \quad \text{at } \{0\}\times Y, \qquad \frac{1+\sigma_2}{2}\Pi_0 \quad \text{at } \{R\}\times Y.$$

Proof. For more on self-adjoint extensions, we refer the reader to the articles [44,62], and books [2,64]. Recall that with a choice of domain $\mathcal{D}\subset H^1([0,R],V)$, we say that

$$G\partial_u:\mathcal{D}\to L^2([0,R],S)$$

is self-adjoint if

$$\left\{\psi\in H^1([0,R],V)|\langle G\partial_u\phi,\psi\rangle=\langle\phi,G\partial_u\psi\rangle\quad\forall\phi\in\mathcal{D}\right\}=\mathcal{D}. \tag{4.2}$$

Let us choose separated boundary conditions for $G\partial_u$; that is, subspaces $L_1,L_2\subset V$ such that

$$\mathcal{D}=\left\{\phi\in H^1([0,R],V)|\phi(0)\in L_1\quad\text{and}\quad\phi(R)\in L_2\right\},$$

and suppose that $G\partial_u$ on the domain $\mathcal{D}$ is self-adjoint in the above sense. Given that $G\partial_u$ is self-adjoint, we shall prove that $\dim V^{+}=\dim V^{-}$, and L_j is exactly the $+1$ eigenspace of an involution σ_j on V; the converse is straightforward and shall be left to the reader. Let us prove this for L_1; the proof for L_2 is similar.

Integrating by parts shows that for $\phi,\psi\in H^1([0,R],V)$, we have the equality (called Green's formula)

$$\langle\phi,G\partial_u\psi\rangle-\langle G\partial_u\phi,\psi\rangle=\langle G\phi(0),\psi(0)\rangle-\langle\phi(R),G\psi(R)\rangle.$$

Thus, by the criterion (4.2) for self-adjointness, we have

$$\left\{\psi\in H^1([0,R],V)|\langle G\phi(0),\psi(0)\rangle-\langle\phi(R),G\psi(R)\rangle=0\quad\forall\phi\in\mathcal{D}\right\}=\mathcal{D}.$$

Taking ϕ,ψ to vanish near $u=R$, by definition of $\mathcal{D}$, it follows that

$$\{w\in V|\langle Gv,w\rangle=0\quad\forall v\in L_1\}=L_1.$$

Now

$$\begin{aligned}\{w\in V|\langle Gv,w\rangle=0\quad\forall v\in L_1\}=L_1 &\iff \{w\in V|\langle v,Gw\rangle=0\quad\forall v\in L_1\}=L_1\\ &\iff G^{-1}L_1^{\perp}=L_1\\ &\iff GL_1=L_1^{\perp}.\end{aligned}$$

Define $\sigma_1:V\to V$ by $\sigma_1:=-1$ on L_1 and $\sigma_1:=1$ on $L_1^{\perp}$. We claim that $\sigma_1G=-G\,\sigma_1$. Indeed, if $v\in L_1$, then

$$\sigma_1Gv=\sigma_1(Gv)=Gv\quad(\text{because } Gv\in L_1^{\perp})=-G\sigma_1v\quad(\text{because } v\in L_1)$$

a similar argument shows that $\sigma_1G=-G\sigma_1$ on $L_1^{\perp}$. Hence, $\sigma_1G=-G\sigma_1$ on all of V. We now show that $\dim V^{+}=\dim V^{-}$. Indeed, if $\phi\in V^{+}$, then $G\phi=\mathrm{i}\phi$, so

$$G\sigma_1\phi = -\sigma_1 G\phi = -\sigma_1(\mathrm{i}\phi) = -\mathrm{i}(\sigma_1\phi).$$

It follows that $\sigma_1 : V^+ \to V^-$ is an isomorphism. In particular, dim $V^+ = $ dim V^-. This completes our proof. □

Because of Proposition 4.1, we now henceforth *assume* that dim $V^+ = $ dim V^-; we fix two involutions σ_1, σ_2 over $\ker(D_Y)$ anticommuting with G, so that $G\eth_u$ acts on the domain

$$\mathcal{D} = \left\{ \phi \in H^1([0,R], V) | \frac{1+\sigma_1}{2}\phi(0) = 0 \quad \text{and} \quad \frac{1+\sigma_2}{2}\phi(R) = 0 \right\}.$$

On $H^1([0,R], V^\perp)$, we can put both local and non-local separated boundary conditions. For many applications non-local conditions which are so-called APS spectral boundary conditions play a significant role [3,4,22,26,29,36,67]. We will consider the following non-local condition afterwards

$$\begin{aligned} \Pi_{\sigma_1} &= \Pi_> + \frac{1+\sigma_1}{2}\Pi_0 \quad \text{at } \{0\} \times Y, \\ \Pi_{\sigma_2} &= \Pi_< + \frac{1+\sigma_2}{2}\Pi_0 \quad \text{at } \{R\} \times Y, \end{aligned} \tag{4.3}$$

where $\Pi_>, \Pi_<$ denote the orthogonal projections onto the positive and negative eigenspaces of D_Y, respectively. The projectors Π_{σ_1} and Π_{σ_2} are called *generalized APS spectral projections*. We denote by $\eth_{\Pi_\sigma}$ the resulting operator with these boundary conditions:

$$\eth_{\Pi_\sigma} := \eth : \mathrm{dom}(\eth_{\Pi_\sigma}) \to L^2([0,R] \times Y, S),$$

where

$$\mathrm{dom}(\eth_{\Pi_\sigma}) := \left\{ \phi \in H^1([0,R] \times Y, S) | \Pi_{\sigma_1}\phi|_{u=0} = 0, \quad \Pi_{\sigma_2}\phi|_{u=R} = 0 \right\}.$$

Then the spectrum of the Dirac type operator $\eth_{\Pi_\sigma}$ consists of discrete real eigenvalues $\{\mu_k\}$.

