FREDHOLM PERTURBATIONS OF DIRAC OPERATORS ON MANIFOLDS WITH CORNERS

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Abstract. A Dirac operator $\eth$ on a compact manifold with corners related to a complete metric on the interior of 'multi-cylindrical' (exact b-) type is Fredholm if and only if the induced Dirac operator on each boundary hypersurface is invertible. In case the boundary codimension is at most two a result of the second author and V. Nistor shows that $\eth$ can be made Fredholm by perturbation with a b-smoothing operator only if the induced Dirac operators on the corners have index zero. In this case we provide explicit Fredholm perturbations and compute the index of the resulting Fredholm operators.

Introduction

The index theorem of Atiyah, Patodi and Singer [1] gives a quite explicit formula for the index of a Dirac operator associated to a manifold 'with cylindrical ends'. By exponential compactification this may be reinterpreted as an index formula in the 'b-category' as discussed extensively in [22]. Thus, the Dirac operators considered are for Clifford bundles $E \longrightarrow X$ over even-dimensional compact manifolds with boundary with metrics which become singular near the boundary components and take the form

$$g = \frac{dx^2}{x^2} + h.$$

Here h is a 2-cotensor smooth up to the boundary and restricting to a metric on it and $x \in C^\infty(X)$ is a defining function for the boundary. Given such geometric data, let $\eth^+ : C^\infty(X, E^+) \longrightarrow C^\infty(X, E^-)$ be the associated Dirac operator. If the metric and connection are of product type near the boundary, then

$$\eth^+ = \Gamma[x\partial_x + \eth_0], \quad \text{in } x < \varepsilon,$$

where Γ is Clifford multiplication by idx/x and $\eth_0$ is the induced Dirac operator on the boundary. Then $\eth^+$ is Fredholm on its natural Sobolev domain if and only if $\eth_0$ is invertible; the index formula of [1] in this case is

$$\operatorname{ind} \eth^+ = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_0), \tag{0.1}$$

where AS is the Atiyah-Singer density and $\eta(\eth_0)$ is the eta invariant of $\eth_0$. Even if $\eth_0$ is not invertible, Atiyah, Patodi and Singer give an index formula which can be interpreted in one of several related ways. One approach is simply to note that on a

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manifold with boundary with exact b-metric, $\eth^+$ is always Fredholm as an operator between weighted Sobolev spaces

$$\eth^+ : x^\alpha H^1_b(X, E^+) \longrightarrow x^\alpha L^2_b(X, E^-), \quad 0 \neq |\alpha| < \delta,\ \delta \text{ small.} \tag{0.2}$$

Here, $H^1_b(X, E^+)$ is the natural domain of $\eth^+$ and δ can be taken to be the smallest absolute value of a non-zero eigenvalue of $\eth_0$. The index formula then becomes

$$\operatorname{ind}_\alpha \eth^+ = \int_X \mathrm{AS} - \frac{1}{2}[\eta(\eth_0) \pm \dim\ker(\eth_0)], \tag{0.3}$$

where the sign is the sign of α. Whilst very natural, this approach fails in higher codimension cases for several related reasons. One difficulty with (0.2) is that it amounts to replacing $\eth^+$ by $x^{-\alpha}\eth^+ x^\alpha$, which has induced boundary operator

$$\eth_0 + \alpha. \tag{0.4}$$

Such a perturbation, by a constant, has the undesirable effect of destroying the structure of $\eth^+$ as an admissible Dirac operator and hence invalidates the local index theorem, which is a key step in the proof of (0.1).

To overcome some of the difficulties with the approach through conjugation, a 'small' perturbation can be used in place of 'α' in (0.4). The space of b-smoothing operators is a natural class of perturbations preserving all the weighted metric Sobolev spaces and such perturbations do not affect the local index theorem. A perturbation $R \in \Psi^{-\infty}_b(X, E^+, E^-)$ may be chosen such that $\eth^+ + R$ is Fredholm on the natural domain of $\eth^+$ and then a direct analogue of (0.1) holds

$$\operatorname{ind}(\eth^+ + R) = \int_X \mathrm{AS} - \frac{1}{2}\widetilde{\eta}(\eth_0 + i\tau - \Gamma N(R)(\tau)). \tag{0.5}$$

Here, we denote by $\widetilde{\eta}$ the generalization of the eta invariant introduced in [23] for suspended families of pseudodifferential operators; $N(R)(\tau)$ is the indicial family of the perturbation.

The third approach to Fredholm perturbations, and the one closest to the methods of [1], is to enlarge the domain of $\eth^+$ to include the 'extended L^2 null space'. This has been systematically studied by Carron [8]. In the case of a manifold with boundary, the enlarged domain is simply

$$\text{E-Dom}(\eth^+) = \{u \in x^{-\delta} H^1_b(X, E^+)\,;\, \eth^+ u \in L^2_b(X, E^-)\}, \quad \delta > 0 \text{ small}$$

and

$$\eth^{\mathrm{E}} : \text{E-Dom}(\eth^+) \longrightarrow L^2_b(X, E^-)$$

is Fredholm. It is relatively easy to relate the indices stemming from these three types of Fredholm problems; namely

$$\operatorname{ind} \eth^{\mathrm{E}} = \operatorname{ind}_\alpha \eth^+ = \operatorname{ind}(\eth^+ + R), \quad \alpha < 0 \text{ small} \tag{0.6}$$

provided $\eth_0 + i\tau - \delta\Gamma N(R)(\tau)$ is invertible for all $0 \leq \delta \leq 1$ and $0 < \operatorname{Im}\tau \leq \delta'$ for some $\delta' > 0$.

The merit of considering Fredholm perturbations R is that the index formula (0.5) is easily evaluated using the approach in [22]. However, a second step is needed to relate the index so obtained to solutions of the unperturbed operator. In the generalization to manifolds with corners here we do not directly relate the index of the perturbed operator to the solutions of the unperturbed operator. However, the formula itself is given in a 'transgressed' form in which the index of the perturbed operator is expressed in terms of unperturbed data, plus a minimal term corresponding to the 'deformation' of the operator.

In this paper we discuss Fredholm perturbations of this type for compact manifolds with corners. If the induced Dirac operators on the corners of codimension two and greater are all invertible, then we can proceed in much the same way as for manifolds with boundary. In particular, conjugation gives a Fredholm operator, cf. (0.2) in the codimension one case, and a direct generalization of (0.3) holds; see Theorem 6.8. Similar relations as in (0.6) also hold. Moreover, the invertibility of these induced operators is also a necessary condition for conjugation to give a Fredholm perturbation; see Theorem 2.1. Unfortunately, these strong invertibility assumptions on the corner Dirac operators are rarely satisfied by geometric operators. For codimension two manifolds with corners, Müller [28] also gives an index formula under these strong invertibility assumptions.

For Dirac operators on manifolds with corners up to codimension two, the weaker assumption that each of the induced Dirac operators on a corner has index zero, is necessary and sufficient for there to exist perturbations $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ such that $\eth^+ + R$ is Fredholm on the natural domain of $\eth^+$. The necessity of this condition follows from work of V. Nistor and the second author. If there is a single corner component the cobordism invariance of the index implies the vanishing of this index. The sufficiency is shown here by the construction of Fredholm perturbations $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ from Lagrangian subspaces of the kernels of the induced Dirac operators on the corners. In future work, using a slightly larger class of perturbations, we expect to remove both the assumption on the Dirac operators on the corners and the limit on the codimension of the boundary.

For simplicity, we shall describe here the form of our results in the special case that X has a single corner component; the general case is treated in the body of the paper. Fix an ordering H_1 and H_2 of the boundary hypersurfaces. If $\eth_M^+$ is the Dirac operator induced on M from H_1 then the Dirac operator induced on M from $\eth_{H_2}$ is

$$i\eth_M^-, \quad \eth_M^- = (\eth_M^+)^*. \tag{0.7}$$

The factor of i and the opposite chirality induced from $\eth_{H_2}$ appear in the 'corner correction term' in the index formula (0.9) below. Set $V = \ker \eth_M^+ \oplus \ker \eth_M^-$. Since $\dim \ker \eth_M^+ = \dim \ker \eth_M^-$, as the index of $\eth_M^+$ is zero, we may choose a unitary isomorphism

$$T^+ : \ker \eth_M^+ \longrightarrow \ker \eth_M^- \tag{0.8}$$

with respect to the L^2 inner products. The $+1$ eigenspace of $T = T^+ \oplus T^-$ on V, where $T^- = (T^+)^*$, defines a subspace $\Lambda_T \subset V$ which is Lagrangian in the sense that $V = \Lambda_T \oplus \Gamma\Lambda_T$, where $\Gamma = \pm i$ on $\ker \eth_M^\pm$. Given such a subspace we construct an operator $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ that preserves the Clifford structure at the boundary faces, has restriction to the corner equal to $-T$ and is such that the indicial families of $\eth + R$ at H_1 and H_2 are invertible for $\tau \in \mathbb{R}$.

Theorem 0.1. *Let X be an even-dimensional compact manifold with two boundary hypersurfaces intersecting in a single corner, let $\eth^+ : C^\infty(X, E^+) \longrightarrow C^\infty(X, E^-)$ be an admissible Dirac operator on X, associated to a product-type b-metric and* *connection on* X, *then*

$$\operatorname{ind}(\eth^+ + R) = \int_X \mathrm{AS} - \frac{1}{2}\sum_{i=1,2}\left\{{}^b\eta(\eth_{H_i}) + \dim\ker \eth_{H_i}\right\} - \frac{1}{2}c(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}). \tag{0.9}$$

In the sum ${}^b\eta(\eth_{H_i})$ is the b-eta invariant of the Dirac operator $\eth_{H_i}$ induced on the hypersurface H_i. The third 'corner correction term' is described as follows. First, $\Lambda_{C_1}, \Lambda_{C_2} \subset V$ are the scattering Lagrangians associated to the operators $\eth_{H_1}$ and $\eth_{H_2}$ respectively. Then

$$\begin{aligned} c(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}) = \dim(\Lambda_T \cap \Lambda_{C_1}) &+ m(\Lambda_T, \Lambda_{C_1}) \\ &+ \dim(\Lambda_{\Gamma T} \cap \Lambda_{C_2}) - m(\Lambda_{\Gamma T}, \Lambda_{C_2}), \end{aligned} \tag{0.10}$$

where $\Lambda_{\Gamma T}$ is the Lagrangian associated to ΓT and

$$m(\Lambda_T, \Lambda_{C_1}) = -\frac{1}{i\pi}\sum_{\substack{e^{i\theta}\in\operatorname{spec}(-T^- C_1^+)\\ \theta\in(-\pi,\pi)}} i\theta,$$

and where a similar formula holds for $m(\Lambda_{\Gamma T}, \Lambda_{C_2})$. The function m was introduced in the work of Lesch-Wojciechowski [14] and C_1^+ is the restriction C_1 to V^+. A formula similar to (0.9) holds for manifolds with any number of corners up to codimension two, as long as the induced Dirac operators on the corners have index zero; see Theorem 6.12.

As already noted, Müller [28] gives an index formula for Dirac operators on manifolds with corners up to codimension two under the assumption that the induced Dirac operators on the corners are invertible. Without this assumption a signature formula was proved in [13] by Hassell, Mazzeo and the second author, using the techniques of analytic surgery [12]. Salomonsen in [31] also gives an index formula without this assumption by considering a related problem on a manifold with boundary with wedge singularities.

In the next two sections we discuss Dirac operators and their induced operators on general manifolds with corners. Then the two classes of perturbations on Dirac operators are examined, corresponding to conjugation by boundary defining functions and to adding b-smoothing operators. The proof of Theorem 0.1 uses the difference of the traces of the heat operators $\exp(-tA^*A)$ and $\exp(-tAA^*)$ where $A = \eth^+ + R$. These heat operators are not trace class in the usual sense, but instead the b-trace functional, described in Section 3, can be applied to these operators; this functional was introduced in [22] for manifolds with boundary. The index of $\eth^+ + R$ can then be computed following the approach in [22] and resulting in a similar formula to (0.5) involving an interior term and boundary eta term. The eta term is further analyzed in Section 4, and then the 'transgressed' index formula is presented in Section 6. In Section 7 we give an application of Theorem 0.1 to derive a splitting formula for the eta invariant. Finally, the appendix contains a treatment of b-pseudodifferential operators and the corresponding heat calculus.

1. Dirac operators on manifolds with corners

In this first section we fix notation. See the appendix for a summary of the basic notions and notations used in the b-geometry on manifolds with corners.

1.1. **Dirac operators.** Let X be a compact manifold with corners and $H_1, \dots, H_N$ a fixed ordering of the boundary hypersurfaces of X. A metric on the b-tangent bundle ${}^bTX$ is said to be *exact* if it takes the form

$$g = \sum_{i=1}^{N} \Big(\frac{d\rho_i}{\rho_i}\Big)^2 + g', \tag{1.1}$$

where $g' \in C^\infty(X, T^*X \otimes T^*X)$ for some choice of boundary defining functions $\{\rho_i\}_{i=1}^N$. If M is a codimension k face defined by $\rho_{i_1}, \dots, \rho_{i_k}$, then $\{\frac{d\rho_{i_j}}{\rho_{i_j}}\}_{j=1}^k$ is an orthonormal set at M, and hence

$${}^bT^*X|_M \equiv {}^bN^*M \oplus {}^bT^*M, \qquad {}^bN^*M \equiv \mathrm{span}_{\mathbb{R}}\Big\{\frac{d\rho_{i_j}}{\rho_{i_j}}\Big\} \tag{1.2}$$

is an orthogonal decomposition. Assume that X is even dimensional. Let E be a Hermitian, $\mathbb{Z}_2$-graded Clifford module over X associated to an exact b-metric g. Thus, $E = E^+ \oplus E^-$ and there is a homomorphism $\sigma : {}^bT^*X \otimes E \longrightarrow E$, $(\xi, e) \mapsto \sigma(\xi)e$, that is odd with respect to the $\mathbb{Z}_2$-grading and satisfies $\sigma(\xi)^2 = |\xi|_g^2$. (Note our sign convention is $\sigma(\xi)^2 = |\xi|_g^2$ and not the usual $\sigma(\xi)^2 = -|\xi|_g^2$.) A *$b$-connection*, ${}^b\nabla \in \mathrm{Diff}_b^1(X, E, {}^bT^*X \otimes E)$, is an operator satisfying

$${}^b\nabla(fe) = df \otimes e + f\,{}^b\nabla e, \quad f \in C^\infty(X),\ e \in C^\infty(X, E).$$

It is called a *Clifford connection* if for any $\alpha \in C^\infty(X, {}^bT^*X)$ and $e \in C^\infty(X, E)$,

$${}^b\nabla\sigma(\alpha)e = \sigma(\nabla\alpha)e + \sigma(\alpha){}^b\nabla e, \tag{1.3}$$

where ∇ is the Levi-Civita connection associated to g. By the structure of the metric in (1.1) and the definition of the Levi-Civita connection, it follows that $\nabla\frac{d\rho_i}{\rho_i} \in \rho\, C^\infty(X, {}^bT^*X \otimes {}^bT^*X)$, where $\rho = \rho_1 \cdots \rho_N$ is a total boundary defining function. As $\rho\,{}^bT^*X \equiv T^*X$, the Levi-Civita connection ∇ is therefore a *true* (as opposed to just a "b-") connection on ${}^bT^*X$,

$$\nabla : C^\infty(X, {}^bT^*X) \longrightarrow C^\infty(X, T^*X \otimes {}^bT^*X).$$

The splitting (1.2) implies that the Levi-Civita connection can be restricted to any face M of X, giving the Levi-Civita connection ∇^M on that face associated to the exact b-metric g_M on M induced from g. The decomposition (1.2) also gives a natural way to define an induced connection ${}^b\nabla^M \in \mathrm{Diff}_b^1(M, E|_M, {}^bT^*M \otimes E|_M)$ from ${}^b\nabla$. By (1.3), this connection is a Clifford connection and it commutes with the homomorphisms $\sigma(\frac{d\rho_{i_j}}{\rho_{i_j}})$ where M is defined by $\rho_{i_1}, \dots, \rho_{i_k}$.

If ${}^b\nabla$ is also $\mathbb{Z}_2$-graded and Hermitian, then the associated *(generalized) Dirac operator*, $\eth \in \mathrm{Diff}_b^1(X, E)$, is the operator

$$\eth = \frac{1}{i}\sigma \cdot {}^b\nabla.$$

Note that $\eth$ is self-adjoint and is odd with respect to the $\mathbb{Z}_2$-grading of E (the proof is essentially the same as in the manifold with boundary case, see [22, Lem. 3.32]). The restrictions of $\eth$ to $C^\infty(X, E^+)$ and $C^\infty(X, E^-)$ are denoted by $\eth^+$ and $\eth^-$ respectively.

1.2. **Induced Dirac operators.** Let $M = H_{i_1} \cap \cdots \cap H_{i_k}$ where $1 \le i_1 < \cdots < i_k \le N$ be a non-trivial intersection. Note that M has possibly several components, each of a face of codimension k. Let $[0,1)^k_x \times M$ be a product decomposition of X near M, where $x_j = \rho_{i_j}$ near $x_j = 0$, and choose an isomorphism $E \cong E|_M$ over this decomposition. Then

$$\eth = \sigma_1\, x_1 D_{x_1} + \cdots + \sigma_k\, x_k D_{x_k} + B_M + B', \qquad D_{x_j} = \frac{1}{i}\partial_{x_j},$$

where B' vanishes on M, $\sigma_j = \sigma\big(\frac{dx_j}{x_j}\big)\big|_M$, and where $B_M = \frac{1}{i}\sigma|_{{}^bT^*M} \cdot {}^b\nabla^M \in \mathrm{Diff}^1_b(M, E|_M)$ is self-adjoint and is odd with respect to the $\mathbb{Z}_2$-grading of E. Moreover, the Clifford module structure and the fact that ${}^b\nabla$ is Clifford imply that

$$\sigma_i^* = \sigma_i; \quad \sigma_i\sigma_j + \sigma_i\sigma_j = 2\delta_{ij}; \quad \sigma_i \circ B_M = -B_M \circ \sigma_i. \tag{1.4}$$

Freezing the coefficients at the boundary and taking the Mellin transform in the normal variables defines the normal operator of $\eth$ at M:

$$N_M(\eth)(\tau) = \sigma_1\tau_1 + \cdots + \sigma_k\tau_k + B_M. \tag{1.5}$$

See the appendix for more on normal operators.

Assume that $k = 2\ell$ is even, where $\ell \in \mathbb{N}$. Define

$$\omega_j = i\sigma_{2j}\,\sigma_{2j-1} : E^+|_M \longrightarrow E^+|_M, \quad j = 1, 2, \ldots, \ell.$$

By (1.4)), for each i, j, $\omega_i\,\omega_j = \omega_j\,\omega_i$, $\omega_i^* = \omega_i$, and $\omega_i^2 = \mathrm{Id}$. Thus, $\{\omega_i\}$ is a commuting family of self-adjoint operators each having eigenvalues ± 1. We define E_M to be the common intersection of the $+1$ eigenspaces of $\omega_1, \ldots, \omega_{\ell-1}$:

$$E_M = \{e \in E^+|_M\,;\, \omega_j e = e,\ j = 1, \ldots, \ell - 1\}.$$

Then E_M is $\mathbb{Z}_2$-graded: $E_M = E_M^+ \oplus E_M^-$, where $E_M^\pm$ are the ± 1 eigenspaces of ω_ℓ. Now assume that $k = 2\ell + 1$ is odd. If $k = 1$, we define $E_M = E^+|_M$. If $\ell \in \mathbb{N}$, we define $E_M = E^+_{M'}|_M$, where $M \subset M'$ with $M' = H_{i_1} \cap \cdots \cap H_{i_{2\ell}}$, and where $E_{M'}$ is already defined since 2ℓ is even.

The definition of these induced bundles may be clearer through the following inductive definition. For a codimension one face, we set $E_M = E^+|_M$. Assume that E_M is defined when M is the intersection of at most $k-1$ boundary hypersurfaces of X. For $M = H_{i_1} \cap \cdots \cap H_{i_k}$, set $M' = H_{i_1} \cap \cdots \cap H_{i_{k-1}}$. Then $E_{M'}$ is already defined. If k is even, we define $E_M = E_{M'}|_M$, and if k is odd, we define $E_M = E^+_{M'}|_M$ where $E^+_{M'}|_M$ is the $+1$ eigenspace of $i\sigma_k\sigma_{k-1}$ on $E_{M'}$.

By (1.4), if $k = 2\ell$ or $k = 2\ell + 1$, where $k \in \mathbb{N}$, then $i\sigma_k B_M\,\omega_j = \omega_j\, i\sigma_k B_M$ for $j = 1, \ldots, \ell - 1$. Thus, we obtain an operator

$$\eth_M = i\sigma_k B_M^+ : C^\infty(M, E_M) \longrightarrow C^\infty(M, E_M). \tag{1.6}$$

Since $\sigma^M = i\sigma_k \cdot \sigma|_{{}^bT^*M}$ is a Clifford action on E_M, $\eth_M = \frac{1}{i}\sigma^M \cdot {}^b\nabla^M$ is a Dirac operator on M. If $k = 2\ell$ is even, then $\omega_\ell \eth_M = -\eth_M \omega_\ell$, so $\eth_M$ is odd with respect to the $\mathbb{Z}_2$-grading of E_M. Thus, for k even, $\eth_M$ is $\mathbb{Z}_2$-graded:

$$\eth_M^\pm : C^\infty(M, E_M^\pm) \longrightarrow C^\infty(M, E_M^\mp).$$

The operator $\eth_M$ defined in (1.6) is called the *induced Dirac operator of* $\eth$ *on* M. If $F \subset M$ is a codimension k face of X (a component of M), then $\eth_F$ is by definition the restriction of $\eth_M$ to F.

Since $\eth_M = i\sigma_k B_M^+$, given $F \in M_{k+1}(X)$ with $F \subset H_{i_1} \cap \cdots \cap H_{i_{k+1}}$, by (1.5), $N_F(\eth_M)(\tau) = i\sigma_k[\sigma_{k+1}\tau + B_F^+]$. Hence, for odd codimension M,

$$N_F(\eth_M)(\tau) = \frac{1}{i}\,\sigma^M\Big(\frac{d\rho_{k+1}}{\rho_{k+1}}\Big)\Big[\,i\tau + \eth_F\,\Big], \quad \sigma^M\Big(\frac{d\rho_{k+1}}{\rho_{k+1}}\Big) = i\sigma_k\sigma_{k+1}. \tag{1.7}$$

If M has even codimension, we must replace $\eth_M$ with $\eth_M^+$. In particular, $N_H(\eth^+)(\tau) = \frac{1}{i}\,\sigma_H[\,i\tau + \eth_H\,]$ for any $H \in M_1(X)$.

Finally, we note that different orderings of the boundary hypersurfaces give rise to induced Dirac operators differing by a Clifford action. For instance, let $M \in M_2(X)$ with $M \subset H_1 \cap H_2$. Then, by (1.7), we have

$$N_M(\eth_{H_1})(\tau) = \Gamma[\,i\tau + \eth_M\,], \qquad \Gamma = i\omega, \tag{1.8}$$

where $\omega = i\sigma_2\sigma_1$ is, by definition, the $\mathbb{Z}_2$-grading of E_M. However, using the fact that $\eth_{H_2} = i\sigma_2 B_{H_2}^+$, a similar computation used to derive (1.7) shows that

$$N_M(\eth_{H_2})(\tau) = \Gamma'[\,i\tau + \eth'_M\,], \qquad \Gamma' = -i\omega,\ \eth'_M = i\omega\,\eth_M. \tag{1.9}$$

The discrepancies between (1.8) and (1.9) will come into play later (see for instance, in Section 6.3).

2. Perturbations of Dirac operators

By Theorem B.2 of the appendix, a Dirac operator $\eth$ is Fredholm if and only if each of its normal operators is invertible for all real parameters. This condition is very restrictive. In this section, we consider two methods to make a Dirac operator Fredholm. The first is by conjugation by boundary defining functions and the second is by adding a b-pseudodifferential operator to it.

2.1. Weighted Sobolev Spaces.

Given a multi-index α on X; that is, an N-tuple of real numbers, we set $\rho^\alpha = \rho_1^{\alpha_1}\cdots\rho_N^{\alpha_N}$. Also, for any $\delta > 0$, we write $0 < |\alpha| < \delta$ to mean that $0 < |\alpha_i| < \delta$ for each $i = 1, \ldots, N$. The following theorem shows that in general one cannot make a Dirac operator Fredholm by considering it on weighted Sobolev spaces.

Theorem 2.1. *There exists a $\delta > 0$, such that for all multi-indices α with $0 < |\alpha| < \delta$,*

$$\eth^+ : \rho^\alpha H_b^1(X, E^+) \longrightarrow \rho^\alpha L_b^2(X, E^-) \tag{2.1}$$

is Fredholm, if and only if, $\ker\eth_M = 0$ for each $M \in M_k(X)$ with $k \geq 2$; in which case, given any two such multi-indices, if we denote the index of the operator (2.1) by $\mathrm{ind}_\alpha\eth^+$, then we have

$$\mathrm{ind}_\alpha\eth^+ - \mathrm{ind}_{\alpha'}\eth^+ = -\sum_{\mathrm{sgn}\,\alpha_H \neq \mathrm{sgn}\,\alpha'_H,\ H\in M_1(X)} \mathrm{sgn}(\alpha_H - \alpha'_H)\cdot\dim\ker\eth_H.$$

In particular, if X is a manifold with boundary, then $\eth^+$ can *always* be made Fredholm by considering it on weighted Sobolev spaces. In Theorem 6.8 we give a formula for the index $\mathrm{ind}_\alpha\eth^+$. This formula is the direct analog of the corresponding formula on manifolds with boundary [22].

Necessity in Theorem 2.1: If $\ker\eth_M = 0$ for each $M \in M_k(X)$ with $k \geq 2$, then by Proposition 2.4 in the next section, it follows that $N_M(\eth)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Thus, the following proposition proves the necessity part of Theorem 2.1.

Proposition 2.2. *Let E and F be vector bundles over a compact manifold with corners X (not necessarily even dimensional) and let $A \in \Psi_b^m(X, E, F)$ be elliptic. Suppose that for each $M \in M_2(X)$, $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then for some $\delta > 0$, for all multi-indices α with $0 < |\alpha| < \delta$,*

$$A : \rho^\alpha H_b^m(X, E) \longrightarrow \rho^\alpha L_b^2(X, F)$$

is Fredholm. Moreover, given any two such multi-indices α, α', we have

$$\operatorname{ind}_\alpha A - \operatorname{ind}_{\alpha'} A = \quad - \sum_{\substack{\operatorname{sgn} \alpha_H \neq \operatorname{sgn} \alpha'_H, \\ z \in \operatorname{spec}_H(A),\, H \in M_1(X)}} \operatorname{sgn}(\alpha_H - \alpha'_H) \cdot \operatorname{rk}_H(z), \tag{2.2}$$

where $\operatorname{spec}_H(A)$ is the set of poles on the real axis of the meromorphic family $N_H(A)(\tau)^{-1}$, and where $\operatorname{rk}_H(z)$ is the rank of the pole at z (see [22, Sec. 5.9]*).*

Proof. Once we prove that A is Fredholm on weighted Sobolev spaces, the "relative index formula" (2.2) is proved just as in the manifolds with boundary case [22, Th. 6.5]. Thus, we need only prove the first statement of the proposition. We work within the context of non-weighted Sobolev spaces. Thus, we prove that $\rho^{-\alpha} A \rho^\alpha : H_b^m(X, E) \longrightarrow L_b^2(X, F)$ is Fredholm. In the statements that follow, Fredholm properties always refer to non-weighted Sobolev spaces.

Since $N_M(A)(\tau)$ is invertible for each $M \in M_2(X)$, by Theorem B.2 of the appendix, for each i, $N_{H_i}(A)(\tau)$ is Fredholm for all $\tau \in \mathbb{R}$. We prove the following statement by induction:

(2.3) For each $j = 1, \ldots, N$, there is a $\delta_j > 0$ such that given any multi-index α with $0 < |\alpha_i| < \delta_j$ for $i = 1, \ldots, j$, the operator $N_{H_i}(\rho^{-\alpha} A \rho^\alpha)(\tau)$ is invertible for all $\tau \in \mathbb{R}$.

Setting $j = N$ and applying Theorem B.2 of the appendix proves the proposition. If $j = 1$, α is any multi-index, and $\rho'_1 = (\rho_2 \cdots \rho_N)|_{H_1}$, then

$$N_{H_1}(\rho^{-\alpha} A \rho^\alpha)(\tau) = (\rho'_1)^{-\alpha} N_{H_1}(A)(\tau - i\alpha_1)(\rho'_1)^\alpha. \tag{2.4}$$

Fix any $r > 0$. Then by Lemma B.6, there is an $r' > 0$ such that the operator $(\rho'_1)^{-\alpha} N_{H_1}(A)(\tau)(\rho'_1)^\alpha$ is invertible for all $\tau \in \mathbb{C}$ such that $|\Re\tau| \geq r'$ if $0 \leq |\alpha|, |\operatorname{Im}\tau| \leq r$. Thus, as $N_{H_1}(A)(\tau)$ is a holomorphic family and Fredholm for $\tau \in \mathbb{R}$, by analytic Fredholm theory and the fact that Fredholm operators form an open set in the bounded operators, it follows that for some $0 < \delta_1 < r$ and for all $0 \leq |\alpha| \leq \delta_1$, the inverse of $(\rho'_1)^{-\alpha} N_{H_1}(A)(\tau)(\rho'_1)^\alpha$ exists for $\tau \in \mathbb{C}$ near $\mathbb{R}$ with the exception of a finite number of poles on the real line. In particular, choosing δ_1 smaller if necessary, by (2.4), for all multi-indices α with $0 < |\alpha| < \delta_1$, it follows that $N_{H_1}(\rho^{-\alpha} A \rho^\alpha)(\tau)$ is invertible for all $\tau \in \mathbb{R}$.

Assume that (2.3) holds for $j < N$. Then repeating the $j = 1$ argument implies that for some $\delta > 0$, $N_{H_{j+1}}(\rho^{-\alpha} A \rho^\alpha)(\tau)$ is invertible for all $\tau \in \mathbb{R}$ with $0 < |\alpha| < \delta$. Let $0 < \delta_{j+1} = \min\{\delta, \delta_1, \ldots, \delta_j\}$. Then (2.3) holds for $j + 1$, so our induction step is finished and our proof is complete. □

Sufficiency in Theorem 2.1: To prove sufficiency, we work in the context of non-weighted Sobolev spaces. Thus, assume that for some $\delta > 0$, $\rho^{-\alpha} \eth^+ \rho^\alpha$ is Fredholm on L_b^2 for all multi-indices α with $0 < |\alpha| < \delta$. We show that $\ker \eth_M = 0$ for each $M \in M_k(X)$ with $k \geq 2$.

