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Zeta regularized determinants for conic manifolds

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Abstract

We study (relative) zeta regularized determinants of Laplace type operators on compact conic manifolds. We establish gluing formulae for relative zeta regularized determinants. For arbitrary self-adjoint extensions of the Laplace–Beltrami operator, we express the relative ζ-determinants for these as a ratio of the determinants of certain finite matrices. For the self-adjoint extensions corresponding to Dirichlet and Neumann conditions, the formula is particularly simple and elegant.

Keywords: Zeta regularized determinants; Conic manifolds; Gluing formulae

1. Introduction

In this paper we study relative zeta regularized determinants of second order regular singular differential operators generalizing Laplace type operators on conic manifolds. Of particular interest is the Laplace–Beltrami operator on a conic manifold and its self-adjoint extensions corresponding to Dirichlet and Neumann conditions. Our main result, Theorem 1.1, gives an explicit formula for the relative ζ-determinants of these self-adjoint extensions of Laplacians in terms of the determinants of certain finite matrices. To concisely state our results, we recall the prerequisite material as it was developed, much of which we later adapt to the context in which we work.

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Throughout this discussion, let M be an n-dimensional connected Riemannian manifold and let $\Delta^{(k)}$ be the Laplace–Beltrami operator acting on k-forms.

When M is compact without boundary, the operator $\Delta^{(k)}$ extends uniquely to a self-adjoint operator acting on sections of the associated bundle of L^2-forms. In addition, the heat operator is trace class. Under these conditions one can associate to $\Delta^{(k)}$ a zeta function which, for $\Re(s) > n/2$, is given by

$$\zeta\big(s, \Delta^{(k)}\big) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}\big(e^{-t\Delta^{(k)}} P_k\big)\, dt, \tag{1.1}$$

where Tr denotes trace and P_k is projection on the orthogonal complement of the null space of $\Delta^{(k)}$. Fundamental results of Seeley [44] imply that the $\zeta(s, \Delta^{(k)})$ extends to a meromorphic function on the complex plane which is regular at zero. Thus, following [42], one can associate to $\Delta^{(k)}$ a zeta regularized determinant

$$\det_\zeta\big(\Delta^{(k)}\big) = e^{-\zeta'(0,\Delta^{(k)})}. \tag{1.2}$$

Introduced to provide an analytic counterpart to an important combinatorial invariant (Reidemeister torsion), it was soon realized that the theory of ζ-determinants could be extended to provide powerful tools in a variety of contexts. The first such extension was to compact manifolds with boundary.

For compact manifolds with boundary, the Laplace–Beltrami operator is no longer essentially self-adjoint. Among self-adjoint extensions, there are two geometrically natural choices: the Dirichlet extension and the Neumann extension. These Dirichlet and Neumann conditions form a pair of complementary boundary conditions. For any such pair it is possible to establish a gluing formula for ζ-determinants of Laplace type operators. More precisely, suppose that M is a closed manifold and that L is an elliptic differential operator on M. Suppose that Γ is a closed codimension one submanifold of M and let M_Γ be the compact manifold with boundary obtained by cutting M along Γ and gluing copies of Γ to the cut. Suppose that B and B' are complementary boundary conditions and the boundary value problem determined by L and B is invertible and admits a principal angle. It is then a theorem of Burghelea, Friedlander and Kappeler [9] that the ζ-determinants of L and L_B satisfy a gluing formula:

$$\frac{\det_\zeta(L)}{\det_\zeta(L_B)} = C \det_\zeta(\mathcal{R}),$$

where the constant C is independent of perturbations of L, B and B' by differential operators of sufficiently small order and the BFK operator $\mathcal{R}$ (the composition of the Poisson operator for L_B and the boundary condition B') is pseudodifferential on Γ (cf. Section 4). Such gluing formulae make it possible to use ζ-determinants to address problems arising in differential topology via “cut and paste” arguments.

When the base manifold M is no longer compact, serious complications arise in the corresponding analysis and there are a number of obstructions to obtaining good results. Among these obstructions is the fact that it is often the case that the associated Laplace operators are not essentially self-adjoint, and there is no canonical choice for which extension should be chosen. Moreover, given an extension, it is rarely the case that the corresponding zeta function is defined and holomorphic at zero, and the required estimates for the behavior of the heat kernel which would facilitate the appropriate regularization are, in general, difficult. In an attempt to address such problems for a large class of interesting examples, Müller [35] (cf. [30]) observed that, in certain circumstances, there may be a corresponding relative theory of determinants associated to natural pairs of operators (e.g., the standard Laplace operator on $\mathbb{R}^n$ and the Laplace operator arising from a compact perturbation of the Euclidean metric). This approach was further developed by Carron [11] to noncompact cases and has led to the relative ζ-determinant formula of Dirac Laplacians with two boundary conditions [43], as well as results for a variety of examples [35].

For manifolds which lack compactness, but for which there is uniform structure at infinity, a good deal is known. Included in this class of results is the early work of Hassell involving the behavior of analytic torsion under analytic surgery [23], and the work of Hassell and Zelditch [24] which provides an analysis of determinants of Laplacians on exterior Euclidean domains. These papers involve a regularization of the heat kernel due to Melrose; the so-called b-heat trace [33]. The theory associated to the b-heat trace has since undergone extensive development and application (cf. [31,41] and references therein).

More recently, Loya and Park (motivated by earlier work of Park and Wojciechowski on the adiabatic decomposition of ζ-determinants [37–40]), have adapted b-trace techniques and the gluing argument of Burghelea, Kappeler and Friedlander to study the decomposition of ζ-determinants for Laplacians on manifolds with cylindrical ends [31]; see Müller and Müller [36] for related work.

The present work exploits a similar circle of ideas to analyze determinants of second order regular singular operators [7] generalizing Laplace type operators on compact conic manifolds. More precisely, suppose that M is a compact Riemannian manifold with boundary. Suppose that Γ is a closed manifold and (referring details to Section 2) that $[0,2]_r \times \Gamma$ is a collar of $\Gamma := \partial M$ over which the metric is of product type $dr^2 + h$ with h a metric on Γ. Let $\Delta : C_c^\infty(M \setminus \Gamma, E) \to C_c^\infty(M \setminus \Gamma, E)$ be a symmetric nonnegative second order differential operator such that over the collar $[0,2]_r \times \Gamma$, Δ takes the form

$$\Delta = -\partial_r^2 + r^{-2} A, \tag{1.3}$$

where A is a Laplace-type operator over Γ satisfying $A \geqslant -1/4$. In Section 2 we explain how the Laplace–Beltrami operator on a conic manifold can be transformed into such an operator. Suppose that M decomposes as

$$M = X \cup Y, \tag{1.4}$$

where $X = [0,1]_r \times \Gamma$ and Y is a compact manifold admitting a collar neighborhood $[1,2]_r \times \Gamma$ of its boundary (we identify the boundary of Y with Γ). We also assume that the induced Dirichlet Laplacian on Y, Δ_Y, is invertible. The main example of an operator Δ satisfying the above hypotheses is the Laplace–Beltrami operator acting on forms over a conic manifold (see Section 2 and [7,13,34]). The self-adjoint extensions of Δ are parameterized by Lagrangian subspaces Λ of an associated finite-dimensional symplectic vector space V (see below and Section 2). Among these Λ, we consider the self-adjoint extensions denoted by D, N, which correspond to the Dirichlet, Neumann conditions, and we denote the resulting self-adjoint extensions of Δ by Δ_{D}, Δ_{N}. For this pair, the relative zeta function $\zeta(s, \Delta_{\mathrm{D}}, \Delta_{\mathrm{N}})$, which is defined as in (1.1) using a relative trace (cf. (3.3)), is regular at $s=0$ (see Theorem 3.2). Hence we can define the relative determinant for $(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}})$ by

$$\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}}) = \exp\bigl(-\zeta'(0, \Delta_{\mathrm{D}}, \Delta_{\mathrm{N}})\bigr).$$

Our main result is a formula for this relative determinant in terms of determinants of certain (finite) matrices. To describe these matrices, we proceed as follows. Define an operator

$$L_{\mathrm{D}} := \sum_j \langle\cdot, v_j\rangle v_j : (\ker \Delta_{\mathrm{D}})\big|_{r=1} \to (\ker \Delta_{\mathrm{D}})\big|_{r=1}, \tag{1.5}$$

where $v_j := u_j|_{r=1}$ with $\{u_j\}$ is an orthonormal basis for the kernel of Δ_{D}. By Theorem 4.5, the sections $\{v_j\}$ are linearly independent in $L^2(\Gamma, E_\Gamma)$, where $E_\Gamma := E|_\Gamma$, so that L_{D} is a positive linear operator on the finite-dimensional vector space $(\ker \Delta_{\mathrm{D}})|_{r=1}$. Substituting Neumann for Dirichlet, define an operator L_{N} similarly.

With $A \geqslant -1/4$ as in (1.3), we will write $\nu_\ell := \sqrt{\lambda_\ell + 1/4} \geqslant 0$, where λ_ℓ are the eigenvalues of A. We denote by Π_V the orthogonal projection onto $V := \bigoplus_{-1/4 \leqslant \lambda_\ell < 3/4} E_\ell$, where E_ℓ is the eigenspace of λ_ℓ. Recalling our assumption that Δ_Y is invertible, we set $\mathcal{N}_{Y,V} := \Pi_V \mathcal{N}_Y \Pi_V$, where $\mathcal{N}_Y$ is the Dirichlet-to-Neumann map for the restriction of Δ to Y. We define a linear map $\mathcal{M}_\nu$ on V by its action on eigenspaces: $\mathcal{M}_\nu := \nu_\ell|_{E_\ell}$. Finally, we say that $\mathcal{N}_Y$ is of *clean type* if $\mathcal{N}_Y$ maps either V or $V^\perp$ into itself. With these conventions, our main result is the following:

Theorem 1.1. *When Δ_Y is invertible and $\mathcal{N}_Y$ is of clean type, the following relative determinant formula holds*:

$$\det{}_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}}) = \prod_{0<\nu_\ell<1} 2^{-2\nu_\ell} \frac{\Gamma(1-\nu_\ell)}{\Gamma(1+\nu_\ell)} \cdot \frac{\det L_{\mathrm{N}}}{\det L_{\mathrm{D}}} \cdot \frac{\det^*(\frac{\mathrm{Id}}{2} + \mathcal{M}_\nu + \mathcal{N}_{Y,V})}{\det^*(\frac{\mathrm{Id}}{2} - \mathcal{M}_\nu + \mathcal{N}_{Y,V})},$$

where $\det^*$ *denotes the determinant over the orthogonal complement of the kernel of the matrix.*

Remark 1.2. There are direct analogs of Theorem 1.1 for a larger class of self-adjoint extensions (extensions of mixed D and N type; cf. Theorem 5.6). There are also extensions of Theorem 1.1 valid for *arbitrary* self-adjoint extensions. The formulae in this case, however, are quite different in form from those appearing in Theorems 1.1 and 5.6 (cf. Theorem 5.10).

Theorem 1.1 means that the relative determinant $\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}})$ depends on data near the cone point given by the ν_ℓ's as well as data of the whole manifold M via L_{D}, L_{N} and $\mathcal{N}_{Y,V}$. We expect this result to have a number of interesting applications; for example, to the study of analytic torsion for conic manifolds.

To prove our result, we adapt much of the machinery referenced above to the context of conic manifolds. We proceed as follows.

In Section 2, we review material involving the analysis of Laplace operators on metric cones, including the parameterization of self-adjoint extensions by Lagrangian subspaces of a symplectic vector space associated to forms which are formally harmonic near the singularity. In Section 3 we define relative zeta functions for pairs of self-adjoint extensions of the Laplace operator and investigate their regularity properties (cf. Theorems 3.1 and 3.2). In Section 4 we develop the gluing formulae we require for our results. More precisely, given a decomposition of M as in (1.4), and a self-adjoint extension Δ_Λ, we define, for $\mu \geqslant 0$ a BFK-operator, $\mathcal{R}_\Lambda(\mu)$, for $\Delta_\Lambda + \mu$. Using the variational argument of [9] and the model problem approach used in [31,32], we establish a gluing formula for relative determinants (Theorem 4.2), a result of independent interest. In Section 5 we specialize to Dirichlet and Neumann boundary conditions, use our gluing formula and an analysis of associated one-dimensional model problems to complete the proof of our main theorem. We also present a relative determinant formula involving self-adjoint extensions of mixed D and N type and also for arbitrary self-adjoint extensions.

2. Self-adjoint extensions

In this section we introduce the notation and parameterizations we will use throughout the remainder of the paper.

2.1. Conic manifolds

A (connected) conic manifold M is a compact connected metric space with a distinguished subset $\Sigma \subset M$ whose elements are called singular points, satisfying:

(1) Σ is a collection of isolated points,
(2) $M \setminus \Sigma$ is a Riemannian manifold,
(3) for each $p \in \Sigma$, there is a neighborhood $U \ni p$ and an isometry

$$I : U \to (0,2]_r \times \Gamma \tag{2.1}$$

with metric

$$g = dr^2 + r^2 h,$$

where Γ is a compact manifold without boundary and h is a metric on Γ.

We refer to the product in (2.1) as the metric cone at p and we write

$$C_p = [0,2] \times \Gamma;$$

see Fig. 1. We denote by $\Delta^{(k)}$ the Laplace–Beltrami operator acting on k-forms with compact support. Our immediate interest is an investigation of the asymptotics of the trace of extensions of the corresponding heat operators. It is a theorem of Cheeger [13] that the effect of the singular set on the asymptotics of the heat trace can be localized to the metric cones C_p. Thus, for our purposes it suffices to assume throughout the article that the singular set consists of a point.

For k away from the middle dimension, $|k - n/2| \geqslant 2$, $\Delta^{(k)}$ is essentially self-adjoint and thus admits a unique self-adjoint extension which acts on the corresponding collection of L^2-forms. This extension coincides with the graph closure of $\Delta^{(k)}$ in L^2, the so-called minimal extension of $\Delta^{(k)}$, the domain of which we denote by $D_{\min}$. When $|k - n/2| < 2$, the Laplace–Beltrami operator is no longer essentially self-adjoint, and it becomes necessary to choose a domain for

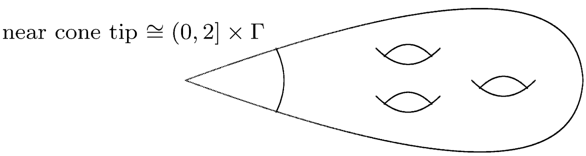

Fig. 1. A conic manifold.

the self-adjoint extension. Any such choice must include the domain of the minimal extension and can be no larger than the domain of the maximal extension which is given by

$$D_{\max} = \{ f \in L^2\colon \forall g \in D_{\min},\ \exists h \in L^2\colon \langle f, \Delta^{(k)} g \rangle = \langle h, g \rangle \}. \tag{2.2}$$

The collection of possible extensions is naturally parameterized by the behavior of forms which are formally harmonic at Σ.