Local boundary conditions for $\eth$ over $V^\perp$ are also quite common [5–7,21,32,69].

4.2. Determinant on the zero mode

To study $\zeta_{\eth^2_{\Pi_\sigma}}(s)$ and $\det_\zeta \eth^2_{\Pi_\sigma}$, we begin by computing the zeta function over the zero mode of D_Y, leaving the nonzero modes to Section 4.3. Thus, over $\ker(D_Y)$, we shall study

$$D := G\eth_u \quad \text{over } [0,R] \times Y$$

with boundary conditions given by vanishing on the $+1$ eigenspaces of σ_1 and σ_2, respectively,

$$\frac{1+\sigma_1}{2}\Pi_0 \quad \text{at } \{0\} \times Y, \qquad \frac{1+\sigma_2}{2}\Pi_0 \quad \text{at } \{R\} \times Y.$$

Observe that

$$D\varphi = \lambda\varphi \iff G\eth_u\varphi = \lambda\varphi \iff \eth_u\varphi = -\lambda G\varphi \iff \varphi(u) = e^{-\lambda G u}\varphi_0,$$

where $\varphi_0 = \varphi(0)$. To get a nice form for ϕ, we put φ_0 in a nice basis. Observe that since σ_1 and σ_2 anticommute with G, it follows that the product $\sigma_1\sigma_2$ is a unitary isomorphism on V^- (and also on V^+), which we denote by $(\sigma_1\sigma_2)_-$: $V^- \to V^-$. Let $\{\varphi_1, \ldots, \varphi_m\}$ be an eigenbasis for $(\sigma_1\sigma_2)_-$ so that

$$\varphi_j \in V^- \quad \text{and} \quad \sigma_1\sigma_2\varphi_j = e^{i\gamma_k}\varphi_j \quad \forall j, \quad \gamma_k \in (-\pi, \pi].$$

Since, σ_1 anticommutes with G, $\{\sigma_1\varphi_1, \ldots, \sigma_1\varphi_m\}$ is a basis for V^+, so that

$$\{\sigma_1\varphi_1, \ldots, \sigma_1\varphi_m, \varphi_1, \ldots, \varphi_m\}$$

is a basis for $\ker(D_Y) = V^+ \oplus V^-$. Moreover, since G preserves $\mathrm{span}\{\sigma_1\varphi_j, \varphi_j\}$ for each j, we shall henceforth fix a j and assume that φ_0 is in $\mathrm{span}\{\sigma_1\varphi_j, \varphi_j\}$:

$$\varphi_0 = a\sigma_1\varphi_j + b\varphi_j.$$

Then,

$$\begin{aligned}\frac{1+\sigma_1}{2}\varphi_0 = 0 &\Longleftrightarrow \sigma_1\varphi_0 = -\varphi_0 \Longleftrightarrow a\varphi_j + b\sigma_1\varphi_j = -a\sigma_1\varphi_j - b\varphi_j \\ &\Longleftrightarrow a = -b \Longleftrightarrow \varphi_0 = \sigma_1\varphi_j - \varphi_j \quad \text{(modulo a constant factor)}.\end{aligned}$$

Thus,

$$\varphi(u) = \mathrm{e}^{-\lambda Gu}\varphi_0 = \mathrm{e}^{-\mathrm{i}\lambda u}\sigma_1\varphi_j - \mathrm{e}^{\mathrm{i}\lambda u}\varphi_j.$$

We now deal with the boundary condition at $u = R$. To do so, note that

$$\sigma_1\sigma_2\varphi_j = \mathrm{e}^{\mathrm{i}\gamma_k}\varphi_j \Rightarrow \sigma_2\varphi_j = \mathrm{e}^{\mathrm{i}\gamma_k}\sigma_1\varphi_j, \quad \sigma_2\sigma_1\varphi_j = \mathrm{e}^{-\mathrm{i}\gamma_k}\varphi_j.$$

Therefore,

$$\begin{aligned}\frac{1+\sigma_2}{2}\varphi(R) = 0 &\Longleftrightarrow \sigma_2\varphi(R) = -\varphi(R) \\ &\Longleftrightarrow \mathrm{e}^{-\mathrm{i}\lambda R}\mathrm{e}^{-\mathrm{i}\gamma_k}\varphi_j - \mathrm{e}^{\mathrm{i}\lambda R}\mathrm{e}^{\mathrm{i}\gamma_k}\sigma_1\varphi_j = -\mathrm{e}^{-\mathrm{i}\lambda R}\sigma_1\varphi_j + \mathrm{e}^{\mathrm{i}\lambda R}\varphi_j \\ &\Longleftrightarrow \mathrm{e}^{2\mathrm{i}\lambda R} = \mathrm{e}^{-\mathrm{i}\gamma_k} \Longleftrightarrow 2\lambda R = -\gamma_k + 2\pi n, \quad n \in \mathbb{Z} \\ &\Longleftrightarrow \left(\lambda R + \frac{\gamma_k}{2}\right) = \pi n, \quad n \in \mathbb{Z} \Longleftrightarrow \sin\left(\lambda R + \frac{\gamma_k}{2}\right) = 0.\end{aligned}$$

Therefore, we can see that

$$\lambda \text{ is an eigenvalue of } D \text{ over } \mathrm{span}\{\sigma_1\varphi_j, \varphi_j\} \Longleftrightarrow \sin\left(\lambda R + \frac{\gamma_k}{2}\right) = 0.$$

It follows that

$$\begin{aligned}&\lambda^2 \text{ is an eigenvalue of } D^2 \text{ over } \mathrm{span}\{\sigma_1\varphi_j, \varphi_j\} \\ &\quad\Longleftrightarrow F_k(\lambda) := \sin\left(\lambda R + \frac{\gamma_k}{2}\right)\sin\left(-\lambda R + \frac{\gamma_k}{2}\right) = 0, \quad \text{if } \gamma_k \neq 0\end{aligned}$$

and

$$\lambda^2 \neq 0 \text{ is an eigenvalue of } D^2 \text{ over } \mathrm{span}\{\sigma_1\varphi_j, \varphi_j\} \Longleftrightarrow F_k(\lambda) := \frac{\sin\lambda R}{\lambda} = 0,$$

if $\gamma_k = 0$.