Let $n' = \operatorname{codim} X$ and let $M \in M_{n'}(X)$. We prove that $\ker \eth_M = 0$. To simplify presentation, assume that $M = H_1 \cap \cdots \cap H_{n'}$ and that M is the single boundary component of $F = H_1 \cap \cdots \cap H_{n'-1}$; the general case is not essentially different. By Theorem B.2, $N_F(\rho^{-\alpha}\eth\rho^{\alpha})(\tau)$ is invertible for all $\tau \in \mathbb{R}^{n'-1}$ and for all $0 < |\alpha| < \delta$. Let $\rho' = (\rho_{n'} \cdots \rho_N)|_F$. Then using notation in Section 1.2, cf. (1.5), we can write

$$\begin{aligned} N_F(\rho^{-\alpha}\eth^{+}\rho^{\alpha})(\tau) &= \sum_{j=1}^{n'-1} \sigma_j(\tau_j - i\alpha_j) + (\rho')^{-\alpha} B_F^{+} (\rho')^{-\alpha} \\ &= \frac{1}{i}\sigma_{n'-1}\Big[\sum_{j=1}^{n'-1} i\beta_j(\tau_j - i\alpha_j) + (\rho')^{-\alpha} D_F (\rho')^{\alpha}\Big], \end{aligned}$$

where $\beta_j = i\sigma_{n'-1}\sigma_j$ and $D_F = i\sigma_{n'-1}B_F^{+}$. The induced operator $\eth_F$ is the operator D_F restricted to sections of E_F (see Section 1.2). Setting $\tau = 0$, we see that

$$\sum_{j=1}^{n'-1} (\beta_j \alpha_j) + (\rho')^{-\alpha} D_F (\rho')^{\alpha} \tag{2.5}$$

is invertible for all $0 < |\alpha| < \delta$. We claim that $(\rho')^{-\alpha} D_F (\rho')^{\alpha}$ is Fredholm for any α with $|\alpha_{n'}| > 0$ sufficiently small. To see this, let $\rho'' = (\rho_{n'+1} \cdots \rho_N)|_M$. Then arguing as in the derivation of (1.7), we obtain

$$N_M((\rho')^{-\alpha} D_F (\rho')^{\alpha})(\tau) = \frac{1}{i}\,\sigma_{n'-1}\sigma_{n'}(\rho'')^{-\alpha}[\,i\tau + \alpha_{n'} + D_M\,](\rho'')^{\alpha}, \tag{2.6}$$

where $D_M = i\sigma_{n'}B_F^{+}$. The induced Dirac operator $\eth_M$ is the operator D_M restricted to sections of E_M. Since $\alpha_{n'} + D_M$ is self-adjoint, (2.6) is invertible for all $\tau \in \mathbb{R}\backslash\{0\}$. Now D_M is Fredholm since it is elliptic and M is compact without boundary. Thus, D_M has discrete spectrum near zero, so in fact, (2.6) is invertible for all $\tau \in \mathbb{R}$ with $|\alpha_{n'}| > 0$ sufficiently small. It follows that $(\rho')^{-\alpha} D_F (\rho')^{\alpha}$ is Fredholm for $|\alpha_{n'}| > 0$ sufficiently small. Thus, since the index in invariant under Fredholm perturbations, the fact that (2.5) is invertible for all $0 < |\alpha| < \delta$ implies that $\rho_{n'}^{\mp\alpha_{n'}} D_F \rho_{n'}^{\pm\alpha_{n'}}$ must have index 0. Hence by the relative index formula (2.2), we have

$$0 - 0 = \operatorname{ind}_{\alpha_{n'}} D_F - \operatorname{ind}_{-\alpha_{n'}} D_F = \operatorname{sgn}\alpha_{n'} \cdot \dim\ker D_M.$$

Thus, $\ker D_M = 0$ and so, $\ker \eth_M = 0$. We remark that the key to proving this was that D_M has discrete spectrum near zero which allowed us to conclude that $\rho_{n'}^{\mp\alpha_{n'}} D_F \rho_{n'}^{\pm\alpha_{n'}}$ is Fredholm. Now setting all multi-indices to zero in (2.6) shows that $N_M(D_F)(\tau)$ is invertible for all $\tau \in \mathbb{R}$. Thus, Theorem B.2 implies that D_F is Fredholm for all $F \in M_{n'-1}(X)$. In particular, D_F has discrete spectrum near zero. Using this fact and going through a similar argument as we did above in showing that $\ker D_M = 0$ for each $M \in M_{n'}(X)$ shows that $\ker D_F = 0$ for each $F \in M_{n'-1}(X)$. Continuing by induction finishes up the sufficiency proof.

2.2. **Compatible operators.** We now consider perturbations of Dirac operators by b-smoothing operators. A natural choice of perturbations are those having similar properties as the Dirac operator with respect to the Clifford action at the boundary faces.

Definition 2.3. An operator $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ is said to be *Clifford compatible* if given any codimension k face $M \subset H_{i_1} \cap \cdots \cap H_{i_k}$, for each $j = 1, \ldots, k$, we have

$$\sigma_j \circ N_M(R)(\tau) = -N_M(R)(\overline{\tau})^* \circ \sigma_j, \qquad \tau \in \mathbb{C}^k, \tag{2.7}$$

where $\sigma_j = \sigma(\frac{d\rho_{i_j}}{\rho_{i_j}})\big|_M$ and when restricted to real parameters, $N_M(R)(\tau)$ is an even function of $\tau \in \mathbb{R}^k$.

In analogy with the definition of induced Dirac operators, we define

$$R_M(\tau) = i\sigma_k N_M(R)(\tau). \tag{2.8}$$

By condition (2.7), it follows that for real τ, $R_M(\tau)$ is self-adjoint and it defines an operator on the induced vector bundle E_M, such that when k is even, $R_M(\tau)$ is odd with respect to the $\mathbb{Z}_2$-grading of E_M.

Proposition 2.4. *Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be Clifford compatible and let $M \in M_k(X)$. Then $N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k \setminus \{0\}$, and is invertible at $\tau = 0$ if and only if $\eth_M + R_M(0)$ is invertible. Moreover,*

$$\eth^+ + R : H_b^1(X, E^+) \longrightarrow L_b^2(X, E^-)$$

is Fredholm if and only if each of the induced operators $\eth_M + R_M(0)$ is invertible, if and only if each of the induced operators $\eth_H + R_H(0)$ on each of the hypersurfaces H of X is invertible.

Proof. Fix any $0 \leq j \leq k$. We prove that $N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k$ with $\tau_j \neq 0$. Indeed, observe that

$$N_M(\eth^+ + R)(\tau) = \sigma_1\tau_1 + \cdots + \sigma_k\tau_k + B_M^+ + N_M(R)(\tau) = \frac{1}{i}\sigma_j[i\tau_j + i\sigma_j A_j(\tau)],$$

where

$$A_j(\tau) = \sigma_1\tau_1 + \cdots + \sigma_{j-1}\tau_{j-1} + \sigma_{j+1}\tau_{j+1} + \cdots + \sigma_k\tau_k + B_M^+ + N_M(R)(\tau).$$

Since $\sigma_i \circ B_M^+ = -B_M^- \circ \sigma_i$ and $\sigma_i \circ N_M(R)(\tau) = -N_M(R)(\tau)^* \circ \sigma_i$ for $\tau \in \mathbb{R}^k$ (see (1.4) and (2.7)), it follows that $i\sigma_j A_j(\tau)$ is self-adjoint for all $\tau \in \mathbb{R}^k$. Thus, $N_M(\eth + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k$ with $\tau_j \neq 0$. It is invertible at $\tau = 0$ if and only if $B_M^+ + N_M(R)(0)$ is invertible, which holds if and only if $\eth_M + R_M(0)$ is invertible. The Fredholm properties of $\eth^+ + R$ follow from Theorem B.2. □

Choosing $R = 0$ gives the following corollary.

Corollary 2.5. *The following are equivalent:*

(1) $\eth^+ : H_b^1(X, E^+) \longrightarrow L_b^2(X, E^-)$ is Fredholm.
(2) For each $H \in M_1(X)$, $\eth_H : H_b^1(H, E_H) \longrightarrow L_b^2(H, E_M)$ is invertible.
(3) For each $M \in M'(X)$, $\eth_M : H_b^1(M, E_M) \longrightarrow L_b^2(M, E_M)$ is invertible.

2.3. **Dirac operators on codimension two manifolds with corners.** The following theorem, due to Melrose and Nistor, characterizes those Dirac operators $\eth$ that can be made Fredholm by perturbation by b-smoothing operators.

Theorem 2.6. *Let X be an even-dimensional compact manifold with corners of codimension two. Then there exists an $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ so that*

$$\eth^+ + R : H_b^1(X, E^+) \longrightarrow L_b^2(X, E^-)$$

is Fredholm if and only if the L^2-index of the positive part of the induced Dirac operator on each codimension two face is zero.

The last statement means that $\operatorname{ind} \eth_M^+ = 0$ for each $M \in M_2(X)$. We remark that this theorem remains valid for any manifold with corners of arbitrary codimension as long as we assume that $\ker \eth_M = 0$ for all $M \in M_k(X)$ with $k \geq 3$. The proof follows essentially the same line of reasoning. In Section 2.4, we show that the signature operator on any $4k$ dimensional manifold with corners of codimension two, and any Dirac operator on a manifold with one codimension two face, satisfy the conditions of this theorem.

Sufficiency in Theorem 2.6: Assume that $\operatorname{ind} \eth_M^+ = 0$ for each $M \in M_2(X)$. To prove the sufficiency in Theorem 2.6, we give a recipe to build a Clifford compatible perturbation from the null spaces of the corner Dirac operators. The basic idea is the following. According to Proposition 2.4, we need an R such that for each boundary face M, $\eth_M + R_M(0)$ is invertible. To construct such an R, we first define a compatible perturbation, say S, supported near the corners, such that for each codimension two face M, $\eth_M + S_M(0)$ is invertible. We then define another compatible perturbation, say S', supported near the boundary hypersurfaces and vanishing at the corners, such that for each boundary hypersurface H, $\eth_H + S_H(0) + S'_H(0)$ is invertible. Then $R = S + S'$ is the required perturbation.

Let $\chi \in C_c^\infty([0,1))$ be a nonnegative function such that $\chi(x) = 1$ for $0 \leq x \leq \frac{1}{2}$ and $\chi(x) = 0$ for $x \geq \frac{3}{4}$. Let φ be an even, real-valued, Schwartz function on $\mathbb{R}$ with $\varphi(0) > 0$. Then the Fourier transform $\widehat{\varphi}(\tau)$ is an even entire function. Define $Q \in \Psi_b^{-\infty}([0,1))$ by defining its Schwartz kernel, which we again denote by Q:

$$Q = \varphi(\log s)\, \chi(x)\chi(x') \frac{dx'}{x'}, \qquad s = \frac{x}{x'}. \tag{2.9}$$

Since φ is even and real valued, Q is self-adjoint, and by definition, $N(Q)(\tau) = \widehat{\varphi}(\tau)$.

Let $M \subset H_{i_1} \cap H_{i_2}$, where $i_1 < i_2$, be a codimension two face. From Section 1.2, recall that $E_M = E^+|_M$, which splits: $E_M = E_M^+ \oplus E_M^-$, where $E_M^\pm$ are the ± 1 eigenspaces of $\omega = i\sigma_2\sigma_1$. Then $\eth_M = i\sigma_2 B_M^+$ is odd with respect to the $\mathbb{Z}_2$-grading of E_M, and by assumption, $\operatorname{ind} \eth_M^+ = 0$. Hence, $\dim\ker \eth_M^+ = \dim\ker \eth_M^-$, so we can choose a unitary, self-adjoint isomorphism T_M on $\ker \eth_M$ that is odd with respect to the $\mathbb{Z}_2$-grading of E_M. To construct such a map, let $T_M^+ : \ker \eth_M^+ \longrightarrow \ker \eth_M^-$ be a unitary isomorphism. Choosing bases of $\ker \eth_M^+$ and $\ker \eth_M^-$ shows that T_M^+ is a finite rank operator. Thus, $T_M^+ \in \Psi^{-\infty}(M, E_M^+, E_M^-)$. Now set $T_M = T_M^+ + T_M^- \in \Psi^{-\infty}(M, E_M)$, where $T_M^- = (T_M^+)^*$. Then T_M is unitary, self-adjoint, and is odd with respect to the $\mathbb{Z}_2$-grading of E_M. Choose a decomposition

$$X \cong [0,1)_x^2 \times M, \tag{2.10}$$

near M, where $x_j = \rho_{i_j}$ near $x_j = 0$, and over which $E^+ \cong E_M$. Using this product decomposition, we define

$$S_M = -\frac{1}{i}\sigma_2\, Q_1^2 Q_2^2\, T_M \in \Psi_b^{-\infty}(X, E^+, E^-), \tag{2.11}$$

where $\sigma_2 = \sigma\big(\frac{dx_2}{x_2}\big)|_M$ and where $Q_i = Q(x_i, x_i')$ with Q given in (2.9). One can check that the operator S_M defined in (2.11) is Clifford compatible. Note that

$$N_M(S_M)(\tau_1, \tau_2) = -\frac{1}{i}\sigma_2\, \widehat{\varphi}(\tau_1)^2 \widehat{\varphi}(\tau_2)^2 T_M. \tag{2.12}$$

Lemma 2.7. *Assume that* $\operatorname{ind} \eth^+_M = 0$ *for each* $M \in M_2(X)$*, and set*

$$S = \sum_{M \in M_2(X)} S_M, \tag{2.13}$$

where each S_M *is defined as in* (2.11) *above by a choice of unitary isomorphism* T_M*. Then* $S \in \Psi_b^{-\infty}(X, E^+, E^-)$ *and for each* $M \in M_2(X)$*,* $N_M(\eth^+ + S)(\tau)$ *is invertible for all* $\tau \in \mathbb{R}^2$*. In particular, for* $0 < |\alpha| < \delta$ *for some* $\delta > 0$*,*

$$\eth^+ + S : \rho^\alpha H^1_b(X, E^+) \longrightarrow \rho^\alpha L^2_b(X, E^-)$$

is Fredholm.

Proof. Since each S_M is b-smoothing, so is S. Moreover, by (2.12), we have $S_M(\tau) = -\widehat{\varphi}(\tau_1)^2 \widehat{\varphi}(\tau_2)^2 T_M$. Thus, by Proposition 2.4, $N_M(\eth^+ + S)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ if and only if $\eth_M + S_M(0) = \eth_M - T_M$ is invertible. But $\eth_M - T_M$ is invertible by construction. The fact that $\eth^+ + S$ is Fredholm on weighted Sobolev spaces follows from Proposition 2.2. □

In Theorem 6.13, we give a formula for the index of the operator $\eth^+ + S$ on weighted Sobolev spaces.

Using the operator $S \in \Psi_b^{-\infty}(X, E^+, E^-)$ in (2.13), we construct an operator R satisfying Theorem 2.6. Let $H \in M_1(X)$ and let $X \cong [0, 1)_x \times H$ near H, where x is the fixed boundary defining function for H near $x = 0$, and over which $E \cong E|_H$. Since S is compatible with $\eth$, by Theorem B.2 and Proposition 2.4, it follows that $N_H(\eth + S)(\tau)$ is a family of Fredholm operators for $\tau \in \mathbb{C}$ near $\mathbb{R}$ and is invertible for all $\tau \in \mathbb{R}$ if and only if $\eth_H + S_H(0)$ is invertible. By Theorem B.8, the orthogonal projection Π_H onto the null space of $\eth_H + S_H(0)$ is an element of $\Psi^{-\infty,\varepsilon}(H, E_H)$ for some $\varepsilon > 0$. As the invertible operators form an open set in the bounded operators between $H^1_b(H, E_H)$ and $L^2_b(H, E_H)$, it follows that $\eth_H + S_H(0) + \Pi_H + K$ is invertible for all operators $K \in \Psi^{-\infty,\varepsilon}(H, E_H)$ with $\|K\| < \delta$ for some $\delta > 0$. Let $K_H \in \Psi^{-\infty,\varnothing}(H, E_H)$ be a self-adjoint smoothing operator such that $\|K - \Pi_H\| < \delta$. Then $\eth_H + S_H(0) + K_H$ is invertible and if

$$S'_H = \frac{1}{i} \sigma_H Q^2 K_H \in \Psi_b^{-\infty}(X, E^+, E^-), \tag{2.14}$$

where $\sigma_H = \sigma\left(\frac{dx}{x}\right)|_H$ and where Q is defined in (2.9), then S'_H is Clifford compatible and by Proposition 2.4, $N_H(\eth + S + S'_H)(\tau)$ is invertible for all $\tau \in \mathbb{R}$. Summing up such operators for each $H \in M_1(X)$ produces an $S' \in \Psi_b^{-\infty}(X, E^+, E^-)$ such that $N_H(\eth + S + S')(\tau)$ is invertible for all $\tau \in \mathbb{R}$ for each $H \in M_1(X)$. Thus, if $R = S + S'$, then $\eth^+ + R$ is Fredholm on L^2_b. This proves the following proposition, which in turn proves sufficiency in Theorem 2.6.

Proposition 2.8. *Suppose that* $\operatorname{ind} \eth^+_M = 0$ *for each* $M \in M_2(X)$*. Then the operator* $R = S + S' \in \Psi_b^{-\infty}(X, E^+, E^-)$ *defined above is such that* $\eth^+ + R$ *is Fredholm on* L^2_b*.*

In Theorem 6.12 we give a formula for the index of the operator $\eth^+ + R$. We remark that we could replace each operator S'_H in (2.14) by any nonzero multiple of it and the operator $\widetilde{R}$ defined in this manner will still be Fredholm. The index of $\eth^+ + \widetilde{R}$ is similar to the formula found in Formula (6.14) of Theorem 6.12, but with appropriate sign changes.

Necessity in Theorem 2.6: The proof of necessity is based on the *degree map*, due to Melrose and Nistor, which we now define. Let $M \subset H_{i_1} \cap H_{i_2}$, $i_1 < i_2$. Recall that, cf. (1.5), $N_M(\eth^+)(\tau) = \sigma_1\tau_1 + \sigma_2\tau_2 + B_M^+$, and $\eth_M = i\sigma_2 B_M^+$ is odd with respect to the $\mathbb{Z}_2$-grading on $E_M = E^+|_M$. If $A(\tau) = N_M(\eth^+)(\tau)$, then by Proposition 2.4, $A(\tau)$ is invertible for all $\tau \in \mathbb{R}^2 \setminus \{0\}$, so $A(\tau)^{-1}dA(\tau)$ exists for all $\tau \in \mathbb{R}^2 \setminus \{0\}$, where d is the differential with respect to $\tau = (\tau_1, \tau_2)$. We define

$$\alpha(\tau) = \overline{\mathrm{Tr}}(A(\tau)^{-1}dA(\tau)),$$

where $\overline{\mathrm{Tr}}$ is the regularized trace with respect to $P = \eth_M^*\eth_M$; that is, $\alpha(\tau)$ is the regular value of the meromorphic function $\mathrm{Tr}(A(\tau)^{-1}dA(\tau)P^{-z})$ at $z = 0$. A short computation shows that

$$d(A(\tau)^{-1}dA(\tau)) = [\, A(\tau)^{-1}\partial_{\tau_2}A(\tau), A(\tau)^{-1}\partial_{\tau_1}A(\tau)\,]\, d\tau_1 \wedge d\tau_2,$$

which implies that

$$d\alpha(\tau) = \overline{\mathrm{Tr}}(\,[\, A(\tau)^{-1}\partial_{\tau_2}A(\tau), A(\tau)^{-1}\partial_{\tau_1}A(\tau)\,]\,)\, d\tau_1 \wedge d\tau_2. \tag{2.15}$$

Thus (see for instance [24]), we have $d\alpha(\tau) = \mathrm{Res}(\gamma(\tau))\, d\tau_1 \wedge d\tau_2$, where $\mathrm{Res}(\gamma(\tau))$ is the Wodzicki residue of $\gamma(\tau) = A(\tau)^{-1}\partial_{\tau_2}A(\tau)\,[\log P, A(\tau)^{-1}\partial_{\tau_1}A(\tau)\,]$. Since the residue depends only on the $-\dim M$ homogeneous component of the local symbol of $\gamma(\tau)$, and since $A(\tau)^{-1}$ has at most a finite rank singularity at $\tau = 0$ (given by the projection onto the null space of $\eth_M$; cf. (2.16) below), it follows that $d\alpha(\tau)$ is smooth for all $\tau \in \mathbb{R}^2$.

Lemma 2.9. *For any $r > 0$, set*

$$\deg(A) = \int_{\mathbb{S}_r} \alpha - \int_{\mathbb{B}_r} d\alpha,$$

where $\mathbb{S}_r = \{\tau \in \mathbb{R}^2\, ;\, |\tau| = r\}$ and $\mathbb{B}_r = \{\tau \in \mathbb{R}^2\, ;\, |\tau| \le r\}$. Then $\deg(A) = \mathrm{ind}\, \eth_M^+$ for any $r > 0$.

Proof. Observe that $A(\tau) = \sigma_1\tau + \sigma_2\tau_2 + B_M^+ = \frac{1}{i}\sigma_2[\omega\tau_1 + i\tau_2 + \eth_M]$, where $\omega = i\sigma_2\sigma_1$ is the $\mathbb{Z}_2$-grading on E_M, and hence $\alpha(\tau) = \overline{\mathrm{Tr}}(a(\tau)^{-1}da(\tau))$, where $a(\tau) = \omega\tau_1 + i\tau_2 + \eth_M$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $(\tau_1, \tau_2) \equiv \tau_1 + i\tau_2$, we can write $a(\tau)$ as a matrix using the $\mathbb{Z}_2$-grading:

$$a(\tau) = \begin{bmatrix} \tau & \eth_M^- \\ \eth_M^+ & -\overline{\tau} \end{bmatrix}.$$

A straightforward computation shows that

$$a(\tau)^{-1}da(\tau) = \begin{bmatrix} \overline{\tau}\,(|\tau|^2 + \eth_M^-\eth_M^+)^{-1}d\tau & (|\tau|^2 + \eth_M^-\eth_M^+)^{-1}\eth_M^- \\ (|\tau|^2 + \eth_M^+\eth_M^-)^{-1}\eth_M^+ & \tau\,(|\tau|^2 + \eth_M^+\eth_M^-)^{-1}d\overline{\tau} \end{bmatrix} \tag{2.16}$$

Let Π_0 be the orthogonal projection onto $\ker \eth_M$. Then the formula (2.16) implies that $\Pi_0^\perp a(\tau)^{-1}da(\tau)\Pi_0^\perp$ is a smooth function of $\tau \in \mathbb{C}$, and that

$$a(\tau)^{-1}da(\tau) = \frac{\Pi_0^+}{\tau}d\tau + \frac{\Pi_0^-}{\overline{\tau}}d\overline{\tau} + \Pi_0^\perp a(\tau)^{-1}da(\tau)\Pi_0^\perp,$$

where $\Pi_0^\pm$ are the orthogonal projections onto $\ker \eth_M^\pm$. It follows that if N^+ is the dimension of the null space of $\eth_M^+$ and N^- is the dimension of the null space of $\eth_M^-$, then

$$\alpha(\tau) = \frac{N^+}{\tau}d\tau + \frac{N^-}{\overline{\tau}}d\overline{\tau} + \beta(\tau),$$

where $\beta(\tau) = \overline{\mathrm{Tr}}(\Pi_0^\perp a(\tau)^{-1} da(\tau)\Pi_0^\perp)$ is a smooth function of $\tau \in \mathbb{C}$. Observe that $d\alpha = d\beta$. Thus, for any $r > 0$, we have

$$\begin{aligned}\deg(A) &= \int_{\mathbb{S}_r} \alpha - \int_{\mathbb{B}_r} d\alpha = \int_{\mathbb{S}_r} \frac{N^+}{\tau} d\tau + \frac{N^-}{\overline{\tau}} d\overline{\tau} + \int_{\mathbb{S}_r} \beta - \int_{\mathbb{B}_r} d\beta \\ &= N^+ - N^- + 0 \\ &= \operatorname{ind} \eth_M^+.\end{aligned}$$

□

Given $R \in \Psi_b^{-\infty}(X, E^+, E^-)$, by Lemma B.6, the operator $A_R(\tau) = N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ sufficiently large. In particular, $A_R(\tau)^{-1} dA_R(\tau)$ exists for $\tau \in \mathbb{R}^2$ sufficiently large. We define

$$\alpha_R(\tau) = \overline{\mathrm{Tr}}(A_R(\tau)^{-1} dA_R(\tau)). \tag{2.17}$$

A similar argument as we did for the case that $R = 0$ shows that $d\alpha_R(\tau)$ is a smooth two-form on all of $\mathbb{R}^2$ (see the discussion around (2.15)).

Proposition 2.10. *For any $r > 0$ sufficiently large so that $\alpha_R(\tau)$ is defined, set*

$$\deg(A_R) = \int_{\mathbb{S}_r} \alpha_R - \int_{\mathbb{B}_r} d\alpha_R.$$

Then $\deg(A_R) = \operatorname{ind} \eth_M^+$ *for any such $r > 0$.*

Proof. Let $R_t \in \Psi_b^{-\infty}(X, E^+, E^-)$ be a smooth family of operators and set $A_t(\tau) = N_M(\eth^+ + R_t)(\tau)$. Let $r > 0$ be such that $\deg(A_t)$ is defined for all t sufficiently near 0. By Lemma 2.9 and the fact that the operators of order $-\infty$ form an affine space, to prove this proposition it suffices to show that $(d/dt)\deg(A_t) = 0$ for t near 0. Set $\alpha_t(\tau) = \overline{\mathrm{Tr}}(A_t(\tau)^{-1} dA_t(\tau))$ and $S_t(\tau) = (d/dt)A_t(\tau) = (d/dt)N_M(R_t)(\tau)$. Then, since $S_t(\tau)$ is a smoothing operator, the commuting properties of the trace imply that

$$\begin{aligned}\frac{d}{dt}\alpha_t(\tau) &= -\mathrm{Tr}(A_t(\tau)^{-1} S_t(\tau) A_t(\tau)^{-1} dA_t(\tau)) + \mathrm{Tr}(A_t(\tau)^{-1} dS_t(\tau)) \\ &= -\mathrm{Tr}(S_t(\tau) A_t(\tau)^{-1} dA_t(\tau) A_t(\tau)^{-1}) + \mathrm{Tr}(A_t(\tau)^{-1} dS_t(\tau)) \\ &= \mathrm{Tr}(S_t(\tau) dA_t(\tau)^{-1}) + \mathrm{Tr}(dS_t(\tau) A_t(\tau)^{-1}) \\ &= d\,\mathrm{Tr}(S_t(\tau) A_t(\tau)^{-1}).\end{aligned}$$

Thus, $(d/dt)\alpha_t(\tau)$ is exact and smooth on all of $\mathbb{R}^2$. It follows that $(d/dt)d\alpha_t(\tau) = 0$ and that $(d/dt)\deg(A_t) = 0$. □

The following lemma finishes the proof of necessity in Theorem 2.6.

Lemma 2.11. *Let $M \in M_2(X)$ and suppose that for some $R \in \Psi_b^{-\infty}(X, E^+, E^-)$, $N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then* $\operatorname{ind} \eth_M^+ = 0$.

Proof. Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be such that $N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then, $A_R(\tau)^{-1}$ exists for all $\tau \in \mathbb{R}^2$. Thus, $\alpha_R(\tau)$ defined in (2.17) is smooth on all of $\mathbb{R}^2$. Hence, by Stokes' Theorem, $\deg(A_R) = 0$, which implies that $\operatorname{ind} \eth_M^+ = 0$. □

General Clifford compatible perturbations: In the sense described in the following proposition, the special Clifford compatible operators defined in Lemma 2.7 form a complete set of Clifford compatible Fredholm perturbations of Dirac operators.

Proposition 2.12. *Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be a Clifford compatible operator such that for each $M \in M_2(X)$, $N_M(\eth^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then there exists a continuous family of Clifford compatible operators $R_t \in \Psi_b^{-\infty}(X, E^+, E^-)$, $0 \leq t \leq 1$, such that R_1 is an operator of the form defined in Lemma 2.7, and such that for each $M \in M_2(X)$, $N_M(R_0)(\tau) = N_M(R)(\tau)$, and $N_M(\eth^+ + R_t)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ and $0 \leq t \leq 1$.*

Proof. Fix $M \in M_2(X)$. The idea of this proof is to reduce the problem to the connectedness of $\mathrm{GL}(k, \mathbb{C})$. To do so, Let $\varphi_j \in C^\infty(M, E_M)$, $j \geq 1$, be an orthonormal basis of $L^2(M, E_M)$ consisting of eigenvectors of $\eth_M$. If $\eth_M \varphi_j = \lambda_j \varphi_j$, then the positive and negative parts of φ_j satisfy $\eth_M^\pm \varphi_j^\pm = \lambda_j \varphi_j^\mp$. For any $k \in \mathbb{N}$, we set $\mathcal{E}_k = \mathrm{span}_{1 \leq j \leq k}\{\varphi_j\}$ and $\mathcal{E}_k^\pm = \mathrm{span}_{1 \leq j \leq k}\{\varphi_j^\pm\}$. Then $\mathcal{E}_k = \mathcal{E}_k^+ \oplus \mathcal{E}_k^-$ and since $\dim \ker \eth_M^+ = \dim \ker \eth_M^-$ by Lemma 2.11, and $\eth_M^+$ is an isomorphism from $\mathcal{E}_k^+ \setminus \ker \eth_M^+$ onto $\mathcal{E}_k^- \setminus \ker \eth_M^-$, it follows that $\dim \mathcal{E}_k^+ = \dim \mathcal{E}_k^-$.