2.2. The Laplace–Beltrami operator near the cone tip

We now describe the Laplacian near the cone tip. Although what we say now seems to be "folklore," we cannot find the details spelled out explicitly in any published source, so we shall outline the details of this important description. We begin with a rescaling trick (cf. Cheeger [12], Brüning and Seeley [7]). There is a natural isomorphism between the space consisting of k-forms on $C_p \setminus \Sigma$ and $C^\infty((0,2]; \Omega^k(\Gamma) \oplus \Omega^{k-1}(\Gamma))$ defined by writing $\phi \in C^\infty(C_p \setminus \Sigma, \Omega^k)$ as

$$\phi = r^{k + \frac{1-n}{2}} \phi_k + r^{k-1+\frac{1-n}{2}} dr \wedge \phi_{k-1}, \tag{2.3}$$

where $\phi_j \in C^\infty((0,2]; \Omega^j(\Gamma))$. (Brüning and Seeley [7, p. 370] put dr to the right of ϕ_{k-1}. This will give a slightly different formula in (2.5); their formula has factors of $(-1)^{k+1}$ in the off diagonal terms.) Given $\phi, \psi \in C_c^\infty(C_p \setminus \Sigma, \Omega^k)$ written using the isomorphism (2.3), it is automatic that

$$\langle \phi, \psi \rangle_g = \int_0^2 \bigl(\langle \phi_k(r), \psi_k(r) \rangle_h + \langle \phi_{k-1}(r), \psi_{k-1}(r) \rangle_h \bigr)\, dr.$$

It follows that (2.3) defines a unitary rescaling map from the bundle of L^2 k-forms on C_p with the cone metric to $L^2((0,2]; \Omega^k(\Gamma) \oplus \Omega^{k-1}(\Gamma))$ with the *usual* metric. With respect to (2.3), the exterior differential d with domain $C^\infty((0,2]; \Omega^k(\Gamma) \oplus \Omega^{k-1}(\Gamma))$ can be written as

$$d = \begin{pmatrix} \frac{1}{r} d_\Gamma & 0 \\ \partial_r + \frac{1}{r}(k + \frac{1-n}{2}) & -\frac{1}{r} d_\Gamma \end{pmatrix},$$

where d_Γ is the exterior differential on Γ. Note that the factors of $\frac{1}{r}$ in front of the d_Γ's and the term $\frac{1}{r}(k + \frac{1-n}{2})$ arise from the dependence of the powers of r in (2.3) on the form degrees. Taking the adjoint of d, we see that d^* with domain $C^\infty((0,2]; \Omega^{k+1}(\Gamma) \oplus \Omega^k(\Gamma))$ with the usual metric can be written as

$$d^* = \begin{pmatrix} \frac{1}{r} d_\Gamma^* & -\partial_r + \frac{1}{r}(k + \frac{1-n}{2}) \\ 0 & -\frac{1}{r} d_\Gamma^* \end{pmatrix}.$$

A short computation shows that over $C^\infty((0,2]; \Omega^k(\Gamma) \oplus \Omega^{k-1}(\Gamma))$, $d^* d$ equals

$$\begin{pmatrix} \frac{d_\Gamma^* d_\Gamma}{r^2} - \partial_r^2 + \frac{1}{r^2}(k + \frac{1-n}{2})(k + 1 + \frac{1-n}{2}) & -\frac{d_\Gamma}{r^2} + \frac{1}{r}\partial_r d_\Gamma - \frac{1}{r^2}(k + \frac{1-n}{2}) d_\Gamma \\ -\frac{1}{r}\partial_r d_\Gamma^* - \frac{1}{r^2}(k + \frac{1-n}{2}) d_\Gamma^* & \frac{d_\Gamma^* d_\Gamma}{r^2} \end{pmatrix}$$

and that dd^* equals

$$\begin{pmatrix} \frac{d_\Gamma d_\Gamma^*}{r^2} & -\frac{1}{r}\partial_r d_\Gamma + \frac{1}{r^2}(k-1+\frac{1-n}{2})d_\Gamma \\ -\frac{d_\Gamma^*}{r^2} + \frac{1}{r}\partial_r d_\Gamma^* + \frac{1}{r^2}(k-1+\frac{1-n}{2})d_\Gamma^* & \frac{d_\Gamma d_\Gamma^*}{r^2} - \partial_r^2 + \frac{1}{r^2}(k-1+\frac{1-n}{2})(k-2+\frac{1-n}{2}) \end{pmatrix}.$$

Adding d^*d and dd^*, we see that mapping $C^\infty((0,2];\Omega^k(\Gamma)\oplus\Omega^{k-1}(\Gamma))$ to itself, the Laplace operator takes the form

$$\Delta^{(k)} = -\partial_r^2 + r^{-2}A_k. \tag{2.4}$$

Here,

$$A_k = \begin{pmatrix} \Delta_\Gamma^{(k)} + (k+\frac{1-n}{2})(k+1+\frac{1-n}{2}) & -2d_\Gamma \\ -2d_\Gamma^* & \Delta_\Gamma^{(k-1)} + (k-1+\frac{1-n}{2})(k-2+\frac{1-n}{2}) \end{pmatrix}, \tag{2.5}$$

where $\Delta_\Gamma^{(k)}$ denotes the Laplace operator acting on k-forms over Γ.

Lemma 2.1. *We have*

$$A_k \geqslant -1/4,$$

where A_k is given in (2.5).

Proof. Let $\phi = \phi_k \oplus \phi_{k-1} \in C^\infty((0,2];\Omega^k(\Gamma)\oplus\Omega^{k-1}(\Gamma))$ and suppose that $A\phi = \lambda\phi$. By definition of A_k, $A_k\phi = \lambda\phi$ is equivalent to

$$\left(\Delta_\Gamma^{(k)} + \left(k+\frac{1-n}{2}\right)\left(k+1+\frac{1-n}{2}\right)\right)\phi_k - 2d_\Gamma\phi_{k-1} = \lambda\phi_k \quad \text{and}$$
$$\left(\Delta_\Gamma^{(k-1)} + \left(k-1+\frac{1-n}{2}\right)\left(k-2+\frac{1-n}{2}\right)\right)\phi_{k-1} - 2d_\Gamma^*\phi_k = \lambda\phi_{k-1}.$$

Applying d_Γ to the first of these equations (we will not need the second equation), we obtain

$$\left(\Delta_\Gamma^{(k+1)} + \left(k+\frac{1-n}{2}\right)\left(k+1+\frac{1-n}{2}\right)\right)(d_\Gamma\phi_k) = \lambda(d_\Gamma\phi_k),$$

where we used that $d_\Gamma\Delta_\Gamma^{(k)} = \Delta_\Gamma^{(k+1)}d_\Gamma$, since both sides equal $d_\Gamma d_\Gamma^* d_\Gamma$. We know that $\Delta_\Gamma^{(k+1)} \geqslant 0$, so

$$\lambda \geqslant \left(k+\frac{1-n}{2}\right)\left(k+1+\frac{1-n}{2}\right) \geqslant -\frac{1}{4},$$

because the function $f(x) = x(x+1)$ achieves its minimum when $x = -1/2$, with minimum value $-1/4$. $\square$

2.3. Self-adjoint extensions and the D and N extensions

We now generalize the above considerations to Brüning and Seeley's [7] category of *regular singular elliptic operators*. Let M be a compact Riemannian manifold with boundary having a collar neighborhood $[0,2]_r \times \Gamma$ of $\Gamma = \partial M$ where the metric is of product type $dr^2 + h$ with h a metric on Γ. Let $\Delta : C_c^\infty(M \setminus \Gamma, E) \to C_c^\infty(M \setminus \Gamma, E)$ be a symmetric second order differential operator such that over the collar $[0,2]_r \times \Gamma$, Δ takes the form

$$\Delta = -\partial_r^2 + r^{-2} A, \tag{2.6}$$

where $A : C^\infty(\Gamma, E_\Gamma) \to C^\infty(\Gamma, E_\Gamma)$ is a Laplace-type operator over Γ such that $A \geqslant -1/4$. As shown in (2.5) and Lemma 2.1, the Laplace–Beltrami operator over a conic manifold can be transformed to this category.

We now describe two natural self-adjoint extensions of Δ. Using the expression for the Laplacian given in (2.6), Cheeger [13] gives a description of the maximal domain on k-forms. More precisely, let us fix k and let

$$-\frac{1}{4} \leqslant \underbrace{\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_q}_{<3/4} < \underbrace{\lambda_{q+1} \leqslant \lambda_{q+2} \leqslant \lambda_{q+3} \leqslant \cdots}_{\geqslant 3/4} \tag{2.7}$$

be the eigenvalues of A with corresponding orthonormal eigenvectors $\{\phi_\ell\}$. Then (cf. Cheeger [13], Mooers [34, Proposition 2.3]), it is straightforward to prove that ϕ is in $D_{\max}$ if and only if ϕ is in H^2 away from the singular set and near the singular set, we can write

$$\phi = \sum_{-1/4 \leqslant \lambda_\ell < 3/4} \left\{ c_\ell^+(\phi) \psi_\ell^+ + c_\ell^-(\phi) \psi_\ell^- \right\} + o\left(r^{3/2}\right), \tag{2.8}$$

where setting $\nu_\ell := \sqrt{\lambda_\ell + \frac{1}{4}} \geqslant 0$, we have

$$\psi_\ell^+ := \begin{cases} \frac{1}{\sqrt{2\nu_\ell}} r^{\frac{1}{2}+\nu_\ell} \phi_\ell & \text{for } \nu_\ell > 0, \\ r^{\frac{1}{2}} \phi_\ell & \text{for } \nu_\ell = 0, \end{cases} \qquad \psi_\ell^- := \begin{cases} \frac{1}{\sqrt{2\nu_\ell}} r^{\frac{1}{2}-\nu_\ell} \phi_\ell & \text{for } \nu_\ell > 0, \\ r^{\frac{1}{2}} \log r \phi_\ell & \text{for } \nu_\ell = 0. \end{cases} \tag{2.9}$$

Moreover, an integration by parts argument shows that for $\phi, \psi \in D_{\max}$, we have

$$\langle \Delta\phi, \psi \rangle - \langle \phi, \Delta\psi \rangle = \sum_{-1/4 \leqslant \lambda_\ell < 3/4} a_\ell \left\{ c_\ell^+(\phi) \overline{c_\ell^-(\psi)} - c_\ell^-(\phi) \overline{c_\ell^+(\psi)} \right\}, \tag{2.10}$$

where $a_\ell = 1$ for $\nu_\ell > 0$ and $a_\ell = -1$ for $\nu_\ell = 0$. Following Mooers [34, p. 9], we can put this computation in a symplectic framework by considering the $2q$-dimensional complex vector space V spanned by the collection $\{\psi_\ell^\pm\}_{\ell=1}^q$:

$$V = \operatorname{span}\left\{\psi_\ell^\pm\right\}_{\ell=1}^q \tag{2.11}$$

and defining an Hermitian symplectic structure $\omega : V \times V \to \mathbb{C}$ by setting

$$\omega(\psi_i^{\pm}, \psi_i^{\mp}) = \begin{cases} \pm 1 & \text{when } \nu_i \neq 0, \\ \mp 1 & \text{when } \nu_i = 0; \end{cases}$$
$$\omega(\psi_i^{\pm}, \psi_j^{\pm}) = 0, \quad \text{otherwise} \tag{2.12}$$

and extending to $V \times V$ linearly in the first factor and conjugate linear in the second factor. Then it follows that self-adjoint realizations of Δ are in one-to-one correspondence with Lagrangian subspaces of V with the Hermitian symplectic structure ω. Explicitly, given a Lagrangian subspace $\Lambda \subset V$, near $r = 0$ elements in the domain for the Λ extension have the form

$$\phi = \sum_{-1/4 \leqslant \lambda_\ell < 3/4} \{c_\ell^+(\phi)\psi_\ell^+ + c_\ell^-(\phi)\psi_\ell^-\} + o(r^{3/2}) \quad (\Lambda \text{ extension}),$$

where

$$\sum_{-1/4 \leqslant \lambda_\ell < 3/4} \{c_\ell^+(\phi)\psi_\ell^+ + c_\ell^-(\phi)\psi_\ell^-\} \in \Lambda. \tag{2.13}$$

The Friedrichs extension, which we shall call the "D extension," is the extension obtained by choosing all the c_ℓ^-'s in (2.8) to vanish. The "D" stands for Dirichlet because in the special case that all the eigenvalues $\lambda_\ell < 3/4$ equal zero, we have

$$\phi = \sum_{\lambda_\ell = 0} \{c_\ell^+(\phi) r \phi_\ell + c_\ell^-(\phi)\phi_\ell\} + o(r^{3/2}),$$

and we see that ϕ satisfies the Dirichlet condition at $r = 0$ if and only if $c_\ell^-(\phi) = 0$ for all ℓ. In the case all the $\lambda_\ell < 3/4$ equal zero, observe that the Neumann boundary condition is obtained by choosing all the $c_\ell^+(\phi)$'s to vanish. In the general case we call the "N extension" the extension obtained by choosing all the $c_\ell^+(\phi)$'s to vanish in (2.8) *except* when $\nu_\ell = 0$, where we require $c_\ell^-(\phi) = 0$ just as for the D extension. In summary, near $r = 0$, elements in the domain for the D extension have the form

$$\phi = \sum_{-1/4 \leqslant \lambda_\ell < 3/4} c_\ell(\phi) r^{\frac{1}{2} + \nu_\ell} \phi_\ell + o(r^{\frac{3}{2}}) \quad (D \text{ extension}),$$

while elements in the domain for the N extension have the form

$$\phi = \sum_{-1/4 \leqslant \lambda_\ell < 3/4} c_\ell(\phi) r^{\frac{1}{2} - \nu_\ell} \phi_\ell + o(r^{\frac{3}{2}}) \quad (N \text{ extension}).$$

In terms of Lagrangian subspaces of V, we have

$$D = \operatorname{span}\{\psi_\ell^+\}, \qquad N = \operatorname{span}\{\psi_\ell^- \text{ for } \nu_\ell \neq 0,\ \psi_\ell^+ \text{ for } \nu_\ell = 0\}. \tag{2.14}$$

We remark that the D and N extensions are the two canonical scale invariant domains of Δ, where scale invariant means that they are invariant under the scaling $r \mapsto cr$ for $c > 0$; for more on self-adjoint extensions, see Gil and Mendoza [20].

3. Usual poles of relative zeta functions

Let $\Lambda \subset V$ be a Lagrangian subspace of the symplectic vector space V defined by (2.8)–(2.12) above. For simplicity, we assume that Λ decomposes "diagonally" as

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_q, \tag{3.1}$$

where Λ_ℓ is a Lagrangian subspace of the two-dimensional space spanned by $\{\psi_\ell^\pm\}$ with respect to the induced symplectic structure such that when $\lambda_\ell = -\frac{1}{4}$, $\Lambda_\ell = \mathrm{span}\{\psi_\ell^+\}$ (both the D and N extensions have this property). Then Λ determines a self-adjoint extension of Δ, which we denote by Δ_Λ. The main result of this section is the following theorem.

Theorem 3.1. *As $t \to 0$,*

$$\mathrm{Tr}\big(e^{-t\Delta_\Lambda}\big) \sim \sum_{\lambda_\ell < 3/4} \sum_{j=0}^{\infty} a_{\Lambda,j,\ell} t^{j\sqrt{\lambda_\ell + 1/4}} + \sum_{j=0}^{\infty} a_j t^{\frac{j-n}{2}} + b \log t,$$

where the coefficients a_j, $j = 0, 1, 2, \ldots,$ and b are independent of the choice of self-adjoint extension, and b depends only on the operator A.