Now we write

$$\zeta_{D^2}(s) = \sum_{\gamma_k=0} \zeta_k(s) + \sum_{\gamma_k \neq 0} \zeta_k(s),$$

where the ζ_k's are the zeta functions for D^2 over span$\{\sigma_1\varphi_k, \varphi_k\}$. The zeta function for the $\gamma_k = 0$ case is handled in (3.2):

$$\sum_{\gamma_k=0} \zeta_k(s) = 2h \frac{R^{2s}}{\pi^{2s}} \zeta_{\mathcal{R}}(2s),$$

where h is the number of γ_k's equal to 0 and where there is a factor of 2 in front because D^2 has each nonzero eigenvalue λ^2 with multiplicity two. Thus, by the calculation found in (3.3), we have

$$e^{-\sum_{\gamma_k=0} \zeta_k'(0)} = (2R)^{2h}. \tag{4.4}$$

We now focus on the $\gamma_k \neq 0$ case. As is now familiar, we write

$$\zeta_k(s) = \frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x, \quad F_k(\lambda) := \sin\left(\lambda R + \frac{\gamma_k}{2}\right) \sin\left(-\lambda R + \frac{\gamma_k}{2}\right).$$

Writing

$$\begin{aligned} F_k(\mathrm{i}x) &:= \sin\left(\mathrm{i}xR + \frac{\gamma_k}{2}\right) \sin\left(-\mathrm{i}xR + \frac{\gamma_k}{2}\right) = -\frac{\left(e^{-xR+\mathrm{i}\frac{\gamma_k}{2}} - e^{xR-\mathrm{i}\frac{\gamma_k}{2}}\right)\left(e^{xR+\mathrm{i}\frac{\gamma_k}{2}} - e^{-xR-\mathrm{i}\frac{\gamma_k}{2}}\right)}{4} \\ &= \frac{(e^{2xR} - e^{i\gamma_k} - e^{-\mathrm{i}\gamma_k} + e^{-2xR})}{4} = \frac{e^{2xR}}{4}\left(1 - 2e^{-2xR}\cos(\gamma_k) + e^{-4xR}\right) \end{aligned}$$

we have

$$\begin{aligned} &\int_1^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x \\ &= \int_1^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left(\frac{e^{2xR}}{4}\left(1 - 2e^{-2xR}\cos(\gamma_k) + e^{-4xR}\right)\right)\mathrm{d}x \\ &= 2R \int_1^\infty x^{-2s}\mathrm{d}x + \int_1^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1 - 2e^{-2xR}\cos(\gamma_k) + e^{-4xR}\right)\mathrm{d}x \\ &= \frac{2R}{2s-1} + \int_1^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1 - 2e^{-2xR}\cos(\gamma_k) + e^{-4xR}\right)\mathrm{d}x. \end{aligned}$$

Therefore, splitting $\zeta_k(s) = \frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x$ into integrals from 0 to 1 and from 1 to ∞, we have

$$\begin{aligned} \zeta_k(s) = &\frac{\sin \pi s}{\pi} \int_0^1 x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log F_k(\mathrm{i}x)\,\mathrm{d}x + \frac{\sin \pi s}{\pi} \frac{2R}{2s-1} \\ &+ \frac{\sin \pi s}{\pi} \int_1^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1 - 2e^{-2xR}\cos(\gamma_k) + e^{-4xR}\right)\mathrm{d}x. \end{aligned}$$

Taking $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}$ and using that $F_k(\mathrm{i}) = \frac{e^{2R}}{4}(1 - 2e^{-2R}\cos(\gamma_k) + e^{-4R})$ and $F_k(0) = \sin^2(\frac{\gamma_k}{2})$, we get

$$\begin{aligned}
\zeta_k'(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x - 2R + \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}x}\log\left(1-2\mathrm{e}^{-2xR}\cos(\gamma_k)+\mathrm{e}^{-4xR}\right)\mathrm{d}x \\
&= \log F_k(\mathrm{i}) - \log F_k(0) - 2R - \log\left(1-2\mathrm{e}^{-2R}\cos(\gamma_k)+\mathrm{e}^{-4R}\right) \\
&= \log\left(\frac{\mathrm{e}^{2R}}{4}\left(1-2\mathrm{e}^{-2R}\cos(\gamma_k)+\mathrm{e}^{-4R}\right)\right) - \log\sin^2\left(\frac{\gamma_k}{2}\right) - 2R \\
&\quad - \log\left(1-2\mathrm{e}^{-2R}\cos(\gamma_k)+\mathrm{e}^{-4R}\right) \\
&= -\log\left(4\sin^2\left(\frac{\gamma_k}{2}\right)\right).
\end{aligned}$$

Therefore,

$$\mathrm{e}^{-\zeta_k'(0)} = 4\sin^2\left(\frac{\gamma_k}{2}\right) = 4\left(\frac{\mathrm{e}^{\mathrm{i}\gamma_k/2}-\mathrm{e}^{-\mathrm{i}\gamma_k/2}}{2\mathrm{i}}\right)^2 = 4\,\frac{2-\mathrm{e}^{\mathrm{i}\gamma_k}-\mathrm{e}^{-\mathrm{i}\gamma_k}}{4}.$$