We first show that there is a continuous family of Clifford compatible operators $S_t \in \Psi_b^{-\infty}(X, E^+, E^-)$, $0 \leq t \leq 1$, supported near M such that $N_M(\eth^+ + S_t)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ and $0 \leq t \leq 1$, $N_M(S_0)(\tau) = N_M(R)(\tau)$, and such that for some $k \in \mathbb{N}$, we have $N_M(S_1)(\tau) : \mathcal{E}_k \longrightarrow \mathcal{E}_k$ and $N_M(S_1)(\tau) = 0$ on $\mathcal{E}_k^\perp$. Indeed, by Lemma 2.4, $N_M(\eth_M^+ + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ if and only if $\eth_M + R_M(0) : H^1(M, E_M) \longrightarrow L^2(M, E_M)$ is invertible. Moreover, by Clifford compatibility, it follows that $\eth_M$ and $R_M(0)$ are odd with respect to the $\mathbb{Z}_2$-grading of E_M. Hence, the restrictions, $\eth_M^\pm + R_M^\pm(0)$, to sections of $E_M^\pm$ are each invertible. Here, $\eth_M^- = (\eth_M^+)^*$ and $R_M^-(\tau) = (R_M^+(\overline{\tau}))^*$. Since the invertible operators form an open set in the bounded operators, there is an $\varepsilon > 0$ such that $\eth^+ + R_M^+(0) + A$ is invertible for all bounded operators A from $H^1(M, E_M^+)$ into $L^2(M, E_M^-)$ with norm less then ε. Observe that there exists a $k \in \mathbb{N}$ such that if we define

$$R_k^+(\tau) = \sum_{i+j \geq k+1} R_{ij}(\tau)\, \varphi_j^- \otimes \varphi_j^+, \quad R_{ij}(\tau) = (R_M^+(\tau)\varphi_j^+, \varphi_i^-),$$

then the norm of $R_k^+(\tau)$ between $H^1(M, E_M^+)$ and $L^2(M, E_M^-)$ is less then ε. Hence,

$$\eth_M^+ + R_M^+(0) - tR_k^+(0) : H^1(M, E_M^+) \longrightarrow L^2(M, E_M^-)$$

is invertible for all $0 \leq t \leq 1$. Let $S_t^\pm(\tau) = R_M^\pm(\tau) - tR_k^\pm(\tau)$, where $R_k^-(\tau) = (R_k^+(\overline{\tau}))^*$ for $\tau \in \mathbb{C}^2$. By definition, we have

$$S_1^+(\tau) = \sum_{i+j \leq k} R_{ij}(\tau)\, \varphi_j^- \otimes \varphi_j^+.$$

Let $S_t(\tau) = S_t^+(\tau) + S_t^-(\tau)$. Then $S_1(\tau) : \mathcal{E}_k \longrightarrow \mathcal{E}_k$ and $S_1(\tau) = 0$ on $\mathcal{E}_k^\perp$. Now choose any $S_t \in \Psi_b^{-\infty}(X, E^+, E^-)$ supported near M such that $N_M(S_t)(\tau) = (1/i)\sigma_2 S_t(\tau)$. Then by construction, S_t is Clifford compatible and S_t satisfies the conditions required.

We now show that there is a continuous family of Clifford compatible operators $T_t \in \Psi_b^{-\infty}(X, E^+, E^-)$, $0 \leq t \leq 1$, supported near M such that $N_M(\eth^+ + T_t)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ and $0 \leq t \leq 1$, $N_M(T_0)(\tau) = N_M(S_1)(\tau)$, and such that, using the notation around (2.9), we have $N_M(T_1)(\tau) = (1/i)\sigma_2\,\hat{\varphi}(\tau_1)^2\hat{\varphi}(\tau_2)^2 S_1(0)$, where $\tau = (\tau_1, \tau_2) \in \mathbb{C}^2$. In fact, just choose any $T_t \in \Psi_b^{-\infty}(X, E^+, E^-)$ supported near M such that $N_M(T_t)(\tau) = (1/i)\sigma_2\,\hat{\varphi}(t\tau_1)^2\hat{\varphi}(t\tau_2)^2 S_1((1-t)\tau)$. This T_t satisfies the conditions required.

We next show that there is a continuous family of Clifford compatible operators $U_t \in \Psi_b^{-\infty}(X, E^+, E^-)$, $0 \le t \le 1$, supported near M such that $N_M(\eth^+ + U_t)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ and $0 \le t \le 1$, $N_M(U_0)(\tau) = N_M(T_1)(\tau)$, and such that U_1 is an operator of the form given in (2.11). Let $T^+ : \ker \eth_M^+ \longrightarrow \ker \eth_M^-$ be any unitary operator. Here, we used that $\dim\ker\eth_M^+ = \dim\ker\eth_M^-$ by Lemma 2.11. Since $\mathrm{GL}(k, \mathbb{C})$ is connected, there exists a smooth family of linear isomorphisms $\tilde{V}_t^+ : \mathcal{E}_k^+ \longrightarrow \mathcal{E}_k^-$, $0 \le t \le 1$, such that $\tilde{V}_0^+ = (\eth_M^+ + S_1^+(0))|_{\mathcal{E}_k^+}$ and $\tilde{V}_1^+ = (\eth_M^+ + T^+)|_{\mathcal{E}_k^+}$. Let $V_t^+ = (\tilde{V}_t^+ - \eth_M^+)|_{\mathcal{E}_k^+}$, so that $(\eth_M^+ + V_t^+)|_{\mathcal{E}_k^+} = \tilde{V}_t^+$, and define $V_t^+ = 0$ on $(\mathcal{E}_k^+)^\perp$. Set $V_t^- = (V_t^+)^*$ and $V_t = V_t^+ + V_t^-$. Note that $V_t : \mathcal{E}_k \longrightarrow \mathcal{E}_k$ and $V_t = 0$ on $\mathcal{E}_k^\perp$. Since $V_t = 0$ on $\mathcal{E}_k^\perp$, it follows that V_t is really just a finite rank operator, and hence, $V_t \in \Psi^{-\infty}(M, E_M)$. Also, by construction, $\eth_M + V_t$ is invertible from $H^1(M, E_M)$ onto $L^2(M, E_M)$ for all $0 \le t \le 1$ such that $V_t = 0$ on $\mathcal{E}_k^\perp$, $V_0 = S_1(0)$, and $V_1 = T$, where $T = T^+ + T^-$ with $T^- = (T^+)^*$. Using the notation around (2.11), we define

$$U_t = \frac{1}{i}\sigma_2\,Q_1^2 Q_2^2\,V_t \in \Psi_b^{-\infty}(X, E^+, E^-).$$

Then U_t satisfies all the properties discussed above.

Finally, if we define

$$R_t = \begin{cases} S_{3t} & 0 \le t \le 1/3, \\ T_{3t-1} & 1/3 \le t \le 2/3, \\ U_{3t-2} & 2/3 \le t \le 1, \end{cases}$$

then $R_t \in \Psi_b^{-\infty}(X, E^+, E^-)$, $0 \le t \le 1$, is a continuous family of Clifford compatible operators supported near M such that $N_M(\eth^+ + R_t)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ and $0 \le t \le 1$, $N_M(R_0)(\tau) = N_M(R)(\tau)$, and R_1 is an operator of the form defined in (2.11). Adding together such operators for each $M \in M_2(X)$ produces an operator satisfying all the requirements of this proposition. □

2.4. **Examples.** We give a couple examples when the index of the positive part of the induced Dirac operator on each codimension two face is always zero.

Proposition 2.13. *Let X be compact, oriented, $4k$ dimensional manifold with corners of codimension two. Then for the signature operator, the index of the positive part of the induced Dirac operator on each codimension two face is zero.*

Proof. We first recall various facts about the signature operator. Here, $E = \wedge {}^bT^*X$ is the full exterior product of ${}^bT^*X$ and $\eth = d + \delta$, where $\delta = d^*$. The $\mathbb{Z}_2$-grading of E is given by $Z = i^{2k}\sigma(dg)$, where σ is the symbol of $\eth$ and dg is the Riemannian volume form. Thus, $E = E^+ \oplus E^-$, where $E^\pm$ are the ± 1 eigenspaces of Z.

Let $M \subset H_{i_1} \cap H_{i_2}$, $i_1 < i_2$, be a codimension two face. We show that $\operatorname{ind}\eth_M^+ = 0$. Now, $E_M = E^+|_M$ has the Clifford action $\sigma^M = i\sigma_2\sigma$ and is $\mathbb{Z}_2$-graded with respect to $\omega = i\sigma_2\sigma_1$. The induced Dirac operator is $\eth_M = \frac{1}{i}\sigma^M\nabla^M$, where ∇^M is the Levi-Civita connection of the induced Riemannian metric g_M on M (cf. Section

1.2). Since M is a closed compact manifold, by the Atiyah-Singer index theorem [3, Th. 4.3], we have

$$\operatorname{ind} \eth_M^+ = \int_M \widehat{A}(M) \cdot \mathrm{ch}'(E_M).$$

Here, $\widehat{A}(M) = \det^{1/2}((R_M/4\pi i)/(\sinh(R_M/4\pi i)))$ is the $\widehat{A}$-genus, with R_M the Riemannian curvature tensor of g_M; and $\mathrm{ch}'(E_M) = 2^{-(2k-1)}\mathrm{tr}(\tau e^{-Q'/8\pi i})$ is the relative Chern character of E_M, where $\tau = i^{2k-1}\sigma^M(dg_M) \cdot \omega$ is the relative $\mathbb{Z}_2$-grading, and $Q' = Q_M - \frac{1}{4}\sigma^M(R_M)$, where Q_M is the curvature of ∇^M on E_M.

By the results of [3, p. 149], $Q' = \frac{1}{4}\sigma'(R_M)$, where σ' is a Clifford action on E_M commuting with σ^M. Also, since $1 = i^{2k}\sigma(dg)$ on E_M, one can show that $\tau = 1$ or $\tau = -1$. Hence, by [3, Lem. 4.4], $\mathrm{ch}'(E_M) = \pm 2^{2k} \det^{1/2}(\cosh(R_M/4\pi i))$. It follows that $\widehat{A}(M) \cdot \mathrm{ch}'(E_M) = \pm\mathcal{L}(M)$, where $\mathcal{L}(M)$ is the L-genus. Thus, $\operatorname{ind} \eth_M^+ = \pm\mathrm{sign}(M)$. Since the signature of any $2\times$odd dimensional closed manifold is zero, and $\dim M = 2(2k-1)$, we have $\mathrm{sign}(M) = 0$. Thus, $\mathrm{ind}(\eth_M^+) = 0$. □

Proposition 2.14. *Let X be an even dimensional compact manifold with corners of codimension two. Suppose that each codimension two face is the boundary of some boundary hypersurface. Then for any Dirac operator on X, the index of the positive part of the induced operator on each codimension two face is zero.*

Proof. This follows from the cobordism invariant of the index for if $M \in M_2(X)$, then M bounds by assumption, so by Theorem 3 of [29, Ch. 17], $\operatorname{ind} \eth_M^+ = 0$. □

Corollary 2.15. *Let X be an even dimensional compact manifold with corners of codimension two with exactly one codimension two face. Then, given any Dirac operator, the index of the positive part of the induced operator on the codimension two face is zero.*

3. Regularized trace

Given a Dirac operator $\eth$ with $\operatorname{ind} \eth_M^+ = 0$ for each $M \in M_2(X)$, by Proposition 2.8, there exists an $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ such that $A = \eth^+ + R$ is Fredholm on L_b^2. The traditional analytic way to determine a formula for $\mathrm{ind}\,(\eth^+ + R)$ is to consider the difference in the traces of e^{-tA^*A} and e^{-tAA^*}. These heat operators are however not trace class. In this section, following [22] in the manifolds with boundary case, we define a regularization of the trace, called the b-trace, that can be used in place of the usual trace. To simplify notation, we drop vector bundles throughout this section. All the results hold for bundles with only notational changes.

3.1. b-trace.

The space $S^0([0,1)^k)$ consists of those functions on $[0,1)^k$ all of whose b-derivatives are bounded on $[0,1)^k$. (See appendix for more on symbol spaces.) Given any $\eta \geq 0$, we define $S^{0,\eta}([0,1)^k)$ as those symbols $u \in S^0([0,1)^k)$ such that given any $1 \leq j \leq k$, we can write $u = v_j + x_j^\eta w_j$, where $v_j \in C^\infty([0,1)^k)$, and where $w_j \in S^0([0,1)^k)$ is continuous, with all b-derivatives, up to $x_j = 0$.

Lemma 3.1. *Let $f \in S^{0,\eta}([0,1)^k)$, where $0 < \eta \leq 1$. Then we can write*

$$f(x_1, \ldots, x_k) = f(0) + \sum_I x_{i_1}^{\eta_{i_1}} \cdots x_{i_\ell}^{\eta_{i_\ell}} f_I(x_I),$$

where the sum is over all $I = (i_1, \ldots, i_\ell), 1 \leq i_1 < \cdots < i_\ell \leq k$, and where for each $I = (i_1, \ldots, i_\ell)$, $x_I = (x_{i_1}, \ldots, x_{i_\ell})$ and $f_I \in S^{0,0}([0,1)^\ell)$.

Moreover, if $\eta = 1$*, then for each* I*,*

$$f_I(x_I) = \int_0^1 \cdots \int_0^1 (\partial_{x_{i_1}} \cdots \partial_{x_{i_\ell}} f \circ g_I)(t_1 x_{i_1}, \ldots, t_\ell x_{i_\ell})\, dt_1 \cdots dt_\ell,$$

where $g_I(x_I) = (y_1, \ldots, y_k)$*, with* $y_i = 0$ *if* $i \notin I$*;* $y_i = x_{i_j}$ *if* $i = i_j$*.*

Proof. We use induction on k. If $k = 1$, then by definition of $S^{0,\eta}([0,1))$,

$$f(x) = f(0) + x^\eta f_1(x), \tag{3.1}$$

where $f_1(x) \in S^{0,0}([0,1))$, and where by the fundamental theorem of calculus, if $\eta = 1$, then $f_1(x) = \int_0^1 (\partial_x f)(tx)dt$. Thus, our lemma is true if $k = 1$. Assume our lemma is true for k; we prove it is true for $k+1$. Applying our lemma to the first k variables of $f(x_1, \ldots, x_{k+1})$ yields

$$f(x_1, \ldots, x_{k+1}) = f(0, x_{k+1}) + \sum_I x_{i_1}^{\eta_{i_1}} \cdots x_{i_\ell}^{\eta_{i_\ell}} f_I(x_I, x_{k+1}),$$

where if $\eta = 1$, then

$$f_I(x_I, x_{k+1}) = \int_0^1 \cdots \int_0^1 (\partial_{x_{i_1}} \cdots \partial_{x_{i_\ell}} f \circ g_I)(t_1 x_{i_1}, \ldots, t_\ell x_{i_\ell}, x_{k+1}) dt_1 \cdots dt_\ell,$$

where $g_I(x_I, x_{k+1}) = (y_1, \ldots, y_k, x_{k+1})$, with $y_i = 0$ if $i \notin I$; $y_i = x_{i_j}$ if $i = i_j$. Applying (3.1) to each of the terms $f(0, x_{k+1})$ and $f_I(x_I, x_{k+1})$ with respect to the variable x_{k+1} proves our lemma for $k+1$. □

We define

$$\mathbb{C}_+^N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N \,;\, \Re z_i > 0 \text{ for all } i\}.$$

Observe that if $A \in \Psi_b^{-\infty}(X)$, then $\rho^z A$, where $\rho^z = \rho_1^{z_1} \cdots \rho_N^{z_N}$, is trace class on $L_b^2(X)$ for all $z \in \mathbb{C}_+^N$ with trace given by $\operatorname{Tr}(\rho^z A) = \int_X (\rho^z A)|_{\Delta_b}$. Here, we identify A with its Schwartz kernel on X_b^2 and we identify Δ_b with X. The function $\mathbb{C}_+^N \ni z \mapsto \operatorname{Tr}(\rho^z A)$ is holomorphic, and as we now show, meromorphic on $\mathbb{C}$.

Proposition 3.2. *Let* $A \in \Psi_b^{-\infty}(X)$*. Then we can write*

$$\operatorname{Tr}(\rho^z A) = \sum_{I,J} \frac{f_I(z_I)}{z_J},$$

where the sum is over multi-indices I *and* J *such that* $I \cup J = \{1, \ldots, N\}$ *and* $I \cap J = \varnothing$*, and where for each* I*,* $f_I : \mathbb{C}^{|I|} \longrightarrow \mathbb{C}$ *is meromorphic, with only simple poles, all on the set* $\{z_I \in \mathbb{C}^{|I|} \,;\, z_i \in -\mathbb{N}$ *for some* $i \in I\}$*. In particular, for each* I*,* f_I *is holomorphic near* $z_I = 0$*.*

Proof. Let $\{\mathcal{U}_i\}$, where $\mathcal{U}_i \cong [0,\varepsilon)^k \times \mathbb{R}_y^{n-k}$ (some $k \geq 0$, where $n = \dim X$), be a covering of $X \cong \Delta_b$ with the appropriate ρ_j's defining the $[0,\varepsilon)$ factors. Let $\{\varphi_i\}$ be a partition of unity of Δ_b subordinate to the cover $\{\mathcal{U}_i\}$. Since $A|_{\Delta_b} = \sum_i \varphi_i A|_{\Delta_b}$, we may assume that $A|_{\Delta_b}$ is supported on some $\mathcal{U}_i$, which we now fix. Denote by $x_1, \ldots, x_k$, the fixed boundary defining functions that define the hypersurfaces of $\mathcal{U}_i$. Then $\rho^z A|_{\Delta_b} = x^w B(w', x, y)|\frac{dx}{x} dy|$, where $w = (z_1, \ldots, z_k)$, $w' = (z_{k+1}, \ldots, z_N)$, and where $B(w', x, y) \in C_c^\infty(\mathcal{U}_i)$ is entire in w', therefore

$$\operatorname{Tr}(\rho^z A) = \int_{[0,1)_x^k \times \mathbb{R}_y^{n-k}} x^w B(w', x, y) \frac{dx}{x} dy.$$

By Lemma 3.1, $B(w',x,y) = B(w',0,y) + \sum_I x_{i_1}\cdots x_{i_\ell} B_I(w',x_I,y)$ for some smooth functions $B_I(w',x_I,y)$. Since for any $a \in \mathbb{C}_+$, $\int_0^1 s^{a-1}ds = 1/a$, we have

$$\begin{aligned}\operatorname{Tr}(\rho^z A) &= \frac{1}{z_1\cdots z_k}\int B(w',0,y)dy \\ &\quad + \sum_{I,J} \frac{1}{z_{j_1}\cdots z_{j_{k-\ell}}}\int x_{i_1}^{z_{i_1}}\cdots x_{i_\ell}^{z_{i_\ell}} B_I(w',x_I,y)dx_I dy,\end{aligned} \tag{3.2}$$

where $I \cup J = (1,\ldots,k)$. By Taylor's theorem, $B_I(w',x_I,y) \sim \sum_\alpha x_I^\alpha B_{I,\alpha}(w',y)$, and so

$$\int x_{i_1}^{z_{i_1}}\cdots x_{i_\ell}^{z_{i_\ell}} B_I(w',x_I,y)dx_I dy \sim \sum_\alpha \frac{1}{z_{i_1}+\alpha_1+1}\cdots\frac{1}{z_{i_\ell}+\alpha_\ell+1}\int B_{I,\alpha}(w',y)dy.$$

This formula, together with (3.2), prove our result. □

In particular, $\operatorname{Tr}(\rho^z A) = \sum_{I,J,|I|\le N} f_I(z_I)/z_J + f(z)$, where $f(z)$ is holomorphic at 0. Thus, the regular value of $\operatorname{Tr}(\rho^z A)$ at $z=0$ is well-defined.

Definition 3.3. Let $A \in \Psi_b^{-\infty}(X)$. Then the regular value of the meromorphic function $\operatorname{Tr}(\rho^z A)$ at $z=0$ is called the *b-trace* of A and is denoted by ${}^b\operatorname{Tr}(A)$.

Remark 3.4. If $A \in \Psi_b^{-\infty,\alpha}(X)$ is in the calculus with bounds with $\alpha_{ff} > 0$, then a proof similar to that of Proposition 3.2 shows that we can write $\operatorname{Tr}(\rho^z A) = \sum_{I,J} f_I(z_I)/z_J$, where the sum is over all $I \cup J = \{1,\ldots,N\}$ with $I \cap J = \varnothing$, and where for each I, $f_I : \{z_I \in \mathbb{C}^{|I|}\,;\, z_i > -\alpha_i \text{ for all } i \in I\} \longrightarrow \mathbb{C}$ is holomorphic. Then ${}^b\operatorname{Tr}(A)$ is defined to be the regular value of $\operatorname{Tr}(\rho^z A)$ at $z=0$, just as in the case $A \in \Psi_b^{-\infty}(X)$.

Proposition 3.5. *If $A \in \rho^\varepsilon \Psi_b^{-\infty,\alpha}(X)$ where $\varepsilon > 0$, then ${}^b\operatorname{Tr}(A) = \operatorname{Tr}(A)$.*

Indeed, in this case, $\operatorname{Tr}(\rho^z A)$ is regular at $z=0$ with value $\operatorname{Tr}(A)$. Thus, the b-trace is a generalization of the usual trace. However, unlike the usual trace, the b-trace does not vanish on commutators, see Theorem 3.7.

3.2. **The trace defect formula.** Let $M = H_{i_1} \cap \cdots \cap H_{i_k}$, where $1 \le i_1 < \cdots < i_k \le N$. Then M is a possible disjoint union of codimension k boundary faces of X. Given $A \in \Psi_b^{-\infty}(X)$, we define

$$\int_{\mathbb{R}^k} {}^b\operatorname{Tr}(\,N_M(A)(\tau)\,)d\tau = \sum_{F\,=\,\text{component of } M} \int_{\mathbb{R}^k} {}^b\operatorname{Tr}(\,N_F(A)(\tau)\,)d\tau.$$

Lemma 3.6. *If $A \in \Psi_b^{-\infty}(X)$, then the regular value of $z_1\cdots z_k \operatorname{Tr}(\rho^z A)$ at $z=0$ is*

$$\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} {}^b\operatorname{Tr}(\,N_M(A)(\tau)\,)d\tau,$$

where $M = H_1 \cap \cdots \cap H_k$ and where the b-trace appearing in the integral is the b-trace on $\Psi_b^{-\infty}(M)$.

Proof. Let $\mathcal{U} = [0,1)^k_x \times M$ be a neighborhood in X near M where $x_i = \rho_i$ near $x_i = 0$. Let $\chi \in C^\infty_c([0,1))$ with $\chi = 1$ near 0 and set $\chi_i = \chi(x_i)$. Observe that $\chi_1 \cdots \chi_k - 1 = \sum_{i=1}^k (\chi_i - 1)\chi_{i+1}\cdots\chi_k$, so

$$A = \chi_1 \cdots \chi_k A - \sum_{i=1}^{k} (\chi_i - 1)\chi_{i+1} \cdots \chi_k A.$$

Since $\chi_i - 1 = 0$ near $x_i = 0$, it follows that $\mathrm{Tr}(\rho^z(\chi_i - 1)\chi_{i+1}\cdots\chi_k A)$ is regular at $z_i = 0$. Hence, the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z A)$ at $z = 0$ is the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z \chi_1 \cdots \chi_k A)$ at $z = 0$. Thus, we may assume that A is supported near M. If $s_j = x_j/x'_j$, where the primed variables denote the coordinates lifted to the right factor of $\mathcal{U}^2$, then (x, s) are coordinates near Δ_b on the $[0,1)^{2k}_b$ factor of $\mathcal{U}^2_b$ where the x's define *ff*. Thus, we can write $A = A(x,s)|\frac{dx'}{x'}dy'|$, where $A(x,s)$ is supported near $x = 0$. If $z = (w, w')$, where $w = (z_1,\dots,z_k)$ and $w' = (z_{k+1},\dots,z_N)$, then $\rho^z = x^w \cdot r^{w'}$ with $r = \rho_{k+1}\cdots\rho_N$, and we can write

$$\mathrm{Tr}(\rho^z A) = \int x^w B(w', x)\frac{dx}{x},$$

where $B(w',x) = \int_M r^{w'} A(x,1)|_{\Delta_b(M)}$. We can now proceed as we did in the proof of Proposition 3.2 to see that the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z A)$ at $z = 0$ is equal to the regular value of $\int_M r^{w'} A(0,1)|_{\Delta_b(M)}$ at $z = 0$. By the Mellin inversion formula, $A(0,1)$ is given by

$$A(0,s)|_{s=1} = \Big(\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} s^{i\tau} N_M(A)(\tau)d\tau\Big)\Big|_{s=1} = \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} N_M(A)(\tau)d\tau.$$

Thus, by the definition of the b-trace on $\Psi^{-\infty}_b(M)$, the regular value of the function $\int_M r^{w'} A(0,1)|_{\Delta_b(M)}$ at $z = 0$ is $\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\,N_M(A)(\tau)\,)d\tau$. □

The formula in the following theorem is called the *trace-defect formula* and it measures the non-commutativity of the b-trace.

Theorem 3.7. *Let $A \in \Psi^m_b(X)$ and $B \in \Psi^{m'}_b(X)$ with either m or m' equal to $-\infty$. Then*

$$(3.3) \qquad {}^b\mathrm{Tr}[A,B] = -\sum_{M\in M_k(X),k\geq 1} \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\,\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(B)(\tau)\,)\,d\tau,$$

where $\mathbf{D}^k_\tau = D_{\tau_1}\cdots D_{\tau_k}$ with $D_{\tau_j} = \frac{1}{i}\partial_{\tau_j}$.

Proof. Observe that $\rho^z[A,B] = [\rho^z, A]B + [A, \rho^z B]$. Since the trace vanishes on commutators, $\mathrm{Tr}([A,\rho^z B]) = 0$ for $z \in \mathbb{C}^N_+$. Thus, ${}^b\mathrm{Tr}[A,B]$ is the regular value of $\mathrm{Tr}([\rho^z, A]B)$ at $z = 0$, which is the regular value of $\mathrm{Tr}(\rho^z C(z))$ at $z = 0$, where $C(z) = \rho^{-z}[\rho^z, A]B = AB - \rho^{-z}A\rho^z B$. Note that $C(0) = 0$ and thus arguing as in Lemma 3.1, we can write $C(z) = \sum_I z_{i_1}\cdots z_{i_k} C_I(z_I)$, where

$$(3.4) \qquad C_I(z_I) = \int_0^1 \cdots \int_0^1 (\partial_{z_{i_1}}\cdots\partial_{z_{i_k}} C \circ g_I)(t_1 z_{i_1},\dots,t_k z_{i_k})dt_1\cdots dt_k,$$

where $g_I(z_I) = (y_1,\dots,y_N)$, with $y_i = 0$ if $i \notin I$; $y_i = z_{i_j}$ if $i = i_j$. Thus, $\mathrm{Tr}(\rho^z C(z)) = \sum_I z_{i_1}\cdots z_{i_k}\mathrm{Tr}(\rho^z C_I(z_I))$, so by Proposition 3.2 and Lemma 3.6, the regular value of $\mathrm{Tr}(\rho^z C(z))$ at $z = 0$ is

$$\sum_{k=1}^{N} \sum_{|I|=k} \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(N_{M_I}(C_I(0))(\tau)) d\tau,$$

where $M_I = H_{i_1} \cap \cdots \cap H_{i_k}$. We are now left to show that

$$\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(N_{M_I}(C_I(0))(\tau)) d\tau = -\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\mathbf{D}_\tau^k N_{M_I}(A)(\tau) N_{M_I}(B)(\tau)) d\tau.$$

To see this, let $s_i = \rho_i/\rho_i'$. Then (3.4) implies that

$$\begin{aligned} C_I(0) &= (\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} C)(0) = -\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} ([\rho^{-z} A \rho^z] B)|_{z=0} \\ &= -\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} ([s^{-z} A] B)|_{z=0} \\ &= -(-1)^k [(\log s_{i_1}) \cdots (\log s_{i_k}) A] B. \end{aligned}$$

Thus,

$$\begin{aligned} N_{M_I}(C_I(0))(\tau) &= -N_{M_I}(\,(-1)^k [(\log s_{i_1}) \cdots (\log s_{i_k}) A] B\,)(\tau) \\ &= -N_{M_I}(\,(-1)^k (\log s_{i_1}) \cdots (\log s_{i_k}) A\,)(\tau)\, N_{M_I}(B)(\tau) \\ &= -D_{\tau_1} \cdots D_{\tau_k} N_{M_I}(A)(\tau)\, N_{M_I}(B)(\tau), \end{aligned}$$

where we used that $N_{M_I}(A)(\tau) = \rho_{i_1}^{-i\tau_1} \cdots \rho_{i_k}^{-i\tau_k} A \rho_{i_1}^{i\tau_1} \cdots \rho_{i_k}^{i\tau_k}|_{M_I}$. □

3.3. ***b*-integral.** We end this section by discussing the b-integral.

Definition 3.8. If $u \in C^\infty(X, \Omega_b)$, then ${}^b\!\int u$ is defined to be the regular value of $\int_X \rho^z u$ at $z = 0$.

A similar argument used in Proposition 3.2 shows that $\int_X \rho^z u$ has a regular value at $z = 0$. The definition of the b-trace and the b-integral imply the following.

Lemma 3.9. *Given any $A \in \Psi_b^{-\infty}(X)$, we have ${}^b\mathrm{Tr}(A) = {}^b\!\int A|_{\Delta_b}$.*

4. Eta invariants on manifolds with boundary, I

4.1. **Eta invariant for perturbed Dirac operators.** Throughout this section, X is an odd-dimensional compact manifold with boundary, and $\eth \in \mathrm{Diff}_b^1(X, E)$ is a Dirac operator associated to an exact b-metric (see Section 1.1).

Lemma 4.1. *Let $P \in \mathrm{Diff}_b^2(X, E)$ be elliptic with a scalar, nonnegative principal symbol. Then given any $R \in \Psi_b^{-\infty}(X, E)$, $e^{-t(P+R)} = e^{-tP} + tR(t)$, where $R(t) \in C^\infty([0,\infty); \Psi_b^{-\infty}(X, E))$.*

Proof. Defining $F(t) = e^{-t(P+R)} - e^{-tP}$, we have $(\partial_t + (P + R))F(t) = -Re^{-tP}$. Hence, as $F(0) = 0$, by Duhamel's Principle, $F(t) = -\int_0^t e^{-(t-s)(P+R)} Re^{-sP}\, ds$. Since $R \in \Psi_b^{-\infty}(X, E)$, by the properties of the heat operator e^{-sP} as described in Appendix C, we have $Re^{-sP} \in C^\infty([0,\infty); \Psi_b^{-\infty}(X, E))$. It follows that $F(t) \in tC^\infty([0,\infty); \Psi_b^{-\infty}(X, E))$. □

Remark 4.2. This lemma holds for manifolds with corners with the same proof.

The formula (4.1) defines the *b-eta invariant* of the Fredholm perturbation $\eth + R$.

Proposition 4.3. *Given any self-adjoint $R \in \Psi_b^{-\infty}(X, E)$ such that $\eth + R$ is Fredholm, the integral*

$$ {}^b\eta(\eth + R) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(\,(\eth + R)e^{-t(\eth+R)^2}\,)\, dt \tag{4.1} $$

is absolutely convergent.

Proof. We first show that the integral is convergent near $t = \infty$. Indeed, by Proposition C.9, the difference $e^{-t(\eth+R)^2} - \Pi_0$, where Π_0 is the orthogonal projection onto the null space of $\eth + R$, is exponentially decreasing in $\Psi_b^{-\infty,\varepsilon}(X, E)$ as $t \to \infty$ for some $\varepsilon > 0$. Thus, $(\eth + R)e^{-t(\eth+R)^2}$ is also exponentially decreasing as $t \to \infty$, so the integral (4.1) is convergent near $t = \infty$.