Before proceeding to the proof of Theorem 3.1, we note that, as a corollary, we obtain a theory of relative zeta functions. More precisely, let $\Lambda_1, \Lambda_2 \subset V$ be Lagrangian subspaces that decompose as in (3.1). Then by Theorem 3.1, as $t \to 0$,

$$\mathrm{Tr}\big(e^{-t\Delta_{\Lambda_1}}\big) - \mathrm{Tr}\big(e^{-t\Delta_{\Lambda_2}}\big) \sim \sum_{\lambda_\ell < 3/4} \sum_{j=0}^{\infty} (a_{\Lambda_1,j,\ell} - a_{\Lambda_2,j,\ell}) t^{j\sqrt{\lambda_\ell + 1/4}}. \tag{3.2}$$

Let Π_i denote the positive spectral projection of Δ_{Λ_i}. Denoting the point spectrum of Δ_{Λ_i} by $\{\mu_{i1}, \mu_{i2}, \ldots\}$, we define the *relative zeta function* for the pair $(\Delta_{\Lambda_1}, \Delta_{\Lambda_2})$ by

$$\begin{aligned}\zeta(s, \Delta_{\Lambda_1}, \Delta_{\Lambda_2}) := {} & \sum_{\mu_{1j} < 0} (\mu_{1j})^{-s} - \sum_{\mu_{2j} < 0} (\mu_{2j})^{-s} \\ & + \frac{1}{\Gamma(s)} \left(\int_0^1 + \int_1^\infty \right) t^{s-1}\, \mathrm{Tr}\big(\Pi_1 e^{-t\Delta_{\Lambda_1}} - \Pi_2 e^{-t\Delta_{\Lambda_2}}\big)\, dt \end{aligned} \tag{3.3}$$

for $s \in \mathbb{C}$ with $\Re(s) \gg 0$. As a consequence of Theorem 3.1, we automatically get

Theorem 3.2. *The relative zeta function $\zeta(s, \Delta_{\Lambda_1}, \Delta_{\Lambda_2})$ extends to be meromorphic on $\mathbb{C}$ with (possible) simple poles on the "unusual" set $\{-j\sqrt{\lambda_\ell + 1/4} \mid -1/4 < \lambda_\ell < 3/4,\ j \in \mathbb{N}\}$. In particular, the relative zeta function is regular at $s = 0$.*

Remark 3.3. By the work of Mooers [34], the relative zeta function $\zeta(s, \Delta_{\Lambda_1}, \Delta_{\Lambda_2})$ is regular at $s = 0$; what is new in Theorem 3.2 is the exact pole structure for diagonal Lagrangians with the form in (3.1).

For the proof of Theorem 3.1, we begin by studying a related model problem. More precisely, we fix a Lagrangian subspace $\varLambda$ of the boundary data V that decomposes as in (3.1). We let $X := [0,1]_r \times \varGamma$ and we consider the operator $\Delta_{X,\varLambda} := -\partial_r^2 + r^{-2}A$ given in (2.6), where we put the Dirichlet condition at $r=1$ and domain $D_\varLambda$ at $r=0$ fixed by $\varLambda$; that is, whose elements have asymptotics at $r=0$ determined by the Lagrangian subspace $\varLambda$.

Proposition 3.4. *As $t \to 0$,*

$$\operatorname{Tr}\bigl(e^{-t\Delta_{X,\varLambda}}\bigr) \sim \sum_{\lambda_\ell < 3/4} \sum_{j=0}^{\infty} a_{\varLambda,j,\ell}\, t^{j\sqrt{\lambda_\ell + 1/4}} + \sum_{j=0}^{\infty} a_j t^{\frac{j-n}{2}} + b \log t,$$

where the coefficients a_j, $j = 0,1,2,\ldots,$ and b are independent of the choice of self-adjoint extension, and b depends only on the operator A.

Proof. Let $\{\lambda_\ell\}$ denote the set of all eigenvalues of A and let $\varPi_V$ be the orthogonal projection of $L^2(\varGamma, E_\varGamma)$ onto the finitely many eigenspaces of A with eigenvalues $\lambda_\ell < 3/4$. We can write

$$\begin{aligned} e^{-t\Delta_{X,\varLambda}} &= \varPi_V e^{-t\Delta_{X,\varLambda}} \varPi_V + \varPi_V^{\perp} e^{-t\Delta_{X,\varLambda}} \varPi_V^{\perp} \\ &= \bigoplus_{\lambda_\ell < 3/4} \varPi_{\lambda_\ell} e^{-t\mathcal{L}_\ell} \varPi_{\lambda_\ell} + \varPi_V^{\perp} e^{-t\Delta_{X,\varLambda}} \varPi_V^{\perp}, \end{aligned}$$

where $\varPi_{\lambda_\ell}$ is the orthogonal projection onto the eigenspace corresponding to λ_ℓ and

$$\mathcal{L}_\ell := -\frac{d^2}{dr^2} + \lambda_\ell r^{-2}$$

over $[0,1]$ with the Dirichlet condition at $r=1$ and with domain at $r=0$ fixed by $\varLambda_\ell$, that is, whose elements have asymptotics at $r=0$ determined by $\varLambda_\ell$. Let us define

$$\Delta' := -\partial_r^2 + r^{-2}A', \quad \text{where } A' := \begin{cases} \frac{3}{4} & \text{over } V, \\ A & \text{over } V^{\perp}, \end{cases}$$

where A is given in (2.5). Then, $\varPi_V^{\perp} e^{-t\Delta_{X,\varLambda}} \varPi_V^{\perp} = \varPi_V^{\perp} e^{-t\Delta'} \varPi_V^{\perp}$, so

$$e^{-t\Delta_{X,\varLambda}} = \bigoplus_{\lambda_\ell < 3/4} \varPi_{\lambda_\ell} e^{-t\mathcal{L}_\ell} \varPi_{\lambda_\ell} + \varPi_V^{\perp} e^{-t\Delta'} \varPi_V^{\perp}.$$

Hence,

$$\operatorname{Tr}\bigl(e^{-t\Delta_{X,\varLambda}}\bigr) = \sum_{\lambda_\ell < 3/4} \operatorname{Tr}\bigl(e^{-t\mathcal{L}_\ell}\bigr) + \operatorname{Tr}\bigl(\varPi_V^{\perp} e^{-t\Delta'} \varPi_V^{\perp}\bigr). \tag{3.4}$$

Now it follows from the work of Cheeger [13] (cf. [7,8,15]) that as $t \to 0$,

$$\operatorname{Tr}\bigl(e^{-t\Delta'}\bigr) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j-n}{2}} + b \log t,$$

for some coefficients a_j, $j = 0, 1, 2, \dots,$ and b, and from the work of Falomir et al. [19] (cf. [17,18]) that as $t \to 0$, we have

$$\operatorname{Tr}\left(e^{-t\mathcal{L}_\ell}\right) \sim \sum_{j=0}^{\infty} a_{\Lambda_\ell, j, \ell} t^{j\sqrt{\lambda_\ell + 1/4}} + \sum_{j=0}^{\infty} b_j t^{\frac{j-1}{2}}. \tag{3.5}$$

Putting these trace expansions into (3.4) we get our result. $\square$

To prove Theorem 3.1, we cut the manifold M at the hypersurface $r = 1$ in the collar $[0, 2]_r \times \Gamma$, giving a decomposition

$$M = X \cup Y,$$

where $X = [0, 1]_r \times \Gamma$ and Y is a manifold with a collar neighborhood $[1, 2]_r \times \Gamma$ near its boundary, which we identify with Γ.

Let $\varrho(r) \in C^\infty([0, \infty))$ be a nondecreasing function such that $\varrho(r) = 0$ for $r \leqslant 1/4$ and $\varrho(r) = 1$ for $r \geqslant 3/4$. Given any real numbers $\alpha < \beta$, we define

$$\varrho_{\alpha,\beta}(r) := \varrho\big((r - \alpha)/(\beta - \alpha)\big). \tag{3.6}$$

Then $\varrho_{\alpha,\beta}(r) = 0$ on a neighborhood of $\{r \leqslant \alpha\}$ and $\varrho_{\alpha,\beta}(r) = 1$ on a neighborhood of $\{r \geqslant \beta\}$. We define

$$\begin{aligned} &\psi_1(r) = \varrho_{1/2,3/4}(r), \qquad \psi_2(r) = 1 - \psi_1(r), \\ &\varphi_1(r) = \varrho_{1/4,1/2}(r), \qquad \varphi_2(r) = 1 - \varrho_{3/4,1}(r). \end{aligned} \tag{3.7}$$

These functions extend either by 0 or 1 to define smooth functions on all of M and $\{\psi_i\}$ forms a partition of unity of M such that $\varphi_i = 1$ on $\operatorname{supp}(\psi_i)$.

Let $\Delta_{X,\Lambda} = -\partial_r^2 + r^{-2}A$ given in (2.6) denote the Laplacian on $X := [0, 1]_r \times \Gamma$, where we put the Dirichlet condition at $r = 1$ and with domain D_Λ at $r = 0$ fixed by Λ, and let Δ' denote the Laplacian on the compact manifold $M \setminus ([0, 1/4)_r \times \Gamma)$ with boundary with the Dirichlet condition at $r = 1/4$. We define

$$E = \varphi_1 e^{-t\Delta'} \psi_1 + \varphi_2 e^{-t\Delta_{X,\Lambda}} \psi_2. \tag{3.8}$$

It follows that E maps into the domain D_Λ of Δ_Λ, and

$$(\partial_t + \Delta_\Lambda)E = K, \quad \text{where}$$
$$K = [\Delta_\Lambda, \varphi_1]e^{-t\Delta'}\psi_1 + [\Delta_\Lambda, \varphi_2]e^{-t\Delta_{X,\Lambda}}\psi_2.$$

Because the supports of $[\Delta_\Lambda, \varphi_i]$ and ψ_i, where $i = 1, 2$, are disjoint, it is straightforward to check that the Schwarz kernel of K is a smooth function on M^2 vanishing to infinite order at

$t=0$ and near the whole left boundary $\partial M \times M$ of M^2. Thus, the heat operator of Δ_Λ is given by (cf. [3, Chapter 2])

$$e^{-t\Delta_\Lambda} = E + K', \quad K' = E * \sum_{j=1}^{\infty} (-1)^j K_j,$$

where $K_1 = K$ and $K_j = K_{j-1} * K$ with $*$ denoting the convolution of kernels:

$$K * K' = \int_0^t K(t-r)K'(r)\,dr = \int_0^t K(r)K'(t-r)\,dr. \tag{3.9}$$

Arguments similar to those found in [3, Chapter 2] show that the Schwarz kernel of K' is a smooth function on M^2 vanishing to infinite order at $t=0$. Therefore, the asymptotics of $\mathrm{Tr}(e^{-t\Delta_\Lambda})$ as $t \to 0$ are the same as those of

$$\mathrm{Tr}(E) = \mathrm{Tr}\big(\varphi_1 e^{-t\Delta'}\psi_1\big) + \mathrm{Tr}\big(\varphi_2 e^{-t\Delta_{X,\Lambda}}\psi_2\big).$$

By the work of Iwasaki [25] (cf. Greiner [22]), the trace $\mathrm{Tr}(\varphi_1 e^{-t\Delta'}\psi_1)$ has the usual expansion as $t \to 0$ in half-integer powers of t. Now our theorem follows directly from Proposition 3.4.

4. The gluing formula of the ζ-determinant

The object of this section is to derive a BFK-type gluing formula for the ζ-determinant over conic manifolds.

4.1. Statement of the gluing formula

For this and the next section, let us fix a Lagrangian subspace $\Lambda \subset V$ of the symplectic vector space V defined by (2.8)–(2.12) such that Λ decomposes diagonally as (3.1) and is *of mixed D and N type* in the following sense. We require (cf. (2.14)) $\Lambda_\ell \subset \mathrm{span}\{\psi_\ell^\pm\}$ to be either $\mathrm{span}\{\psi_\ell^+\}$ or $\mathrm{span}\{\psi_\ell^-\}$, except when $\lambda_\ell = -1/4$, in which case we only choose $\mathrm{span}\{\psi_\ell^+\}$. Let Δ_Λ denote the Laplace-type operator Δ with domain D_Λ corresponding to the fixed Λ. We are mostly interested when Λ is given by D or N.

We cut the manifold M at the hypersurface $r=1$ in the product neighborhood $[0,2]_r \times \Gamma$, giving a decomposition (see Fig. 2)

$$M = X \cup Y,$$

where $X = [0,1]_r \times \Gamma$ and Y is a manifold with a collar neighborhood $[1,2]_r \times \Gamma$ near its boundary, which we identify with Γ.

Let $\Delta_{X,\Lambda}$ be Δ with domain the restriction of D_Λ to X and with the Dirichlet condition at $r=1$.

Lemma 4.1. *$\Delta_{X,\Lambda}$ is invertible.*

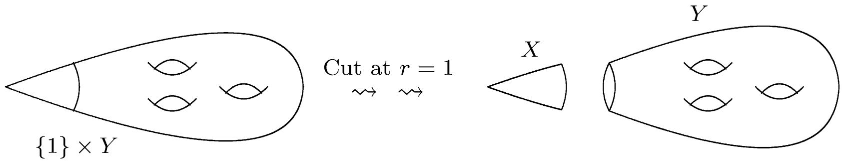

Fig. 2. Cutting M into X and Y.

Proof. Given $\phi \in L^2(X, E)$, it is straightforward to check that $\Delta\phi = 0$ if and only if

$$\phi = \sum_{\lambda_\ell=-1/4} \left\{c_\ell^+(\phi) r^{\frac{1}{2}}\phi_\ell + c_\ell^-(\phi) r^{\frac{1}{2}} \log r \phi_\ell\right\} + \sum_{-1/4<\lambda_\ell<3/4} \left\{c_\ell^+(\phi) r^{\frac{1}{2}+\nu_\ell}\phi_\ell + c_\ell^-(\phi) r^{\frac{1}{2}-\nu_\ell}\phi_\ell\right\} + \sum_{\lambda_\ell \geqslant \frac{3}{4}} c_\ell(\phi) r^{\frac{1}{2}+\nu_\ell}\phi_\ell.$$

Requiring ϕ to vanish at $r = 1$, we obtain

$$\phi = \sum_{\lambda_\ell=-1/4} c_\ell^-(\phi) r^{\frac{1}{2}} \log r \phi_\ell + \sum_{-1/4<\lambda_\ell<3/4} \left\{c_\ell^+(\phi) r^{\frac{1}{2}+\nu_\ell}\phi_\ell + c_\ell^-(\phi) r^{\frac{1}{2}-\nu_\ell}\phi_\ell\right\},$$

where $c_\ell^+(\phi) + c_\ell^-(\phi) = 0$ for $-1/4 < \lambda_\ell < 3/4$. By definition of Λ, $\phi \in D_\Lambda$ implies that $\phi \equiv 0$. This completes our proof. $\square$

Assume that Δ_Y, the induced Dirichlet Laplacian on Y, is also invertible; for example, this condition is satisfied when Δ is the Laplace–Beltrami operator.

Let $\mu \geqslant 0$ and $\mathcal{R}_\Lambda(\mu)$ denote the BFK operator for $\Delta_\Lambda + \mu$ cut at $r = 1$ defined as follows: for any $\mu \in [0, \infty)$ and $\varphi \in C^\infty(\Gamma, E_\Gamma)$, we can choose a smooth function $\phi(\mu) = (\phi_1(\mu), \phi_2(\mu)) \in C^\infty(X \setminus \Sigma, E) \oplus C^\infty(Y, E)$ with asymptotics at Σ fixed by Λ and is continuous at $r = 1$ with value φ such that $(\Delta_\Lambda + \mu)\phi(\mu) = 0$ off of Γ. Then $\mathcal{R}_\Lambda(\mu)$ is defined by

$$\mathcal{R}_\Lambda(\mu)\varphi := \partial_r\phi_1(\mu)|_{r=1} - \partial_r\phi_2(\mu)|_{r=1}. \tag{4.1}$$

Thus, $\mathcal{R}_\Lambda(\mu)$ is simply the sum of the Dirichlet-to-Neumann operators of the restrictions of Δ_Λ to X and Y. For simplicity, we denote $\mathcal{R}_\Lambda(0)$ by $\mathcal{R}_\Lambda$.

Using the definition provided by (3.3), one can define a relative zeta function $\zeta(s, \Delta_\Lambda, \Delta_{X,\Lambda})$, which is regular at $s = 0$ (see the analysis in Section 3). We define

$$\det{}_\zeta(\Delta_\Lambda, \Delta_{X,\Lambda}) := \exp\left(\frac{d}{ds}\bigg|_{s=0} \zeta(s, \Delta_\Lambda, \Delta_{X,\Lambda})\right).$$

We shall prove the following theorem.