Combining this formula with the $\gamma_k = 0$ case in (4.4), we get

$$\det{}_\zeta D^2 = (2R)^{2h}\cdot\prod_{\gamma_k\neq 0} 4\cdot\frac{2-\mathrm{e}^{\mathrm{i}\gamma_k}-\mathrm{e}^{-\mathrm{i}\gamma_k}}{4} = (2R)^{2h}4^{\frac{h_Y}{2}-h}\cdot\prod_{\gamma_k\neq 0}\frac{2-\mathrm{e}^{\mathrm{i}\gamma_k}-\mathrm{e}^{-\mathrm{i}\gamma_k}}{4},$$

where h is the number of γ_k's equal to 0 and $h_Y = \dim\ker(D_Y)$. Thus, we have proved

Lemma 4.2. *The zeta function $\zeta_{D^2}(s)$ has a meromorphic extension to the whole complex plane with only a single pole at $s = 1$, and*

$$\det{}_\zeta D^2 = R^{2h}2^{h_Y}\det{}^*\left(\frac{2\mathrm{Id}-(\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4}\right)$$

with h the number of $(+1)$-eigenvalues of $(\sigma_1\sigma_2)_-$, $h_Y = \dim\ker(D_Y)$, and where $\det^*$ means to take the determinant on the orthogonal complement of the kernel.

4.3. Determinant on the nonzero mode

We now find the zeta function for the square of

$$\eth = G(\partial_u + D_Y) \quad \text{over} \quad [0,R]\times Y$$

on the orthogonal complement of $\ker(D_Y)$ with the boundary conditions

$$\Pi_> \quad \text{at} \quad \{0\}\times Y, \qquad \Pi_< \quad \text{at} \quad \{R\}\times Y.$$

Let $\{\phi_k\}$ be an eigenbasis of the positive eigenvectors of D_Y, so that $\{G\phi_k\}$ is an eigenbasis of the negative eigenvectors of D_Y,

$$D_Y\phi_k = \mu_k\phi_k \quad \text{and} \quad D_YG\phi_k = -\mu_kG\phi_k.$$

Let us fix a k; then with respect to the basis $\{\phi_k, G\,\phi_k\}$, we can write

$$\eth = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\left(\partial_u + \begin{pmatrix}\mu_k & 0 \\ 0 & -\mu_k\end{pmatrix}\right) = \begin{pmatrix} 0 & -\partial_u+\mu_k \\ \partial_u+\mu_k & 0\end{pmatrix}$$

over $\mathrm{span}\{\phi_k, G\phi_k\}$. Hence, the eigenvalues of $\eth$ over $\mathrm{span}\{\phi_k, G\phi_k\}$ are obtained by solving

$$\begin{pmatrix} 0 & -\partial_u + \mu_k \\ \partial_u + \mu_k & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \lambda\begin{pmatrix} f \\ g \end{pmatrix} \Longleftrightarrow \begin{array}{c} -g'(u) + \mu_k g(u) = \lambda f(u) \\ f'(u) + \mu_k f(u) = \lambda g(u) \end{array}$$

with $f(0) = 0$ (from the boundary condition $\Pi_>$) and $g(R) = 0$ (from the boundary condition $\Pi_<$). To solve these equations, apply $-\partial_u + \mu_k$ to the second line and use the first line to get

$$(-\partial_u + \mu_k)(f'(u) + \mu_k f(u)) = \lambda(-\partial_u + \mu_k)g(u) \Longleftrightarrow -f''(u) + \mu_k^2 f(u) = \lambda^2 f(u).$$

Solving this equation for $f(u)$ and using that $f(0) = 0$, we see that

$$f(u) = \sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) \quad \text{(modulo a constant)}.$$

Now the equation $f'(u) + \mu_k f(u) = \lambda g(u)$ implies that

$$\sqrt{\mu_k^2 - \lambda^2}\cosh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) + \mu_k \sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) = \lambda g(u).$$

Therefore, from the condition $g(R) = 0$, and the fact that $\lambda = 0$ cannot be an eigenvalue as can be easily checked, we conclude that

$$\begin{aligned} &\lambda \text{ or } -\lambda \text{ is an eigenvalue of } \eth \text{ over span}\{\phi_k, G\phi_k\} \\ &\Longleftrightarrow \lambda^2 \text{ is an eigenvalue of } \eth^2 \text{ over span}\{\phi_k, G\phi_k\} \\ &\Longleftrightarrow F_k(\lambda) := \cosh\left(\sqrt{\mu_k^2 - \lambda^2}R\right) + \mu_k \frac{\sinh\left(\sqrt{\mu_k^2 - \lambda^2}R\right)}{\sqrt{\mu_k^2 - \lambda^2}} = 0. \end{aligned}$$

Now to evaluate the zeta function of $\eth^2$ over span$\{\phi_k, G\phi_k\}$:

$$\zeta_k(s) = \frac{2\sin \pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x$$

we need to find $F_k(\mathrm{i}x)$. To do so, observe that

$$\begin{aligned} F_k(\mathrm{i}x) &= \cosh\left(\sqrt{\mu_k^2 + x^2}R\right) + \mu_k \frac{\sinh(\sqrt{\mu_k^2 + x^2}R)}{\sqrt{\mu_k^2 + x^2}} \\ &= \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}} + \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}}{2} + \frac{\mu_k}{\sqrt{\mu_k^2 + x^2}}\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}} - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}}{2} \\ &= \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\left(1 + \frac{\mu_k}{\sqrt{\mu_k^2 + x^2}}\right) + \frac{\mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}}{2}\left(1 - \frac{\mu_k}{\sqrt{\mu_k^2 + x^2}}\right) \\ &= \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\left(1 + \frac{\mu_k}{\sqrt{\mu_k^2 + x^2}}\right)\left(1 + \mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\frac{1 - \frac{\mu_k}{\sqrt{\mu_k^2+x^2}}}{1 + \frac{\mu_k}{\sqrt{\mu_k^2+x^2}}}\right). \end{aligned}$$

Simplifying the right-hand formula, we get

$$F_k(\mathrm{i}x) = \frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\left(1 + \frac{\mu_k}{\sqrt{\mu_k^2 + x^2}}\right)\left(1 + \frac{x^2\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}}{\left(\sqrt{\mu_k^2 + x^2} + \mu_k\right)^2}\right).$$