If $R' = \eth R + R\eth + R^2 \in \Psi_b^{-\infty}(X, E)$, then $e^{-t(\eth+R)^2} = e^{-t(\eth^2+R')}$. Thus, by Lemma 4.1, $e^{-t(\eth+R)^2} = e^{-t\eth^2} + tT(t)$, where $T(t) \in C^\infty([0, \infty); \Psi_b^{-\infty}(X, E))$. It follows that

$$ (\eth + R)e^{-t(\eth+R)^2} = \eth e^{-t\eth^2} + S(t), $$

where $S(t) = Re^{-t\eth^2} + t(\eth + R)T(t)$. Observe that $S(t) \in C^\infty([0, \infty); \Psi_b^{-\infty}(X, E))$. This implies that ${}^b\mathrm{Tr}\,(S(t)) \in C^\infty([0, \infty)_t)$, and therefore $t^{-1/2}\,{}^b\mathrm{Tr}\,(S(t))$ is integrable near $t = 0$. By [22, Th. 8.36], ${}^b\mathrm{Tr}\,(\eth e^{-t\eth^2}) \in t^{1/2}C^\infty([0, \infty)_t)$. Thus, $t^{-1/2}\,{}^b\mathrm{Tr}\,((\eth + R)e^{-t(\eth+R)^2})$ is also integrable near $t = 0$. □

The following proposition shows that the b-eta invariant is also defined for $R = 0$.

Proposition 4.4. *The integral*

$$ {}^b\eta(\eth) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(\eth e^{-t\eth^2}) dt $$

is absolutely convergent; it defines the b-eta invariant of $\eth$.

Proof. This proposition is [22, Prop. 9.16], in which it is proved that ${}^b\mathrm{Tr}(\eth e^{-t\eth^2}) = O(t^{-1})$ as $t \to \infty$, and vanishes to order $t^{1/2}$ at $t = 0$. In particular, the integral defining the b-eta invariant of $\eth$ converges. □

Under certain conditions, we can relate ${}^b\eta(\eth + R)$ and ${}^b\eta(\eth)$, see Theorem 5.1.

4.2. **Perturbations of Dirac operators.** We now describe a particular perturbation R making $\eth$ Fredholm. Let $Y = \partial X = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_q$, where $Y_1, \ldots, Y_q$ are the connected components of Y, and let x be the fixed boundary defining function for Y. We denote $E|_Y$ by E_0 and we assume that $X \cong [0, 1)_x \times Y$ near Y over which $E \cong E_0$. As in the discussion around (1.5) in Section 1.2 for the even-dimensional case, we can write

$$ N_Y(\eth)(\tau) = \sigma\tau + B, \tag{4.2} $$

where $B \in \mathrm{Diff}^1(Y, E_0)$ is self-adjoint and $\sigma = \sigma(\frac{dx}{x})|_Y$ are such that $\sigma^* = \sigma$, $\sigma^2 = 1$, and $\sigma \circ B = -B \circ \sigma$. Note that E_0 is $\mathbb{Z}_2$-graded: $E_0 = E_0^+ \oplus E_0^-$, where $E_0^\pm$ are the ± 1 eigenspaces of $\omega_0 = -\sigma$.

As in (1.6) of Section 1.1, we define the induced Dirac operator on Y by $\eth_0 = i\sigma B \in \mathrm{Diff}^1(Y, E_0)$. Observe that $\omega_0 \circ \eth_0 = -\eth_0 \circ \omega_0$ and so $\eth_0$ is odd with respect to the $\mathbb{Z}_2$-grading of E_0. Thus, we can define $\eth_0^\pm : C^\infty(Y, E_0^\pm) \longrightarrow C^\infty(Y, E_0^\mp)$. Also, we can write (4.2) as

$$ N_Y(\eth)(\tau) = \Gamma[i\tau + \eth_0], \quad \Gamma = i\omega_0 = -i\sigma. \tag{4.3} $$

We denote the restriction of $\eth_0$ to $C^\infty(Y_i, E_0)$ by $\eth_0^i$. Then for each i, $\eth_0^i$ is odd with respect to the $\mathbb{Z}_2$-grading of $E_0|_{Y_i}$ given by ω_0^i, the restriction of ω_0 to $E_0|_{Y_i}$. We *assume* that for each i,

$$\operatorname{ind}(\eth_0^i)^+ = 0.$$

(This is true for instance if Y has only one component by the cobordism invariance of the index.) Now as in the discussion around (2.11) in Section 2.3, we can choose $T_i^+ \in \Psi^{-\infty}(Y_i, E_0^+, E_0^-)$ such that $T_i^+ : \ker(\eth_0^i)^+ \longrightarrow \ker(\eth_0^i)^-$ is a unitary isomorphism. Set $T_i = T_i^+ + T_i^- \in \Psi^{-\infty}(Y_i, E_0)$, where $T_i^- = (T_i^+)^{-1}$. Then T_i is self-adjoint and is odd with respect to the $\mathbb{Z}_2$-grading of $E_0|_{Y_i}$. Define

$$T = T_1 + \cdots + T_q \in \Psi^{-\infty}(Y, E_0). \tag{4.4}$$

With respect to the product decomposition $[0, 1)_x \times Y$, we define

$$R = -\frac{1}{i}\sigma Q^2 T = -\Gamma Q^2 T \in \Psi_b^{-\infty}(X, E), \tag{4.5}$$

where Q is the self-adjoint operator given in (2.9), which we now recall how to define. Let $\chi \in C_c^\infty([0, 1))$, where $\chi \geq 0$, $\chi(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, and $\chi(x) = 0$ for $x \geq \frac{3}{4}$. Let φ be an even, real-valued, Schwartz function on $\mathbb{R}$ with $\varphi(0) > 0$. We define $Q \in \Psi_b^{-\infty}([0, 1))$ by defining its Schwartz kernel, again denoted by Q:

$$Q = \varphi(\log s)\, \chi(x)\chi(x')\frac{dx'}{x'}, \qquad s = \frac{x}{x'}. \tag{4.6}$$

Since φ is even, Q is self-adjoint, and by definition, we have $N(Q)(\tau) = \widehat{\varphi}(\tau)$, where the Fourier transform $\widehat{\varphi}(\tau)$ is an even entire function. By Proposition 2.4, it follows that $\eth + R$ is Fredholm. Thus, the b-eta invariant ${}^b\eta(\eth + R)$ given by (4.1) is defined.

Note that $R \in \Psi_b^{-\infty}(X, E)$ since $T \in \Psi^{-\infty}(Y, E_0)$ is "diagonal" in the sense that in (4.4), T is written as the sum of individual operators on each component of Y. For any arbitrary element of $\Psi^{-\infty}(Y, E_0)$, the definition (4.5) would not define a b-pseudodifferential operator in the sense of Appendix A. In fact, it would define an operator in the "overblown" b-calculus, see [19, Sec. 4.6], which is very similar to the usual b-calculus, but allows "non-diagonal" elements. However, we remark that all the results in this section hold for any $T = T^+ + T^-$, where $T^+ : \ker \eth_0^+ \longrightarrow \ker \eth_0^-$ is a unitary isomorphism, and where $T^- = (T^+)^*$. For this reason, the overblown b-calculus is the natural class of operators in studying perturbed Dirac operators on manifolds with boundary.

4.3. **Variation of the eta.** Suppose that $R = R(r)$ defined by (4.5) (where T is of the form (4.4)) depends smoothly on a parameter $r \in [0, 1]$. In particular, we have

$$N_Y(\eth + R)(\tau) = \Gamma[i\tau + \eth_0 + R_0], \qquad R_0(r) = -\widehat{\varphi}_r(\tau)^2\, T(r), \tag{4.8}$$

where φ_r and each unitary isomorphism $T_i^+(r) : \ker(\eth_0^i)^+ \longrightarrow \ker(\eth_0^i)^-$ making up $T(r)$ depend smoothly on r. We now investigate the variation of the eta invariant of $\eth + R(r)$. To do so, we need two lemmas, in which we denote $\eth + R(r)$ by $A(r)$.

Lemma 4.5. *With $\dot{A} = \frac{d}{dr}A$, we have*

$$\frac{d}{dr}[t^{-1/2}\, {}^b\mathrm{Tr}(Ae^{-tA^2})] = \frac{d}{dt}[2t^{1/2}\, {}^b\mathrm{Tr}(\dot{A}e^{-tA^2})] + \alpha_r(t),$$

where

$$\alpha_r(t) = -t^{-1/2}\int_0^t {}^b\mathrm{Tr}[Ae^{-(t-s)A^2}, \dot{A}Ae^{-sA^2}]ds.$$

Proof. The same arguments, which are based on Duhamel's principle, used in the proof of Proposition 8.39 of [22] show that

$$\frac{d}{dr}[t^{-1/2}\,{}^b\mathrm{Tr}(Ae^{-tA^2})] = \frac{d}{dt}[2t^{1/2}\,{}^b\mathrm{Tr}(\dot{A}e^{-tA^2})] - \tag{4.9}$$
$$t^{-1/2}\int_0^t {}^b\mathrm{Tr}[Ae^{-(t-s)A^2}, \dot{A}Ae^{-sA^2}]ds - t^{-1/2}\int_0^t {}^b\mathrm{Tr}[A^2e^{-(t-s)A^2}, \dot{A}e^{-sA^2}]ds.$$

Thus, our lemma is proved once we show that the last term of this equation is zero. To see this, note that $\eth_0$ and R_0 anti-commute with Γ (as they are both odd with respect to the $\mathbb{Z}_2$-grading of E_0) and $R_0^2 = \widehat{\varphi}_r(\tau)^4\,\Pi_0$, where Π_0 is the projection onto the null space of $\eth_0$. Thus,

$$N_Y(A)(\tau)^2 = \tau^2 + (\eth_0 + R_0)^2 = \tau^2 + \eth_0^2 + \widehat{\varphi}_r(\tau)^4\,\Pi_0.$$

Now an elementary computation shows that for any $u \in \mathbb{R}$,

$$e^{-uN_Y(A)(\tau)^2} = e^{-u\tau^2}[e^{-u\eth_0^2} + (e^{-u\,\widehat{\varphi}_r(\tau)^4} - 1)\Pi_0]. \tag{4.10}$$

It follows that $N_Y(A^2e^{-(t-s)A^2})(\tau)$ is an even function of τ and, since $N_Y(\dot{A})(\tau) = \Gamma\dot{R}_0$, $N_Y(\dot{A}e^{-sA^2})(\tau)$ is also an even function of τ. Thus, the integrand in the trace-defect formula (3.3) for ${}^b\mathrm{Tr}[A^2e^{-(t-s)A^2}, \dot{A}e^{-sA^2}]$ is odd in the indicial parameter and hence this b-trace vanishes. Therefore, the last term in (4.9) is zero. □

Lemma 4.6. *Given $r_0, r_1 \in [0,1]$, we have*

$${}^b\eta(\eth + R(r_1)) - {}^b\eta(\eth + R(r_0)) = \lim_{t\to\infty}\Big\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_{r_0}^{r_1} {}^b\mathrm{Tr}(\dot{A}(r)e^{-tA(r)^2})\,dr\Big\}$$
$$+ \frac{1}{i\pi}\int_{r_0}^{r_1} \mathrm{Tr}(\dot{T}^+(r)T^+(r)^{-1})\,dr.$$

Proof. By definition of the b-eta invariant in (4.1), we have

$${}^b\eta(\eth + R(r_1)) - {}^b\eta(\eth + R(r_0)) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_{r_0}^{r_1}\frac{d}{dr}t^{-1/2}\,{}^b\mathrm{Tr}(Ae^{-tA^2}\,)\,dr\,dt.$$

By Lemma 4.5, for any $0 < \varepsilon < a$, we have

$$\int_\varepsilon^a \frac{d}{dr}[t^{-1/2}\,{}^b\mathrm{Tr}(Ae^{-tA^2})]dt = 2t^{1/2}\,{}^b\mathrm{Tr}(\dot{A}e^{-tA^2})\Big|_{t=\varepsilon}^{t=a} + \int_\varepsilon^a \alpha_r(t)\,dt.$$

Taking $a \to \infty$ and $\varepsilon \to 0$ shows that

$${}^b\eta(\eth + R(r_1)) - {}^b\eta(\eth + R(r_0)) = \lim_{t\to\infty}\Big\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_{r_0}^{r_1} {}^b\mathrm{Tr}(\dot{A}(r)e^{-tA(r)^2})\,dr\Big\}$$
$$- \lim_{t\to 0}\Big\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_{r_0}^{r_1} {}^b\mathrm{Tr}(\dot{A}(r)e^{-tA(r)^2})\,dr\Big\} + \int_0^\infty \alpha_r(t)\,dt.$$

Since $\dot{A} = \dot{R}(r) \in \Psi_b^{-\infty}(X,E)$, it follows that $2t^{1/2}\,{}^b\mathrm{Tr}(\dot{A}e^{-tA^2}) \to 0$ as $t \downarrow 0$. We show that $\frac{1}{\sqrt{\pi}}\int_0^\infty \alpha_r(t)\,dt = \frac{1}{i\pi}\mathrm{Tr}(\dot{T}^+(r)T^+(r)^{-1})$, which finishes the proof of this lemma. Now by the trace-defect formula (3.3), $\frac{1}{\sqrt{\pi}}\int_0^\infty \alpha_r(t)\,dt$ is given by

$$\frac{1}{2\pi^{3/2}i}\int_0^\infty t^{-1/2}\int_0^t\int_{\mathbb{R}} \operatorname{Tr}(\,\partial_\tau N_Y(Ae^{-(t-s)A^2})(\tau)\cdot N_Y(\dot{A}Ae^{-sA^2})(\tau)\,)\,d\tau\,ds\,dt \tag{4.11}$$

Since $N_Y(A)(\tau)=\Gamma[i\tau+\eth_0+R_0]$, for any $u\in\mathbb{R}$, we have

$$N_Y(A)(\tau)e^{-uN_Y(A)(\tau)^2} = i\Gamma\tau e^{-uN_Y(A)(\tau)^2} + \Gamma(\eth_0+R_0)e^{-uN_Y(A)(\tau)^2}. \tag{4.12}$$

Thus,

$$\begin{aligned}\partial_\tau[N_Y(A)(\tau)e^{-uN_Y(A)(\tau)^2}] = i\Gamma e^{-uN_Y(A)(\tau)^2} + i\Gamma\tau\partial_\tau e^{-uN_Y(A)(\tau)^2} + \\ \Gamma\partial_\tau R_0 e^{-uN_Y(A)(\tau)^2} + \Gamma(\eth_0+R_0)\partial_\tau e^{-uN_Y(A)(\tau)^2}.\end{aligned} \tag{4.13}$$

By (4.12), and three facts: $N_Y(\dot{A})(\tau)=\Gamma\dot{R}_0$, $\Gamma\dot{R}_0\Gamma=\dot{R}_0$ (as $\Gamma^2=-1$ and Γ anti-commutes with R_0), and $\dot{R}_0\eth_0=0$, we have

$$N_Y(\dot{A})(\tau)N_Y(A)(\tau)e^{-uN_Y(A)(\tau)^2} = i\tau\dot{R}_0 e^{-uN_Y(A)(\tau)^2} + \dot{R}_0R_0e^{-uN_Y(A)(\tau)^2}. \tag{4.14}$$

Since R_0 and $e^{-uN_Y(A)(\tau)^2}$ are both even in τ, the first two terms on the right of (4.13) are even, while the the last two terms on the right of (4.13) are odd. Also, the first term of (4.14) is odd, while the last term of (4.14) is even. Thus,

$$\partial_\tau N_Y(Ae^{-(t-s)A^2})(\tau)\cdot N_Y(\dot{A}Ae^{-sA^2})(\tau) = \gamma_1+\gamma_2 \quad \text{modulo odd in } \tau,$$

where

$$\begin{aligned}\gamma_1 = i\Gamma e^{-(t-s)N_Y(A)(\tau)^2}\dot{R}_0R_0e^{-sN_Y(A)(\tau)^2} + \\ i\Gamma\tau\partial_\tau e^{-(t-s)N_Y(A)(\tau)^2}\cdot\dot{R}_0R_0e^{-sN_Y(A)(\tau)^2}\end{aligned}$$

and

$$\begin{aligned}\gamma_2 = i\Gamma\tau\partial_\tau R_0\cdot e^{-(t-s)N_Y(A)(\tau)^2}\dot{R}_0e^{-sN_Y(A)(\tau)^2} + \\ i\Gamma(\eth_0+R_0)\tau\partial_\tau e^{-(t-s)N_Y(A)(\tau)^2}\cdot\dot{R}_0e^{-sN_Y(A)(\tau)^2}.\end{aligned}$$

Hence, as the odd terms in τ will integrate out to zero in the inside integral of (4.11), we are left to compute

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \alpha_r(t)\,dt = \frac{1}{2\pi^{3/2}i}\int_0^\infty t^{-1/2}\int_0^t\int_{\mathbb{R}}[\operatorname{Tr}(\gamma_1)+\operatorname{Tr}(\gamma_2)]d\tau ds dt.$$

Now by (4.10), for any linear map L on $\ker\eth_0$, we have

$$e^{-(t-s)N_Y(A)(\tau)^2}Le^{-sN_Y(A)(\tau)^2} = e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}L,$$

and

$$\tau\partial_\tau e^{-(t-s)N_Y(A)(\tau)^2}\cdot Le^{-sN_Y(A)(\tau)^2} = -(t-s)\tau\partial_\tau(\tau^2+\widehat{\varphi}_r(\tau)^4)e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}L.$$

Applying these two identities to γ_1 and γ_2 we obtain

$$\gamma_1 = e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}i\Gamma\dot{R}_0R_0 - (t-s)\tau\partial_\tau(\tau^2+\widehat{\varphi}_r(\tau)^4)e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}i\Gamma\dot{R}_0R_0$$

and

$$\gamma_2 = e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}i\Gamma\tau\partial_\tau R_0\cdot\dot{R}_0 - (t-s)\tau\partial_\tau(\tau^2+\widehat{\varphi}_r(\tau)^4)e^{-t\tau^2-t\widehat{\varphi}_r(\tau)^4}i\Gamma R_0\dot{R}_0.$$

Since R_0 anti-commutes with Γ, $\mathrm{Tr}(\Gamma R_0 \cdot \dot{R}_0) = \mathrm{Tr}(\dot{R}_0 \cdot \Gamma R_0) = -\mathrm{Tr}(\Gamma \dot{R}_0 R_0)$, and similarly, $\mathrm{Tr}(\Gamma \tau\partial_\tau R_0 \cdot \dot{R}_0) = -\mathrm{Tr}(\Gamma \dot{R}_0 \tau\partial_\tau R_0)$. Thus,

$$\mathrm{Tr}(\gamma_1) + \mathrm{Tr}(\gamma_2) = e^{-t\tau^2 - t\widehat{\varphi}_r(\tau)^4} i\Big[\mathrm{Tr}(\Gamma \dot{R}_0 R_0) - \mathrm{Tr}(\Gamma \dot{R}_0 \tau\partial_\tau R_0)\Big].$$

Now observe that $\dot{R}_0 = \frac{d}{dr}\widehat{\varphi}_r(\tau)^2 T + \widehat{\varphi}_r(\tau)^2 \dot{T}$ and $\tau\partial_\tau R_0 = \tau\partial_\tau[\widehat{\varphi}_r(\tau)^2]\cdot T$. Thus, as $T^2 = \Pi_0$ where Π_0 is the projection onto the null space of $\eth_0$, and $\mathrm{Tr}(\Gamma\Pi_0) = 0$, we have

$$\mathrm{Tr}(\gamma_1) + \mathrm{Tr}(\gamma_2) = e^{-t\tau^2 - t\widehat{\varphi}_r(\tau)^4} i\Big[\widehat{\varphi}_r(\tau)^4 - \widehat{\varphi}_r(\tau)^2 \tau\partial_\tau[\widehat{\varphi}_r(\tau)^2]\Big]\,\mathrm{Tr}(\Gamma \dot{T} T).$$

Since $\int_0^\infty t^{1/2} e^{-ta} dt = a^{-3/2}\frac{\sqrt{\pi}}{2}$ for any $a > 0$, we have

$$\int_0^\infty t^{-1/2}\int_0^t [\mathrm{Tr}(\gamma_1) + \mathrm{Tr}(\gamma_2)] ds\, dt = \frac{\sqrt{\pi}}{2}\frac{i(\widehat{\varphi}_r(\tau)^4 - \widehat{\varphi}_r(\tau)^2\tau\partial_\tau[\widehat{\varphi}_r(\tau)^2]\,)}{(\tau^2 + \widehat{\varphi}_r(\tau)^4)^{3/2}} \cdot \mathrm{Tr}(\Gamma\dot{T}T).$$

Integrating the right-hand side of this equation over $\tau \in \mathbb{R}$, using integration by parts, we get

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \alpha_r(t)\, dt = \frac{1}{2\pi^{3/2} i}\int_0^\infty t^{-1/2}\int_0^t\int_{\mathbb{R}} [\mathrm{Tr}(\gamma_1) + \mathrm{Tr}(\gamma_2)] d\tau ds dt = \frac{1}{2\pi}\mathrm{Tr}(\Gamma\dot{T}T).$$

Finally, to finish the proof of Theorem 4.7, we just need to prove that

$$\mathrm{Tr}(\Gamma\, \dot{T}(r) T(r)) = -2i\,\mathrm{Tr}(\dot{T}^+(r) T^+(r)^{-1}).$$

Since $T = T^+ + T^-$, where $T^+ : \ker \eth_0^+ \longrightarrow \ker \eth_0^-$ is a unitary isomorphism and where $T^- = (T^+)^*$, and since ω_0 defines the $\mathbb{Z}_2$-grading on E_0, we have

$$\mathrm{Tr}(\omega_0 \dot{T} T) = \mathrm{Tr}(\omega_0 \dot{T}^+ T^-) + \mathrm{Tr}(\omega_0 \dot{T}^- T^+) = -\mathrm{Tr}(\dot{T}^+ T^-) + \mathrm{Tr}(\dot{T}^- T^+).$$

However, as $T^+T^- = (T^-)^{-1}T^-$ is the projection onto $\ker \eth_0^-$, which is constant in r, it follows that $\dot{T}^+ T^- + T^+ \dot{T}^- = 0$. Thus, $\mathrm{Tr}(\dot{T}^- T^+) = -\mathrm{Tr}(\dot{T}^+ \dot{T}^-)$, and so

$$\mathrm{Tr}(\Gamma\dot{T}T) = i\,\mathrm{Tr}(\omega_0\dot{T}T) = -2i\,\mathrm{Tr}(\dot{T}^+ T^-) = -2i\,\mathrm{Tr}(\dot{T}^+ (T^+)^{-1}).$$

□

We now prove our variation formula.

Theorem 4.7. *Let $R(r) \in \Psi_b^{-\infty}(X, E)$, $r \in [0,1]$, be a smooth family of self-adjoint perturbations satisfying* (4.8) *and suppose that $\eth + R(r)$ is Fredholm for $r \in [0,1]$ and the null space of $\eth + R(r)$ has the same dimension for each $r \in [0,1]$. Then ${}^b\eta(\eth + R(r))$ is smooth for $r \in [0,1]$ and*

$$\frac{d}{dr}{}^b\eta(\eth + R(r)) = \frac{1}{i\pi}\mathrm{Tr}(\dot{T}^+(r)T^+(r)^{-1}),$$

where $\dot{T}^+(r) = \frac{d}{dr}T^+(r)$.

Proof. Since $\eth + R(r)$ is Fredholm and its null space has constant dimension by assumption, the proof used in Proposition 8.39 of [22] shows that $2t^{1/2}\,{}^b\mathrm{Tr}(\dot{A}e^{-tA^2}) \to 0$ as $t \to \infty$. Our theorem is proved. □

A class of perturbations which always admit constant dimensional null spaces is provided in Theorem 4.13.

4.4. **Lagrangian subspaces.** We now recall some results concerning Lagrangian subspaces, which will be used to define classes of perturbations such that the null space of $\eth + R$ and the b-eta invariant ${}^b\eta(\eth + R)$ are computable in terms of unperturbed data. We continue to use the notation of Section 4.1. Set

$$V = \ker \eth_0 = V_1 \oplus \cdots \oplus V_q, \quad V_i = \ker \eth_0^i.$$

If $(\cdot,\cdot)$ is the inner product on $L^2(Y, E_0)$, then $\Omega(v, w) = (\omega_0 v, w)$, $v, w \in V$, is a (complex) symplectic structure on V, where $\omega_0 = -\sigma$ is the $\mathbb{Z}_2$-grading of $E_0 = E|_{x=0}$. Let Π_0 be the orthogonal projection of $L^2(Y, E_0)$ onto V. Then there exists a canonical Lagrangian subspace Λ_C of V given by the "limiting values of extended L^2-solutions of $\eth u = 0$", also called the *scattering Lagrangian*:

$$\Lambda_C = \{\Pi_0(u|_Y)\,;\, u \in C^\infty(X, E) + \bigcap_{\varepsilon>0} x^\varepsilon H_b^\infty(X, E) \text{ and } \eth u = 0\}.$$

Let Π_C be the orthogonal projection onto Λ_C. Then [27],

$$C = 2\Pi_C - \mathrm{Id} \tag{4.15}$$

is a unitary isomorphism of V satisfying $C\omega_0 = -\omega_0 C$ and $C^2 = \mathrm{Id}$, and is such that $\Lambda_C = \{u \in V\,;\, Cu = u\}$. In particular, $\Pi_C = \frac{1}{2}(C + \mathrm{Id})$.

Definition 4.8. A unitary isomorphism $T : V \longrightarrow V$ is *Lagrangian with respect to* ω_0 if $T\omega_0 = -\omega_0 T$ and $T^2 = \mathrm{Id}$. The set of such isomorphisms is denoted by $\mathcal{L}(V)$. An element $T \in \mathcal{L}(V)$ is *diagonal* if $T = T_1 \oplus \cdots \oplus T_q$, where $T_i : V_i \longrightarrow V_i$ is (necessarily) Lagrangian with respect to ω_0^i, the restriction of ω_0 to V_i.

Observe that if $T^+ : V^+ \longrightarrow V^-$, where $V^+ = \ker \eth_0^+$ and $V^- = \ker \eth_0^-$, is a unitary isomorphism and $T^- = (T^+)^{-1}$, then $T = T^+ + T^-$ is Lagrangian with respect to ω_0. Conversely, every $T \in \mathcal{L}(V)$ arises in this way.

Note that every $T \in \mathcal{L}(V)$ has eigenvalues ± 1. Since $T\omega_0 = -\omega_0 T$, ω_0 is an isomorphism between the ± 1 eigenspaces. We denote the $+1$ eigenspace by Λ_T. This subspace is called the *Lagrangian subspace associated to* T and it can be identified as the image of the orthogonal projection $\Pi_T = \frac{1}{2}(T + \mathrm{Id})$. The -1 eigenspace of T is the orthogonal complement $\Lambda_T^\perp$, and can be identified as the image of the orthogonal projection $\Pi_T^\perp = \frac{1}{2}(\mathrm{Id} - T)$. Two elements $S, T \in \mathcal{L}(V)$ are said to be *transversal* if $\Lambda_S \cap \Lambda_T = 0$.

Lemma 4.9. *Two elements $S, T \in \mathcal{L}(V)$ are transversal if and only if for all $v \in V$ with $v \neq 0$, we have $Sv \neq Tv$. In other words, S and T are transversal if and only if the unitary matrix $U = ST$ does not have a $+1$ eigenvalue.*

Proof. It is clear that if $Sv \neq Tv$ for all $v \in V$ with $v \neq 0$, then $\Lambda_S \cap \Lambda_T = 0$. Conversely, let S and T be transversal and suppose that $Sv = Tv$ for some $v \in V$. We show that $v = 0$. Indeed, define $w = \omega_0(v - Sv) = \omega_0(v - Tv)$. Then, $Sw = w$ and $Tw = w$. Since S and T are transversal, $w = 0$. Thus, $Sv = v$ and $Tv = v$. Again, as S and T are transversal, $v = 0$. □

Proposition 4.10. *Let $T \in \mathcal{L}(V)$. Then the set of $S \in \mathcal{L}(V)$ that are transversal to T is open, dense, and connected in $\mathcal{L}(V)$. Moreover, there exists a diagonal $S \in \mathcal{L}(V)$ that is transversal to T.*

Proof. Since Λ_T is a closed subspace of V and since the eigenvectors corresponding to the eigenvalue $+1$ of $S \in \mathcal{L}(V)$ depend smoothly on S, if $S \in \mathcal{L}(V)$ and $\Lambda_S \cap \Lambda_T = 0$, then $\Lambda_R \cap \Lambda_T = 0$ for all $R \in \mathcal{L}(V)$ sufficiently close to S.

We now prove density. Let $S \in \mathcal{L}(V)$. Let $\Lambda_S = \operatorname{span}\{u_1, \dots, u_s\}$ and let $v_j = \omega_0 u_j$. Then $\{u_j, v_j\}$ is an orthogonal decomposition of V into the ± 1 eigenspaces of S. Suppose that $\Lambda_S \cap \Lambda_T = \operatorname{span}\{u_1, \dots, u_r\}$. For $\theta \in [0, 2\pi]$, we define

$$w_j^\theta = \cos\theta \cdot u_j + i \sin\theta \cdot v_j, \quad z_j^\theta = \omega_0 w_j^\theta = \cos\theta \cdot v_j + i\sin\theta \cdot u_j, \;\; 1 \le j \le r.$$

It follows that $\{w_j^\theta, z_j^\theta, u_k, v_k\}$, where $1 \le j \le r$ and $r+1 \le k \le s$, are mutually orthogonal unit vectors of V. Define

$$S_\theta w_j^\theta = w_j^\theta,\; S_\theta z_j^\theta = -z_j^\theta,\; 1 \le j \le r\,;\; S_\theta u_k = u_k,\; S_\theta v_k = -v_k,\; r+1 \le k \le s.$$

Then S_θ is a unitary isomorphism of V that is Lagrangian with respect to ω_0. By construction, $S_0 = S$ and for $0 < \theta < \pi$, S_θ is transversal to S. Thus, as $\theta \downarrow 0$, S_θ is a family of unitary isomorphisms of V, Lagrangian with respect to ω_0, converging to S. Observe that if S is diagonal, then by construction, S_θ is also diagonal.

We are left to prove connectedness. Let $R, S \in \mathcal{L}(V)$ be transversal to T. By Lemma 4.8, T^+R^- and T^+S^- do not have a $+1$ eigenvalue and thus, there exists a path of unitary operators U_t^- on V^- with no $+1$ eigenvalue such that $U_0^- = T^+R^-$ and $U_1^- = T^+S^-$. Define $U_t^+ = T^-(U_t^-)^*T^+$ and acting on $V = V^+ \oplus V^-$, set

$$U_t = \begin{bmatrix} U_t^+ & 0 \\ 0 & U_t^- \end{bmatrix}.$$

It is now straightforward to check that $T_t = TU_t$ defines a path in $\mathcal{L}(V)$ such that T_t is transversal to T for each $t \in [0,1]$ and is such that $T_0 = R$ and $T_1 = S$. ☐

4.5. **Null spaces of Perturbed Dirac operators.** We first analyze the null spaces of one-dimensional operators. For any $\alpha \in \mathbb{R}$ and interval $[0, a]$, we denote by $\hat{H}_b^\alpha([0,a], V)$, the space of functions that are in $H_b^\alpha([0,a], V)$ near $x = 0$, and are in $H^\alpha([0,a], V)$ away from $x = 0$. We set $\hat{L}_b^2([0,1], V) = \hat{H}_b^0([0,a], V)$.