Theorem 4.2. *If Δ_Y is invertible, then the following gluing formula holds*:

$$\frac{\det_\zeta(\Delta_\Lambda, \Delta_{X,\Lambda})}{\det_\zeta \Delta_Y} = C\frac{\det_\zeta \mathcal{R}_\Lambda}{\det L_\Lambda}, \tag{4.2}$$

where L_Λ is the linear operator defined as in (1.5) *and C is a constant determined by the restriction of Δ_Λ near $\{r=1\}\times\Gamma$, in particular, is* independent *of the choice of Lagrangian $\Lambda\subset V$ satisfying* (3.1).

We outline the main steps in the argument, which follow [9] in spirit.

We begin by proving

$$\frac{\det_\zeta(\Delta_\Lambda+\mu,\Delta_{X,\Lambda}+\mu)}{\det_\zeta(\Delta_Y+\mu)}\det_\zeta\mathcal{R}_\Lambda(\mu)^{-1}=C(\mu), \tag{4.3}$$

where $C(\mu)$ is independent of the choice of Lagrangian $\Lambda\subset V$ (see Section 4.4). We establish (4.3) by showing that the variation of the log of the left-hand side is related to the variation of the log of a relative determinant of a certain *model problem* defined away from the conic point.

Having established (4.3), we prove that as $\mu\to 0$,

$$\log\det_\zeta(\Delta_\Lambda+\mu,\Delta_{X,\Lambda}+\mu)=h_\Lambda\log\mu+\log\det_\zeta(\Delta_\Lambda,\Delta_{X,\Lambda})+o(1), \tag{4.4}$$

where h_Λ is the dimension of the null space of Δ_Λ.

Also, as $\mu\to 0$ we have

$$\log\det_\zeta(\Delta_Y+\mu)=\log\det_\zeta\Delta_Y+o(1), \tag{4.5}$$

which is valid because Δ_Y is invertible.

Finally, we prove that as $\mu\to 0$

$$\log\det_\zeta\mathcal{R}_\Lambda(\mu)=h_\Lambda\log\mu-\log\det L_\Lambda+\log\det_\zeta\mathcal{R}_\Lambda+o(1). \tag{4.6}$$

Now combining (4.3)–(4.6), we see that

$$\begin{aligned}&h_\Lambda\log\mu+\log\det_\zeta(\Delta_\Lambda,\Delta_{X,\Lambda})-\log\det_\zeta\Delta_Y\\&\quad-(h_\Lambda\log\mu-\log\det L_\Lambda+\log\det_\zeta\mathcal{R}_\Lambda)=\log C+o(1).\end{aligned}$$

Cancelling $h_\Lambda\log\mu$ and taking $\mu\to 0$, we get our final result putting $C:=\lim_{\mu\to 0}C(\mu)$.

4.2. The BFK operator

In this subsection, we study some properties of the BFK operator $\mathcal{R}_\Lambda(\mu)$ defined in (4.1). We begin with

Proposition 4.3. *$\mathcal{R}_\Lambda(\mu)$ is a smooth function of $\mu\in[0,\infty)$.*

Proof. $\mathcal{R}_\Lambda(\mu)$ is the sum of Dirichlet-to-Neumann (DN) operators over X and Y. Since Y is a smooth manifold with boundary and Δ_Y is invertible, the DN operator over Y is a smooth function of $\mu\in[0,\infty)$ as shown in [9, p. 49]. Thus, we are left to show that the DN operator over X, which we denote by $\mathcal{N}_{X,\Lambda}(\mu)$, is smooth in $\mu\in[0,\infty)$. To this end, let $\{\lambda_\ell\}$ denote the set of all eigenvalues of A and let Π_{λ_ℓ} denote the orthogonal projection of $L^2(\Gamma,E_\Gamma)$ onto the

λ_ℓ eigenspace of A. Then we can write

$$\Delta_{X,\Lambda}(\mu) = \bigoplus_\ell \Pi_{\lambda_\ell} \mathcal{L}_\ell(\mu) \Pi_{\lambda_\ell},$$

where $\mathcal{L}_\ell(\mu) := \mathcal{L}_\ell + \mu$ with

$$\mathcal{L}_\ell := -\frac{d^2}{dr^2} + \lambda_\ell r^{-2}$$

defined over [0, 1] with the Dirichlet condition at $r = 1$ and with domain at $r = 0$ determined by Λ_ℓ for $\lambda_\ell < 3/4$; when $\lambda_\ell \geqslant 3/4$, $\mathcal{L}_\ell$ is essentially self-adjoint. Therefore,

$$\mathcal{N}_{X,\Lambda}(\mu) = \bigoplus_\ell \Pi_{\lambda_\ell} \mathcal{N}_\ell(\mu) \Pi_{\lambda_\ell}, \tag{4.7}$$

where $\mathcal{N}_\ell(\mu)$ is the DN operator corresponding to $\mathcal{L}_\ell + \mu$. Thus, for each ℓ, we just have to prove that $\mathcal{N}_\ell(\mu)$ is a smooth function of $\mu \in [0, \infty)$. Consider first the case that $\lambda_\ell \neq -1/4$. Then $\mathcal{L}_\ell(\mu)\phi = 0$ if and only if

$$\phi = c_\ell^+(\phi) r^{1/2} I_{\nu_\ell}\big(\mu^{1/2} r\big) + c_\ell^-(\phi) r^{1/2} I_{-\nu_\ell}\big(\mu^{1/2} r\big), \tag{4.8}$$

where $I_{\pm\nu_\ell}(x)$ are modified Bessel functions and $\nu_\ell := \sqrt{\lambda_\ell + 1/4}$. Using the asymptotics near $x = 0$ (see [1, p. 375])

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)} \left(1 + \frac{x^2}{4(1+\nu)} + \frac{x^4}{32(1+\nu)(2+\nu)} + \cdots \right), \tag{4.9}$$

we see that if $\lambda_\ell \geqslant 3/4$, that is, $\nu_\ell \geqslant 1$, then we must have $c_\ell^-(\phi) = 0$ in order for ϕ to be in L^2. Therefore, if $\lambda_\ell \geqslant 3/4$, then given $\varphi \in \mathbb{C}$, we have $\mathcal{L}_\ell(\mu)\phi = 0$ with $\phi|_{r=1} = \varphi$ if and only if

$$\phi = \frac{\varphi}{I_{\nu_\ell}(\mu^{1/2})} r^{1/2} I_{\nu_\ell}\big(\mu^{1/2} r\big).$$

From this formula and the asymptotics (4.9) it is obvious that $\mathcal{N}_\ell(\mu)\varphi = (\partial_r \phi)|_{r=1}$ is a smooth function of $\mu \in [0, \infty)$. This same exact argument works to prove that $\mathcal{N}_\ell(\mu)$ is smooth in $\mu \in [0, \infty)$ in the case that $-1/4 < \lambda_\ell < 3/4$ and $\Lambda_\ell = \operatorname{span}\{\psi_\ell^+\}$. In the case that $-1/4 < \lambda_\ell < 3/4$ and $\Lambda_\ell = \operatorname{span}\{\psi_\ell^-\}$, we must take $c_\ell^+(\phi) = 0$ in (4.8) and then given $\varphi \in \mathbb{C}$, $\mathcal{L}_\ell(\mu)\phi = 0$ with ϕ in the domain of $\mathcal{L}_\ell$ and $\phi|_{r=1} = \varphi$ if and only if

$$\phi = \frac{\varphi}{I_{-\nu_\ell}(\mu^{1/2})} r^{1/2} I_{-\nu_\ell}\big(\mu^{1/2} r\big).$$

Again using the asymptotics (4.9) it is obvious that $\mathcal{N}_\ell(\mu)\varphi = (\partial_r \phi)|_{r=1}$ is a smooth function of $\mu \in [0, \infty)$. Finally, suppose that $\lambda_\ell = -1/4$. In this case, $\mathcal{L}_\ell(\mu)\phi = 0$ if and only if

$$\phi = c_\ell^+(\phi) r^{1/2} I_0\big(\mu^{1/2} r\big) + c_\ell^-(\phi) r^{1/2} K_0\big(\mu^{1/2} r\big),$$

where $I_0(x)$, $K_0(x)$ are modified Bessel functions. For $\lambda_\ell = -1/4$, by assumption on Λ we have $\Lambda_\ell = \operatorname{span}\{\psi_\ell^+\}$, therefore, in order for ϕ to be in the domain of $\mathcal{L}_\ell$ we must have $c_\ell^-(\phi) = 0$. Hence, given $\varphi \in \mathbb{C}$, $\mathcal{L}_\ell(\mu)\phi = 0$ with ϕ in the domain of $\mathcal{L}_\ell$ and $\phi|_{r=1} = \varphi$ if and only if

$$\phi = \frac{\varphi}{I_0(\mu^{1/2})} r^{1/2} I_0\big(\mu^{1/2} r\big).$$

Using the asymptotics (4.9) with $\nu = 0$, it is obvious that $\mathcal{N}_\ell(\mu)\varphi = (\partial_r\phi)|_{r=1}$ is a smooth function of $\mu \in [0, \infty)$. This completes our proof. □

In the following theorem, γ denotes the restriction operator to $r = 1$ and γ^* is the adjoint of γ, which is also given by $\gamma^* = (\cdot \otimes \delta_\Gamma)$, where δ_Γ is the delta distribution concentrated on the hypersurface $\{r = 1\} \cong \Gamma$.

Theorem 4.4. *For $\mu > 0$, $\mathcal{R}_\Lambda(\mu)$ is a positive definite first order elliptic classical pseudodifferential operator and for $\mu > 0$,*

$$\mathcal{R}_\Lambda(\mu)^{-1} = \gamma(\Delta_\Lambda + \mu)^{-1}\gamma^*.$$

The proof of this result is similar to the proof of [28, Lemma 3.3] or [31, Theorem A.2], so we omit it.

We now analyze $\mathcal{R}_\Lambda = \mathcal{R}_\Lambda(0)$. By standard analytic Fredholm theory,

$$(\Delta_\Lambda + \mu)^{-1} = \frac{1}{\mu}\sum_j \langle\cdot, u_j\rangle u_j + Q(\mu), \tag{4.10}$$

where $\{u_j\}$ is an orthonormal basis for the kernel of Δ_Λ on D_Λ and $Q(\mu)$ is a pseudodifferential operator of order -2 away from $r = 0$ depending continuously on μ (even at $\mu = 0$). The proof of the following theorem is very similar to the proofs of [31, Lemma A.3 and Theorem A.4], with the appropriate translations from the cylindrical end case considered in [31] to the conic case considered here.

Theorem 4.5. *The sections $\{v_j := u_j|_{r=1}\}$ are linearly independent in $L^2(\Gamma, E_\Gamma)$ and the kernel of $\mathcal{R}_\Lambda = \mathcal{R}_\Lambda(0)$ is exactly the subspace $V_0 = \operatorname{span}\{v_j\} \subset L^2(\Gamma, E_\Gamma)$. The BFK operator $\mathcal{R}_\Lambda = \mathcal{R}_\Lambda(0)$ is a nonnegative self-adjoint first order elliptic classical pseudodifferential operator such that*

$$\mathcal{R}_\Lambda = \begin{cases} 0 & \textit{on } V_0, \\ A^{-1} & \textit{on } V_0^\perp, \end{cases}$$

where $A = P^\perp \gamma Q(0)\gamma^* P^\perp$ with P the orthogonal projection onto V_0.

4.3. Difference of resolvents

Recall our situation: M is a manifold with a collar neighborhood $[0, 2]_r \times \Gamma$ of its boundary over which the metric takes the form

$$dr^2 + h,$$

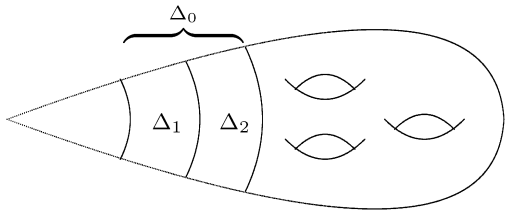

Fig. 3. The maps $\Delta_0, \Delta_1, \Delta_2$.

where h is a metric on Γ. Thus,

$$M = X \cup Y, \quad X = [0,1]_r \times \Gamma,$$

where Y has a collar neighborhood $Z := [1,2]_r \times \Gamma$. Let us *fix* $0 < a < 1$ and $0 < b < 2$, and define

$$M_0 := [a,b] \times \Gamma, \qquad M_1 := [a,1] \times \Gamma, \qquad M_2 := [1,b] \times \Gamma.$$

For $j = 0,1,2$, let Δ_j denote the Laplacian on M_j with the Dirichlet boundary condition at the boundaries of M_j; see Fig. 3. The importance of these operators is that they are *independent* of any choice of Λ to get the self-adjoint extension operator Δ_Λ. The goal of this subsection is to compare the determinants on M, X, and Y to those on M_0, M_1, and M_2. For this, we set

$$\Delta_\Lambda(\mu) = \Delta_\Lambda + \mu, \qquad \Delta_{X,\Lambda}(\mu) = \Delta_{X,\Lambda} + \mu, \qquad \Delta_Y(\mu) = \Delta_Y + \mu$$

and for $j = 0,1,2$, $\Delta_j(\mu) = \Delta_j + \mu$. Let $\mathcal{R}_0(\mu)$ denote the BFK operator for the split manifold $M_0 = M_1 \cup M_2$. Thus, for any $\mu \in [0,\infty)$ and $\varphi \in C^\infty(\Gamma, E_\Gamma)$, we can choose a smooth function $\phi(\mu) = (\phi_1(\mu), \phi_2(\mu)) \in C^\infty(M_1, E) \oplus C^\infty(M_2, E)$ that is continuous at $r = 1$ with value φ vanishing at $r = a, b$ such that $(\Delta_0 + \mu)\phi(\mu) = 0$ off of Γ; then,

$$\mathcal{R}_0(\mu)\varphi := \partial_r \phi_1(\mu)|_{r=1} - \partial_r \phi_2(\mu)|_{r=1}.$$

In the following lemma we compare the operators over M, X, Y, to the model operators.

Lemma 4.6. *For $\mu > 0$, the following differences of operators*

$$\Delta_\Lambda(\mu)^{-1} - \Delta_{X,\Lambda}(\mu)^{-1} - \Delta_Y(\mu)^{-1} - \left(\Delta_0(\mu)^{-1} - \Delta_1(\mu)^{-1} - \Delta_2(\mu)^{-1}\right) \quad \text{and}$$
$$\mathcal{R}_\Lambda(\mu)^{-1} - \mathcal{R}_0(\mu)^{-1}$$

are smoothing.