Substituting the right-hand side into $F_k(\mathrm{i}x)$, we see that

$$\begin{aligned}\zeta_k(s) =& \frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x\\ =& \frac{2\sin\pi s}{\pi}\left(\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\right)\mathrm{d}x+\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{\mu_k}{\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x\right.\\ &\left.+\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{x^2\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}}{\left(\sqrt{\mu_k^2+x^2}+\mu_k\right)^2}\right)\mathrm{d}x\right).\end{aligned}$$

By Lemma A.1, the first integral is equal to

$$\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}}{2}\right)\mathrm{d}x = R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}.$$

The second integral is not so easy to find, but can be done as follows:

$$\begin{aligned}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{\mu_k}{\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x &= \int_0^\infty x^{-2s}\frac{-\frac{1}{2}\mu_k(\mu_k^2+x^2)^{-\frac{3}{2}}(2x)}{1+\frac{\mu_k}{\sqrt{\mu_k^2+x^2}}}\mathrm{d}x\\ &= -\mu_k\int_0^\infty x^{-2s+1}\frac{(\mu_k^2+x^2)^{-1}}{\sqrt{\mu_k^2+x^2}+\mu_k}\mathrm{d}x\\ &= -\mu_k^{-2s}\int_0^\infty x^{-2s+1}(1+x^2)^{-1}\cdot\frac{1}{\sqrt{1+x^2}+1}\mathrm{d}x \qquad (x\mapsto\mu_k x)\\ &= -\mu_k^{-2s}\int_0^\infty x^{-2s+1}(1+x^2)^{-1}\cdot\frac{\sqrt{1+x^2}-1}{x^2}\mathrm{d}x\\ &= -\mu_k^{-2s}\left(\int_0^\infty x^{-2s-1}\frac{1}{(1+x^2)^{\frac{1}{2}}}-\int_0^\infty x^{-2s-1}\frac{1}{1+x^2}\mathrm{d}x\right).\end{aligned}$$

Recall the formula (A.1):

$$\int_0^\infty x^{-2s+2a-1}\frac{1}{(1+x^2)^w}\mathrm{d}x = \frac{\Gamma(a-s)\Gamma(s-a+w)}{2\Gamma(w)}.$$

Thus,

$$\begin{aligned}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{\mu_k}{\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x &= -\mu_k^{-2s}\left(\frac{\Gamma(-s)\Gamma(s+\frac{1}{2})}{2\Gamma(\frac{1}{2})}-\frac{\Gamma(-s)\Gamma(s+1)}{2\Gamma(1)}\right)\\ &= -\frac{1}{2}\mu_k^{-2s}\left(\frac{\Gamma(-s)\Gamma(s+\frac{1}{2})}{\sqrt{\pi}}-\Gamma(-s)\Gamma(s+1)\right).\end{aligned}$$

By the Reflection Formula (A.2), we have

$$\Gamma(-s)\Gamma(s+1) = \Gamma(-s)\Gamma(1-(-s)) = \frac{\pi}{\sin\pi(-s)} = -\frac{\pi}{\sin\pi s}$$

so

$$\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{\mu_k}{\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x=-\mu_k^{-2s}\left(-\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s+1)}+1\right).$$

Therefore,

$$\zeta_k(s)=\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x=R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}-\mu_k^{-2s}\left(1-\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s+1)}\right)$$
$$+\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{x^2\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}}{\left(\sqrt{\mu_k^2+x^2}+\mu_k\right)^2}\right)\mathrm{d}x.$$

In conclusion, we have proved the following theorem (the meromorphy statements follow just as in Theorem 3.1):

Theorem 4.3. *Denoting by $\widetilde{ð}$ the restriction of ð to $\ker(D_Y)^\perp$ with the boundary conditions given by $\Pi_>,\Pi_<$, we have*

$$\zeta_{\widetilde{ð}^2}(s)=R\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}\zeta_{D_Y^2}\left(s-\frac{1}{2}\right)-\frac{1}{2}\left(1-\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s+1)}\right)\zeta_{D_Y^2}(s)+2\sum_{\mu_k>0}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}$$
$$\times\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{x^2\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}}{\left(\sqrt{\mu_k^2+x^2}+\mu_k\right)^2}\right)\mathrm{d}x.$$

In particular, $\zeta_{\widetilde{ð}^2}(s)$ has a meromorphic extension to the whole complex plane except for poles at $s=\frac{n-k}{2}$ where $k=0,1,2,3,\ldots$ and $\frac{n-k}{2}\notin\{0\}\cup-\mathbb{N}$. In particular, $\zeta_{\widetilde{ð}^2}(s)$ has the double poles at $s=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots$ and the simple poles at other places.

Using known values of $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ from [1, p. 259], we find that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\frac{1}{2}\left(1-\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s+1)}\right)=\log 2.$$

Therefore,

$$\zeta'_{\widetilde{ð}^2}(0)=\frac{R}{2\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{D_Y^2}\left(s-\frac{1}{2}\right)\right)-\log 2\zeta_{D_Y^2}(0)+2\sum_{\mu_k>0}\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{x^2\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}}{\left(\sqrt{\mu_k^2+x^2}+\mu_k\right)^2}\right)\mathrm{d}x$$
$$=\frac{R}{2\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{D_Y^2}\left(s-\frac{1}{2}\right)\right)-\log 2\zeta_{D_Y^2}(0).$$

Thus, we have

$$\mathrm{e}^{-\zeta'_{\widetilde{ð}^2}(0)}=\mathrm{e}^{CR}2^{\zeta_{D_Y^2}(0)},\qquad C=-\frac{1}{2\sqrt{\pi}}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{D_Y^2}\left(s-\frac{1}{2}\right)\right).$$