Proposition 4.11. *Let $T, S \in \mathcal{L}(V)$ and set $D(T,S) = \Gamma(x\partial_x) + R$, where R is defined in terms of T by* (4.5) *and with domain given by*

$$\operatorname{Dom}(D(T,S)) = \{v \in \hat{H}_b^1([0,1], V)\,;\, \Pi_S^\perp v|_{x=1} = 0\},$$

where $\Pi_S = \frac{1}{2}(S + \mathrm{Id})$ is the orthogonal projection onto Λ_S (so $\Pi_S^\perp = \frac{1}{2}(\mathrm{Id} - S)$ is the orthogonal projection onto $\Lambda_S^\perp$). Then

$$D(T,S) : \operatorname{Dom}(D(T,S)) \longrightarrow \hat{L}_b^2([0,1], V)$$

is Fredholm and $\dim\ker D(T,S) = \dim(\Lambda_T \cap \Lambda_S)$.

Proof. For simplicity, we denote $D(T,S)$ by B. To prove that B is Fredholm, we construct a parametrix for it. We first work near $x = 0$. The formula (4.5) for R shows that near $x = x' = 0$, the Schwartz kernel of R is the same as the Schwartz kernel of the operator S with kernel defined by

$$S = -\varphi(\log(x/x'))^2 \, \frac{dx'}{x'} \cdot \Gamma T.$$

See (4.5) and (4.6) for the various notations. We define an operator G_1 by its Schwartz kernel:

$$G_1 = -\Gamma \frac{1}{2\pi} \int_{\mathbb{R}} (x/x')^{i\tau} (i\tau + \hat{\varphi}(\tau)^2 T)^{-1} d\tau \, \frac{dx'}{x'}.$$

One can check that G_1 maps $L_b^2([0,\infty)_x, V)$ into $H_b^1([0,\infty)_x, V)$, and that

$$G_1 \Gamma(x\partial_x + S) = \mathrm{Id}, \quad \Gamma(x\partial_x + S) G_1 = \mathrm{Id}.$$

We now work near $x = 1$. Since R is supported near $x = 0$, $B = \Gamma(x\partial_x)$ near $x = 1$. Consider the change of variables $s = -\log x$. Then, $x = 1$ corresponds to $s = 0$ and the interval $[0,1]_x$ transforms to $[0,\infty)_s$. Hence, near $s = 0$, $B = D$, where $D = -\Gamma\partial_s$. Consider the operator D with domain

$$\mathrm{Dom}(D) = \{v \in H^1_{loc}([0,\infty)_s, V)\, ;\, \Pi_S^{\perp} v|_{s=0} = 0\}.$$

Let $u = v + w \in L^2_c([0,\infty)_s, V)$ (compactly supported L^2 functions), where v takes values in Λ_S and w takes values in $\Lambda_S^{\perp} \equiv \Gamma\Lambda_S$. Define

$$G_2 u(s) = \Gamma\Big\{ \int_0^s v(r)dr \Big\} - \Gamma\Big\{ \int_s^\infty w(r)dr \Big\}.$$

One can check that G_2 maps $L^2_c([0,\infty)_s, V)$ into $\mathrm{Dom}(D)$ and that

$$DG_2 = \mathrm{Id} \ \text{ on } L^2_c([0,\infty)_s, V), \quad G_2 D = \mathrm{Id} \ \text{ on } H^1_c([0,\infty)_s, V).$$

Now let $\rho(z) \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $\rho(z) = 0$ for $z \le 1/4$ and $\rho(z) = 1$ for $z \ge 3/4$. Given real numbers $\alpha < \beta$, define $\rho_{\alpha,\beta}(z) = \rho((z-\alpha)/(\beta-\alpha))$. Then $\rho_{\alpha,\beta}(z) = 0$ on a neighborhood of $\{z \le \alpha\}$ and $\rho_{\alpha,\beta}(z) = 1$ on a neighborhood of $\{z \le \beta\}$. For simplicity, assume that R is supported on $[0,1/4]$. (If R is supported on an interval larger than $[0,1/4]$, then the subscripts of the ρ's below would be slightly more complicated.) We define

$$\psi_1(x) = 1 - \rho_{3/8,4/8}(x), \quad \psi_2(x) = 1 - \psi_1(x),$$
$$\varphi_1(x) = 1 - \rho_{4/8,5/8}(x), \quad \varphi_2(x) = \rho_{2/8,3/8}(x).$$

Then $\{\psi_i\}$ form a partition of unity of $[0,1]$ and $\varphi_i = 1$ on $\mathrm{supp}(\psi_i)$. We define

$$G = \varphi_1(x) G_1 \psi_1(x') + \varphi_2(x) G_2 \psi_2(x').$$

A straightforward verification, using the properties of G_1 and G_2 already stated, shows that G maps $\hat{L}^2_b([0,1], V)$ into $\mathrm{Dom}(B)$ and that

$$B\,G = \mathrm{Id} + K, \quad G\,B = \mathrm{Id} + K',$$

where K and K' are compact operators. It follows that B is Fredholm.

We now prove that $\dim\ker B = \dim(\Lambda_T \cap \Lambda_S)$. Let $\{u^0_j, u_j\}$ be a basis of Λ_T where $\{u^0_j\}$ is a basis of $\Lambda_T \cap \Lambda_S$. Then T decomposes as follows:

$$V = U^0 \oplus U \oplus W, \quad T = \mathrm{Id} \oplus \mathrm{Id} \oplus -\mathrm{Id},$$

where $U^0 = \mathrm{span}\{u^0_j\}$, $U = \mathrm{span}\{u_j\}$, and $W = \Gamma\Lambda_T$. We show that there are exactly $\dim(\Lambda_T \cap \Lambda_S)$ non-trivial solutions to the boundary value problem

$$(x\partial_x - Q^2 T)v = 0, \quad \Pi_S^{\perp} v|_{x=1} = 0 \tag{4.16}$$

if v takes values in U^0, and has no solutions otherwise. This proves the lemma. First suppose that v takes values in W. Let $\{w_j\}$ be a basis of W and write $v = \sum f_j(x) w_j$. Then by (4.16), for each j we have

$$(x\partial_x + Q^2) f_j(x) = 0. \tag{4.17}$$

Since $x\partial_x$ and Q are real, we may assume that f_j is real. Note that $f_j(0) = 0$ for each j since by assumption, v must be square integrable with respect to the measure dx/x and therefore must vanish to some power of x at $x = 0$ [22]. Thus, multiplying (4.17) by $f_j \frac{dx}{x}$ and using the fact that $\int_0^1 \partial_x f_j \cdot f_j dx = \frac{1}{2} f_j(1)^2$ as $f_j(0) = 0$, and that Q is self-adjoint, we obtain

$$\frac{1}{2} f_j(1)^2 + \int |Q f_j|^2 \frac{dx}{x} = 0. \tag{4.18}$$

Thus, $f_j(1) = 0$ and $Q f_j = 0$. In particular, as $Q f_j = 0$, by (4.17), f_j must be constant. As $f_j(0) = f_j(1) = 0$, f_j must be the constant 0.

Now suppose that v takes values in U. Since $U \cap \Lambda_S = 0$ and since $v(1) \in \Lambda_S$, we have $v(1) = 0$. Writing $v = \sum f_j u_j$, by (4.16), we have $(x\partial_x - Q^2) f_j(x) = 0$ for each j. Assuming that f_j is real, a similar argument used to prove (4.18) shows that

$$\frac{1}{2} f_j(1)^2 - \int |Q f_j|^2 \frac{dx}{x} = 0.$$

Since $v(1) = 0$, $f_j(1) = 0$, and thus, $Q f_j = 0$. Arguing as in the previous case shows that f_j must be the constant 0.

Thus, we are left with the case that v takes values in U^0. Write $v = \sum_j f_j u_j^0$. Then by (4.16), $B f_j = 0$ for each j, where B is the 1-dimensional operator

$$B = x\partial_x - Q^2 \quad \text{on } [0,1].$$

We prove that $\dim\ker B = 1$ on $\hat{H}_b^1([0,1])$; this finishes the proof of the proposition. We first observe that

$$B^* = -x\partial_x - Q^2.$$

Thus, the same argument used to prove that there are no solutions to (4.17) proves that $\ker B^* = \{0\}$. Since B is Fredholm, it follows that $\operatorname{ind} B = \dim\ker B$. We show that $\operatorname{ind} B = 1$. To see this, consider equation (4.6) for Q:

$$Q = \varphi(\log s)\, \chi(x)\chi(x') \frac{dx'}{x'}, \qquad s = \frac{x}{x'},$$

where $\chi \in C_c^\infty([0,1))$ is such that $\chi \geq 0$, $\chi(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, and $\chi(x) = 0$ for $x \geq \frac{3}{4}$, and where φ is an even, real-valued, Schwartz function on $\mathbb{R}$ with $\varphi(0) > 0$. Since $N(Q)(\tau) = \widehat{\varphi}(\tau)$, if

$$Q_1 = \varphi(\log s)^2\, \chi(x)\chi(x') \frac{dx'}{x'},$$

then $N(Q^2)(\tau) = N(Q_1)(\tau)$. Hence, by Theorem B.1 in the appendix, it follows that $Q^2 - Q_1$ is compact. Since the index is invariant under compact perturbations, we have $\operatorname{ind} B = \operatorname{ind}(x\partial_x - Q_1)$. Consider the following deformation of Q_1 defined by:

$$Q_t u = \frac{\chi(x)}{2\pi} \int e^{ix\tau}\, \widehat{\varphi}(t\tau)^2\, \widehat{\chi u}(\tau)\, d\tau, \qquad t \in [0,1].$$

Then $Q_t\big|_{t=1} = Q_1$ and, since $N(x\partial_x - Q_t)(\tau) = -i\tau - \widehat{\varphi}(t\tau)^2$ is invertible for $0 \leq t \leq 1$ and for all $\tau \in \mathbb{R}$, by Theorem B.2 in the appendix, it follows that $x\partial_x - Q_t$ is a continuous family of Fredholm operators for each $0 \leq t \leq 1$. Setting $t = 0$, we have $\operatorname{ind} B = \operatorname{ind}(x\partial_x - r\chi^2)$ where $r = \widehat{\varphi}(0)^2$. Again using the fact that the index is stable under compact perturbations, we can replace χ^2 with H, where $H(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, and $H(x) = 0$ for $x > \frac{1}{2}$ and conclude that $\operatorname{ind} B = \operatorname{ind} \tilde{B}$, where $\tilde{B} = x\partial_x - rH$. Since $\tilde{B}^* = -x\partial_x - rH$, the same argument used to prove that there are no solutions to (4.17) proves that $\ker \tilde{B}^* = \{0\}$. Suppose that $\tilde{B}f = 0$. Then

$$x\partial_x f - rH(x)f = 0,$$

and thus, for some $c \in \mathbb{C}$, $f = c\,x^r$ for $x \le \frac{1}{2}$ and $f = c(\frac{1}{2})^r$ for $x > \frac{1}{2}$. Therefore $\dim\ker \tilde{B} = 1$, which implies that $\operatorname{ind} B = 1$. □

Looking over the proof of this proposition, we find that we actually established the following stronger statement.

Corollary 4.12. *With the same hypotheses as Proposition 4.11, a non-trivial element $v \in \hat{H}^1_b([0,1], V)$ satisfies $[\Gamma(x\partial_x) + R]v = 0$ only if v takes values in Λ_T. Moreover, given any $v_0 \in \Lambda_S$, the boundary value problem*

$$v \in \operatorname{Dom}(D(T,S)), \quad D(T,S)v = 0, \quad v|_{x=1} = v_0,$$

has a non-trivial solution if and only if $v_0 \in \Lambda_T \cap \Lambda_S$, in which case, the solution is unique and also takes values in $\Lambda_T \cap \Lambda_S$. Thus, the boundary values in $\Lambda_T \cap \Lambda_S$ parameterize $\ker D(T,S)$.

As an easy consequence of Proposition 4.11, we prove the following result, which shows that there are many smooth families of perturbations $R(r)$ of the form (4.5) such that $\eth + R(r)$ has constant dimensional null space for $r \in [0,1]$.

Theorem 4.13. *Assume that on a collar $X \cong [0,1)_x \times Y$ of the boundary, $\eth$ is a product:*

$$\eth = \Gamma[x\partial_x + \eth_0].$$

If $T \in \mathcal{L}(V)$ is diagonal and $R \in \Psi_b^{-\infty}(X,E)$ is of product type, defined by (4.5) *with respect to the same product $X \cong [0,1)_x \times Y$, then*

$$\ker(\eth + R) \equiv \ker\eth \oplus \ker D(T,C) \qquad (\textit{kernels on } L^2_b),$$

where C is the scattering matrix in (4.15) *and $D(T,C)$ is the operator in Proposition 4.11 with $S = C$. In particular,*

$$\dim\ker(\eth + R) = \dim\ker\eth + \dim(\Lambda_T \cap \Lambda_C).$$

Proof. Suppose that $u \in \ker\eth$. Let $\varphi_j \in C^\infty(Y, E_0)$ be the eigenvectors of $\eth_0$ with corresponding eigenvalues $\lambda_j \in \mathbb{R}$. Then on the product decomposition, $X \cong [0,1)_x \times Y$, we can write $u = \sum_j f_j(x)\varphi_j(y)$ for some $f_j \in L^2_b([0,1))$. Since $\eth = \Gamma[x\partial_x + \eth_0]$ on the collar and since $\eth u = 0$, one concludes that $f_j(x) = 0$ if $\lambda_j \ge 0$, and $f_j(x) = c_j x^{-\lambda_j}$ if $\lambda_j < 0$, where c_j is a constant. Thus, $u = \sum_{\lambda_j<0} c_j x^{-\lambda_j}\varphi_j(y)$ on the collar. Since T acts only on V, and since R is supported on the collar, it follows that $Ru = 0$. Thus, $(\eth + R)u = 0$ and therefore $u \in \ker(\eth + R)$.

Suppose that $u \in \ker(\eth + R) \setminus \ker\eth$. Since $\eth + R = \Gamma[x\partial_x + \eth_0] + R$ and R acts only on the null space of $\eth_0$, just as in the previous paragraph, on the collar we can write $u = v(x,y) + \sum_{\lambda_j<0} c_j x^{-\lambda_j}\varphi_j(y)$, where $v(x,y) \ne 0$ takes values in V and $[\Gamma(x\partial_x) + R]v = 0$. Since R is supported on $[0,1)$, $v(x,y)$ must be constant off the support of R. Define $\widetilde{v} = u$ off of the collar and $\widetilde{v} = v(1,y) + \sum_{\lambda_j<0} x^{-\lambda_j} c_j\varphi_j(y)$ on the collar. Then $\widetilde{v} \in C^\infty(X,E) + \bigcap_{\varepsilon>0} x^\varepsilon H^\infty_b(X,E)$ and $\eth\widetilde{v} = 0$. Thus, by definition of the scattering Lagrangian, $v(1,y) \in \Lambda_C$. Thus, v is a non-trivial solution to the boundary value problem

$$[\Gamma(x\partial_x) + R]v = 0, \quad v|_{x=1} \in \Lambda_C.$$

By Proposition 4.11 (see also Corollary 4.12), there are exactly $\dim(\Lambda_T \cap \Lambda_C)$ independent solutions to this boundary value problem, occurring only when $v \in \Lambda_T \cap \Lambda_C$. It follows that $\ker(\eth + R) \setminus \ker \eth \equiv \Lambda_T \cap \Lambda_C$. □

4.6. **Eta invariant of the one-dimensional operator.** Let $T, S \in \mathcal{L}(V)$ and let $B = D(T, S)$ be the operator in Proposition 4.11. We now consider the b-eta invariant of B:

$$ {}^b\eta(B) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(Be^{-tB^2})\, dt. \tag{4.19}$$

Here, the heat operator e^{-tB^2} can be constructed explicitly using very similar arguments as in the (more elaborate) development presented after expression (5.9); and hence, will not be reproduced here. The b-trace ${}^b\mathrm{Tr}(Be^{-tB^2})$ is defined (see Definition 3.3) as the regular value at $z = 0$ of the meromorphic function $\mathrm{Tr}(x^z Be^{-tB^2})$ at $z = 0$. The fact that the b-trace exists follows from properties of the Schwartz kernel of e^{-tB^2} derived from its construction. The goal of this section is to prove that

$$ {}^b\eta(D(T, S)) = m(\Lambda_T, \Lambda_S), $$

where

$$ m(\Lambda_T, \Lambda_S) = -\frac{1}{i\pi} \sum_{\substack{e^{i\theta} \in \mathrm{spec}(-T^- S^+) \\ \theta \in (-\pi, \pi)}} i\theta. \tag{4.20}$$

If T and S are transversal, then $m(\Lambda_T, \Lambda_S) = -\frac{1}{i\pi}\mathrm{tr}(\log(-T^- S^+))$, where the logarithm is defined by its standard branch. By Lemma 4.9, $T^- S^+$ has no $+1$ eigenvalue if T and S are transversal, so $\log(-T^- S^+)$ is defined. The function $m(\Lambda_T, \Lambda_S)$ was first introduced in the work of Lesch and Wojciechowski [14]. It is nicely related to the Maslov index [6], [7]. To prove that ${}^b\eta(D(T, S)) = m(\Lambda_T, \Lambda_S)$, we begin with the following lemma.

Lemma 4.14. *If $T \in \mathcal{L}(V)$, then ${}^b\eta(D(T, -T)) = 0$.*

Proof. For simplicity, denote $D(T, -T)$ by D_T. Since $\Pi_{-T}^\perp = \Pi_T$, we have

$$ \mathrm{Dom}(D_T) = \{v \in \hat{H}_b^1([0, 1], V)\, ;\, \Pi_T v|_{x=1} = 0\}, \tag{4.21}$$

therefore

$$ \mathrm{Dom}(D_T^2) = \{v \in \hat{H}_b^2([0, 1], V)\, ;\, \Pi_T v|_{x=1} = 0,\ \Pi_T D_T v|_{x=1}\}. $$

The heat kernel $e^{-tD_T^2}$ takes an initial condition v to a function u_t that satisfies

$$ (\partial_t + D_T^2)u_t = 0; \qquad u_0 = v, \quad \Pi_T u_t|_{x=1} = 0, \quad \Pi_T D_T u_t|_{x=1} = 0. $$

Near $x = 1$, $R = 0$, so at $x = 1$ we have $\Pi_T D_T = \Pi_T \Gamma x \partial_x = \Gamma \Pi_T^\perp \partial_x$. Thus,

$$ (\partial_t + D_T^2)u_t = 0; \qquad u_0 = v, \quad \Pi_T u_t|_{x=1} = 0, \quad \Pi_T^\perp \partial_x u_t|_{x=1} = 0. $$

The heat operator of this heat equation is described as follows. Let $D_\pm$ be the scalar operators

$$ D_+ = \Gamma(x\partial_x + Q^2), \quad D_- = \Gamma(x\partial_x - Q^2), $$

where $Q \in \Psi_b^{-\infty}([0, 1))$ is the one-dimensional operator given in (4.6) and denote the corresponding solution operators to the following scalar heat equations by $e^{-tD_+^2}$

and $e^{-tD_-^2}$, respectively: Given an initial condition v, the functions $u_t^\pm = e^{-tD_\pm^2} v$ satisfy

$$(\partial_t + D_+^2)u_t^+ = 0, \qquad u_0^+ = v,\ (\partial_x u_t^+)|_{x=1} = 0;$$
$$(\partial_t + D_-^2)u_t^- = 0, \qquad u_0^- = v,\ u_t^-|_{x=1} = 0.$$

Since $R = -\Gamma Q^2 T$, we have $D_T = D_+$ on $\Lambda_T^\perp$ and $D_T = D_-$ on Λ_T, so

$$e^{-tD_T^2} = e^{-tD_+^2}\ \Pi_T^\perp + e^{-tD_-^2}\ \Pi_T,$$

Also, since $\Gamma : \Lambda_T \longrightarrow \Lambda_T^\perp$, we have $\mathrm{Tr}(\Gamma \Pi_T^\perp) = 0$ and $\mathrm{Tr}(\Gamma\Pi_T) = 0$. In particular,

$$^b\mathrm{Tr}(\Gamma x\partial_x e^{-tD_T^2}) = 0.$$

Furthermore, as $T\Pi_T = \Pi_T$ and $T\Pi_T^\perp = -\Pi_T^\perp$, we obtain

$$\mathrm{Tr}(\Gamma T\Pi_T) = \mathrm{Tr}(\Gamma\Pi_T) = 0, \quad \mathrm{Tr}(\Gamma T\Pi_T^\perp) = -\mathrm{Tr}(\Gamma\Pi_T^\perp) = 0.$$

Thus, ${}^b\mathrm{Tr}(Re^{-tD_T^2}) = 0$. Hence, ${}^b\mathrm{Tr}(D_T e^{-tD_T^2}) = 0$, and so ${}^b\eta(D_T) = 0$. □

As the proof that ${}^b\eta(D(T,S)) = m(\Lambda_T, \Lambda_S)$ for $T, S \in \mathcal{L}(V)$ is a bit detailed, we first give an outline of its proof. Fix $0 < a < 1$ such that R is supported completely on $[0, a)$. In order to use ${}^b\eta(D(T,-T))$ to help us calculate ${}^b\eta(D(T,S))$, the idea is to analytically separate $[0,1]$ into two parts: $[0,a]$ and $[a,1]$. On the interval $[0,a]$, we denote $\Gamma(x\partial_x) + R$ by D_1, which has the domain given in (4.21) with 1 replaced by a throughout the expression. On the interval $[a,1]$, we have $\Gamma(x\partial_x) + R = \Gamma(x\partial_x)$, which we denote by D_2, and on this interval, we put the boundary conditions

$$\mathrm{Dom}(D_2) = \{v \in H^1([a,1], V)\,;\, \Pi_T^\perp v|_{x=a} = 0,\ \Pi_S^\perp v|_{x=1} = 0\}. \tag{4.22}$$

The operator D_2 appears in, for instance, [14] and [7]. However, others usually put $s = \log x$ so that if $a' = \log a$, then

$$D_2 = \Gamma\partial_s, \quad \mathrm{Dom}(D_2) = \{v \in H^1([a',0]_s, V)\,;\, \Pi_T^\perp v|_{s=a'} = 0,\ \Pi_S^\perp v|_{s=0} = 0\}.$$

Observe that D_2 is not degenerate on $[a,1]_x$ (or on $[a',0]_s$ in the variable s): it is a true (as opposed to a "b-") elliptic operator on $[a,1]_x$. In [14], it is shown that $\eta(D_2) = m(\Lambda_T, \Lambda_S)$. Here, $\eta(D_2)$ is the usual eta invariant of D_2 defined by the integral (4.19) but with the usual trace replacing the b-trace. We show that ${}^b\eta(D(T,S))$ separates into two parts: ${}^b\eta(D(T,S)) = {}^b\eta(D_1) + \eta(D_2)$. The computation in Lemma 4.14 shows that ${}^b\eta(D_1) = 0$, and hence, as $\eta(D_2) = m(\Lambda_T, \Lambda_S)$, it follows that ${}^b\eta(D(T,S)) = m(\Lambda_T, \Lambda_S)$. Our proof is finished.

Thus, we are left to prove that ${}^b\eta(D(T,S)) = {}^b\eta(D_1) + \eta(D_2)$. The crux of the idea is to "twist apart" the subintervals $[0,a]$ and $[a,1]$, of $[0,1]$, from each other. The technique to do so goes back to Vishik [32], and was later applied by Brüning and Lesch [5] to study the eta invariant.

Acting on a pair of elements $(v_1, v_2) \in V \oplus V$ considered as a column vector, for each $\theta \in [0, \pi/4]$ we define (cf. [5, Sec. 3])

$$Q(\theta) = \begin{bmatrix} \cos^2\theta & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix} \otimes \Pi_T + \begin{bmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta \end{bmatrix} \otimes \Pi_T^\perp.$$

A straightforward computation shows that $Q(\theta)$ is an orthogonal projection:

$$Q(\theta)^2 = Q(\theta), \quad Q(\theta)^* = Q(\theta),$$

and that

$$Q(0) = \begin{bmatrix} \Pi_T & 0 \\ 0 & \Pi_T^\perp \end{bmatrix}, \quad Q(\pi/4) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We denote by B_θ, the operator $\Gamma(x\partial_x) + R$ with domain

$$\begin{aligned} \operatorname{Dom}(B_\theta) = \{u \in \hat{H}_b^1([0,a],V) \oplus H^1([a,1],V)\ ; \\ Q(\theta)u|_{x=a} = 0,\ \Pi_S^\perp u|_{x=1} = 0\}. \end{aligned} \tag{4.23}$$

Here, $\Gamma(x\partial_x) + R$ acts as $[\Gamma(x\partial_x) + R] \oplus \Gamma(x\partial_x)$ on the domain of B_θ since $R = 0$ on $[a,1]$. For each $\theta \in [0,\pi/4]$, the heat operator $e^{-tB_\theta^2}$ can be constructed explicitly using very similar arguments as detailed in Section 5.3 after Expression (5.9). Moreover, the b-eta invariant of B_θ can then be defined in terms of $e^{-tB_\theta^2}$ using the integral (4.19). As the proofs of the these facts concerning the heat operator of B_θ^2 are much simpler than those presented in Section 5.3, we will take these facts for granted, and leave the interested reader to check Section 5.3 for the details. We now compare B_θ for $\theta = 0$ and $\theta = \pi/4$.

If $u = (u_1,u_2) \in \operatorname{Dom}(B_{\pi/4})$, then $Q(\pi/4)(u_1,u_2)|_{x=a} = 0$ if and only if $u_1|_{x=a} = u_2|_{x=a}$. Thus, u considered as a function on $[0,1]$ itself, is continuous across $x = a$. For this reason, $Q(\pi/4)$ represents the *continuous transmission condition.* In particular, it follows that

$$\operatorname{Dom}(B_{\pi/4}) = \hat{H}_b^1([0,1],V),$$

and that $B_{\pi/4} = D(T,S)$, where $D(T,S)$ is the operator in Proposition 4.11. Thus,

$${}^b\eta(B_{\pi/4}) = {}^b\eta(D(T,S)).$$

We now consider B_0, which according to the formula for $Q(0)$, has domain

$$\begin{aligned} \operatorname{Dom}(B_0) = \{(u_1,u_2) \in & \hat{H}_b^1([0,a],V) \oplus H^1([a,1],V)\ ; \\ & \Pi_T u_1|_{x=a} = 0,\ \Pi_T^\perp u_2|_{x=a} = 0,\ \Pi_S^\perp u_2|_{x=1} = 0\}. \end{aligned}$$

Thus, $\operatorname{Dom}(B_0)$ decomposes into two parts: $\operatorname{Dom}(B_0) = \operatorname{Dom}(D_1) \oplus \operatorname{Dom}(D_2)$, where $D_1 = \Gamma(x\partial_x) + R$ with $\operatorname{Dom}(D_1) = \{u \in \hat{H}_b^1([0,a],V)\,;\,\Pi_T u|_{x=a} = 0\}$, and where $D_2 = \Gamma(x\partial_x)$ has domain given in (4.22). It follows that the heat operators decompose:

$$e^{-tB_0^2} = e^{-tD_1^2} \oplus e^{-tD_2^2}, \quad B_0 e^{-tB_0^2} = D_1 e^{-tD_1^2} \oplus D_2 e^{-tD_2^2}.$$

Hence, ${}^b\eta(B_0)$ also splits into two pieces:

$${}^b\eta(B_0) = {}^b\eta(D_1) + \eta(D_2).$$

Since ${}^b\eta(D_1) = 0$ by Lemma 4.14 and $\eta(D_2) = m(\Lambda_T,\Lambda_S)$ by [14], we have ${}^b\eta(B_0) = \eta(D_2) = m(\Lambda_T,\Lambda_S)$. In summary, the projector $Q(\theta)$ "rotates" from the boundary condition that separates $[0,a]$ and $[a,1]$ from each other, and which produces the operators D_1 and D_2, to the continuous transmission condition, which gives the original operator $D(T,S)$. We shall prove that ${}^b\eta(B_\theta)$ is in fact constant in the parameter θ. It follows that

$${}^b\eta(D(T,S)) = {}^b\eta(B_{\pi/4}) = {}^b\eta(B_0) = m(\Lambda_T,\Lambda_S).$$

Thus, it remains to prove that ${}^b\eta(B_\theta)$ is constant.

Proposition 4.15. *Let $T, S \in \mathcal{L}(V)$ and let B_θ be the operator $\Gamma(x\partial_x) + R$ with domain* (4.23)*. Then the b-eta invariant ${}^b\eta(B_\theta)$ is constant in the parameter $\theta \in [0, \pi/4]$. In particular, setting $\theta = 0$ and $\theta = \pi/4$, we have*

$$^b\eta(D(T,S)) = m(\Lambda_T, \Lambda_S),$$

where $m(\Lambda_T, \Lambda_S)$ is defined in (4.20)*.*

Proof. To prove that ${}^b\eta(B_\theta)$ is constant, we use similar arguments as those found in Section 4.3 concerning the variation of the eta invariant. We start by showing that $\dim\ker B_\theta$ is constant in $\theta \in [0, \pi/4]$. Thus, let $u = (u_1, u_2) \in \ker B_\theta$. Then $B_\theta u_1 = [\Gamma(x\partial_x) + R]u_1 = 0$ and $B_\theta u_2 = \Gamma(x\partial_x u_2) = 0$, and $Q(\theta)(u_1, u_2)|_{x=a} = 0$ and $\Pi_S^\perp u_2|_{x=1} = 0$. Since $[\Gamma(x\partial_x)+R]u_1 = 0$, by Corollary 4.12, we have $u_1(x) \in \Lambda_T$ for all $x \in [0, a]$. In particular, $u_1|_{x=a} \in \Lambda_T$; that is, $\Pi_T^\perp u_1|_{x=a} = 0$. Since $Q(\theta)(u_1, u_2)|_{x=a} = 0$, by the definition of $Q(\theta)$, we have

$$\cos\theta\, u_1|_{x=a} = \sin\theta\, \Pi_T u_2|_{x=a}, \quad \cos\theta\, \Pi_T^\perp u_2|_{x=a} = 0. \tag{4.24}$$

Since $\theta \in [0, \pi/4]$, the second equation in (4.24) implies that $u_2|_{x=a} \in \Lambda_T$. On the other hand, as $B_\theta u_2 = \Gamma(x\partial_x u_2) = 0$, it follows that u_2 is constant. Moreover, since $\Pi_S^\perp u_2|_{x=1} = 0$, we must have $u_2 \in \Lambda_S$. Thus, $u_2 \in \Lambda_T \cap \Lambda_S$. Now the first equation in (4.24), plus Corollary 4.12, imply that if $\theta = 0$, then $u_1 = 0$, and if $\theta \in (0, \pi/4]$, then u_1 is completely determined by the value of $u_2 \in \Lambda_T \cap \Lambda_S$. Therefore, $\dim\ker B_\theta = \dim(\Lambda_T \cap \Lambda_S)$ for all $\theta \in [0, \pi/4]$, and hence, $\dim\ker B_\theta$ is constant. We can now proceed as in the proof of Theorem 5.7, which is provided Section 5.3, to prove that ${}^b\eta(B_\theta)$ is constant. However, the proof of our current problem is not as involved, since in this case we are dealing with a one-dimensional operator. As all the details are given to prove Theorem 5.7, to avoid repeating the essentially the same arguments, we appeal to Section 5.3 for the remaining details of this proof. □

5. Eta invariants on manifolds with boundary, II

5.1. **Main result.** Throughout this section, X is an odd-dimensional compact manifold with boundary, and $\eth \in \mathrm{Diff}_b^1(X, E)$ is a Dirac operator associated to an exact b-metric (see Section 1.1). We refer the reader to Sections 4.3, 4.4, and 4.5 for the various notations in this section. Our aim is to prove the following theorem.