Proof. Recall that for real numbers $\alpha < \beta$, the function $\varrho_{\alpha,\beta}(r)$ in (3.6) has the property that $\varrho_{\alpha,\beta}(r) = 0$ on a neighborhood of $\{r \leqslant \alpha\}$ and $\varrho_{\alpha,\beta}(r) = 1$ on a neighborhood of $\{r \geqslant \beta\}$. Let us choose real numbers a_1, a_2, b_1, b_2 such that

$$a < a_1 < a_2 < 1 < b_1 < b_2 < b.$$

We define

$$\psi_1(r) := 1 - \varrho_{a_1,a_2}(r), \qquad \psi_2(r) := \varrho_{b_1,b_2}(r), \qquad \psi_0(r) := 1 - \psi_1(r) - \psi_2(r) \quad \text{and}$$
$$\varphi_1(r) := 1 - \varrho_{a_2,1}(r), \qquad \varphi_2(r) := \varrho_{1,b_1}(r), \qquad \varphi_0(r) := \varrho_{a,a_1}(r) - \varrho_{b_2,b}(r).$$

The functions $\{\psi_i\}$, $\{\varphi_i\}$ extend either by 0 or 1 to define smooth functions on all of M and $\{\psi_i\}$ forms a partition of unity of M such that $\varphi_i = 1$ on $\operatorname{supp}(\psi_i)$. Now to prove this lemma, we first claim that each of the following equalities hold modulo smoothing:

$$\begin{aligned} \Delta_\Lambda(\mu)^{-1} &= \varphi_1 \Delta_{X,\Lambda}(\mu)^{-1}\psi_1 + \varphi_0\Delta_0(\mu)^{-1}\psi_0 + \varphi_2\Delta_Y(\mu)^{-1}\psi_2, \\ \Delta_{X,\Lambda}(\mu)^{-1} &= \varphi_1 \Delta_{X,\Lambda}(\mu)^{-1}\psi_1 + \varphi_0\Delta_1(\mu)^{-1}\psi_0, \\ \Delta_Y(\mu)^{-1} &= \varphi_0\Delta_2(\mu)^{-1}\psi_0 + \varphi_2\Delta_Y(\mu)^{-1}\psi_2. \end{aligned} \tag{4.11}$$

For instance, let us verify the first claim in (4.11); the other claims are verified using a similar argument. Define

$$Q(\mu) = \varphi_1 \Delta_{X,\Lambda}(\mu)^{-1}\psi_1 + \varphi_0\Delta_0(\mu)^{-1}\psi_0 + \varphi_2\Delta_Y(\mu)^{-1}\psi_2.$$

Then observe that $\Delta_\Lambda(\mu)Q(\mu) = \mathrm{Id} + K(\mu)$, where

$$K(\mu) = \big[\Delta_\Lambda(\mu), \varphi_1\big]\Delta_{X,\Lambda}(\mu)^{-1}\psi_1 + \big[\Delta_\Lambda(\mu), \varphi_0\big]\Delta_0(\mu)^{-1}\psi_0 + \big[\Delta_\Lambda(\mu), \varphi_2\big]\Delta_Y(\mu)^{-1}\psi_2.$$

Because the support of $[\Delta_\Lambda(\mu), \varphi_i]$ and ψ_i are separated by some positive length, it follows that $K(\mu)$ is smoothing. A similar argument works to prove that the other equalities in (4.11) hold modulo smoothing. From (4.11), it follows that modulo smoothing,

$$\Delta_\Lambda(\mu)^{-1} - \Delta_{X,\Lambda}(\mu)^{-1} - \Delta_Y(\mu)^{-1} = \varphi_0\Delta_0(\mu)^{-1}\psi_0 - \varphi_0\Delta_1(\mu)^{-1}\psi_0 - \varphi_0\Delta_2(\mu)^{-1}\psi_0. \tag{4.12}$$

On the other hand, very similar arguments used to establish (4.11) shows that modulo smoothing,

$$\begin{aligned} \Delta_0(\mu)^{-1} &= \varphi_1\Delta_1(\mu)^{-1}\psi_1 + \varphi_0\Delta_0(\mu)^{-1}\psi_0 + \varphi_2\Delta_2(\mu)^{-1}\psi_2, \\ \Delta_1(\mu)^{-1} &= \varphi_1\Delta_1(\mu)^{-1}\psi_1 + \varphi_0\Delta_1(\mu)^{-1}\psi_0, \\ \Delta_2(\mu)^{-1} &= \varphi_0\Delta_2(\mu)^{-1}\psi_0 + \varphi_2\Delta_2(\mu)^{-1}\psi_2. \end{aligned}$$

Then, modulo smoothing,

$$\Delta_0(\mu)^{-1} - \Delta_1(\mu)^{-1} - \Delta_2(\mu)^{-1} = \varphi_0\Delta_0(\mu)^{-1}\psi_0 - \varphi_0\Delta_1(\mu)^{-1}\psi_0 - \varphi_0\Delta_2(\mu)^{-1}\psi_0.$$

Comparing this with (4.12) proves the first statement of our lemma. Conjugating the first formula in (4.11) with γ_0 and γ_0^* we get our second statement. □

4.4. Variation of log-det

We now state some variational results.

Proposition 4.7. *For $\mu > 0$, we have*

$$\frac{d}{d\mu}\big(\log\det_\zeta\big(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\big) - \log\det_\zeta \Delta_Y(\mu)$$
$$- \log\det_\zeta \Delta_0(\mu) - \log\det_\zeta \Delta_1(\mu) - \log\det_\zeta \Delta_2(\mu)\big)$$
$$= \operatorname{Tr}\big(\Delta_\Lambda(\mu)^{-1} - \Delta_{X,\Lambda}(\mu)^{-1} - \Delta_Y(\mu)^{-1} - \big(\Delta_0(\mu)^{-1} - \Delta_1(\mu)^{-1} - \Delta_2(\mu)^{-1}\big)\big) \tag{4.13}$$

and

$$\frac{d}{d\mu}\big(\log\det_\zeta \mathcal{R}_\Lambda(\mu) - \log\det_\zeta \mathcal{R}_0(\mu)\big)$$
$$= \operatorname{Tr}\left(\mathcal{R}_\Lambda(\mu)^{-1}\frac{d}{d\mu}\mathcal{R}_\Lambda(\mu) - \mathcal{R}_0(\mu)^{-1}\frac{d}{d\mu}\mathcal{R}_0(\mu)\right). \tag{4.14}$$

Proof. From [2, Proposition 2.9], we have

$$\partial_\mu\big[\log\det_\zeta\big(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\big) - \log\det_\zeta \Delta_Y(\mu)$$
$$- \log\det_\zeta \Delta_0(\mu) - \log\det_\zeta \Delta_1(\mu) - \log\det_\zeta \Delta_2(\mu)\big]$$
$$= \operatorname{Tr}\big(\Delta_\Lambda(\mu)^{-1-s} - \Delta_{X,\Lambda}(\mu)^{-1-s} - \Delta_Y(\mu)^{-1-s}$$
$$- \big(\Delta_0(\mu)^{-1-s} - \Delta_1(\mu)^{-1-s} - \Delta_2(\mu)^{-1-s}\big)\big)\big|_{s=0}. \tag{4.15}$$

Here the right-hand side of (4.15) means that we evaluate the meromorphic extension of the difference of zeta functions at $s = 0$. It follows from Lemma 4.6 that we can put $s = 0$ in the right-hand side (4.15), and when we do we get exactly (4.13). A similar proof can be used to derive (4.14). We only remark that the operator on the right-hand side of (4.14) is smoothing by Lemma 4.6. □

Using this proposition together with the proof of [31, Lemma 5.1] or [28, Corollary 3.9], we can equate (4.13) and (4.14) to prove that for $\mu > 0$ we have

$$\frac{d}{d\mu}\big(\log\det_\zeta\big(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\big) - \log\det_\zeta \Delta_Y(\mu)$$
$$- \log\det_\zeta \Delta_0(\mu) - \log\det_\zeta \Delta_1(\mu) - \log\det_\zeta \Delta_2(\mu)\big)$$
$$= \frac{d}{d\mu}\big(\log\det_\zeta \mathcal{R}_\Lambda(\mu) - \log\det_\zeta \mathcal{R}_0(\mu)\big).$$

In particular, for $\mu \in \mathbb{R}^+$, there exists a constant K such that

$$\frac{\det_\zeta(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu))}{\det_\zeta \Delta_Y(\mu)}\det_\zeta \mathcal{R}_\Lambda(\mu)^{-1} = K\frac{\det_\zeta \Delta_0(\mu)}{\det_\zeta \Delta_1(\mu)\cdot\det_\zeta \Delta_2(\mu)}\det_\zeta \mathcal{R}_0(\mu)^{-1}. \tag{4.16}$$

By [9, Theorem A.3], if we take logarithms of both sides of (4.16) and then take $\mu \to \infty$, the coefficients of the asymptotics of log-determinants are given in terms of local data, that is, symbols of the difference of operators given in Lemma 4.6. Hence, if we use Lemma 4.6, we get

$$K = 1.$$

Thus, we obtain the following proposition.

Proposition 4.8. *For* $\mu > 0$*,*

$$\frac{\det_\zeta(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu))}{\det_\zeta \Delta_Y(\mu)} \det_\zeta \mathcal{R}_\Lambda(\mu)^{-1} = C(\mu),$$

where $C(\mu)$ *is the right-hand side of* (4.16) *with* $K = 1$*, which is independent of the choice of Lagrangian* Λ*.*

Notice that $\Delta_0, \Delta_1, \Delta_2$, and $\mathcal{R}_0$ are defined away from the conic singularity as we see in Fig. 3; this is why $C(\mu)$ is defined independent of the choice of Lagrangian Λ.

4.5. Limit as $\mu \to 0$ *and the conclusion of Theorem 4.2*

We now take $\mu \to 0$ in Proposition 4.8. Since $\Delta_0, \Delta_1, \Delta_2$ are invertible operators defined with Dirichlet boundary conditions on smooth manifolds with boundary, it follows that $C(\mu)$ is continuous at $\mu = 0$:

$$\lim_{\mu \to 0} C(\mu) = C := \frac{\det_\zeta \Delta_0}{\det_\zeta \Delta_1 \cdot \det_\zeta \Delta_2} \det_\zeta \mathcal{R}_0^{-1}. \tag{4.17}$$

Next, consider the following proposition.

Proposition 4.9. *For* $\mu > 0$ *near* 0*, we have*

$$\det_\zeta\big(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\big) = \mu^{h_\Lambda} \cdot \det_\zeta(\Delta_\Lambda, \Delta_{X,\Lambda})\big(1 + o(1)\big),$$

where $h_\Lambda = \dim\ker \Delta_\Lambda$.

Proof. Let $\{\mu_j\}$ denote the eigenvalues of Δ_Λ and Π_0 (Π_+) denote the orthogonal projection onto the zero (positive) eigenspace(s) of Δ_Λ. Then $\Pi = \Pi_0 + \Pi_+$ is the orthogonal projection onto the nonnegative eigenspaces of Δ_Λ and we can write

$$\operatorname{Tr}\big(\Pi e^{-t(\Delta_\Lambda + \mu)}\big) = e^{-t\mu} \operatorname{Tr}\big(\Pi e^{-t\Delta_\Lambda}\big) = e^{-t\mu} h_\Lambda + e^{-t\mu} \operatorname{Tr}\big(\Pi_+ e^{-t\Delta_\Lambda}\big).$$

Hence, by definition of the relative zeta function for $(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu))$, for $\mu > 0$ small we have

$$\zeta\big(s, \Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\big) = \sum_{\mu_j < 0} (\mu_j + \mu)^{-s} - \sum_{\mu_{X,j} < 0} (\mu_{X,j} + \mu)^{-s} + \mu^{-s} h_\Lambda$$
$$+ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\mu} \operatorname{Tr}\big(\Pi_+ e^{-t\Delta_\Lambda} - \Pi_{X,+} e^{-t\Delta_{X,\Lambda}}\big)\, dt,$$

where notations with the subscript X denote the corresponding eigenvalues and projections for $\Delta_{X,\Lambda}$. Since $\mathrm{Tr}(\Pi_+ e^{-t\Delta_\Lambda} - \Pi_{X,+} e^{-t\Delta_{X,\Lambda}})$ vanishes exponentially as $t \to \infty$ and $e^{-t\mu} = 1 + \mathcal{O}(t\mu)$, it follows that the sum of the first and third terms on the right side equals $\zeta(s, \Delta_\Lambda, \Delta_{X,\Lambda}) + \mathcal{O}(\mu)$. Thus, differentiating both sides with respect to s, we find that as $\mu \to 0^+$,

$$\log \det_\zeta \bigl(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu)\bigr) = h_\Lambda \log \mu + \log \det_\zeta (\Delta_\Lambda, \Delta_{X,\Lambda}) + o(1),$$

which completes our proof. $\square$

Before finishing the proof of Theorem 4.2, we need a proposition.

Proposition 4.10. *As $\mu \to 0$, we have*

$$\det_\zeta \mathcal{R}_\Lambda(\mu) = \mu^{h_\Lambda} \cdot \frac{\det_\zeta \mathcal{R}_\Lambda}{\det L_\Lambda} \bigl(1 + o(1)\bigr),$$

where $h_\Lambda = \dim \ker \Delta_\Lambda$ and L_Λ is the linear operator defined as in (1.5) *in terms of* $\ker \Delta_\Lambda$ *instead of* $\ker \Delta_{\mathrm{D}}$.

Proof. If $P : L^2(\Gamma, E_\Gamma) \to V_0$ denotes the orthogonal projection onto $V_0 := (\ker \Delta_\Lambda)|_{r=1}$, then

$$\zeta\bigl(s, \mathcal{R}_\Lambda(\mu)\bigr) = \mathrm{Tr}\bigl(\mathcal{R}_\Lambda(\mu)^{-s}\bigr) = \mathrm{Tr}\bigl(P\mathcal{R}_\Lambda(\mu)^{-s}\bigr) + \mathrm{Tr}\bigl(P^\perp \mathcal{R}_\Lambda(\mu)^{-s}\bigr).$$

Proposition 4.3 along with Theorem 4.5 implies that as $\mu \to 0$,

$$-\frac{d}{ds}\bigg|_{s=0} \mathrm{Tr}\bigl(P^\perp \mathcal{R}_\Lambda(\mu)^{-s}\bigr) = \log \det_\zeta \mathcal{R}_\Lambda + o(1).$$

Since $\mathrm{Tr}(P\mathcal{R}_\Lambda(\mu)^{-s}) = \mathrm{Tr}(P\mathcal{R}_\Lambda(\mu)^{-s} P)$, it follows that

$$-\frac{d}{ds}\bigg|_{s=0} \mathrm{Tr}\bigl(P\mathcal{R}_\Lambda(\mu)^{-s}\bigr) = -\log \det\bigl(P\mathcal{R}_\Lambda(\mu)^{-1} P\bigr).$$

Thus, we are left to prove that, as $\mu \to 0^+$,

$$\det\bigl(P\mathcal{R}_\Lambda(\mu)^{-1} P\bigr) = \mu^{-h_\Lambda}\bigl(\det L_\Lambda + o(1)\bigr).$$

To prove this, we note that by (4.10), we have

$$P\mathcal{R}_\Lambda(\mu)^{-1} P = \frac{1}{\mu} \sum \langle \cdot, v_j \rangle v_j + Q'(\nu),$$

where $Q'(\mu) = P\gamma_0 Q(\mu) \gamma_0^* P$ is an operator that depends continuously on $\mu \in [0, \infty)$. This implies that

$$\det\bigl(P\mathcal{R}_\Lambda(\mu)^{-1} P\bigr) = \mu^{-h_\Lambda} \det\bigl(L_\Lambda + \mu Q'(\mu)\bigr), \tag{4.18}$$

from which our result easily follows. $\square$

Now taking $\mu \to 0$ in

$$\frac{\det_\zeta(\Delta_\Lambda(\mu), \Delta_{X,\Lambda}(\mu))}{\det_\zeta \Delta_Y(\mu)} \det_\zeta \mathcal{R}_\Lambda(\mu)^{-1} = C(\mu),$$

and using (4.17) and Propositions 4.9 and 4.10, we get Theorem 4.2.

5. Relative determinant formulae

In this section we derive a formula for the relative determinant of Δ with the D and N extensions: $\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}})$.