Combining this formula with Lemma 4.2, we get

Theorem 4.4. *The zeta function $\zeta_{ð^2_{\Pi_\sigma}}(s)$ has a meromorphic extension to the whole complex plane except for poles at $s=\frac{n-k}{2}$, where $k=0,1,2,3,\ldots$ and $\frac{n-k}{2}\notin\{0\}\cup-\mathbb{N}$. In particular, $\zeta_{ð^2_{\Pi_\sigma}}(s)$ has the double poles at $s=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots$ and the simple poles at other places. For the ζ-determinant of $ð^2_{\Pi_\sigma}$, we have*

$$\det{}_\zeta \eth^2_{\Pi_\sigma} = R^{2h} e^{CR} 2^{\zeta_{D_Y^2}(0)+h_Y} \det{}^* \left(\frac{2\mathrm{Id} - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4} \right),$$

where h *is the number of* $(+1)$*-eigenvalues of* $(\sigma_1\sigma_2)_-$, $h_Y = \dim\ker(D_Y)$ *and* $C = -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{D_Y^2}(s-\frac{1}{2})\right)$.

This theorem was proved in [53] using the method of adiabatic decomposition [23,60,61]. The contour integral method presented here to prove Theorem 4.4 is much simpler and more direct than the proof presented in [53]. For further results in a similar context see [43,54]. We also remark that the constant C in [53,54] was accidentally stated incorrectly as $C = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}(\Gamma(s)^{-1}\zeta_{D_Y^2}(s-1/2))$.

4.4. Chiral boundary conditions

We now consider our operator

$$\eth = G(\partial_u + D_Y) \quad \text{over} \quad [0,R]\times Y$$

with chiral Dirichlet boundary conditions. Over $\ker(D_Y)$, just as before, we fix involutions σ_1, σ_2 anticommuting with G and we put boundary conditions given by vanishing on the $+1$ eigenspaces of σ_1 and σ_2, respectively:

$$\frac{1+\sigma_1}{2}\Pi_0 \quad \text{at} \quad \{0\}\times Y, \qquad \frac{1+\sigma_2}{2}\Pi_0 \quad \text{at} \quad \{R\}\times Y.$$

Over $\ker(D_Y)^\perp$, we put chiral Dirichlet conditions:

For $\phi = \phi_+ + \phi_-$ with $\phi_\pm \in H^1([0,R]\times Y, S^\pm)$, we require $\phi_+(0) = 0$ and $\phi_-(R) = 0$. Note that, by Proposition 4.1, we *cannot* use these "chiral" boundary conditions over $\ker(D_Y)$. (In fact, one can check that these "chiral" conditions are not even elliptic boundary conditions over $\ker(D_Y)$!)

We denote by $\eth_{\mathcal{D},\sigma}$ the resulting operator with the above boundary conditions.

By Lemma 4.2, we know the zeta-determinant of $\eth^2_{\mathcal{D},\sigma}$ over V, so we shall now focus on $V^\perp$. As with the non-local APS conditions, let $\{\phi_k\}$ be an eigenbasis of the positive eigenvectors of D_Y, so that $\{G\phi_k\}$ is an eigenbasis of the negative eigenvectors of D_Y,

$$D_Y\phi_k = \mu_k\phi_k \quad \text{and} \quad D_Y G\phi_k = -\mu_k G\phi_k.$$

Let us fix a k, and define

$$\phi_k^+ := \phi_k - \mathrm{i}G\phi_k \in C^\infty(Y,S^+), \qquad \phi_k^- := \phi_k + \mathrm{i}G\phi_k \in C^\infty(Y,S^-).$$

Then observe that

$$G\phi_k^\pm = G\phi_k \mp \mathrm{i}G^2\phi_k = G\phi_k \pm \mathrm{i}\phi_k = \pm\mathrm{i}(\phi_k \mp \mathrm{i}G\phi_k) = \pm\mathrm{i}\phi_k^\pm$$

and

$$D_Y\phi_k^\pm = \mu_k\phi_k \mp \mathrm{i}(-\mu_k)G\phi_k = \mu_k(\phi_k \pm \mathrm{i}G\phi_k) = \mu_k\phi_k^\mp.$$

Therefore, with respect to the basis $\{\phi_k^+, \phi_k^-\}$, we can write

$$\eth = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\left(\partial_u + \begin{pmatrix} 0 & \mu_k \\ \mu_k & 0 \end{pmatrix}\right) = \begin{pmatrix} \mathrm{i}\partial_u & \mathrm{i}\mu_k \\ -\mathrm{i}\mu_k & -\mathrm{i}\partial_u \end{pmatrix}$$

over span$\{\phi_k, G\phi_k\}$. Hence, the eigenvalues of $\eth$ over span$\{\phi_k^+, \phi_k^-\}$ are obtained by solving

$$\begin{pmatrix} \mathrm{i}\partial_u & \mathrm{i}\mu_k \\ -\mathrm{i}\mu_k & -\mathrm{i}\partial_u \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix} \iff \begin{array}{c} \mathrm{i}f'(u) + \mathrm{i}\mu_k g(u) = \lambda f(u) \\ -\mathrm{i}\mu_k f(u) - \mathrm{i}g'(u) = \lambda g(u) \end{array}$$

with $f(0) = 0$ (from the boundary condition $\phi_+(0) = 0$) and $g(R) = 0$ (from the boundary condition $\phi_-(R) = 0$). To solve these equations, apply $\mathrm{i}\partial_u$ to the first line $\mathrm{i}f'(u) + \mathrm{i}\mu_k g(u) = \lambda f(u)$ and multiply $\mathrm{i}\mu_k$ to the second line $-\mathrm{i}\mu_k f(u) - \mathrm{i}g'(u) = \lambda g(u)$, then add, and then use the first line to get

$$-f''(u) + \mu_k^2 f(u) = \lambda(\mathrm{i}f'(u) + \mathrm{i}\mu_k g(u)) \Rightarrow -f''(u) + \mu_k^2 f(u) = \lambda^2 f(u).$$