Theorem 5.1. *Assume that on a collar $X \cong [0,1)_x \times Y$ of the boundary, $\eth$ is a product:*

$$\eth = \Gamma[x\partial_x + \eth_0].$$

Let $T \in \mathcal{L}(V)$ be diagonal and let $R \in \Psi_b^{-\infty}(X, E)$ be of product type, defined by (4.5) *with respect to the same product $X \cong [0,1)_x \times Y$. Then*

$$^b\eta(\eth + R) = {}^b\eta(\eth) + m(\Lambda_T, \Lambda_C),$$

where C is the scattering matrix given in (4.15) *and $m(\Lambda_T, \Lambda_C)$ is defined in* (4.20)*.*

The idea of this proof is similar to that of Proposition 4.15. In the first step, we analytically "twist off" the null mode $V = \ker\eth_0$ on the end that carries R in order to separate ${}^b\eta(\eth+R)$ into two pieces: an eta invariant on the null mode on the collar involving the one-dimensional perturbed Dirac operator $D(T, C)$ (see Proposition 4.11 with $S = C$), plus an eta invariant on the rest of the manifold independent of R. In the second step, we show that the eta invariant independent of R equals

${}^b\eta(\eth)$. As the proof of Theorem 5.1 is quite long, we break it up into various parts. We set up these steps in Section 5.2, and the proofs of these steps are provided in Sections 5.3 and 5.4.

5.2. **The program to prove Theorem 5.1.** Here we set up the initial steps to prove Theorem 5.1. By an appropriate scaling of the normal variable, we may assume that our odd-dimensional manifold with boundary X has a collar neighborhood $X \cong [0,e]_x \times Y$ near $\partial X = Y$ over which $E \cong E_0 = E|_{x=0}$,

$$\eth = \Gamma[x\partial_x + \eth_0],$$

and $x > e$ off this collar. Let $T \in \mathcal{L}(V)$ be diagonal and let $R \in \Psi_b^{-\infty}(X,E)$ be of product type, defined by (4.5); thus, R is defined by (4.5) with respect to the product $X \cong [0,e]_x \times Y$. We assume that R is supported on the "subcollar" $[0,e^{-1})_x \times Y$. This assumption does not effect the dimension of the null space of $\eth + R$ by Theorem 4.13, and hence neither the b-eta invariant of $\eth + R$ by Theorem 4.7. As mentioned already, to prove Theorem 5.1, we first separate ${}^b\eta(\eth{+}R)$ into two pieces: an eta invariant on the null mode on the collar involving the one-dimensional perturbed Dirac operator, plus an eta invariant on the rest of the manifold that does not depend on R.

To begin this program, we start by separating X at $x = 1$ into two halves as follows. Let

$$M = [0,1]_x \times Y, \quad N = \{p \in X \, ; \, x(p) \geq 1\}.$$

Since $X \cong [0,e]_x \times Y$, we have

$$N \cong [1,e]_x \times Y \quad \text{near } \partial N = Y,$$

and gluing together M and N, identifying the sets where $x = 1$ in the obvious way, reproduces the original manifold X. Note that the Dirac operator $\eth$ induces in a canonical way, operators on M and N, and the b-pseudodifferential operator R is supported completely on M near $x = 0$.

Acting on pairs (v_1, v_2) regarded as a column vector, where $v_1, v_2 \in L^2(Y, E_0)$, we define

$$P = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then P is an orthogonal projection:

$$P^2 = P, \quad P^* = P,$$

and $P(v_1, v_2) = 0$ if and only if $v_1 = v_2$. For any $\alpha \in \mathbb{R}$, we denote by $\hat{H}_b^\alpha(M, E_0)$, the space of functions that are in $H_b^\alpha(M, E_0)$ near $x = 0$, and are in $H^\alpha(M, E_0)$ away from $x = 0$. Set $A = \eth + R$ and denote by A_P, the operator A with domain

$$\mathrm{Dom}(A_P) = \{u \in \hat{H}_b^1(M, E_0) \oplus H^1(N, E) \; ; \; Pu|_{x=1} = 0\}.$$

To define $u|_{x=1}$ we use the fact that for any manifold Z, restricting a function in $H^1(Z)$ to a hypersurface S defines a continuous map $H^1(Z) \longrightarrow H^{1/2}(S)$. Let $u = (u_1, u_2) \in \mathrm{Dom}(A_P)$. Then by definition of P, we have $P(u_1, u_2)|_{x=1} = 0$ if and only if $u_1|_{x=1} = u_2|_{x=1}$. Thus, u considered as a function on X itself, is continuous across $x = 1$. For this reason, P represents the *continuous transmission condition.* In particular, by standard Sobolev space theory on manifolds with boundary, see for instance [4], it follows that

$$\mathrm{Dom}(A_P) \equiv H_b^1(X, E).$$

Thus, $e^{-tA_P^2} = e^{-tA^2}$ where e^{-tA^2} is the heat operator of A^2 in the usual sense (see Appendix C), and so

$$ {}^b\eta(A_P) = {}^b\eta(\eth + R). \tag{5.1} $$

To relate ${}^b\eta(\eth + R)$ to ${}^b\eta(\eth)$, the idea is to "twist off" the null mode on the end that carries R. To do this, we follow Brüning and Lesch [5] and Vishik [32].

Let Π_C be the orthogonal projection onto the scattering Lagrangian. Acting on pairs of functions (v_1, v_2) considered as a column vector, where $v_1, v_2 \in L^2(Y, E_0)$, for each $\theta \in [0, \pi/4]$, we define, cf. [5, Sec. 3],

$$ P(\theta) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \Pi_0^\perp + Q(\theta), \quad 0 \le \theta \le \pi/4, \tag{5.2} $$

where Π_0 is the orthogonal projection onto $V = \ker \eth_0$, and where

$$ Q(\theta) = \begin{bmatrix} \cos^2\theta & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix} \otimes \Pi_C^\perp + \begin{bmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta \end{bmatrix} \otimes \Pi_C. \tag{5.3} $$

The projection $Q(\theta)$ acts entirely on the null mode V, $P(\theta)$ is the continuous transmission condition on $V^\perp$, and by a straightforward computation $P(\theta)$ is an orthogonal projection:

$$ P(\theta)^2 = P(\theta), \quad P(\theta)^* = P(\theta), $$

such that

$$ P(0) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \Pi_0^\perp + \begin{bmatrix} \Pi_C^\perp & 0 \\ 0 & \Pi_C \end{bmatrix}, \quad P(\pi/4) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. $$

In particular, $P(\pi/4)$ is just the continuous transmission boundary condition. Denote by A_θ, the operator $A = \eth + R$ with domain

$$ \mathrm{Dom}(A_\theta) = \{u \in \hat{H}_b^1(M, E_0) \oplus H^1(N, E)\ ;\ P(\theta)u|_{x=1} = 0\}. \tag{5.4} $$

Elements of $\mathrm{Dom}(A_\theta)$ can be described as follows. Let $u = (u_1, u_2) \in \mathrm{Dom}(A_\theta)$ and write $u_1 = v_1 + w_1$ and, on the collar $N \cong [1, e]_x \times Y$, $u_2 = v_2 + w_2$, where v_i, respectively w_i, is the orthogonal projection of u_i onto V, respectively $V^\perp$. Then by definition of $P(\theta)$, we have $P(\theta)(u_1, u_2)|_{x=1} = 0$ if and only if $w_1|_{x=1} = w_2|_{x=1}$ and $Q(\theta)(v_1, v_2)|_{x=1} = 0$. Thus, elements of $\mathrm{Dom}(A_\theta)$ are continuous across $x = 1$ in $V^\perp$; it is only on V where the boundary condition $P(\theta)$ has any effect. In particular, gluing together w_1 and w_2 at $x = 1$, on the product decomposition $X \cong [0, e]_x \times Y$ an element $u \in \mathrm{Dom}(A_\theta)$ can be written in the form $u = v + w$, where $w \in \Pi_0^\perp \hat{H}_b^1([0, e] \times Y, E_0)$, and where $v = (v_1, v_2)$ with $v_1 \in \hat{H}_b^1([0, 1], V)$ and $v_2 \in H^1([1, e], V)$ such that $Q(\theta)(v_1, v_2)|_{x=1} = 0$.

In Section 5.3, for each $\theta \in [0, \pi/4]$ we show that the b-eta invariant of A_θ can be defined in terms of the heat operator by the usual integral (4.1) and we show that the variation of ${}^b\eta(A_\theta)$ in θ is zero. Let us consider the b-eta invariant at the end points. By (5.1) we have ${}^b\eta(A_{\pi/4}) = {}^b\eta(\eth + R)$. To compute ${}^b\eta(A_0)$, note that by the formula for $P(0)$, we can write

$$ \mathrm{Dom}(A_0) = \{(u_1, u_2) \in \Pi_0 \hat{H}_b^1(M, E_0) \oplus \left[\Pi_0^\perp \hat{H}_b^1(M, E_0) \oplus H^1(N, E)\right]\ ; \\ \Pi_C^\perp u_1|_{x=1} = 0,\ \hat{\Pi} u_2|_{x=1} = 0\}, $$

where

$$ \hat{\Pi} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \Pi_0^\perp + \begin{bmatrix} 0 & 0 \\ 0 & \Pi_C \end{bmatrix}. $$

Thus, $\mathrm{Dom}(A_0)$ decomposes into two parts:

$$\mathrm{Dom}(A_0) = \mathrm{Dom}(D(T,C)) \oplus \mathrm{Dom}(D),$$

where $D(T,C)$ is the one-dimensional operator $\Gamma(x\partial_x)+R$ with domain (see Proposition 4.11 with $S = C$)

$$\mathrm{Dom}(D(T,C)) = \{u \in \Pi_0 \hat{H}^1_b(M,E_0) = \hat{H}^1_b([0,1],V)\,;\, \Pi_C^\perp u|_{x=1} = 0\},$$

and where D is the Dirac operator $\eth$ with domain

$$\mathrm{Dom}(D) = \{u \in \Pi_0^\perp \hat{H}^1_b(M,E_0) \oplus H^1(N,E)\ ;\ \hat{\Pi} u|_{x=1} = 0\}. \tag{5.5}$$

In particular, the heat operators also decompose:

$$e^{-tA_0^2} = e^{-tD(T,C)^2} \oplus e^{-tD^2}, \quad A_0 e^{-tA_0^2} = D(T,C)e^{-tD(T,C)^2} \oplus De^{-tD^2}, \tag{5.6}$$

and hence, ${}^b\eta(A_0)$ also splits into two pieces:

$${}^b\eta(A_0) = {}^b\eta(D(T,C)) + {}^b\eta(D),$$

where ${}^b\eta(D(T,C))$ is the b-eta invariant of the one-dimensional perturbed Dirac operator $D(T,C)$ and where ${}^b\eta(D)$ is the b-eta invariant of the operator D. To relate ${}^b\eta(A_{\pi/4}) = {}^b\eta(\eth + R)$ and ${}^b\eta(A_0) = {}^b\eta(D(T,C)) + {}^b\eta(D)$, we first prove that $\dim\ker A_\theta$ is constant in order to understand the variation of ${}^b\eta(A_\theta)$.

Proposition 5.2. *For any $\theta \in [0,\pi/4]$, we have*

$$\ker A_\theta \equiv \ker \eth \oplus \ker D(T,C) \qquad \textit{(kernels on } L^2_b\textit{)},$$

where C is the scattering matrix in (4.15) *and $D(T,C)$ is the operator in Proposition 4.11 with $S = C$. In particular,*

$$\dim\ker A_\theta = \dim\ker\eth + \dim(\Lambda_T \cap \Lambda_C).$$

Proof. Arguing in a similar manner as in the proof of Theorem 4.13, one can show that $\ker\eth$ can be considered a subspace of $\ker A_\theta$ for each $\theta \in [0,\pi/4]$.

We now show that $\ker A_\theta \setminus \ker\eth \equiv \ker D(T,C)$. Let $\varphi_j \in C^\infty(Y,E_0)$ be the eigenvectors corresponding to the non-zero eigenvalues $\lambda_j \in \mathbb{R}$ of $\eth_0$. Then as in the proof of Theorem 4.13, one shows that if $u \in \ker A_\theta \setminus \ker\eth$, then on the product decomposition $X \cong [0,e]_x \times Y$ we can write $u = v + w$, where w is of the form $w = \sum_{\lambda_j<0} c_j x^{-\lambda_j}\varphi_j(y)$, and where $v = (v_1,v_2) \neq 0$ with $v_1 \in \hat{H}^1_b([0,1],V)$ and $v_2 \in H^1([1,e],V)$ such that $[\Gamma(x\partial_x)+R]v_1 = 0$, $\Gamma(x\partial_x)v_2 = 0$, $Q(\theta)(v_1,v_2)|_{x=1} = 0$, and $v_2|_{x=1} \in \Lambda_C$. Since $\Gamma(x\partial_x)v_2 = 0$, v_2 must be a constant vector in Λ_C. Now using the definition of $Q(\theta)$ in (5.3) and the fact that $Q(\theta)(v_1|_{x=1},v_2) = 0$ and that $v_2 \in \Lambda_C$ so that $\Pi_C^\perp v_2 = 0$, we have $\cos^2\theta\,\Pi_C^\perp v_1|_{x=1} = 0$. Since $0 \leq \theta \leq \pi/4$, we must have $\Pi_C^\perp v_1|_{x=1} = 0$ and so, $v_1 \in \mathrm{Dom}(D(T,C))$. Using the definition of $Q(\theta)$ once more, we must have $\sin\theta\, v_1|_{x=1} = \cos\theta\, v_2$. It follows that $\ker A_\theta \setminus \ker\eth$ is isomorphic to the space of nontrivial solutions to $D(T,C)v_1 = 0$ such that $\sin\theta\, v_1|_{x=1} = \cos\theta\, v_2$ where $v_2 \in \Lambda_C$. We now analyze this boundary value problem for $\theta \in [0,\pi/4]$.

First, setting $\theta = 0$ in $\sin\theta\, v_1|_{x=1} = \cos\theta\, v_2$ gives $v_2 = 0$. Thus, the only requirement on v_1 is that it be in $\ker D(T,C)$. Thus, $\ker A_0 \setminus \ker\eth \equiv \ker D(T,C)$ and by Proposition 4.11, $\dim(\ker A_0 \setminus \ker\eth) = \dim(\Lambda_T \cap \Lambda_C)$.

Assume now that $\theta \in (0,\pi/4]$. Then, by Corollary 4.12, there exists a unique solution v_1 to the problem

$$D(T,C)v_1 = 0, \quad \sin\theta\, v_1|_{x=1} = \cos\theta\, v_2,$$

if and only if $v_2 \in \Lambda_T \cap \Lambda_C$. Hence, $\ker A_\theta \setminus \ker \eth \equiv \ker D(T,C)$ and $\dim(\ker A_\theta \setminus \ker \eth) = \dim(\Lambda_T \cap \Lambda_C)$ for all $\theta \in (0, \pi/4]$. □

The remaining steps to prove Theorem 5.1 are as follows.

The program to prove Theorem 5.1:

(Step 1) First we prove that the b-eta invariant ${}^b\eta(A_\theta)$ is constant for $\theta \in [0, \pi/4]$. In particular, equating the invariants for $\theta = 0$ and $\theta = \pi/4$, we obtain

$$ {}^b\eta(\eth + R) = {}^b\eta(D(T,C)) + {}^b\eta(D), $$

where ${}^b\eta(D(T,C))$ is the b-eta invariant of the one-dimensional operator $D(T,C)$, and where ${}^b\eta(D)$ is the b-eta invariant of the operator $\eth$ with domain given in (5.5). In particular, by Proposition 4.15, we have

$$ {}^b\eta(\eth + R) = m(\Lambda_T, \Lambda_C) + {}^b\eta(D). $$

(Step 2) Second, we prove that ${}^b\eta(D) = {}^b\eta(\eth)$.

Combining these two steps proves Theorem 5.1. In Section 5.3 we work out Step 1 and then in Section 5.4 we complete Step 2.

5.3. **Rotating boundary conditions.** If A_θ is the operator $A = \eth + R$ with domain given in (5.4), we first prove that for each $\theta \in [0, \pi/4]$, the b-eta invariant of A_θ can be defined by the usual integral:

$$ {}^b\eta(A_\theta) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})\, dt. $$

To prove this, we show that the b-trace of $A_\theta e^{-tA_\theta^2}$ can be defined, and then we show that the integral defining ${}^b\eta(A_\theta)$ converges. Finally, we show that ${}^b\eta(A_\theta)$ is constant, which establishes Theorem 5.7.

To begin this program, we start by describing the domain of A_θ is a slightly different way. Recall that $X \cong [0,e]_x \times Y$ near its boundary over which all our structures are of product type, and that $x > e$ off this collar. Let $\hat{X}$ denote the manifold

$$ \hat{X} = M \sqcup N, \qquad M = [0,1]_x \times Y, \qquad N = \{p \in X\,;\, x(p) \geq 1\}. $$

Here, $\sqcup$ means "disjoint union". Observe that $N \cong [1,e]_x \times Y$ near $\partial N \cong Y$. Note that $\partial \hat{X}$ consists of two parts: the boundary Y coming from $x = 0$ and the boundary $Y \sqcup Y$ coming from $x = 1$. Over the second boundary coming from $x = 1$, we have a collar decomposition

$$ \hat{X} \cong [0,1]_r \times (Y \sqcup Y), \quad r = \begin{cases} -\log x & \text{if} \quad e^{-1} \leq x \leq 1 \\ \log x & \text{if} \quad 1 \leq x \leq e. \end{cases} \tag{5.7} $$

The bundle E extends naturally to a vector bundle over $\hat{X}$ such that $E \cong E_0$ over M and over the collar (5.7). Smooth sections of E over $\hat{X}$ are just pairs (u_1, u_2), where u_1 and u_2 are smooth sections over M and N, respectively.

Note that $\eth$ and R define operators on $\hat{X}$. We assume that R is supported completely on the subcollar $[0, e^{-1})_x \times Y$ of M so that over the collar (5.7) above, $A = \eth$. Since $\eth = \Gamma[x\partial_x + \eth_0]$ over $[0,e]_x \times Y$, and since $x\partial_x = -\partial_r$ if $r = -\log x$

and $x\partial_x = \partial_r$ if $r = \log x$, acting on a pair (u_1, u_2) where u_1 and u_2 are sections over M and N respectively, over the collar (5.7) we have

$$\eth = \hat{\Gamma}[\partial_r + \hat{\eth}_0], \qquad \hat{\Gamma} = \begin{bmatrix} -\Gamma & 0 \\ 0 & \Gamma \end{bmatrix}, \quad \hat{\eth}_0 = \begin{bmatrix} -\eth_0 & 0 \\ 0 & \eth_0 \end{bmatrix}. \tag{5.8}$$

It follows that A_θ can be considered an operator on $\hat{X}$ with domain

$$\mathrm{Dom}(A_\theta) = \{u \in \hat{H}_b^1(\hat{X}, E) \ ; \ P(\theta)u|_{r=0} = 0\},$$

where $P(\theta)$ is defined in (5.2) and where for any $\alpha \in \mathbb{R}$, $\hat{H}_b^\alpha(\hat{X}, E)$ consists of those functions in $H_b^\alpha(\hat{X}, E)$ near $x = 0$, and in the usual Sobolev space $H^\alpha(\hat{X}, E)$ away from $x = 0$. If $\hat{L}_b^2(\hat{X}, E) = \hat{H}_b^0(\hat{X}, E)$, then A_θ defines a continuous linear map

$$A_\theta : \mathrm{Dom}(A_\theta) \longrightarrow \hat{L}_b^2(\hat{X}, E).$$

Note that A is only a b-operator near the boundary of $\hat{X}$ at $x = 0$. By (5.8), near the boundary of $\hat{X}$ at $r = 0$, A is just a usual elliptic differential operator: smooth up to $r = 0$ and not degenerate there.

To show that the b-eta invariant of A_θ can be defined, we first need to show that the b-trace ${}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})$ can be defined. Thus, we need to understand the Schwartz kernel of $e^{-tA_\theta^2}$. To do so, we construct the heat operator directly. By definition,

$$\mathrm{Dom}(A_\theta^2) = \{u \in \hat{H}_b^1(\hat{X}, E) \ ; \ P(\theta)u|_{r=0} = 0, \ P(\theta)A_\theta u|_{r=0} = 0\},$$

so $e^{-tA_\theta^2}$ is the unique operator such that given an initial condition $v \in \hat{L}_b^2(\hat{X}, E)$, the function $u_t = e^{-tA_\theta^2}v$ solves

$$(\partial_t + A_\theta^2)u_t = 0; \quad u_0 = v, \quad P(\theta)u_t|_{r=0} = 0, \quad P(\theta)A_\theta u_t|_{r=0} = 0. \tag{5.9}$$

The idea to construct the heat operator is simple: we just glue together the heat operator H_1 of A_θ^2 away from $r = 0$ and the heat operator H_2 of A_θ^2 near $r = 0$. Actually, this method will only construct a parametrix for the heat operator, but one that is very close to the genuine heat operator.

Away from $r = 0$, we have $\hat{X} \equiv X$, the original manifold. Thus, let $H_1 = e^{-tA^2}$, where e^{-tA^2} is the heat operator of $A = \eth + R$ in the usual sense. The heat operator e^{-tA^2} is described in Appendix C. Near $r = 0$, $R = 0$, and so, $A = \eth$. The boundary condition $P(\theta)$ is the continuous transmission condition on $V^\perp = (\ker \eth_0)^\perp$, and so on $V^\perp$, we can identify elements of $\mathrm{Dom}(A_\theta)$ as sections on X rather than on $\hat{X}$. Thus, on $V^\perp$ near $r = 0$, the heat operator of A_θ^2 should be close to the heat operator of $\eth^2$ projected onto $V^\perp$. For this reason, as $\eth^2 = (xD_x)^2 + \eth_0^2$ on the collar of X, where $D_x = i^{-1}\partial_x$, we define

$$H_2^\perp = \Pi_0^\perp e^{-t(xD_x)^2} e^{-t\eth_0^2} \Pi_0^\perp, \tag{5.10}$$

where $e^{-t(xD_x)^2}$ is the heat operator of $(xD_x)^2$ on $[0, \infty)_x$ given by

$$e^{-t(xD_x)^2}(x, x') = \frac{1}{\sqrt{4\pi t}} e^{-t(\log x - \log x')^2/4t}.$$

Consider now the heat operator of A_θ^2 near $r = 0$ on V. Directly from the definition of $P(\theta)$ in (5.2) and of $\hat{\Gamma}$ in (5.8), one can easily check that

$$P(\theta)\hat{\Gamma} = \hat{\Gamma}(\mathrm{Id} - P(\theta)). \tag{5.11}$$

Now on the collar $\hat{X} \cong [0, 1]_r \times (Y \sqcup Y)$, we have $A = \hat{\Gamma}(\partial_r + \hat{\eth}_0)$, where $\hat{\Gamma}$ and $\hat{\eth}_0$ are given in (5.8). Thus, projected onto V, we see that $A = \hat{\Gamma}\partial_r$, $P(\theta) = Q(\theta)$ where

$Q(\theta)$ is given in (5.3), and by (5.11), $P(\theta)A = \hat{\Gamma}(\mathrm{Id} - Q(\theta))\partial_r$. Thus, as $A^2 = D_r^2$ on V near $r = 0$, where $D_r = i^{-1}\partial_r$, the heat operator of A_θ^2 projected onto V should be close to the heat operator on the infinite cylinder $[0,\infty)_r \times (Y \sqcup Y)$ fixed by

$$(\partial_t + D_r^2)u_t = 0; \quad u_0 = \Pi_0 v, \quad Q(\theta)u_t|_{r=0} = 0, \quad (\mathrm{Id} - Q(\theta))D_r u_t|_{r=0} = 0.$$

Using standard Laplace transform techniques [9], the solution operator of this heat equation can be computed explicitly, and equals

$$H_2^\theta = \Pi_0 \frac{1}{\sqrt{4\pi t}} \Big\{ e^{-(r-r')^2/4t} + (\mathrm{Id} - 2Q(\theta))e^{-(r+r')^2/4t} \Big\} \Pi_0. \tag{5.12}$$

Let $\rho(z) \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $\rho(z) = 0$ for $z \leq 1/4$ and $\rho(z) = 1$ for $z \geq 3/4$. Given any real numbers $\alpha < \beta$, we define

$$\rho_{\alpha,\beta}(z) = \rho((z-\alpha)/(\beta-\alpha)). \tag{5.13}$$

Then $\rho_{\alpha,\beta}(z) = 0$ on a neighborhood of $\{z \leq \alpha\}$ and $\rho_{\alpha,\beta}(z) = 1$ on a neighborhood of $\{z \leq \beta\}$.

Recall that r is the variable on the collar $\hat{X} \cong [0,1]_r \times (Y \sqcup Y)$. We define

$$\begin{aligned} \psi_1(r) &= \rho_{1/2,3/4}(r), & \psi_2(r) &= 1 - \psi_1(r), \\ \varphi_1(r) &= \rho_{1/4,1/2}(r), & \varphi_2(r) &= 1 - \rho_{3/4,1}(r). \end{aligned} \tag{5.14}$$

These functions extend either by 0 or 1 to define smooth functions on all of $\hat{X}$ and $\{\psi_i\}$ forms a partition of unity of $\hat{X}$ such that $\varphi_i = 1$ on $\mathrm{supp}(\psi_i)$. We define

$$E_\theta = \varphi_1 H_1 \psi_1 + \varphi_2 H_2^\perp \psi_2 + \varphi_2 H_2^\theta \psi_2. \tag{5.15}$$

It follows that

$$(\partial_t + A_\theta^2)E_\theta = K_\theta,$$

where

$$K_\theta = [A^2, \varphi_1]H_1\psi_1 + [\eth^2, \varphi_2]H_2^\perp \psi_2 + [\eth^2, \varphi_2]H_2^\theta \psi_2.$$

It is straightforward to check that the Schwartz kernel of K_θ is a smooth function on $\hat{X}^2$ vanishing to infinite order at $t = 0$ and at the boundary hypersurfaces of $\hat{X}^2$ coming from the boundary $x = 0$ in $\hat{X}$, and vanishing near the whole left boundary $\partial\hat{X} \times \hat{X}$ of $\hat{X}^2$. Thus, the heat operator of A_θ^2 is given by (cf. [2])

$$e^{-tA_\theta^2} = E_\theta + K_\theta', \quad K_\theta' = E_\theta * \sum_{j=1}^\infty (-1)^j K_j,$$

where $K_1 = K_\theta$ and $K_j = K_{j-1} * K_\theta$ with $*$ denoting the convolution of kernels:

$$K * K' = \int_0^t K(t-r)\, K'(r)dr = \int_0^t K(r)\, K'(t-r)dr. \tag{5.16}$$

Arguments similar to those found in [3, Ch. 2] or [22, p. 269] show that the Schwartz kernel of K_θ' is a smooth function on $\hat{X}^2$ vanishing to infinite order at $t = 0$ and also at the boundary hypersurfaces of $\hat{X}^2$ coming from the boundary $x = 0$ in $\hat{X}$.

From the properties of E_θ and K_θ', it follows that for each $t > 0$, the regularized trace ${}^b\mathrm{Tr}(e^{-tA_\theta^2})$ exists (see Definition 3.3) as the regular value of the meromorphic function $\mathrm{Tr}(x^z e^{-tA_\theta^2})$ at $z = 0$. (Here, we use only the boundary defining function x; we are not concerned with the boundary at $r = 0$ because the kernel of $e^{-tA_\theta^2}$ is smooth up to this face, as follows from the properties of E_θ and K_θ'.) Similarly, ${}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})$ is defined. Since K_θ vanishes to infinite order $t = 0$, the asymptotics of $e^{-tA_\theta^2}$ as $t \downarrow 0$ are the same as those of E_θ as $t \downarrow 0$. Now the asymptotics of E_θ are determined from those of H_1, $H_2^\perp$, and H_2^θ, which are easily computed.

Lemma 5.3. *Let $n = \dim X$. Then for any $\theta \in [0, \pi/4]$, as $t \downarrow 0$, we have*

(1) ${}^b\mathrm{Tr}(e^{-tA_\theta^2}) \sim \sum_{j=0}^\infty a_j(\theta)\, t^{(j-n)/2}$*;*

(2) ${}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2}) \sim \sum_{j=1}^\infty b_j(\theta)\, t^{j/2}$*.*

Proof. The asymptotics of ${}^b\mathrm{Tr}(e^{-tA_\theta^2})$ as $t \downarrow 0$ are the same as those of

$$ {}^b\mathrm{Tr}(E_\theta) = {}^b\mathrm{Tr}(\varphi_1 H_1 \psi_1) + {}^b\mathrm{Tr}(\varphi_2 H_2^\perp \psi_2) + {}^b\mathrm{Tr}(\varphi_2 H_2^\theta \psi_2). $$

By Lemma C.10, we have ${}^b\mathrm{Tr}(\varphi_1 H_1 \psi_1) \sim \sum_{j=0}^\infty a_j t^{(j-n)/2}$ as $t \downarrow 0$. Directly from the formula (5.10) for $H_2^\perp$, we have ${}^b\mathrm{Tr}(\varphi_2 H_2^\perp \psi_2) \sim \sum_{j=0}^\infty a'_j\, t^{(j-n)/2}$ as $t \downarrow 0$. Also, directly from the formula (5.12) for H_2^θ, we have ${}^b\mathrm{Tr}(\varphi_2 H_2^\theta \psi_2) \sim \sum_{j=0}^\infty a''_j\, t^{(j-1)/2}$ as $t \downarrow 0$. Thus, (1) is proved.