5.1. The D extension: Model case

For $\lambda \geqslant -1/4$, consider the following operator:

$$\mathcal{L} := -\frac{d^2}{dr^2} + \lambda r^{-2} \quad \text{over } [0, R],$$

where we impose the Dirichlet condition at $r = R$. If $\lambda \geqslant 3/4$, it is not necessary to impose any boundary condition at $r = 0$ to get a self-adjoint extension, but if not, we need a boundary condition to get a self-adjoint extension of $\mathcal{L}$. In this subsection, we choose the Friedrichs extension, i.e., the D extension defined in Section 2.3. We denote by $\mathcal{L}_{\mathrm{D}}$ this self-adjoint extension. Let us recall

$$\nu := \sqrt{\lambda + 1/4} \geqslant 0. \tag{5.1}$$

Now putting $\mu = z^2$ with $z \in \mathbb{R}^+$, two solutions of $(\mathcal{L} + z^2) f = 0$ satisfying the boundary conditions at $r = 0, r = R$ are given by (see, for instance, Callias [10, p. 360])

$$\phi(r) = r^{1/2} I_\nu(zr), \qquad \psi(r) = r^{1/2}\big(K_\nu(zr) I_\nu(zR) - K_\nu(zR) I_\nu(zr)\big),$$

respectively. Here I_ν, K_ν denote the modified Bessel functions. Then it is known that the resolvent kernel $(\mathcal{L}_{\mathrm{D}} + z^2)^{-1}(r, r')$ is given by (see [16])

$$\big(\phi'(r)\psi(r) - \phi(r)\psi'(r)\big)^{-1} \phi(r)\psi(r') \quad \text{if } r \leqslant r'.$$

Using the relation

$$I_\nu'(x) K_\nu(x) - I_\nu(x) K_\nu'(x) = \frac{1}{x}, \tag{5.2}$$

we can derive

$$\big(\mathcal{L}_{\mathrm{D}} + z^2\big)^{-1}(r, r') = (rr')^{1/2} I_\nu(zr)\big(K_\nu(zr') - K_\nu(zR) I_\nu(zR)^{-1} I_\nu(zr')\big)$$

for $r \leqslant r'$. Then, by a straightforward computation (cf. [19, Appendix B]), we obtain

$$\operatorname{Tr}\big((\mathcal{L}_{\mathrm{D}} + z^2)^{-1}\big) = \int_0^R \big(\mathcal{L}_{\mathrm{D}} + z^2\big)^{-1}(r, r)\, dr = (2z)^{-1}\big(-\nu z^{-1} + R I_\nu'(zR) I_\nu(zR)^{-1}\big). \tag{5.3}$$

Combining this with the equality

$$\frac{d}{dz}\log\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)=2z\,\mathrm{Tr}\big(\big(\mathcal{L}_{\mathrm{D}}+z^2\big)^{-1}\big),$$

which can be proved following the proof of Proposition 4.7, see especially (4.15), we get

$$\frac{d}{dz}\log\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)=\frac{d}{dz}\log\big(z^{-\nu}I_\nu(zR)\big).$$

Integrating, we see that for some constant c, we have

$$\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)=c\big(z^{-\nu}I_\nu(zR)\big),\quad\text{or}$$
$$\log\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)=\log c+\log\big(z^{-\nu}I_\nu(zR)\big).$$

To determine the constant c, we need to know the constant terms of the asymptotics of $\log\det_\zeta(\mathcal{L}_{\mathrm{D}}+z^2)$ and $\log(z^{-\nu}I_\nu(zR))$ as $z\to\infty$. For $\log\det_\zeta(\mathcal{L}_{\mathrm{D}}+z^2)$, we have:

Lemma 5.1. *The constant term in the asymptotics of* $\log\det_\zeta(\mathcal{L}_{\mathrm{D}}+z^2)$ *as* $z\to\infty$ *is trivial.*

Proof. By (3.5), we have the following asymptotics as $t\to 0$:

$$\mathrm{Tr}\big(e^{-t\mathcal{L}_{\mathrm{D}}}\big)\sim\sum_j a_j t^{\xi_j},\tag{5.4}$$

where $\xi_j\to\infty$ (the exact values of a_j, ξ_j are not important). As $z\to\infty$, we can disregard the long-time behavior of $\mathrm{Tr}(e^{-t\mathcal{L}_{\mathrm{D}}})$, which decays exponentially, therefore as $z\to\infty$, we have

$$\begin{aligned}\log\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)&\sim-\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Big(\sum_j a_j t^{\xi_j}\Big)e^{-tz^2}\,dt\\&=-\frac{d}{ds}\bigg|_{s=0}\sum_j a_j z^{-2(s+\xi_j)}\Gamma(s)^{-1}\Gamma(s+\xi_j).\end{aligned}$$

From this, the conclusion follows. □

For $\log(z^{-\nu}I_\nu(zR))$, we use the following asymptotics as $x\to\infty$ (see [10, p. 361] or [1, p. 377]),

$$I_\nu(x)\sim\frac{e^x}{\sqrt{2\pi x}}\bigg(1-\frac{4\nu^2-1}{8x}+\frac{(4\nu^2-1)(4\nu^2-9)}{2(8x)^2}+\mathcal{O}\big(x^{-3}\big)\bigg).\tag{5.5}$$

This implies that the constant term as $z\to\infty$ in the asymptotics of $\log(z^{-\nu}I_\nu(zR))$ is $-\log\sqrt{2\pi R}$. Hence, we conclude that $c=\sqrt{2\pi R}$ and

$$\det_\zeta\big(\mathcal{L}_{\mathrm{D}}+z^2\big)=\sqrt{2\pi R}\big(z^{-\nu}I_\nu(zR)\big).\tag{5.6}$$

To get $\det_\zeta \mathcal{L}_\mathrm{D}$, we use the asymptotics [1, p. 375]

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)}\left(1 + \frac{x^2}{4(1+\nu)} + \frac{x^4}{32(1+\nu)(2+\nu)} + \cdots\right), \tag{5.7}$$

as $x \to 0$. Then, taking the limit $x \to 0$ in (5.6), we obtain:

Proposition 5.2. *The following determinant formula holds*:

$$\det_\zeta \mathcal{L}_\mathrm{D} = \frac{\sqrt{\pi}\, 2^{1/2-\nu} R^{1/2+\nu}}{\Gamma(1+\nu)}. \tag{5.8}$$

Using a different method, Lesch [29] derived this formula (for the interval [0, 1]).

5.2. The N extension: Model case

As before, with $\lambda \geqslant -1/4$ we shall work with the model operator

$$\mathcal{L} := -\frac{d^2}{dr^2} + \lambda r^{-2} \quad \text{over } [0, R],$$

but now let us consider the N extension as discussed in Section 2.3. In the case that $\lambda = -1/4$, the N and D extensions are the same, so we shall assume that $\lambda > -1/4$. We denote this self-adjoint extension by $\mathcal{L}_\mathrm{N}$. In this case, one can check that the resolvent kernel of $(\mathcal{L}_\mathrm{N} + z^2)$ is given by

$$\big(\mathcal{L}_\mathrm{N} + z^2\big)^{-1}(r, r') = (rr')^{1/2} I_{-\nu}(zr)\big(K_\nu(zr') - K_\nu(zR) I_\nu(zR)^{-1} I_\nu(zr')\big)$$

for $r \leqslant r'$. Then a straightforward computation following the derivation of (5.6) gives

$$\det_\zeta\big(\mathcal{L}_\mathrm{N} + z^2\big) = \sqrt{2\pi R}\big(z^\nu I_{-\nu}(zR)\big). \tag{5.9}$$

To get $\det_\zeta \mathcal{L}_\mathrm{N}$, we use the asymptotics as $x \to 0$ in (5.7) to obtain the proposition:

Proposition 5.3. *The following determinant formula holds*

$$\det_\zeta \mathcal{L}_\mathrm{N} = \frac{\sqrt{\pi}\, 2^{1/2+\nu} R^{1/2-\nu}}{\Gamma(1-\nu)}. \tag{5.10}$$

It is worth mentioning that this formula, to the best of our knowledge, is new. We remark that the determinant formulas (5.8) and (5.10) agree when $\nu = 0$ (both equal $\sqrt{2\pi R}$) exactly as they should.

5.3. Relative determinant formula

Let us recall that Δ_{D}, Δ_{N} denote the self-adjoint extensions determined by the D extension, N extension at the conical point. Applying the gluing formula in Theorem 4.2, we obtain

$$\frac{\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{X,D})}{\det_\zeta \Delta_Y} = C\frac{\det_\zeta \mathcal{R}_{\mathrm{D}}}{\det L_{\mathrm{D}}}, \qquad \frac{\det_\zeta(\Delta_{\mathrm{N}}, \Delta_{X,N})}{\det_\zeta \Delta_Y} = C\frac{\det_\zeta \mathcal{R}_{\mathrm{N}}}{\det L_{\mathrm{N}}},$$

where C is a constant, independent of the choice of self-adjoint extension, $\mathcal{R}_{\mathrm{D}}, \mathcal{R}_{\mathrm{N}}$ denote the BFK operators and $L_{\mathrm{D}}, L_{\mathrm{N}}$ the operators defined in (1.5) for $\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}}$, respectively. Combining these formulas, we obtain

$$\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}}) = \det_\zeta(\Delta_{X,D}, \Delta_{X,N}) \cdot \frac{\det L_{\mathrm{N}}}{\det L_{\mathrm{D}}} \cdot \frac{\det_\zeta \mathcal{R}_{\mathrm{D}}}{\det_\zeta \mathcal{R}_{\mathrm{N}}}. \tag{5.11}$$

Let $\{\lambda_\ell\}$ denote the set of all eigenvalues of A and recall that Π_V denotes the orthogonal projection onto $V \subset L^2(\Gamma, E_\Gamma)$. We can write

$$\Delta_{X,D} = \Pi_V \Delta_{X,D} \Pi_V + \Pi_V^\perp \Delta_{X,D} \Pi_V^\perp = \bigoplus_{-1/4 \leqslant \lambda_\ell < 3/4} \Pi_{\lambda_\ell} \mathcal{L}_{\ell,D} \Pi_{\lambda_\ell} + \Pi_V^\perp \Delta_{X,D} \Pi_V^\perp,$$

where Π_{λ_ℓ} is the orthogonal projection onto the eigenspace corresponding to λ_ℓ and

$$\mathcal{L}_\ell := -\frac{d^2}{dr^2} + \lambda_\ell r^{-2}$$

over $[0, 1]$ with the Dirichlet condition at $r = 1$ and $\mathcal{L}_{\ell,D}$ represents the D extension of $\mathcal{L}_\ell$. We can write $\Delta_{X,N}$ in a similar manner. Observe that $\Pi_V^\perp \Delta_{X,D} \Pi_V^\perp \equiv \Pi_V^\perp \Delta_{X,N} \Pi_V^\perp$ since the eigenspaces with eigenvalues $\geqslant 3/4$ give rise to an essentially self-adjoint operator. Hence,

$$\det_\zeta(\Delta_{X,D}, \Delta_{X,N}) = \prod_{-1/4 \leqslant \lambda_\ell < 3/4} \det_\zeta(\mathcal{L}_{\ell,D}, \mathcal{L}_{\ell,N}). \tag{5.12}$$

Recalling $\nu_\ell := \sqrt{\lambda_\ell + 1/4}$, by (5.8) and (5.10), we have

$$\det_\zeta(\Delta_{X,D}, \Delta_{X,N}) = \prod_{0 < \nu_\ell < 1} 2^{-2\nu_\ell} \frac{\Gamma(1 - \nu_\ell)}{\Gamma(1 + \nu_\ell)}.$$

Thus,

$$\det_\zeta(\Delta_{\mathrm{D}}, \Delta_{\mathrm{N}}) = \frac{\det L_{\mathrm{N}}}{\det L_{\mathrm{D}}} \prod_{0 < \nu_\ell < 1} 2^{-2\nu_\ell} \frac{\Gamma(1 - \nu_\ell)}{\Gamma(1 + \nu_\ell)} \cdot \frac{\det_\zeta \mathcal{R}_{\mathrm{D}}}{\det_\zeta \mathcal{R}_{\mathrm{N}}}. \tag{5.13}$$

Now we reduce the ratio $\det_\zeta \mathcal{R}_{\mathrm{D}} / \det_\zeta \mathcal{R}_{\mathrm{N}}$ to the ratio of finite-dimensional matrices under the condition that $\mathcal{N}_Y$ is of clean type. Here, $\mathcal{N}_Y$ is the Dirichlet-to-Neumann map for the restriction of Δ to Y and of clean type means that $\mathcal{N}_Y$ maps either V or $V^\perp$ into itself. Recall that $\mathcal{R}_{\mathrm{D}}$ and $\mathcal{R}_{\mathrm{N}}$ are sums of Dirichlet-to-Neumann maps:

$$\mathcal{R}_{\mathrm{D}} = \mathcal{N}_{X,D} + \mathcal{N}_Y, \qquad \mathcal{R}_{\mathrm{N}} = \mathcal{N}_{X,N} + \mathcal{N}_Y,$$

where $\mathcal{N}_{X,D}, \mathcal{N}_{X,N}$ denote the Dirichlet-to-Neumann maps of $\Delta_{X,D}$ and $\Delta_{X,N}$, respectively. Note that from (4.7), $\mathcal{N}_{X,D}, \mathcal{N}_{X,N}$ preserve the eigenspaces of A. It follows that with respect to the decomposition $L^2(\Gamma, E_\Gamma) = V \oplus V^\perp$, $\mathcal{R}_{\mathrm{D}}$ and $\mathcal{R}_{\mathrm{N}}$ have block matrix decompositions of the form

$$\mathcal{R}_{\mathrm{D}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathcal{R}_{\mathrm{N}} = \begin{pmatrix} a' & b \\ c & d \end{pmatrix}.$$

Then $\mathcal{N}_Y$ is of clean type if and only if $b = 0$ or $c = 0$.

Proposition 5.4. *Assume that $\mathcal{N}_Y$ is of clean type. Then we have*

$$\frac{\det_\zeta \mathcal{R}_{\mathrm{D}}}{\det_\zeta \mathcal{R}_{\mathrm{N}}} = \frac{\det^* a}{\det^* a'},$$

where $\det^*$ *denotes the determinant over the orthogonal complement of the kernel of the matrix.*

Proof. Note that

$$\zeta(s, \mathcal{R}_{\mathrm{D}}, \mathcal{R}_{\mathrm{N}}) := \zeta(s, \mathcal{R}_{\mathrm{D}}) - \zeta(s, \mathcal{R}_{\mathrm{N}}) = \frac{1}{2\pi i} \int_\Gamma \lambda^{-s} \operatorname{Tr}\big((\mathcal{R}_{\mathrm{D}} - \lambda)^{-1} - (\mathcal{R}_{\mathrm{N}} - \lambda)^{-1}\big)\, d\lambda,$$

where Γ is an appropriate contour in the complex plane. Then

$$\frac{\det_\zeta \mathcal{R}_{\mathrm{D}}}{\det_\zeta \mathcal{R}_{\mathrm{N}}} = \exp\big(-\zeta'(0, \mathcal{R}_{\mathrm{D}}, \mathcal{R}_{\mathrm{N}})\big).$$

On the other hand,

$$\frac{\det^* a}{\det^* a'} = \exp\big(-\zeta'(0, a, a')\big),$$

where $\zeta(s, a, a')$ denotes the relative ζ-function of the pair (a, a'). So, it suffices to prove that under our assumption, we have

$$\zeta(s, \mathcal{R}_{\mathrm{D}}, \mathcal{R}_{\mathrm{N}}) = \zeta(s, a, a').$$

To prove this, observe that

$$\begin{aligned}
(\mathcal{R}_{\mathrm{D}} - \lambda)^{-1} - (\mathcal{R}_{\mathrm{N}} - \lambda)^{-1} &= (\mathcal{R}_{\mathrm{D}} - \lambda)^{-1}\big((\mathcal{R}_{\mathrm{N}} - \lambda) - (\mathcal{R}_{\mathrm{D}} - \lambda)\big)(\mathcal{R}_{\mathrm{N}} - \lambda)^{-1} \\
&= (\mathcal{R}_{\mathrm{D}} - \lambda)^{-1}(\mathcal{R}_{\mathrm{N}} - \mathcal{R}_{\mathrm{D}})(\mathcal{R}_{\mathrm{N}} - \lambda)^{-1} \\
&= (\mathcal{R}_{\mathrm{D}} - \lambda)^{-1} \begin{pmatrix} a' - a & 0 \\ 0 & 0 \end{pmatrix} (\mathcal{R}_{\mathrm{N}} - \lambda)^{-1}.
\end{aligned}$$

Case 1. Assume that $b=0$. Then one can check that

$$(\mathcal{R}_{\rm D}-\lambda)^{-1}=\begin{pmatrix}(a-\lambda)^{-1} & 0\\ A(\lambda) & (d-\lambda)^{-1}\end{pmatrix},\quad A(\lambda)=-(d-\lambda)^{-1}c(a-\lambda)^{-1},\quad \text{and}$$
$$(\mathcal{R}_{\rm N}-\lambda)^{-1}=\begin{pmatrix}(a'-\lambda)^{-1} & 0\\ A'(\lambda) & (d-\lambda)^{-1}\end{pmatrix},\quad A'(\lambda)=-(d-\lambda)^{-1}c(a'-\lambda)^{-1}.$$

Using these formulas, a short computation shows that

$$\begin{aligned}(\mathcal{R}_{\rm D}-\lambda)^{-1}-(\mathcal{R}_{\rm N}-\lambda)^{-1}&=(\mathcal{R}_{\rm D}-\lambda)^{-1}\begin{pmatrix}a'-a & 0\\ 0 & 0\end{pmatrix}(\mathcal{R}_{\rm N}-\lambda)^{-1}\\ &=\begin{pmatrix}(a-\lambda)^{-1}(a'-a)(a'-\lambda)^{-1} & 0\\ A(\lambda)(a'-a)(a'-\lambda)^{-1} & 0\end{pmatrix}.\end{aligned}$$

It follows that

$$\operatorname{Tr}\big((\mathcal{R}_{\rm D}-\lambda)^{-1}-(\mathcal{R}_{\rm N}-\lambda)^{-1}\big)=\operatorname{Tr}\big((a-\lambda)^{-1}-(a'-\lambda)^{-1}\big),$$

which prove that $\zeta(s,\mathcal{R}_{\rm D},\mathcal{R}_{\rm N})=\zeta(s,a,a')$ in the case when $b=0$.