Solving this equation for $f(u)$ and using that $f(0) = 0$, we see that

$$f(u) = \sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) \quad \text{(modulo a constant).}$$

Now the equation $\mathrm{i}f'(u) + \mathrm{i}\mu_k g(u) = \lambda f(u)$ implies that

$$\mathrm{i}\sqrt{\mu_k^2 - \lambda^2}\cosh\left(\sqrt{\mu_k^2 - \lambda^2}u\right) + \mathrm{i}\mu_k g(u) = \lambda \sinh\left(\sqrt{\mu_k^2 - \lambda^2}u\right).$$

Substituting $u = R$, using that $g(R) = 0$, and then squaring both sides, we see that

$$-(\mu_k^2 - \lambda^2)\cosh^2\left(\sqrt{\mu_k^2 - \lambda^2}R\right) = \lambda^2 \sinh^2\left(\sqrt{\mu_k^2 - \lambda^2}R\right).$$

Using that $\cosh^2 z = 1 + \sinh^2 z$, we obtain

$$-(\mu_k^2 - \lambda^2) - \mu_k^2 \sinh^2\left(\sqrt{\mu_k^2 - \lambda^2}R\right) = 0.$$

It is easily checked that λ cannot equal $\pm\mu_k$, therefore we conclude that

$$\lambda \text{ or } -\lambda \text{ is an eigenvalue of } \eth \text{ over span}\{\phi_k^+, \phi_k^-\} \iff \lambda^2 \text{ is an eigenvalue of } \eth^2$$

$$\text{over span}\{\phi_k^+, \phi_k^-\} \iff F_k(\lambda) := 1 + \left(\frac{\mu_k \sinh(\sqrt{\mu_k^2 - \lambda^2}R)}{\sqrt{\mu_k^2 - \lambda^2}}\right)^2 = 0.$$

As should be obvious by now, to evaluate the zeta function of $\eth^2$ over span$\{\phi_k^+, \phi_k^-\}$:

$$\zeta_k(s) = \frac{2\sin \pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x$$

we need to find $F_k(\mathrm{i}x)$. To do so, observe that

$$
\begin{aligned}
F_k(\mathrm{i}x) =& 1 + \frac{\mu_k^2}{\mu_k^2 + x^2}\sinh^2\left(\sqrt{\mu_k^2 + x^2}R\right) \\
=& 1 + \frac{\mu_k^2}{\mu_k^2 + x^2}\left(\frac{\mathrm{e}^{R\sqrt{\mu_k^2+x^2}} - \mathrm{e}^{-R\sqrt{\mu_k^2+x^2}}}{2}\right)^2 \\
=& 1 + \frac{\mu_k^2}{4(\mu_k^2 + x^2)}\left(\mathrm{e}^{2R\sqrt{\mu_k^2+x^2}} - 2 + \mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}}\right) \\
=& \frac{\mu_k^2 \mathrm{e}^{2R\sqrt{\mu_k^2+x^2}}}{4(\mu_k^2 + x^2)}\left(1 + \left(2 + \frac{4x^2}{\mu_k^2}\right)\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}} + \mathrm{e}^{-4R\sqrt{\mu_k^2+x^2}}\right).
\end{aligned}
$$

Substituting this formula for $F_k(\mathrm{i}x)$ into $\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log F_k(\mathrm{i}x)\,\mathrm{d}x$, we see that

$$
\begin{aligned}
\zeta_k(s) = & \frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mathrm{e}^{2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x - \frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mu_k^2 + x^2\right)\mathrm{d}x \\
& + \frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1 + \left(2 + \frac{4x^2}{\mu_k^2}\right)\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}} + \mathrm{e}^{-4R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.
\end{aligned}
$$

By Lemma A.1, the first integral is equal to

$$
\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mathrm{e}^{2R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x = 2R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}
$$

and the second integral is equal to

$$
\frac{2\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mu_k^2 + x^2\right)\mathrm{d}x = 2\mu_k^{-2s}.
$$

Summing over all k, we have proved the following theorem (the meromorphy statements follow just as in Theorem 3.1):

Theorem 4.5. *Denoting by* $\widetilde{\eth}_{\mathcal{D}}$ *the restriction of* $\eth_{\mathcal{D},\sigma}$ *to* $\ker(D_Y)^\perp$ *with the chiral Dirichlet conditions, we have*

$$
\begin{aligned}
\zeta_{\widetilde{\eth}_{\mathcal{D}}^2}(s) = & R\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\zeta_{D_Y^2}\left(s - \frac{1}{2}\right) - \zeta_{D_Y^2}(s) \\
& + 2\sum_{\mu_k>0}\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1 + \left(2 + \frac{4x^2}{\mu_k^2}\right)\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}} + \mathrm{e}^{-4R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x.
\end{aligned}
$$

In particular, $\zeta_{\widetilde{\eth}_{\mathcal{D}}^2}(s)$ *has a meromorphic extension to the whole complex plane except for simple poles at* $s = \frac{n-k}{2}$ *where* $k = 0, 1, 2, \ldots$ *and* $\frac{n-k}{2} \notin \{0\} \cup -\mathbb{N}$.