We now prove (2). In this case, the asymptotics of ${}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})$ as $t \downarrow 0$ are the same as those of ${}^b\mathrm{Tr}(AE_\theta)$. Observe that

$$ \begin{aligned} AE_\theta = &\varphi_1 A H_1 \psi_1 + \varphi_2 \eth H_2^\perp \psi_2 + \varphi_2 \eth H_2^\theta \psi_2 \\ &+ [A, \varphi_1] H_1 \psi_1 + [\eth, \varphi_2] H_2^\perp \psi_2 + [\eth, \varphi_2] H_2^\theta \psi_2. \end{aligned} $$

The Schwartz kernels of the three last operators vanish on the diagonal, and hence

$$ {}^b\mathrm{Tr}(AE_\theta) = {}^b\mathrm{Tr}(\varphi_1 A H_1 \psi_1) + {}^b\mathrm{Tr}(\varphi_2 \eth H_2^\perp \psi_2) + {}^b\mathrm{Tr}(\varphi_2 \eth H_2^\theta \psi_2). $$

By Proposition 4.3, we have ${}^b\mathrm{Tr}(\varphi_1 A H_1 \psi_1) \sim \sum_{j=1}^\infty b_j t^{j/2}$ as $t \downarrow 0$. Directly from the formula (5.10) for $H_2^\perp$, we see that

$$ {}^b\mathrm{Tr}(\varphi_2 \eth H_2^\perp \psi_2) = {}^b\mathrm{Tr}(\varphi_2 \Gamma [x\partial_x + \eth_0] \Pi_0^\perp e^{-t(xD_x)^2} e^{-t\eth_0^2} \Pi_0^\perp \psi_2). $$

We claim that this expression is zero for all t. Indeed, since $\Gamma \eth_0 = -\eth_0 \Gamma$, if φ is an eigenvector of $\eth_0$ with non-zero eigenvalue λ, then $\Gamma\varphi$ is an eigenvector of $\eth_0$ with eigenvalue $-\lambda$. Therefore choosing an orthonormal basis $\{\varphi_j\}$ of the eigenvectors of $\eth_0$ with positive eigenvalues, $\{\varphi_j, \Gamma\varphi_j\}$ is an orthonormal basis of $V^\perp$. Since $\Pi_0^\perp e^{-t\eth_0^2} \Pi_0^\perp$ is diagonal with respect to this basis, it follows that

$$ \mathrm{Tr}(\Pi_0^\perp \Gamma e^{-t\eth_0^2} \Pi_0^\perp) = 0 \quad \text{and} \quad \mathrm{Tr}(\Pi_0^\perp \Gamma \eth_0 e^{-t\eth_0^2} \Pi_0^\perp) = 0, $$

as the traces involve off-diagonal operators. Thus, ${}^b\mathrm{Tr}(\varphi_2 \eth H_2^\perp \psi_2) = 0$. We also claim that ${}^b\mathrm{Tr}(\varphi_2 \eth H_2^\theta \psi_2) = 0$. Indeed, since $[\partial_r e^{-t(r-r')^2/4t}]|_{r=r'} = 0$, from the formula (5.12) for H_2^θ, we have

$$ {}^b\mathrm{Tr}(\varphi_2 \eth H_2^\theta \psi_2) = \frac{1}{\sqrt{4\pi t}} {}^b\mathrm{Tr}(\varphi_2 \hat\Gamma \partial_r \Pi_0 (\mathrm{Id} - 2Q(\theta)) e^{-(r+r')^2/4t} \Pi_0 \psi_2). $$

By (5.11), it follows that $\hat\Gamma(\mathrm{Id} - 2Q(\theta)) = -(\mathrm{Id} - 2Q(\theta))\hat\Gamma$. Thus, $\mathrm{tr}(\hat\Gamma(\mathrm{Id} - 2Q(\theta))) = 0$. It follows that ${}^b\mathrm{Tr}(\varphi_2 \eth H_2^\theta \psi_2) = 0$. Our proof is now complete. □

In particular, this lemma shows that $t^{-1/2}\, {}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})$ is integrable near $t = 0$. To show that it is integrable near $t = \infty$, we need to understand the long-time behavior of ${}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})$. To do so, we need to extend the calculus with bounds found in Appendix A.2 to our current setting. Intuitively, as $A_\theta = \eth + R$ is only a b-operator at the boundary $x = 0$ in $\hat X$, and is otherwise non-degenerate, the calculus with bounds in the present case should consist of operators that are smoothing on $\hat X \setminus \{x = 0\}$, and near $x = 0$ in $\hat X$ are given by Definitions (A.1) and (A.2) in the appendix. Explicitly, these operators are defined as follows. Given $\varepsilon > 0$, let α_ε be the multi-index on $\hat{X}^2$ that associates the number ε to the left and right boundary hypersurfaces of $\hat{X}^2$ coming from the boundary $x = 0$ in $\hat{X}$, and 0 to the other boundary hypersurfaces. Then the space $\hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$ is the subspace of operators in $\Psi^{-\infty,\alpha_\varepsilon}(\hat{X}, E)$ (see Definition (A.1)) whose Schwartz kernels define smooth densities on the space $\{(p, p') \in \hat{X}^2\,;\, x(p) \neq 0 \text{ or } x(p') \neq 0\}$. Given $\varepsilon > 0$, let β_ε be the multi-index on $\hat{X}^2_b$ that associates the number ε to the left, right, and front face boundary hypersurfaces of $\hat{X}^2_b$ coming from the boundary $x = 0$ in $\hat{X}$, and 0 to the other boundary hypersurfaces. Then the space $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ is the subspace of operators in $\Psi_b^{-\infty,\beta_\varepsilon}(\hat{X}, E)$ (see Definition (A.2)) whose Schwartz kernels define smooth densities on the space $\{(p, p') \in \hat{X}^2\,;\, x(p) \neq 0 \text{ or } x(p') \neq 0\}$.

Proposition 5.4 (cf. Proposition C.9)**.** *Let $\theta \in [0, \pi/4]$. Then we can write*

$$e^{-tA_\theta^2} = \Pi_\theta + R_\theta(t), \quad t > 0, \tag{5.17}$$

where for some $\varepsilon > 0$, $\Pi_\theta \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$ is the finite rank projection onto $\ker A_\theta$, and where $R_\theta(t) \in \hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ is such that as $t \to \infty$, $R_\theta(t) \to 0$ exponentially in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$. In particular,

$$^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2}) \to 0 \quad \textit{exponentially as } t \to \infty.$$

Finally, the pointwise trace of $A_\theta e^{-tA_\theta^2}$ on the diagonal satisfies

$$\mathrm{tr}(A_\theta e^{-tA_\theta^2}) = x^\varepsilon f_\theta(t)\, dg,$$

where $f_\theta(t) \in C^0(\hat{X})$ and vanishes exponentially as $t \to \infty$.

Proof. To prove this proposition, we first prove that for some $\delta > 0$, the resolvent $(A_\theta^2 - \lambda)^{-1}$ is meromorphic on $\mathbb{C} \setminus [\delta, \infty)$ with finite rank residues. Here, A_θ^2 has domain

$$\mathrm{Dom}(A_\theta^2) = \{u \in \hat{H}^2_b(\hat{X}, E)\ ;\ P(\theta)u|_{r=0} = 0,\ P(\theta)A_\theta u|_{r=0} = 0\}.$$

The resolvent construction is very similar to the construction of e^{-tA_θ}: We glue together the resolvent R_1 of A_θ^2 away from $r = 0$ and the resolvent R_2 of A_θ^2 near $r = 0$. This procedure will construct a parametrix for $A_\theta^2 - \lambda$.

First, away from $r = 0$, we have $\hat{X} \equiv X$. So here, we take $R_1(\lambda) = (A^2 - \lambda)^{-1}$. On $V^\perp$, the boundary condition $P(\theta)$ is the transmission condition, so on $V^\perp$ we can consider elements of $\mathrm{Dom}(A_\theta^2)$ as sections on X. Thus, as $\eth^2 = (xD_x)^2 + \eth_0^2$ on sections of $V^\perp$, for the second step in the resolvent construction we define

$$R_2^\perp = \Pi_0^\perp((xD_x)^2 + \eth_0^2 - \lambda)^{-1}\Pi_0^\perp, \tag{5.18}$$

where $((xD_x)^2 + \eth_0^2 - \lambda)^{-1}$ denotes the resolvent of $(xD_x)^2 + \eth_0^2$ on $[0, \infty)_x \times Y$. Lastly, we define a parametrix on V. On the collar $\hat{X} \cong [0, 1]_r \times (Y \sqcup Y)$, $A = \hat{\Gamma}[\partial_r + \hat{\eth}_0]$ where $\hat{\Gamma}$ and $\hat{\eth}_0$ are given in (5.8) and projected onto V, we have $A = \hat{\Gamma}\partial_r$ and $P(\theta) = Q(\theta)$, where $Q(\theta)$ is given in (5.3). By (5.11), we have $Q(\theta)\hat{\Gamma} = \hat{\Gamma}(\mathrm{Id} - Q(\theta))$. Therefore, as $A^2 = D_r^2$ on V near $r = 0$, where $D_r = i^{-1}\partial_r$, we shall consider D_r^2 with domain

$$\mathrm{Dom}(D_r^2) = \{u \in \Pi_0 H^2_{loc}([0, \infty)_r \times (Y \sqcup Y), E_0)\ ;\ Q(\theta)u|_{r=0} = 0, (\mathrm{Id} - Q(\theta))D_r u|_{r=0} = 0\}. \tag{5.19}$$

The resolvent of D_r^2 with this domain can be found explicitly using elementary techniques from ordinary differential equations, and is described as follows. Since $\cos z$ is even, we can write $\cos z =: f(z^2)$, where $f(z)$ is an entire function with $f(0) = 1$. Also, since $(\sin z)/z$ is even, we can write $(\sin z)/z = g(z^2)$, where $g(z)$ is an entire function with $g(0) = 1$. Given $u \in L^2_c([0,\infty)_r, V)$ (compactly supported L^2 functions), we define $G_D(\lambda)$ to be the operator

$$G_D(\lambda)u(r) = \Big(\int_0^\infty f(\lambda\varrho^2)\, u(\varrho) d\varrho \Big) \cdot r\, g(\lambda\, r^2) - \int_0^r (r-\varrho) g(\lambda(r-\varrho)^2)\, u(\varrho) d\varrho.$$

If $\lambda \in \mathbb{C} \setminus [0,\infty)$, then in terms of cosine and sine, we have

$$G_D(\lambda)u(r) = \Big(\int_0^\infty \cos(\sqrt{\lambda}\,\varrho)\, u(\varrho) d\varrho \Big) \cdot \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, r) - \frac{1}{\sqrt{\lambda}} \int_0^r \sin(\sqrt{\lambda}(r-\varrho))\, u(\varrho) d\varrho.$$

One easily checks that $G_D(\lambda)$ maps $Q(\theta) L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0)$ into $\mathrm{Dom}(D_r^2)$ and that

$$(D_r^2 - \lambda) G_D(\lambda) = \mathrm{Id} \quad \text{on} \quad Q(\theta) L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0),$$
$$G_D(\lambda)(D_r^2 - \lambda) = \mathrm{Id} \quad \text{on} \quad Q(\theta) H^2_c([0,\infty)_r \times (Y \sqcup Y), E_0) \cap \mathrm{Dom}(D_r^2).$$

Given $u \in L^2_c([0,\infty)_r, V)$, we define $G_N(\lambda)$ to be the operator

$$G_N(\lambda)u(r) = \Big(\int_0^\infty \varrho\, g(\lambda\varrho^2)\, u(\varrho) d\varrho \Big) \cdot f(\lambda\, r^2) - \int_0^r (r-\varrho) g(\lambda(r-\varrho)^2)\, u(\varrho) d\varrho.$$

(This formula can also be written directly in terms of cosine and sine.) One checks that $G_N(\lambda)$ maps $(\mathrm{Id} - Q(\theta))\Pi_0 L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0)$ into $\mathrm{Dom}(D_r^2)$, and

$$(D_r^2 - \lambda) G_N(\lambda) = \mathrm{Id} \quad \text{on} \quad (\mathrm{Id} - Q(\theta))\Pi_0 L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0),$$
$$G_N(\lambda)(D_r^2 - \lambda) = \mathrm{Id} \quad \text{on} \quad (\mathrm{Id} - Q(\theta))\Pi_0 H^2_c([0,\infty)_r \times (Y \sqcup Y), E_0) \cap \mathrm{Dom}(D_r^2).$$

Finally, if we define

$$R_2^\theta(\lambda) = \Pi_0 \Big\{ Q(\theta) G_D(\lambda) + (\mathrm{Id} - Q(\theta)) G_N(\lambda) \Big\} \Pi_0,$$

then $R_2^\theta(\lambda)$ maps $L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0)$ into $\mathrm{Dom}(D_r^2)$ and satisfies

$$(D_r^2 - \lambda) R_2^\theta(\lambda) = \mathrm{Id} \quad \text{on} \quad \Pi_0 L^2_c([0,\infty)_r \times (Y \sqcup Y), E_0),$$
$$R_2^\theta(\lambda)(D_r^2 - \lambda) = \mathrm{Id} \quad \text{on} \quad \Pi_0 H^2_c([0,\infty)_r \times (Y \sqcup Y), E_0) \cap \mathrm{Dom}(D_r^2).$$

We are now ready to construct a parametrix for $A_\theta^2 - \lambda$. Let φ_1, φ_2, ψ_1, and ψ_2 be the functions found in (5.14), and define

$$G_\theta(\lambda) = \varphi_1 R_1(\lambda)\psi_1 + \varphi_2 R_2^\perp(\lambda)\psi_2 + \varphi_2 R_2^\theta(\lambda)\psi_2. \tag{5.20}$$

By analytic Fredholm theory, see Theorem B.9, for some $\varepsilon > 0$, $R_1(\lambda)$ takes values in $\Psi_b^{-2,\varepsilon}(X,E)$ and is holomorphic on $\mathbb{C} \setminus [0,\infty)$ and meromorphic on a small neighborhood of 0 with finite rank singularities. If σ_0 is the smallest absolute value of the non-zero eigenvalues of $\eth_0$, then $\varphi_2 R_2^\perp(\lambda)\psi_2$ is supported near $r = 0$ and is holomorphic on $\mathbb{C} \setminus [\sigma_0^2, \infty)$ with values in $\varphi_2 \Psi^{-2}(X,E)\psi_2$, where $\Psi^m(X,E)$ denotes the usual space of pseudodifferential operators on the interior of X. By the construction of $G_D(\lambda)$ and $G_N(\lambda)$, the operator $R_2^\theta(\lambda)$ is entire.

Observe that

$$(A_\theta^2 - \lambda) G_\theta(\lambda) = \mathrm{Id} + K_\theta(\lambda), \tag{5.21}$$

$$G_\theta(\lambda)(A_\theta^2 - \lambda) = \mathrm{Id} + K'_\theta(\lambda), \tag{5.22}$$

where

$$K_\theta(\lambda) = [A^2, \varphi_1]R_1(\lambda)\psi_1 + [\eth^2, \varphi_2]R_2^\perp(\lambda)\psi_2 + [\eth^2, \varphi_2]R_2^\theta(\lambda)\psi_2.$$
$$K'_\theta(\lambda) = \varphi_1 R_1(\lambda)[\psi_1, A^2] + \varphi_2 R_2^\perp(\lambda)[\psi_2, \eth^2] + \varphi_2 R_2^\theta(\lambda)[\psi_2, \eth^2]$$

From the explicit descriptions of φ_i, ψ_i, $R_1(\lambda)$, $R_2^\perp(\lambda)$, and $R_2^\theta(\lambda)$, it follows that $K_\theta(\lambda), K'_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$, and are holomorphic on $\mathbb{C} \setminus [0,\infty)$ and meromorphic on a small neighborhood of 0. Let $\lambda_0 \in \mathbb{C} \setminus [0,\infty)$. Then, as A_θ^2 is self-adjoint, $(A_\theta^2 - \lambda_0)^{-1}$ exists. Define

$$\begin{aligned} K &= (A_\theta^2 - \lambda_0)^{-1} K_\theta(\lambda_0) \\ &= G_\theta(\lambda_0)K_\theta(\lambda_0) - K'_\theta(\lambda_0)(A_\theta^2 - \lambda_0)^{-1}K_\theta(\lambda_0), \end{aligned}$$

where we used (5.21) and (5.22). Since $K_\theta(\lambda), K'_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$, we have $K \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$. It follows that if we define $G'_\theta(\lambda) = G_\theta(\lambda) - K$, then

$$(A_\theta^2 - \lambda)G'_\theta(\lambda) = \mathrm{Id} + K''_\theta(\lambda),$$

where $K''_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$, and has the same meromorphic properties as $K_\theta(\lambda)$, but is such that $K''_\theta(\lambda_0) = 0$. Thus, by analytic Fredholm theory [22, Sec. 5.3], $(\mathrm{Id} + K''_\theta(\lambda))^{-1}$ is meromorphic on $\mathbb{C} \setminus [\delta,\infty)$ for some $\delta > 0$ with finite rank residues. It follows that $(A_\theta^2 - \lambda)^{-1}$ is meromorphic on $\mathbb{C} \setminus [\delta,\infty)$ with finite rank residues. Since A_θ^2 is self-adjoint, by standard arguments [22, Ch. 6], these poles are all simple and lie on $[0,\delta)$, with the residue at $\lambda = 0$ given by minus the orthogonal projection onto the null space of A_θ.

To prove the decomposition (5.17), we write $e^{-tA_\theta^2}$ as the contour integral $e^{-tA_\theta^2} = \frac{i}{2\pi}\int_\Upsilon e^{-t\lambda}(A_\theta^2 - \lambda)^{-1} d\lambda$, where Υ is a contour of the form $a + \{\lambda \in \mathbb{C}\,;\, \arg(\lambda) = \pm\pi/4\}$, where $a < 0$. The above analysis of $(A_\theta^2 - \lambda)^{-1}$ and the standard arguments imply that we can shift the contour Υ to a new one Υ' that corresponds to $a > 0$ sufficiently small, such that

$$e^{-tA_\theta^2} = \Pi_\theta + R_\theta(t), \qquad R_\theta(t) = \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}(A_\theta^2 - \lambda)^{-1} d\lambda. \tag{5.23}$$

It remains to show that $R_\theta(t) \in \hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ and is such that as $t \to \infty$, $R_\theta(t) \to 0$ exponentially in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$. Unfortunately, we cannot prove these facts because the parametrix $R_2^\theta(\lambda)$ grows exponentially as $|\lambda| \to \infty$ on the contour! For this reason, we need to substitute another parametrix for $R_2^\theta(\lambda)$ that decays as $|\lambda| \to \infty$ in order to extract precise information about the kernel of $R_\theta(t)$.

Therefore, instead of the domain (5.19), we consider D_r^2 with domain

$$\begin{aligned} \mathrm{Dom}(D_r^2) = \{u \in \Pi_0 H^2([0,\infty)_r \times (Y \sqcup Y), E_0)\,; \\ Q(\theta)u|_{r=0} = 0,\ (\mathrm{Id} - Q(\theta))D_r u|_{r=0} = 0\}. \end{aligned}$$

The heat operator $H_2^\theta(t)$ for this domain is given in (5.12). The resolvent of D_r^2 can be written in terms of the heat operator via the Laplace transform:

$$\tilde{R}_2^\theta(\lambda) = \int_0^\infty e^{t\lambda} H_2^\theta(t)\, dt \ \longleftrightarrow\ H_2^\theta(t) = \frac{i}{2\pi}\int e^{-t\lambda}\tilde{R}_2^\theta(\lambda)\, d\lambda.$$

Using standard Laplace transform techniques [9], we find that

$$\tilde{R}_2^\theta(\lambda) = \Pi_0 \frac{i}{2\sqrt{\lambda}}\Big\{e^{i|r-r'|\sqrt{\lambda}} + (\mathrm{Id} - 2Q(\theta))e^{i(r+r')\sqrt{\lambda}}\Big\}\Pi_0, \tag{5.24}$$

where $\sqrt{\lambda}$ is the standard branch of the square root. Observe that $\tilde{R}_2^\theta(\lambda) \to 0$ *exponentially* as $|\lambda| \to \infty$ in any closed angle of $\mathbb{C}$ not intersecting the positive real axis. However, because of $\sqrt{\lambda}$, we construct a parametrix for $A_\theta^2 - \lambda^2$ with $\operatorname{Im}\lambda > 0$, instead of a parametrix for $A_\theta^2 - \lambda$. Using similar notation as in (5.20), we define

$$\tilde{G}_\theta(\lambda) = \varphi_1 R_1(\lambda^2)\psi_1 + \varphi_2 R_2^\perp(\lambda^2)\psi_2 + \varphi_2 \tilde{R}_2^\theta(\lambda^2)\psi_2.$$

For concreteness, let us take $\Lambda = \{\lambda \in \mathbb{C}\,;\, \varepsilon_0 \le \arg(\lambda) \le \pi - \varepsilon_0\}$, where $0 < \varepsilon_0 < \pi/4$ is fixed, to be our spectral parameter domain. Then by analytic Fredholm theory, see Theorem B.9, replacing $\varepsilon > 0$ with a smaller value if necessary, we may assume that $R_1(\lambda^2) \in \Psi_b^{-2,\varepsilon}(X,E)$ is meromorphic for $\lambda \in \Lambda_\varepsilon$, where

$$\Lambda_\varepsilon = \{\lambda \in \mathbb{C}\,;\, \operatorname{Im}\lambda \ge 0,\ \lambda^2 \in \Lambda\} \cup \{\lambda \in \mathbb{C}\,;\, |\lambda^2| \le \varepsilon\},$$

with finite rank singularities for $\lambda^2 \in \mathbb{R}$ with $\lambda \in \Lambda_\varepsilon$, and as before, $\varphi_2 R_2^\perp(\lambda^2)\psi_2$ is holomorphic for $\lambda \in \Lambda_\varepsilon$ with values in $\Psi^{-2}(X,E)$ supported near $r = 0$. Finally, $\tilde{R}_2^\theta(\lambda^2)$ is meromorphic for $\lambda \in \mathbb{C}$ with only a simple pole at $\lambda = 0$, and vanishes exponentially as $|\lambda| \to \infty$ in Λ_ε. By Remark C.8, the operator $R_1(\lambda^2)$ decays like λ^{-2} in $\Psi_b^{0,\varepsilon}(X,E)$, and $\varphi_2 R_2^\perp(\lambda^2)\psi_2$ decays like λ^{-2} in $\varphi_2\Psi^0(X,E)\psi_2$.

Now,

$$(A_\theta^2 - \lambda^2)\tilde{G}_\theta(\lambda) = \mathrm{Id} + \tilde{K}_\theta(\lambda),$$

where

$$\tilde{K}_\theta(\lambda) = [A^2, \varphi_1]R_1(\lambda^2)\psi_1 + [\eth^2, \varphi_2]R_2^\perp(\lambda^2)\psi_2 + [\eth^2, \varphi_2]\tilde{R}_2^\theta(\lambda^2)\psi_2.$$

Similarly as for the first parametrix G_θ, it follows that $\tilde{K}_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X},E)$ is meromorphic on Λ_ε. However, now $\tilde{K}_\theta(\lambda) \to 0$ like λ^{-2} as a bounded operator on $\hat{L}_b^2(\hat{X},E)$ as $|\lambda| \to \infty$ in Λ_ε. Thus, $\mathrm{Id} + \tilde{K}_\theta(\lambda)$ is invertible on $\hat{L}_b^2(\hat{X},E)$ for $|\lambda|$ sufficiently large in Λ_ε, so by analytic Fredholm theory, see [22, Sec. 5.3], $(\mathrm{Id} + K_\theta(\lambda))^{-1}$ is meromorphic on Λ_ε with finite rank residues, and moreover, we can write $(\mathrm{Id} + \tilde{K}_\theta(\lambda))^{-1} = \mathrm{Id} + \tilde{K}'_\theta(\lambda)$, where $\tilde{K}'_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X},E)$. In particular, $(A_\theta^2 - \lambda^2)^{-1} = \tilde{G}_\theta(\lambda)(\mathrm{Id} + \tilde{K}_\theta(\lambda))^{-1} = \tilde{G}_\theta(\lambda) + H_\theta(\lambda)$, where $H_\theta(\lambda) = \tilde{G}_\theta(\lambda)\tilde{K}'_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X},E)$ is meromorphic on Λ_ε.

We are now ready to prove the decomposition (5.17). Going back to the contour integral (5.23), we have

$$\begin{aligned} R_\theta(t) &= \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}(A_\theta^2 - \lambda)^{-1} d\lambda \\ &= \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}\tilde{G}_\theta(\sqrt{\lambda}) d\lambda + \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda} H_\theta(\sqrt{\lambda}) d\lambda, \end{aligned}$$

where we assume that $\sqrt{\lambda} \in \Lambda_\varepsilon$ for $\lambda \in \Upsilon'$ and for such λ, $\sqrt{\lambda}$ avoids the poles of $\tilde{G}_\theta$ and H_θ. Since $H_\theta(\lambda) \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X},E)$, it follows that $\frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}H_\theta(\sqrt{\lambda})d\lambda \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X},E)$ and vanishes exponentially as $t \to \infty$. By the definition of $\tilde{G}_\theta(\lambda)$, we have

$$\begin{aligned} \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}\tilde{G}_\theta(\sqrt{\lambda}) d\lambda &= \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}\varphi_1 R_1(\lambda)\psi_1 d\lambda \\ &\quad + \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}\varphi_2 R_2^\perp(\lambda)\psi_2 d\lambda + \frac{i}{2\pi}\int_{\Upsilon'} e^{-t\lambda}\varphi_2\tilde{R}_2^\theta(\lambda)\psi_2 d\lambda. \end{aligned}$$

Arguments similar to those found in Proposition C.9 of the appendix show that the first two terms on the right are in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X},E)$ and vanish exponentially in

$\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ as $t \to \infty$. On the other hand, a straightforward computation using the explicit formula for $\tilde{R}_2^\theta(\lambda)$ in (5.24) can be used to verify that the third term on the right is also in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ and vanishes exponentially in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ as $t \to \infty$. The decomposition (5.17) is now proved.

Finally, we prove that $\mathrm{tr}(A_\theta e^{-tA_\theta^2}) = x^\varepsilon f_\theta(t) dg$, where $f_\theta(t) \in C^0(\hat{X})$ and vanishes exponentially as $t \to \infty$ in $C^0(\hat{X})$. The decomposition (5.17) and the definition of $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ imply that in the interior of X, as $t \to \infty$, $\mathrm{tr}(A_\theta e^{-tA_\theta^2}) \to 0$ exponentially in $C^\infty(\hat{X}, \Omega)$. Thus, it suffices to work on the subset $M = [0,1]_x \times Y_y$ of $\hat{X}$ near $x = 0$. In this neighborhood, by definition of $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$, omitting b-density factors we can write

$$\mathrm{tr}(A_\theta e^{-tA_\theta^2}) = f_0(t, y) + x^\varepsilon f_1(t, x, y),$$

where $f_0(t, y)$ takes values in $C^\infty(Y)$ and vanishes to exponential order as $t \to \infty$, and where $f_1(t, x, y)$ takes values in $S^{0,0}([0,1)_x \times Y)$ and vanishes to exponential order as $t \to \infty$. Here, $S^{0,0}([0,1)_x \times Y)$ is of the space of functions which, with all b-derivatives, are continuous on $[0,1)_x \times Y$. Hence, it suffices to prove that for any fixed $t > 0$, $f_0(t, y) = \mathrm{tr}(A_\theta e^{-tA_\theta^2})|_{x=0} = 0$. To see this, note that near $x = 0$, $A_\theta e^{-tA_\theta^2}$ is approximated by its normal operator:

$$A_\theta e^{-tA_\theta^2} = \frac{1}{2\pi} \int_{\mathbb{R}} (x/x')^{i\tau} N(Ae^{-tA^2})(\tau) d\tau \, \frac{dx'}{x'} + xH(x).$$

Thus, it suffices to show that $\mathrm{tr}(N(Ae^{-tA^2})(\tau)) = 0$. By the definition of R, see (4.5), we have $N(R)(\tau) = -\hat{\varphi}(\tau)^2\, \Gamma T$. Also, from (4.10), we have

$$e^{-tN(A)(\tau)^2} = e^{-t\tau^2}[e^{-t\eth_0^2} + (e^{-t\,\hat{\varphi}(\tau)^4} - 1)\Pi_0].$$

Thus,

$$N(Ae^{-tA^2})(\tau) = \Gamma[i\tau + \eth_0 - \hat{\varphi}(\tau)^2\, T] e^{-t\tau^2}[e^{-t\eth_0^2} + (e^{-t\,\hat{\varphi}(\tau)^4} - 1)\Pi_0].$$

Now using the fact that $\mathrm{tr}(\Gamma) = 0$ (since Γ has eigenvalues $\pm i$ with eigenspaces of the same dimension) and $\mathrm{tr}(\Gamma T) = 0$ (since $\Gamma T = -T\Gamma$), this equation shows that $\mathrm{tr}(N(Ae^{-tA^2})(\tau)) = 0$. Our proof is now complete. □

Remark 5.5. Since for $\theta = 0$, the operator splits: $A_0 = D(T, C) \oplus D$, with a corresponding splitting of the heat operators (see discussion around (5.6)), the results of this proposition hold for each of $e^{-tD(T,C)^2}$ and e^{-tD^2}.

This proposition plus (2) of Lemma 5.3, imply that the following integral defining ${}^b\eta(A_\theta)$ is absolutely convergent:

$${}^b\eta(A_\theta) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2})\, dt,$$

which shows that the b-eta invariant of A_θ is well-defined. As a corollary of the proof of Proposition 5.4, we obtain the following result.

Corollary 5.6. *There exists a $\delta > 0$ such that for any $\theta \in [0, \pi/4]$, the spectrum of A_θ in $(-\delta, \delta)$ consists of finitely many eigenvalues of finite multiplicity.*

We are now ready to prove Step 1 in our program to proving Theorem 5.1.