Case 2. Assume that $c=0$. Then one can check that

$$(\mathcal{R}_{\rm D}-\lambda)^{-1}=\begin{pmatrix}(a-\lambda)^{-1} & B(\lambda)\\ 0 & (d-\lambda)^{-1}\end{pmatrix},\quad B(\lambda)=-(a-\lambda)^{-1}b(d-\lambda)^{-1},\quad \text{and}$$
$$(\mathcal{R}_{\rm N}-\lambda)^{-1}=\begin{pmatrix}(a'-\lambda)^{-1} & B'(\lambda)\\ 0 & (d-\lambda)^{-1}\end{pmatrix},\quad B'(\lambda)=-(a'-\lambda)^{-1}b(d-\lambda)^{-1}.$$

Using these formulas, a short computation shows that

$$\begin{aligned}(\mathcal{R}_{\rm D}-\lambda)^{-1}-(\mathcal{R}_{\rm N}-\lambda)^{-1}&=(\mathcal{R}_{\rm D}-\lambda)^{-1}\begin{pmatrix}a'-a & 0\\ 0 & 0\end{pmatrix}(\mathcal{R}_{\rm N}-\lambda)^{-1}\\ &=\begin{pmatrix}(a-\lambda)^{-1}(a'-a)(a'-\lambda)^{-1} & (a-\lambda)^{-1}(a'-a)B'(\lambda)\\ 0 & 0\end{pmatrix}.\end{aligned}$$

It follows that

$$\operatorname{Tr}\big((\mathcal{R}_{\rm D}-\lambda)^{-1}-(\mathcal{R}_{\rm N}-\lambda)^{-1}\big)=\operatorname{Tr}\big((a-\lambda)^{-1}-(a'-\lambda)^{-1}\big),$$

which prove that $\zeta(s,\mathcal{R}_{\rm D},\mathcal{R}_{\rm N})=\zeta(s,a,a')$ in the case when $c=0$. □

Denote the Dirichlet-to-Neumann maps of $\mathcal{L}_{\ell,D},\mathcal{L}_{\ell,N}$ over $[0,1]$ by $\mathcal{N}_{\ell,D},\mathcal{N}_{\ell,N}$; then we prove:

Lemma 5.5. *We have*

$$\mathcal{N}_{\ell,D}=1/2+\nu_\ell,\qquad \mathcal{N}_{\ell,N}=1/2-\nu_\ell.$$

Proof. Assume for the moment that $\nu_\ell := \sqrt{\lambda_\ell + 1/4} > 0$. Then an elementary computation shows that $\mathcal{L}_\ell\phi = 0$ if and only if

$$\phi = c_1 r^{\frac{1}{2}+\nu_\ell} + c_2 r^{\frac{1}{2}-\nu_\ell}.$$

For the D extension, we require $c_2 = 0$ and for the N extension, $c_1 = 0$. Therefore, given $\varphi \in \mathbb{C}$, the unique solutions of $\mathcal{L}_{\ell,D}\phi_{\mathrm{D}} = 0$, $\mathcal{L}_{\ell,N}\phi_{\mathrm{N}} = 0$ with value φ at $r = 1$ are

$$\phi_{\mathrm{D}} = \varphi r^{\frac{1}{2}+\nu_\ell}, \qquad \phi_{\mathrm{N}} = \varphi r^{\frac{1}{2}-\nu_\ell}.$$

Now using the definitions $\mathcal{N}_{\ell,D}\varphi := \frac{d}{dr}|_{r=1}\phi_{\mathrm{D}}$ and $\mathcal{N}_{\ell,N}\varphi := \frac{d}{dr}|_{r=1}\phi_{\mathrm{N}}$ completes the proof for the case that $\nu_\ell > 0$. In the case that $\nu_\ell = 0$, the D and N extensions agree and $\phi_{\mathrm{D}} = \phi_{\mathrm{N}} = \varphi r^{1/2}$. Taking derivatives completes the proof in this case too. □

Combining (5.13) with Proposition 5.4 and Lemma 5.5, we finally complete the proof of Theorem 1.1.

5.4. Relative formulas for mixed D and N type

Fix a Lagrangian subspace $\Lambda \subset V$ of *mixed D and N type* in the sense that

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_q,$$

where (cf. (2.14)) $\Lambda_\ell \subset \operatorname{span}\{\psi_\ell^\pm\}$ equals either $\operatorname{span}\{\psi_\ell^+\}$ or $\operatorname{span}\{\psi_\ell^-\}$, except when $\lambda_\ell = -1/4$, in which case we only choose $\operatorname{span}\{\psi_\ell^+\}$ (for instance, both the D and N extensions have this property). Let Δ_Λ denote the Laplace type operator Δ with domain D_Λ corresponding to Λ. We put

$$P_\Lambda := \bigl\{\ell\colon\ \Lambda_\ell = \operatorname{span}\bigl\{\psi_\ell^+\bigr\}\bigr\};$$

for example, if $\Lambda = D$, then $P_{\mathrm{D}} = \{1,\dots,q\}$ and in the case $\Lambda = N$, we have $P_{\mathrm{N}} = \{1,\dots,q_0\}$, where q_0 is the multiplicity of the $-\frac{1}{4}$ eigenvalue of the operator A in (2.6). The set P_Λ represents a type of "perversity" in the spirit of Cheeger, Goresky and MacPherson [14,21]. Define L_Λ as in (1.5) using $\ker\Delta_\Lambda$ and put $\mathcal{M}_\Lambda := \nu_\ell$ on E_ℓ if $\ell \in P_\Lambda$ and $\mathcal{M}_\Lambda := -\nu_\ell$ on E_ℓ if $\ell \notin P_\Lambda$. Then following the same argument in Section 5.3, it is straightforward to prove the following generalization of Theorem 1.1.

Theorem 5.6. *When Δ_Y is invertible and $\mathcal{N}_Y$ is of clean type, the following relative determinant formula holds*:

$$\det\nolimits_\zeta(\Delta_\Lambda, \Delta_{\mathrm{N}}) = \prod_{\ell\in P_\Lambda} 2^{-2\nu_\ell}\frac{\Gamma(1-\nu_\ell)}{\Gamma(1+\nu_\ell)} \cdot \frac{\det L_{\mathrm{N}}}{\det L_\Lambda} \cdot \frac{\det^*(\frac{\mathrm{Id}}{2} + \mathcal{M}_\Lambda + \mathcal{N}_{Y,V})}{\det^*(\frac{\mathrm{Id}}{2} - \mathcal{M}_\nu + \mathcal{N}_{Y,V})}.$$

5.5. *Relative formulas for arbitrary self-adjoint extensions*

We now generalize Theorems 1.1 and 5.6 to the case of arbitrary self-adjoint extensions.

Before doing so, we need to give an explicit description of self-adjoint extensions and then discuss the corresponding zeta functions. Let $E_\ell^\pm := \operatorname{span}_{\mathbb{C}}\{\psi_\ell^\pm\}$, where $\psi_\ell^\pm$ are given around (2.9) in Section 2.3. Then we can identify

$$V := \bigoplus_{-1/4\leqslant\lambda_\ell<3/4} E_\ell^+ \oplus E_\ell^- = \Bigl(\bigoplus_{-1/4\leqslant\lambda_\ell<3/4} E_\ell^+\Bigr) \oplus \Bigl(\bigoplus_{-1/4\leqslant\lambda_\ell<3/4} E_\ell^-\Bigr) \cong \mathbb{C}^q \oplus \mathbb{C}^q = \mathbb{C}^{2q}.$$

Using elementary symplectic linear algebra (see [26, Section 3]) one can show that a subspace $\Lambda \subset V$ is Lagrangian if and only if there exists $q \times q$ complex matrices $\mathcal{A}$ and $\mathcal{B}$ such that the rank of the $q \times 2q$ matrix $(\mathcal{A}\ \mathcal{B})$ is q, $\mathcal{A}'\mathcal{B}^*$ is self-adjoint where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns multiplied by -1, and

$$\Lambda = \bigl\{v \in \mathbb{C}^{2q}\colon\ (\mathcal{A} \quad \mathcal{B})\, v = 0\bigr\}, \tag{5.14}$$

where we identify $\Lambda \subset V$ with its image under the isomorphism $V \cong \mathbb{C}^{2q}$.

Given such a Lagrangian $\Lambda \subset V$ we can form the zeta function $\zeta(s, \Delta_\Lambda)$. The main result of [26] shows that this zeta function, in general, has "exotic" singularities such as poles of arbitrary finite order and even logarithmic singularities of arbitrary finite order, and it also gives an algebraic-combinatorial algorithm that finds these singularities explicitly. This algorithm is described as follows.

Step 1. Let $\mathcal{A}$ and $\mathcal{B}$ be as in (5.14) and define the function

$$p(x, y) := \det \begin{pmatrix} \begin{array}{cccc|c} & & \mathcal{A} & & \mathcal{B} \\ x\,\mathrm{Id}_{q_0} & 0 & 0 & 0 & \\ 0 & \tau_1 y^{2\nu_1} & 0 & 0 & \\ & & & & \mathrm{Id}_q \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} & \end{array} \end{pmatrix}, \tag{5.15}$$

where Id_k denotes the $k \times k$ identity matrix and where

$$\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}}, \qquad \tau_j = 2^{2\nu_j} \frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}, \quad j = 1, \dots, q_1,$$

with q_0 the multiplicity of the $-1/4$ eigenvalue of A, $q_1 := q - q_0$, and $\{\lambda_j\}$ the eigenvalues of A. Expanding the determinant, we can write $p(x, y)$ as a finite sum

$$p(x, y) = \sum a_{j\alpha} x^j y^{2\alpha},$$

where the α's are linear combinations of $\nu_1, \dots, \nu_{q_1}$ and the $a_{j\alpha}$'s are constants. Let α_0 be the smallest of all α's with $a_{j\alpha} \neq 0$ and let j_0 be the smallest of all j's amongst the $a_{j\alpha_0} \neq 0$. Then factoring out the term $a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0}$ in $p(x, y)$ we can write $p(x, y)$ in the form

$$p(x, y) = a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0} \Bigl(1 + \sum b_{k\beta} x^k y^{2\beta}\Bigr) \tag{5.16}$$

for some constants $b_{k\beta}$ (equal to $a_{k\beta}/a_{j_0\alpha_0}$). Note that the β's are nonnegative but the k's can be negative.

Step 2. Second, putting $z = \sum b_{k\beta} x^k y^{2\beta}$ into the power series $\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$ and *formally* expanding, we can write

$$\log\Bigl(1 + \sum b_{k\beta} x^k y^{2\beta}\Bigr) = \sum c_{\ell\xi} x^{\ell} y^{2\xi} \tag{5.17}$$

for some constants $c_{\ell\xi}$. By construction, the ξ's appearing in (5.17) are nonnegative, countable, and approach $+\infty$ unless $\beta = 0$ is the only β in (5.16), in which case only $\xi = 0$ occurs in (5.17). Also, for a fixed ξ, the ℓ's with $c_{\ell\xi} \neq 0$ are bounded below.

Step 3. Third, for each ξ appearing in (5.17), define

$$p_\xi := \min\{\ell \leqslant 0 \mid c_{\ell\xi} \neq 0\} \quad \text{and} \quad \ell_\xi := \min\{\ell > 0 \mid c_{\ell\xi} \neq 0\}, \tag{5.18}$$

whenever the sets $\{\ell \leqslant 0 \mid c_{\ell\xi} \neq 0\}$ and $\{\ell > 0 \mid c_{\ell\xi} \neq 0\}$, respectively, are nonempty. Let $\mathscr{P}$, respectively $\mathscr{L}$, denote the set of ξ values for which the respective sets are nonempty. Finally, put $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The following theorem is the main result in [26, Theorem 2.1].

Theorem 5.7. *The ζ-function $\zeta(s, \Delta_\Lambda)$ extends from $\Re s > n/2$ to a meromorphic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, $\zeta(s, \Delta_\Lambda)$ can be written in the form*

$$\zeta(s, \Delta_\Lambda) = \zeta_{\mathrm{reg}}(s, \Delta_\Lambda) + \zeta_{\mathrm{sing}}(s, \Delta_\Lambda),$$

where $\zeta_{\mathrm{reg}}(s, \Delta_\Lambda)$ has possible "regular" poles at the "usual" locations $s = \frac{n-k}{2}$ with $s \notin -\mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$, and where $\zeta_{\mathrm{sing}}(s, \Delta_\Lambda)$ has the following expansion:

$$\zeta_{\mathrm{sing}}(s, \Delta_\Lambda) = \frac{\sin(\pi s)}{\pi} \Biggl\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in \mathscr{P}} \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}} + \sum_{\xi \in \mathscr{L}} g_\xi(s) \log(s+\xi) \Biggr\}, \tag{5.19}$$

where j_0 appears in (5.16), γ is the Euler–Mascheroni constant, and $f_\xi(s)$ and $g_\xi(s)$ are entire functions of s such that

$$f_\xi(-\xi) = (-1)^{|p_\xi|+1} c_{p_\xi \xi} \frac{|p_\xi|!}{2^{|p_\xi|}} \xi \quad \textit{and}$$

$$g_\xi(s) = \begin{cases} c_{\ell_0,0} \frac{2^{\ell_0}}{(\ell_0-1)!} s^{\ell_0} + \mathcal{O}(s^{\ell_0+1}) & \textit{if } \xi = 0, \\ -c_{\ell_\xi \xi} \frac{\xi 2^{\ell_\xi}}{(\ell_\xi-1)!} (s+\xi)^{\ell_\xi - 1} + \mathcal{O}((s+\xi)^{\ell_\xi}) & \textit{if } \xi > 0. \end{cases}$$

Remark 5.8. The regular part $\zeta_{\mathrm{reg}}(s, \Delta_\Lambda)$ will only have possible poles at $s = \frac{n-k}{2} \notin -\mathbb{N}$ and the residue of $\zeta_{\mathrm{reg}}(s, \Delta_\Lambda)$ at $s = 0$ is given by

$$\operatorname{Res}_{s=0} \zeta_{\mathrm{reg}}(s, \Delta_\Lambda) = -\frac{1}{2} \operatorname{Res}_{s=-1/2} \zeta(s, A);$$

in particular, this pole is independent of the choice of self-adjoint extension and it vanishes if $\zeta(s, A)$ is in fact analytic at $s = -1/2$. The expansion (5.19) means that for any $N \in \mathbb{N}$,

$$\zeta_{\text{sing}}(s,\Delta_\Lambda)=\frac{\sin(\pi s)}{\pi}\Biggl\{(j_0-q_0)e^{-2s(\log 2-\gamma)}\log s+\sum_{\xi\in\mathcal{P},\,\xi\leqslant N}\frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}}$$
$$+\sum_{\xi\in\mathcal{L},\,\xi\leqslant N}g_\xi(s)\log(s+\xi)\Biggr\}+F_{\text{N}}(s),$$

where $F_{\text{N}}(s)$ is holomorphic for $\Re s\geqslant -N$. Note that the leading terms as $s\to 0$ are contained in $\zeta_{\text{reg}}(s,\Delta_\Lambda)$ and the first term of $\zeta_{\text{sing}}(s,\Delta_\Lambda)$.