Therefore,

$$
\begin{aligned}
\zeta'_{\widetilde{\eth}_{\mathcal{D}}^2}(0) = & \frac{R}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{D_Y^2}\left(s - \frac{1}{2}\right)\right) - \zeta'_{D_Y^2}(0) \\
& + 2\sum_{\mu_k>0}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\log\left(1 + \left(2 + \frac{4x^2}{\mu_k^2}\right)\mathrm{e}^{-2R\sqrt{\mu_k^2+x^2}} + \mathrm{e}^{-4R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x \\
= & -2CR - \zeta'_{D_Y^2}(0) - 2\sum_{\mu_k>0}\log\left(1 + 2\mathrm{e}^{-2R\mu_k} + \mathrm{e}^{-4R\mu_k}\right) \\
= & -2CR - \zeta'_{D_Y^2}(0) - 2\sum_{\mu_k>0}\log\left(1 + \mathrm{e}^{-2R\mu_k}\right)^2,
\end{aligned}
$$

where $C = -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{D_Y^2}(s-\frac{1}{2})\right)$. Thus, we have

$$e^{-\zeta'_{\tilde{\eth}^2}(0)} = \det D_Y^2 e^{2CR} \prod_{\mu_k>0} \left(1 + e^{-2R\mu_k}\right)^4.$$

Combining this formula with Lemma 4.2, we get

Theorem 4.6. *The zeta function $\zeta_{\eth^2_{\mathcal{D},\sigma}}(s)$ has a meromorphic extension to the whole complex plane except for simple poles at $s = \frac{n-k}{2}$, where $k = 0, 1, 2, \ldots$ and $\frac{n-k}{2} \notin \{0\} \cup -\mathbb{N}$ and*

$$\det{}_\zeta \eth^2_{\mathcal{D},\sigma} = R^{2h} e^{2CR} 2^{h_Y} \det D_Y^2 \prod_{\mu_k>0} \left(1 + e^{-2R\mu_k}\right)^4 \cdot \det{}^* \left(\frac{2\mathrm{Id} - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4}\right),$$

where h is the number of $(+1)$-eigenvalues of $(\sigma_1\sigma_2)_-$, $h_Y = \dim\ker(D_Y)$ and $C = -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\left(\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_{D_Y^2}(s-\frac{1}{2})\right)$.

5. Conclusions

In this article, we have performed an analysis of zeta functions of Laplace and Dirac-type operators over finite cylinders $M = [0, R] \times Y$, where different boundary conditions have been imposed at the boundary. The general structure of the results is that answers for operators over M are expressed in terms of data coming solely from Y. As long as the manifold Y is not specified this is the best that can be achieved.

All problems considered are solved in a unified framework, in which the starting point is a contour integral representation of the zeta function involving an implicit eigenvalue equation for the Laplacian or the Dirac operator on M. The method uses the Argument Principle, or Cauchy's formula, and the analysis proceeds along very natural and elegant lines. It is clearly not restricted to the present setting, but whenever eigenvalues are solutions of implicit equations, this strategy can be applied.

Appendix A. Some simple formulas

Lemma A.1. *We have*

$$\frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left(e^{R\sqrt{\mu_k^2+x^2}}\right) \mathrm{d}x = R\mu_k^{-2s+1} \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}$$

and

$$\frac{\sin \pi s}{\pi} \int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x} \log\left((\mu_k^2 + x^2)^a\right) \mathrm{d}x = a\mu_k^{-2s}.$$

Proof. Without the factor $\frac{\sin \pi s}{\pi}$, the first integral is equal to

$$\int_0^\infty x^{-2s} \frac{\mathrm{d}}{\mathrm{d}x}\left(R\sqrt{\mu_k^2+x^2}\right) \mathrm{d}x = R\int_0^\infty x^{-2s} \frac{x}{\sqrt{\mu_k^2+x^2}} \mathrm{d}x = R\int_0^\infty x^{-2s+1} \frac{1}{(\mu_k^2+x^2)^{\frac{1}{2}}} \mathrm{d}x$$
$$= R\mu_k^{-2s+1} \int_0^\infty x^{-2s+1} \frac{1}{(1+x^2)^{\frac{1}{2}}} \mathrm{d}x.$$

Modifying the well-known formula for the Beta function (cf. [1, p. 258])

$$\int_0^\infty x^{-2s+2a-1}\frac{1}{(1+x^2)^w}\,\mathrm{d}x = \frac{\Gamma(a-s)\Gamma(s-a+w)}{2\Gamma(w)} \tag{A.1}$$

and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [1, p. 255], we see that

$$\int_0^\infty x^{-2s+1}\frac{1}{(1+x^2)^{\frac{1}{2}}}\mathrm{d}x = \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\Gamma(\frac{1}{2})} = \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}}.$$

Therefore,

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x = R\mu_k^{-2s+1}\frac{\sin\pi s}{\pi}\frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}}.$$

By the Reflection Formula [1, p. 256]

$$\frac{\pi}{\sin\pi s} = \Gamma(s)\Gamma(1-s) \Rightarrow \frac{\sin\pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)} \tag{A.2}$$

we see that

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mathrm{e}^{R\sqrt{\mu_k^2+x^2}}\right)\mathrm{d}x = R\mu_k^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}$$

as claimed.

Now the second integral is equal to

$$\begin{aligned} a\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\mu_k^2+x^2\right)\mathrm{d}x &= 2a\int_0^\infty x^{-2s+1}\frac{1}{\mu_k^2+x^2}\,\mathrm{d}x \\ &= 2a\mu_k^{-2s}\int_0^\infty x^{-2s+1}\frac{1}{1+x^2}\,\mathrm{d}x. \end{aligned}$$

By (A.1), we see that

$$\int_0^\infty x^{-2s+1}\frac{1}{(1+x^2)^1}\,\mathrm{d}x = \frac{\Gamma(1-s)\Gamma(s)}{2\Gamma(1)} = \frac{\Gamma(1-s)\Gamma(s)}{2}.$$

Therefore, by the Reflection Formula,

$$\frac{\sin\pi s}{\pi}\int_0^\infty x^{-2s}\frac{\mathrm{d}}{\mathrm{d}x}\log\left((\mu_k^2+x^2)^a\right)\mathrm{d}x = 2a\mu_k^{-2s}\frac{\sin\pi s}{\pi}\frac{\Gamma(1-s)\Gamma(s)}{2} = a\mu_k^{-2s}.$$

This proves the second integral and completes our proof. □

Note added in proof

Results related to Theorems 3.1 and 3.2 have also been obtained by Dowker and Apps in [73,74].

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