For a general self-adjoint extension, Theorem 5.7 shows that the $\zeta(s,\Delta_\Lambda)$ may not only have the usual simple pole at $s=0$ (from $\zeta_{\text{reg}}(s,\Delta_\Lambda)$) but also a logarithmic singularity at $s=0$. Thus, the zeta function, in general, has a logarithmic singularity at $s=0$ except for certain self-adjoint extensions; for example, of mixed D and N type. In particular, the usual definition of the zeta-regularized determinant or even the relative determinant is ill-defined via taking the derivative of $\zeta(s,\Delta_\Lambda)$ at $s=0$. However, we can still associate a natural definition of a determinant by subtracting off the singularities. With this in mind, let us define

$$\zeta_0(s,\Delta_\Lambda):=\zeta(s,\Delta_\Lambda)-\frac{c}{s}-(j_0-q_0)s\log s,$$

where $c:=\operatorname{Res}_{s=0}\zeta_{\text{reg}}(s,\Delta_\Lambda)=-(1/2)\operatorname{Res}_{s=-1/2}\zeta(s,A)$. The term c/s cancels the possible pole of $\zeta_{\text{reg}}(s,\Delta_\Lambda)$ at $s=0$ and by the explicit formula (5.19) for $\zeta_{\text{sing}}(s,\Delta_\Lambda)$, the term $(j_0-q_0)s\log s$ cancels the logarithmic singularity of $\zeta_{\text{sing}}(s,\Delta_\Lambda)$ at $s=0$ up to a term that is $\mathcal{O}(s^2\log s)$ at $s=0$. It follows that $\lim_{s\to 0^+}\zeta_0'(s,\Delta_\Lambda)$ exists. Therefore, we can define

$$\det{}_\zeta(\Delta_\Lambda):=\exp\Bigl(-\lim_{s\to 0^+}\zeta_0'(s,\Delta_\Lambda)\Bigr).$$

This definition of course agrees with the standard definition in case $\zeta(s,\Delta_\Lambda)$ is regular at $s=0$. In Theorem 5.10 below we get an explicit formula for the relative determinant

$$\det{}_\zeta(\Delta_\Lambda,\Delta_{\text{N}}).$$

Before doing so, we need the following theorem, proved in [27, Theorem 2.3] and using methods from [4–6], which gives an explicit formula for the relative determinants of the model operators.

Theorem 5.9. *For a Lagrangian $\Lambda\subset V$ given in terms of matrices $\mathcal{A}$ and $\mathcal{B}$ as in* (5.14)*, such that the operator $\Delta_{X,\Lambda}$ is invertible, we have*

$$\det{}_\zeta(\Delta_{X,\Lambda},\Delta_{X,N})=\frac{(-2e^\gamma)^{q_0-j_0}}{a_{j_0\alpha_0}}\det\begin{array}{c}\begin{array}{cc}\quad\mathcal{A}\quad & \quad\mathcal{B}\quad\end{array}\\ \begin{pmatrix}\mathrm{Id}_{q_0} & 0 & 0 & 0\\ 0 & \mathrm{Id}_{q_1} & 0 & \mathrm{Id}_{q_1}\end{pmatrix}\end{array},$$

where j_0 appears in (5.16)*, $a_{j_0\alpha_0}$ is the coefficient in* (5.16)*, q_0 is the multiplicity of the $-1/4$ eigenvalue of A, $q_1:=q-q_0$, γ is the Euler–Mascheroni constant, and where Id_k denotes the $k\times k$ identity matrix.*

This formula gives an explicit formula for the relative determinant of the model operator because the constant $a_{j_0\alpha_0}$ is always explicitly computable for any given $\mathcal{A}$ and $\mathcal{B}$ determining a Lagrangian $L \subset V$.

Now fix a Lagrangian $\Lambda \subset V$ such that the operator $\Delta_{X,\Lambda}$ is invertible. Then following the identical arguments that lead up to (5.11), one can show that

$$\det{}_\zeta(\Delta_\Lambda, \Delta_{\mathrm{N}}) = \det{}_\zeta(\Delta_{X,\Lambda}, \Delta_{X,N}) \cdot \frac{\det L_{\mathrm{N}}}{\det L_\Lambda} \cdot \frac{\det_\zeta \mathcal{R}_\Lambda}{\det_\zeta \mathcal{R}_{\mathrm{N}}},$$

where L_Λ is defined as in (1.5) using $\ker \Delta_\Lambda$. Indeed, this only requires slight modifications of the arguments in the gluing formulas that were used to derive (5.11); the only nontrivial fact one needs to check is that Proposition 4.3 holds for general Lagrangians $\Lambda \subset V$ such that $\Delta_{X,\Lambda}$ is invertible, but one can show this using the explicit description of elements in the domain of Δ_Λ found in [26, Section 4.2]. Therefore, by Theorem 5.9, we have

$$\det{}_\zeta(\Delta_\Lambda, \Delta_{\mathrm{N}}) = \frac{(-2e^\gamma)^{q_0 - j_0}}{a_{j_0\alpha_0}} \det \begin{pmatrix} \multicolumn{2}{c}{\mathcal{A}} & \multicolumn{2}{c}{\mathcal{B}} \\ \mathrm{Id}_{q_0} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_{q_1} & 0 & \mathrm{Id}_{q_1} \end{pmatrix} \frac{\det L_{\mathrm{N}}}{\det L_\Lambda} \cdot \frac{\det_\zeta \mathcal{R}_\Lambda}{\det_\zeta \mathcal{R}_{\mathrm{N}}}.$$

Recall that $\mathcal{R}_\Lambda$ and $\mathcal{R}_{\mathrm{N}}$ are sums of Dirichlet-to-Neumann maps:

$$\mathcal{R}_\Lambda = \mathcal{N}_{X,\Lambda} + \mathcal{N}_Y, \qquad \mathcal{R}_{\mathrm{N}} = \mathcal{N}_{X,N} + \mathcal{N}_Y = \frac{\mathrm{Id}}{2} - \mathcal{M}_\nu + \mathcal{N}_Y,$$

where $\mathcal{N}_{X,\Lambda}$ denotes the Dirichlet-to-Neumann map of $\Delta_{X,\Lambda}$, where $\mathcal{M}_\nu$ is the linear map on V defined by $\mathcal{M}_\nu := \nu_\ell|_{E_\ell}$ and where $\mathcal{N}_{Y,V} := \Pi_V \mathcal{N}_Y \Pi_V$ with $\mathcal{N}_Y$ the Dirichlet-to-Neumann map for the restriction of Δ to Y. Finally, note that $\mathcal{N}_{X,\Lambda}$ preserves V, therefore using Proposition 5.4, we obtain the following theorem.

Theorem 5.10. *Let $\Lambda \subset V$ be Lagrangian such that the operator $\Delta_{X,\Lambda}$ is invertible, and assume that Δ_Y is invertible and $\mathcal{N}_Y$ is of clean type. Then the following relative determinant formula holds*:

$$\det{}_\zeta(\Delta_\Lambda, \Delta_{\mathrm{N}}) = \frac{(-2e^\gamma)^{q_0 - j_0}}{a_{j_0\alpha_0}} \det \begin{pmatrix} \multicolumn{2}{c}{\mathcal{A}} & \multicolumn{2}{c}{\mathcal{B}} \\ \mathrm{Id}_{q_0} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_{q_1} & 0 & \mathrm{Id}_{q_1} \end{pmatrix} \frac{\det L_{\mathrm{N}}}{\det L_\Lambda} \cdot \frac{\det^*(\mathcal{N}_{X,\Lambda} + \mathcal{N}_{Y,V})}{\det^*(\frac{\mathrm{Id}}{2} - \mathcal{M}_\nu + \mathcal{N}_{Y,V})},$$

where the first term on the right is an explicitly computable number explained in Theorem 5.9, *and where $\mathcal{N}_{Y,V} := \Pi_V \mathcal{N}_Y \Pi_V$.*

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1992. Reprint of the 1972 ed.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry III, Math. Proc. Cambridge Philos. Soc. 79 (1976) 71–99.

[3] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer, New York, 1992.

[4] M. Bordag, B. Geyer, K. Kirsten, E. Elizalde, Zeta function determinant of the Laplace operator on the D-dimensional ball, Comm. Math. Phys. 179 (1) (1996) 215–234.

[5] M. Bordag, K. Kirsten, S. Dowker, Heat-kernels and functional determinants on the generalized cone, Comm. Math. Phys. 182 (2) (1996) 371–393.
[6] M. Bordag, K. Kirsten, E. Elizalde, Heat kernel coefficients of the Laplace operator on the D-dimensional ball, J. Math. Phys. 37 (1) (1996) 895–916.
[7] J. Brüning, R. Seeley, The resolvent expansion for second order regular singular operators, J. Funct. Anal. 73 (2) (1987) 369–429.
[8] J. Brüning, R. Seeley, An index theorem for first order regular singular operators, Amer. J. Math. 110 (1988) 659–714.
[9] D. Burghelea, L. Friedlander, T. Kappeler, Meyer–Vietoris-type formula for determinants of elliptic differential operators, J. Funct. Anal. 107 (1) (1992) 34–65.
[10] C. Callias, The heat equation with singular coefficients I. Operators of the form $-d^2/dx^2 + \kappa/x^2$ in dimension 1, Comm. Math. Phys. 88 (3) (1983) 357–385.
[11] G. Carron, Déterminant relatif et fonction Xi, Amer. J. Math. 124 (2002) 307–352.
[12] J. Cheeger, Hodge theory of complex cones, in: Analysis and Topology on Singular Spaces, II, III, Luminy, 1981, Astérisque 101 (1983) 118–134.
[13] J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Differential Geom. 18 (4) (1983) 575–657.
[14] J. Cheeger, M. Goresky, R. MacPherson, L^2-cohomology and intersection homology of singular algebraic varieties, in: Seminar on Differential Geometry, in: Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, NJ, 1982, pp. 303–340.
[15] A.W. Chou, The Dirac operator on spaces with conical singularities and positive scalar curvatures, Trans. Amer. Math. Soc. 289 (1) (1985) 1–40.
[16] N. Dunford, J. Schwarz, Linear Operators I, II, III, Interscience, New York, 1963.
[17] H. Falomir, P.A.G. Pisani, A. Wipf, Pole structure of the Hamiltonian ζ-function for a singular potential, J. Phys. A 35 (26) (2002) 5427–5444.
[18] H. Falomir, M.A. Muschietti, P.A.G. Pisani, R. Seeley, Unusual poles of the ζ-functions for some regular singular differential operators, J. Phys. A 36 (39) (2003) 9991–10010.
[19] H. Falomir, M.A. Muschietti, P.A.G. Pisani, On the resolvent and spectral functions of a second order differential operator with a regular singularity, J. Math. Phys. 45 (12) (2004) 4560–4577.
[20] J.B. Gil, G.A. Mendoza, Adjoints of elliptic cone operators, Amer. J. Math. 125 (2) (2003) 357–408.
[21] M. Goresky, R. MacPherson, Intersection homology theory, Topology 19 (2) (1980) 135–162.
[22] P. Greiner, An asymptotic expansion for the heat equation, Arch. Ration. Mech. Anal. 41 (1971) 163–218.
[23] A. Hassell, Analytic surgery and analytic torsion, Comm. Anal. Geom. 6 (2) (1998) 255–289.
[24] A. Hassell, S. Zelditch, Determinants of Laplacians in exterior domains, Int. Math. Res. Not. 18 (1999) 971–1004.
[25] C. Iwasaki, The asymptotic expansion of the fundamental solution for parabolic initial–boundary value problems and its application, Osaka J. Math. 31 (3) (1994) 663–728.
[26] K. Kirsten, P. Loya, J. Park, Exotic expansions and pathological properties of ζ-functions on conic manifolds, preprint, 2005.
[27] K. Kirsten, P. Loya, J. Park, ζ-Determinants of general self-adjoint extensions of Laplace-type operators on generalized cones, preprint, 2005.
[28] Y. Lee, Mayer–Vietoris formula for determinants of elliptic operators of Laplace–Beltrami type (after Burghelea, Friedlander and Kappeler), Differential Geom. Appl. 7 (4) (1997) 325–340.
[29] M. Lesch, Determinants of regular singular Sturm–Liouville operators, Math. Nachr. 194 (1998) 139–170.
[30] S. Levit, U. Smilansky, A theorem on infinite products of eigenvalues of Sturm–Liouville type operators, Proc. Amer. Math. Soc. 65 (1977) 299–305.
[31] P. Loya, J. Park, Decomposition of the ζ-determinant for the Laplacian on manifolds with cylindrical end, Illinois J. Math. 48 (2004) 1279–1303.
[32] P. Loya, J. Park, On the gluing problem for the spectral invariants of Dirac operators, Adv. Math. 202 (2006) 401–450.
[33] R.B. Melrose, The Atiyah–Patodi–Singer Index Theorem, A.K. Peters, 1993.
[34] E. Mooers, Heat kernel asymptotics on manifolds with conic singularities, J. Anal. Math. 78 (1999) 1–36.
[35] W. Müller, Relative zeta functions, relative determinants and scattering theory, Comm. Math. Phys. 192 (2) (1998) 309–347.
[36] J. Müller, W. Müller, Regularized determinants of Laplace type operators, analytic surgery and relative determinants, preprint, 2004.
[37] J. Park, K.P. Wojciechowski, Adiabatic decomposition of the ζ-determinant of the Dirac Laplacian 1: The case of invertible tangential operator, with an appendix by Y. Lee, Comm. Partial Differential Equations 27 (2002) 1407–1435.

[38] J. Park, K.P. Wojciechowski, Scattering theory and adiabatic decomposition of the ζ-determinant of the Dirac Laplacian, Math. Res. Lett. 9 (2002) 17–25.
[39] J. Park, K.P. Wojciechowski, Adiabatic decomposition of the ζ-determinant and the Dirichlet to Neumann operator, J. Geom. Phys. 55 (2005) 241–266.
[40] J. Park, K.P. Wojciechowski, Adiabatic decomposition of the ζ-determinant and scattering theory, Michigan Math. J. 54 (2006) 207–238.
[41] P. Piazza, Determinant bundles, manifolds with boundary and surgery, II: Spectral sections and surgery rules for anomalies, Comm. Math. Phys. 193 (1) (1998) 105–124.
[42] D.B. Ray, I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. 7 (1971) 145–210.
[43] S. Scott, Zeta determinants on manifolds with boundary, J. Funct. Anal. 192 (2002) 112–185.
[44] R. Seeley, The resolvent of an elliptic boundary problem, Amer. J. Math. 91 (1969) 889–920.