Theorem 5.7. *The b-eta invariant* ${}^b\eta(A_\theta)$ *is constant for* $\theta \in [0, \pi/4]$. *In particular, equating the invariants for* $\theta = 0$ *and* $\theta = \pi/4$, *we obtain*

$${}^b\eta(\eth + R) = {}^b\eta(D(T,C)) + {}^b\eta(D),$$

where ${}^b\eta(D(T,C))$ *is the b-eta invariant of the one-dimensional perturbed Dirac operator* $D(T,C)$, *and where* ${}^b\eta(D)$ *is the b-eta invariant of the operator* $\eth$ *with domain given in* (5.5). *In particular, by Proposition 4.15, we have*

$${}^b\eta(\eth + R) = m(\Lambda_T, \Lambda_C) + {}^b\eta(D).$$

Proof. To show that ${}^b\eta(A_\theta)$ is constant, that is, $(d/d\theta){}^b\eta(A_\theta) = 0$, we use similar arguments as found in Section 4.3. The fact that the domains of the A_θ's are changing with θ cause problems when examining the derivative $(d/d\theta)A_\theta$. To circumvent this difficultly, we employ a trick used by Lesch and Wojciechowski in [14]. Let $U(\theta)$ be the unitary operator

$$U(\theta) = e^{iT(\theta)} = \begin{bmatrix} \cos\theta & -C\sin\theta \\ C\sin\theta & \cos\theta \end{bmatrix}, \qquad T(\theta) = -\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}\theta,$$

where we interpret $U(\theta) = \mathrm{Id}$ on $V^\perp$. Then one can check that for each θ, we have

$$P(\theta) = U(\theta)P(0)U(\theta)^*.$$

Recall that r is the variable on the collar $\hat{X} \cong [0,1]_r \times (Y \sqcup Y)$. Let $\psi(r) \in C_c^\infty([0,1))$ be a nonnegative function supported near $r = 0$ with $\psi(r) = 1$ near $r = 0$ and let Ψ_θ be the operator on the collar $[0,1]_r \times (Y \sqcup Y)$ defined by

$$\Psi_\theta u = e^{i\psi(r)T(\theta)}u.$$

Since $\psi(r)$ is supported near $r = 0$, $\Psi_\theta u = u$ outside a neighborhood of $r = 0$. Thus, Ψ_θ extends as the identity operator off the collar to define an operator on $\hat{L}^2_b(\hat{X}, E)$. Since $\psi(0) = 1$, we have

$$\Psi_\theta u|_{r=0} = e^{iT(\theta)}u|_{r=0} = U(\theta)u|_{r=0}.$$

Thus, as $P(\theta) = U(\theta)P(0)U(\theta)^*$ and $\mathrm{Dom}(A_\theta)$ is the space of $u \in \hat{H}^1_b(\hat{X}, E)$ with $P(\theta)u|_{r=0} = 0$, it follows that

$$\Psi_\theta : \mathrm{Dom}(A_\theta) \longrightarrow \mathrm{Dom}(A_0).$$

Hence, if we define $\tilde{A}_\theta = \Psi_\theta^* A_\theta \Psi_\theta$, where $\Psi_\theta^* = e^{-i\psi(r)T(\theta)}$, then $\tilde{A}_\theta$ has constant domain $\mathrm{Dom}(A_0)$. Explicitly, we find that

$$\tilde{A}_\theta = A + i\psi'(r)\hat{\Gamma}\, T(\theta),$$

where we assumed that $\psi(r)$ is supported sufficiently near $r = 0$ so that $R = 0$ on the support of $\psi(r)$. The heat operators of A_θ^2 and $\tilde{A}_\theta^2$ are related by

$$e^{-t\tilde{A}_\theta^2} = \Psi_\theta^* e^{-tA_\theta^2}\Psi_\theta.$$

To see this, one checks that $\Psi_\theta^* e^{-tA_\theta^2}\Psi_\theta$ is the solution operator to the heat equation for $\tilde{A}_\theta^2$. By uniqueness, the claimed equality must hold. It follows that

$${}^b\mathrm{Tr}(\tilde{A}_\theta e^{-t\tilde{A}_\theta^2}) = {}^b\mathrm{Tr}(\Psi_\theta^* A_\theta \Psi_\theta \Psi_\theta^* e^{-tA_\theta^2}\Psi_\theta) = {}^b\mathrm{Tr}(\Psi_\theta^* A_\theta e^{-tA_\theta^2}\Psi_\theta) = {}^b\mathrm{Tr}(A_\theta e^{-tA_\theta^2}),$$

so ${}^b\eta(\tilde{A}_\theta) = {}^b\eta(A_\theta)$. We now show that ${}^b\eta(\tilde{A}_\theta)$ is constant. Indeed, the same arguments used in the proofs of Lemmas 4.5 and 4.6 can be used to show that for any $\theta_0, \theta_1 \in [0, \pi/4]$, we have

$$\begin{aligned} {}^b\eta(\tilde{A}_{\theta_1}) - {}^b\eta(\tilde{A}_{\theta_0}) = \lim_{t\to\infty} \left\{ \frac{2t^{1/2}}{\sqrt{\pi}} \int_{\theta_0}^{\theta_1} {}^b\mathrm{Tr}\Big(\frac{d\tilde{A}_\theta}{d\theta} e^{-t\tilde{A}_\theta^2} \Big)\, d\theta \right\} \\ - \lim_{t\to 0} \left\{ \frac{2t^{1/2}}{\sqrt{\pi}} \int_{\theta_0}^{\theta_1} {}^b\mathrm{Tr}\Big(\frac{d\tilde{A}_\theta}{d\theta} e^{-t\tilde{A}_\theta^2} \Big)\, d\theta \right\}. \end{aligned} \tag{5.25}$$

Here, unlike Lemmas 4.5 and 4.6, there are no boundary terms since $(d/d\theta)\tilde{A}_\theta = i\psi'(r)\hat{\Gamma}T'(\theta)$ is supported away from $x = 0$. By Proposition 5.4, we have $e^{-tA_\theta^2} = \Pi_\theta + R_\theta(t)$, where for some $\varepsilon > 0$, $\Pi_\theta \in \hat{\Psi}^{-\infty,\varepsilon}(\hat{X}, E)$ is the finite rank projection onto $\ker A_\theta$, and $R_\theta(t) \in \hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ vanishes exponentially in $\hat{\Psi}_b^{-\infty,\varepsilon}(\hat{X}, E)$ as $t \to \infty$. Conjugating by Ψ_θ, it follows that $e^{-t\tilde{A}_\theta^2} = \tilde{\Pi}_\theta + \tilde{R}_\theta(t)$, where $\tilde{\Pi}_\theta$ is the finite rank projection onto $\ker \tilde{A}_\theta$ and $\tilde{R}_\theta(t)$ vanishes exponentially as $t \to \infty$. Since $\ker \tilde{A}_\theta \cong \ker A_\theta$ has constant dimension by Proposition 5.2, the proof of Proposition 8.39 of [22] can be used to show that the first term on the right of (5.25) is equal to zero. We claim that $t^{1/2}\,{}^b\mathrm{Tr}((d\tilde{A}_\theta/d\theta)e^{-t\tilde{A}_\theta^2})$ vanishes as $t \downarrow 0$. To see this, observe that

$${}^b\mathrm{Tr}\Big(\frac{d\tilde{A}_\theta}{d\theta} e^{-t\tilde{A}_\theta^2} \Big) = i\, {}^b\mathrm{Tr}(\psi'(r)\hat{\Gamma}T'(\theta)e^{-t\tilde{A}_\theta^2}) = -i\, {}^b\mathrm{Tr}\Big(\psi'(r) \begin{bmatrix} 0 & -\Gamma C \\ \Gamma C & 0 \end{bmatrix} e^{-tA_\theta^2} \Big).$$

As in the proof of Lemma 5.3, the asymptotics of $t^{1/2}\,{}^b\mathrm{Tr}((d\tilde{A}_\theta/d\theta)e^{-t\tilde{A}_\theta^2})$ as $t \downarrow 0$ are the same as those of

$$-i\, {}^b\mathrm{Tr}\Big(\psi'(r) \begin{bmatrix} 0 & -\Gamma C \\ \Gamma C & 0 \end{bmatrix} E_\theta(t) \Big)$$

as $t \downarrow 0$, where $E_\theta(t)$ is given in (5.15). Since the factor in front of $E_\theta(t)$ lies in $V = \ker \eth_0$ and since $\psi(r)$ is supported near $r = 0$, it follows from the decomposition (5.15) of $E_\theta(t)$ that

$$-i\, {}^b\mathrm{Tr}\Big(\psi'(r) \begin{bmatrix} 0 & -\Gamma C \\ \Gamma C & 0 \end{bmatrix} E_\theta(t) \Big) = -i\, {}^b\mathrm{Tr}\Big(\psi'(r) \begin{bmatrix} 0 & -\Gamma C \\ \Gamma C & 0 \end{bmatrix} H_2^\theta(t) \Big), \tag{5.26}$$

where $H_2^\theta(t)$ is given in (5.12). Using the explicit description of $H_2^\theta(t)$ given in (5.12), it is straightforward to verify that the right hand side of (5.26) vanishes exponentially as $t \downarrow 0$. Thus, the right hand side of (5.25) is 0, and so, ${}^b\eta(\tilde{A}_{\theta_0}) = {}^b\eta(\tilde{A}_{\theta_1})$. Our proof of Theorem 5.7 is now complete. □

5.4. **Stretching the cylinder.** We begin by describing the idea to prove Step 2 in our program to establishing Theorem 5.1. In this step we show that ${}^b\eta(D) = {}^b\eta(\eth)$, where recall that D is the operator $\eth$ with domain given in (5.5):

$$\mathrm{Dom}(D) = \{u \in \Pi_0^\perp \hat{H}_b^1(M, E_0) \oplus H^1(N, E)\ ;\ \hat{\Pi}u|_{x=1} = 0\}.$$

We follow Douglas and Wojciechowski [11] and Müller [27, Sec. 7]. Recall that $X \cong [0, e]_x \times Y$ near its boundary over which all our structures are of product type. For each $a \in [0, \infty)$, let $M_a = [e^{-a}, 1]_x \times Y$. Note that as $a \to \infty$, M_a "approaches"

$M = [0,1]_x \times Y$. Let D_a be the Dirac operator $\eth$ with domain

$$\mathrm{Dom}(D_a) = \{u \in \Pi_0^\perp H^1(M_a, E_0) \oplus H^1(N, E) \; ; \; \Pi_+ u|_{x=e^{-a}} = 0, \; \hat{\Pi} u|_{x=1} = 0\},$$

where Π_+ is the projection onto the eigenspaces of $\eth_0$ with positive eigenvalues, and where

$$\hat{\Pi} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \Pi_0^\perp + \begin{bmatrix} 0 & 0 \\ 0 & \Pi_C \end{bmatrix}.$$

There are two key points that make the analysis of D_a substantially simpler than the analysis of A_θ considered previously. The first is that D_a is no longer degenerate; it is a true elliptic operator. Indeed, if $s = \log x$, then the product decomposition $[e^{-a}, e]_x \times Y$ in X, consisting of M_a and the collar $[1, e]_x \times Y$ of N glued along $x = 1$, transforms to $[-a, 1]_s \times Y$ and $\eth$ takes the product form

$$\eth = \Gamma[\partial_s + \eth_0]$$

over this decomposition. The fact that D_a is no longer degenerate implies that it has discrete spectrum. The second point is that we are keeping the null mode V fixed at the boundary of the "compact part" N. Only $V^\perp$ is allowed to "approach" $x = 0$ as $a \to \infty$. This fact implies, see Proposition 5.8, that the non-zero spectrum of the operators D_a is bounded away from 0 uniformly in $a \in [0, \infty)$.

The proof that ${}^b\eta(D) = {}^b\eta(\eth)$ proceeds as follows: We show that for any $a \in [0, \infty)$, the heat operators $e^{-tD_a^2}$ and $D_a e^{-tD_a^2}$ are of *trace class*, and that the trace $t^{-1/2}\mathrm{Tr}(D_a e^{-tD_a^2})$ is integrable on $[0, \infty)_t$. Hence, the eta invariant

$$\eta(D_a) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, \mathrm{Tr}(D_a e^{-tD_a^2})\, dt \tag{5.27}$$

is defined. We then show that for all $a \in [0, \infty)$, the eta invariant $\eta(D_a)$ is constant. Note that D_0 is the Dirac operator $\eth$ with the "augmented" APS condition:

$$\mathrm{Dom}(D_0) = \{u \in H^1(N, E) \; ; \; (\Pi_+ + \Pi_C)u|_{x=1} = 0\}.$$

In [27], Müller proved that $\eta(D_0) = {}^b\eta(\eth)$, the b-eta invariant of the Dirac operator on the original manifold X. Thus, $\eta(D_a) = {}^b\eta(\eth)$ for all $a \in [0, \infty)$. Finally, we show that

$$\lim_{a \to \infty} \eta(D_a) = {}^b\eta(D).$$

This ends the proof that ${}^b\eta(D) = {}^b\eta(\eth)$.

We begin by constructing the heat operator for D_a^2. To do so, we first define appropriate parametrices and follow the ideas used to produce the heat operator for A_θ^2 found around (5.15). Since N has a collar $N \cong [1, e]_x \times Y$ near $\partial N \cong Y$ over which all our structures are products, this manifold along with all its geometric structures can be "doubled" across $x = 1$ (for this double construction, see [4]). Let H_1 be the heat operator for the double of $\eth^2$.

Changing coordinates to $s = \log x$, the product decomposition $[e^{-a}, e]_x \times Y$ in X transforms to $[-a, 1]_s \times Y$, where $[-a, 0]_s \times Y = M_a$ and $[0, 1]_s \times Y$ is the collar of N. Also, $\eth$ takes the product form $\eth = \Gamma[\partial_s + \eth_0]$ over this decomposition. Let $\{\varphi_j\}$ be an orthonormal basis for the eigenspaces of $\eth_0$ with *positive* eigenvalues. Then $\{\Gamma\varphi_j\}$ is an orthonormal basis for the eigenspaces of $\eth_0$ with negative eigenvalues.

Let H_2^a be the heat operator for $\Pi_0^\perp\{\Gamma[\partial_s + \eth_0]\}^2\Pi_0^\perp = \Pi_0^\perp\{D_s^2 + \eth_0^2\}\Pi_0^\perp$, where $D_s = i^{-1}\partial_s$, over the infinite cylinder $[-a,\infty)_s \times Y$ with domain

$$\{u \in \Pi_0^\perp H^1([-a,\infty)_s \times Y, E_0)\ ;\ \Pi_+ u|_{s=-a} = 0,\ \Pi_+\{\Gamma[\partial_s + \eth_0]u\}|_{s=-a} = 0\}.$$

Using standard Laplace transform methods [9, pp. 357–8], this heat operator is of the form $H_2^a(t,s,y,s',y') = H(t,s+a,y,s'+a,y')$, where $H(t,s,y,s',y')$ is described in [2, Sec. 2]:

$$\begin{aligned} H(t,s,y,s',y') = &\sum_j \frac{e^{-\lambda_j^2 t}}{\sqrt{4\pi t}}[e^{-(s-s')^2/4t} - e^{-(s+s')^2/4t}]\,\varphi_j(y)\otimes\varphi_j(y') \\ &+ \sum_j \Big\{ \frac{e^{-\lambda_j^2 t}}{\sqrt{4\pi t}}[e^{-(s-s')^2/4t} + e^{-(s+s')^2/4t}] \\ &- \lambda_j e^{\lambda_j(s+s')}\mathrm{erfc}\Big(\frac{s+s'}{2\sqrt{t}} + \lambda_j\sqrt{t}\Big)\Big\}\Gamma\varphi_j(y)\otimes\Gamma\varphi_j(y'), \end{aligned} \tag{5.28}$$

where $\mathrm{erfc}(x)$ is the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\,d\xi.$$

Finally, let H_2^C be the heat operator for $\Pi_0\{\Gamma[\partial_s + \eth_0]\}^2\Pi_0 = \Pi_0 D_s^2 \Pi_0$ over the infinite cylinder $[0,\infty)_s \times Y$ with domain

$$\{u \in \Pi_0 H^1([0,\infty)_s \times Y, E_0)\ ;\ \Pi_C u|_{s=0} = 0,\ \Pi_C\{\Gamma\partial_s u\}|_{s=0} = 0\}.$$

Then, cf. (5.12), we have

$$H_2^C = \Pi_0 \frac{1}{\sqrt{4\pi t}}\Big\{e^{-(s-s')^2/4t} + (\mathrm{Id} - 2\Pi_C)e^{-(s+s')^2/4t}\Big\}\Pi_0. \tag{5.29}$$

Define

$$\begin{aligned} \psi_1(s) &= \rho_{1/4,1/2}(s), \quad \psi_2(s) = 1 - \psi_1(s), \\ \varphi_1(s) &= \rho_{0,1/4}(s), \quad \varphi_2(s) = 1 - \rho_{1/2,3/4}(s), \end{aligned}$$

where the function $\rho_{\alpha,\beta}$ is defined in (5.13). Each of these functions extend either by 0 or 1 to define smooth functions on all of X. We define

$$E_a = \varphi_1 H_1 \psi_1 + \varphi_2 H_2^a \psi_2 + \varphi_2 H_2^C \psi_2, \tag{5.30}$$

where we understand that $s \geq -a$ in the Schwartz kernel of H_2^a and $s \geq 0$ in the Schwartz kernel for H_2^C. With this understanding, E_a defines, in a completely natural way, a map from $\Pi_0^\perp \hat{L}_b^2(M_a, E_0) \oplus L^2(N, E)$ into $\mathrm{Dom}(D_a)$ and its Schwartz kernel is smooth on the manifold $(\hat{X}_a)^2$, where $\hat{X}_a$ is defined as

$$\hat{X}_a = M_a \sqcup N, \quad M_a = [-a,0]_s \times Y, \quad N = \{p \in X\ ;\ s(p) \geq 0\}.$$

As for the manifold $\hat{X}$ studied previously, the boundary of $\hat{X}_a$ consists of two parts: the boundary Y coming from $s = -a$, and another boundary $Y \sqcup Y$ coming from $s = 0$. This latter boundary has a collar similar to that of $\hat{X}$ found in (5.7). In terms of the manifold $\hat{X}_a$, the domain of D_a can be written as

$$\mathrm{Dom}(D_a) = \{u \in \hat{H}_b^1(\hat{X}_a, E)\ ;\Pi_0 u|_{M_a} = 0,\ \Pi_+ u|_{s=-a} = 0,\ \hat{\Pi} u|_{s=0} = 0\}.$$

Now,

$$(\partial_t + D_a^2)E_a = K_a,$$

where

$$K_a = [\eth^2, \varphi_1]H_1\psi_1 + [\eth^2, \varphi_2]H_2^a\psi_2 + [\eth^2, \varphi_2]H_2^C\psi_2.$$

Note that the Schwartz kernel of K_a is a smooth function on $(\hat{X}_a)^2$ vanishing to infinite order at $t = 0$ and at the boundary hypersurfaces of $(\hat{X}_a)^2$ coming from the boundary $s = -a$ in $\hat{X}_a$, and vanishing near the whole left boundary $\partial\hat{X}_a \times \hat{X}_a$ of $(\hat{X}_a)^2$. Then the heat operator of D_a^2 is given by

$$e^{-tD_a^2} = E_a + E_a * F_a, \quad F_a = \sum_{j=1}^{\infty}(-1)^j K_j,$$

where $K_1 = K_a$ and $K_j = K_{j-1} * K_a$, with $*$ denoting the convolution of kernels as in (5.16). Arguments similar to those found in [3, Ch. 2] or [22, p. 269] show that the Schwartz kernel of $E_a * F_a$ is a smooth function on $(\hat{X}_a)^2$ vanishing to infinite order at $t = 0$ and at the boundary hypersurfaces of $(\hat{X}_a)^2$ coming from the boundary $s = -a$ in $\hat{X}_a$. Directly from the properties of E_a and $E_a * F_a$, it follows that the heat operators $e^{-tD_a^2}$ and $D_a e^{-tD_a^2}$ have smooth Schwartz kernels on $(\hat{X}_a)^2$, and hence, are of trace class for each $t > 0$. In the following proposition we collect various properties of the heat operator.

Proposition 5.8. *For each $a \in [0,\infty)$, the heat operators $e^{-tD_a^2}$ and $D_a e^{-tD_a^2}$ have smooth Schwartz kernels, and hence, are of trace class for each $t > 0$. Moreover, if $n = \dim X$, then*

1. *As $t \downarrow 0$, $\mathrm{Tr}(e^{-tD_a^2}) \sim \sum_{j=0}^{\infty} a_j(a)\, t^{(j-n)/2}$;*
2. *As $t \downarrow 0$, $\mathrm{Tr}(D_a e^{-tD_a^2}) \sim \sum_{j=1}^{\infty} b_j(a)\, t^{j/2}$;*
3. *As $t \to \infty$, $\mathrm{Tr}(D_a e^{-tD_a^2}) \to 0$ exponentially.*
4. *D_a has pure point spectrum. Moreover, the non-zero spectrum of D_a^2 has a uniform positive lower bound for all $a \in [0,\infty)$.*

Proof. Statements (1) and (2) are proved using similar arguments found in Lemma 5.3. To see (3) and the first part of (4), observe that since the heat operator $e^{-tD_a^2}$ is of trace class, standard arguments (see [3, Sec. 2.6]) show that D_a has pure point spectrum. If $\{\lambda_j = \lambda_j(a)\}$ denotes the eigenvalues of D_a, then

$$\mathrm{Tr}(D_a e^{-tD_a^2}) = \sum_{\lambda_j \neq 0} \lambda_j e^{-t\lambda_j^2}, \tag{5.31}$$

which immediately gives statement (3). The fact that the non-zero spectrum of D_a^2 has a uniform positive lower bound for all $a \in [0,\infty)$ follows from the fact that D has discrete spectrum near 0 by Corollary 5.6 with θ set to 0 (see also Remark 5.5), and then using essentially the same arguments found in the proof of Theorem 6.1 of [11]. As to avoid duplicating their arguments, we omit the details. □

In particular, the integral (5.27) defining the eta invariant of D_a converges. As an application of this proposition, we relate the eta invariant $\eta(D_a)$ to the original definition as defined by Atiyah, Patodi, and Singer [2]. Given $z \in \mathbb{C}$, Proposition 5.8 implies that the function

$$\eta(z, D_a) = \frac{1}{\Gamma((z+1)/2)} \int_0^\infty t^{(z-1)/2}\, \mathrm{Tr}(D_a e^{-tD_a^2})\, dt$$

converges for $\Re z > -1/2$. Note that $\eta(0, D_a) = \eta(D_a)$. If $\{\lambda_j = \lambda_j(a)\}$ denotes the eigenvalues of D_a, then using (5.31), a short computation shows that

$$\eta(z, D_a) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sgn} \lambda_j}{|\lambda_j|^z}.$$

Thus, formally speaking $\eta(D_a) = \eta(0, D_a) =$ "$\sum_{\lambda_j \neq 0} \operatorname{sgn} \lambda_j$". Hence, $\eta(D_a)$ is a measure of the spectral asymmetry of D_a. We now prove that $\eta(D_a) = {}^b\eta(\eth)$.

Lemma 5.9. *For all $a \in [0, \infty)$, we have $\eta(D_a) = {}^b\eta(\eth)$.*

Proof. We first show that $\dim \ker D_a$ is constant, and then, following the proof of Theorem 4.7, we show that $\eta(D_a)$ is independent of $a \in [0, \infty)$. Our proof is now finished since Müller [27] showed that $\eta(D_0) = {}^b\eta(\eth)$.

Recall that $M_a = [-a, 0]_s \times Y$, that $[0, 1]_s \times Y$ is a collar of N near ∂N, and that $\eth$ takes the product form $\eth = \Gamma[\partial_s + \eth_0]$ over these decompositions. Given $u \in \operatorname{Dom}(D_a)$, we have $u = (u_1, u_2) \in \Pi_0^\perp H^1(M_a, E_0) \oplus H^1(N, E)$ where $\Pi_+ u|_{s=-a} = 0$ and $\hat{\Pi} u|_{s=0} = 0$. Suppose that $D_a u = 0$. Let $\{\varphi_j\}$ be the eigenvectors of $\eth_0$ corresponding to nonzero eigenvalues $\lambda_j \in \mathbb{R}$ of $\eth_0$. Since $D_a u = 0$, it follows that $u_1 = \sum_j a_j e^{-\lambda_j s} \varphi_j(y)$ and, on the collar $[0, 1]_s \times Y$ of N, $u_2 = v + \sum_j b_j e^{-\lambda_j s} \varphi_j(y)$ where $a_j, b_j \in \mathbb{C}$ and $v \in V = \ker \eth_0$. Since $\Pi_+ u|_{s=-a} = 0$ and $\hat{\Pi} u|_{s=0} = 0$, we must have $a_j = 0$ and $b_j = 0$ for $\lambda_j > 0$, $a_j = b_j$ if $\lambda_j < 0$, and $\Pi_C v = 0$. Hence, $u_1 = \sum_{\lambda_j < 0} a_j e^{-\lambda_j s} \varphi_j(y)$ and $u_2 = v + \sum_{\lambda_j < 0} a_j e^{-\lambda_j s} \varphi_j(y)$ where $\Pi_C v = 0$. Thus, we conclude that all the spaces $\ker D_a$ are canonically isomorphic to each other for each $a \in [0, \infty)$. Hence, $\dim \ker D_a$ is constant. It is not needed for later, but one can show that $\ker D_0 \equiv \ker \eth$ (the L^2_b null space of $\eth$ on the original manifold X), cf. Theorem 4.13.

To show that $\eta(D_a)$ is constant, we define a transformation that gives each D_a a common domain. For each $a \in [0, \infty)$, define

$$\psi_a(s) = s - a + a\rho_{1/3,1/2}(s),$$

where $\rho_{\alpha,\beta}$ is defined in (5.13). Observe that $\psi_a : [0, 1] \longrightarrow [-a, 1]$ and

$$\psi_a(s) = s - a \quad \text{if } 0 \le s \le 1/3, \qquad \psi_a(s) = s \quad \text{if } 1/2 \le s \le 1. \tag{5.32}$$

Note that $\psi_a'(s) = 1 + a\rho_{1/3,1/2}'(s) \ge 1$ since $\rho_{1/3,1/2}(s)$ is nondecreasing. Thus, $\psi_a(s)$ is a diffeomorphism. Let $\varphi_a : [-a, 1] \longrightarrow [0, 1]$ be the inverse of $\psi_a(s)$. Now define

$$\Psi_a : \operatorname{Dom}(D_a) \longrightarrow \operatorname{Dom}(D_0)$$

as follows. Let $u = (u_1, u_2) \in \Pi_0^\perp H^1(M_a, E_0) \oplus H^1(N, E)$ be in the domain of D_a. Then we define $\Psi_a u \in H^1(N, E)$ by $\Psi_a u = u_2$ off the collar $[0, 1]_s \times Y$ of N, and on the collar $[0, 1]_s \times Y$ of N, we define

$$\Psi_a u(s, y) = \begin{cases} \Pi_0^\perp u_1(\psi_a(s), y) + \Pi_0 u_2(s, y) & \text{if } 0 \le s \le \varphi_a(0); \\ \Pi_0^\perp u_2(\psi_a(s), y) + \Pi_0 u_2(s, y) & \text{if } \varphi_a(0) \le s \le 1. \end{cases}$$

It is easy to verify that $\Psi_a u \in \operatorname{Dom}(D_0)$; that is, $\Psi_a u \in H^1(N, E)$ and it satisfies the boundary condition: $(\Pi_+ + \Pi_C)\Psi_a u|_{s=0} = 0$. Thus, Ψ_a transforms the operators D_a into a smooth family of operators $\tilde{D}_a$ with constant domain $\operatorname{Dom}(D_0)$ via

$$\tilde{D}_a = \Psi_a \, D_a \, \Psi_a^{-1} : \operatorname{Dom}(D_0) \longrightarrow L^2(N, E).$$

In particular, for each $a \in [0,\infty)$, $\eta(\tilde{D}_a) = \eta(D_a)$. We now show that $\eta(D_a)$ is constant. Indeed, the same arguments used in the proofs of Lemmas 4.5 and 4.6 can be used to show that for any $a_0, a_1 \in [0,\infty)$, we have

$$\eta(D_{a_1}) - \eta(D_{a_0}) = \lim_{t\to\infty}\left\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_{a_0}^{a_1}\mathrm{Tr}\Big(\frac{d\tilde{D}_a}{da}e^{-t\tilde{D}_a^2}\Big)\,da\right\} - \lim_{t\to 0}\left\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_{a_0}^{a_1}\mathrm{Tr}\Big(\frac{d\tilde{D}_a}{da}e^{-t\tilde{D}_a^2}\Big)\,da\right\}. \tag{5.33}$$

Since $\ker \tilde{D}_a = \Psi_a \ker D_a$, the dimension $\dim\ker\tilde{D}_a$ is constant, so the proof of Proposition 8.39 of [22] can be used to show that the first term on the right of (5.33) is equal to zero. By (5.32), it follows that $\tilde{D}_a = \Gamma[\partial_s + \eth_0]$ on the subcollar $[0,1/3]_s \times Y$ of N near ∂N and that $\tilde{D}_a = \eth$ for $s \geq 1/2$. Thus, the pertubation $\tilde{D}_a$ differs from $\eth$ only on the subcollar $[1/3,1/2]_s \times Y$ of N, and so $\mathrm{tr}((d\tilde{D}_a/da)e^{-t\tilde{D}_a^2})$ is supported on $[1/3,1/2]_s \times Y$ of N. Hence, as the small time heat trace asymptotics are local, the asymptotics of $t^{1/2}\mathrm{Tr}((d\tilde{D}_a/da)e^{-t\tilde{D}_a^2})$ as $t \downarrow 0$ are exactly the same as the corresponding heat trace asymptotics of the following problem on a finite cylinder. Let $D(a) = \Gamma[\partial_s + \eth_0]$ have domain

$$\mathrm{Dom}(D(a)) = \{u \in \Pi_0^{\perp} H^1([-a,1]\times Y, E_0)\ ;\ \Pi_+ u|_{s=-a} = 0,\ \Pi_- u|_{s=1} = 0\}.$$

Lesch and Wojciechowski [14], amongst others, have analyzed eta invariants of such operators on finite cylinders. It is easy to check that $\dim\ker D(a) = 0$ for all a. If we define

$$\tilde{D}(a) = \Psi_a\, D(a)\, \Psi_a^{-1} : \mathrm{Dom}(D(0)) \longrightarrow L^2([0,1]_s \times Y, E_0),$$

then $\eta(\tilde{D}(a)) = \eta(D(a))$ and $\dim\ker\tilde{D}(a) = 0$. Hence, the same argument used to prove (5.33), plus the fact that the small time asymptotics of $t^{1/2}\mathrm{Tr}((d\tilde{D}_a/da)e^{-t\tilde{D}_a^2})$ and $t^{1/2}\mathrm{Tr}((d\tilde{D}(a)/da)e^{-t\tilde{D}(a)^2})$ are the same, imply that

$$\eta(D_{a_1}) - \eta(D_{a_0}) = \eta(D(a_1)) - \eta(D(a_0)).$$

To compute $\eta(D(a))$ explicitly, observe that if $U = \Pi_+ - \Pi_-$, then

$$U : \mathrm{Dom}(D(a)) \longrightarrow \mathrm{Dom}(D(a))$$

and $\Gamma U = -U\Gamma$. Hence, $UD(a) = -D(a)U$, therefore $D(a)$ has symmetric spectrum. It follows that $\eta(D(a)) = 0$ for all $a \in [0,\infty)$. Thus, $\eta(D_{a_1}) = \eta(D_{a_0})$ and our proof is complete. □

We now proceed to show that ${}^b\eta(\eth) = \lim_{a\to\infty}\eta(D_a) = {}^b\eta(D)$, where D is the operator $\eth$ with domain (5.5). To do so, we need to compare the heat operator $e^{-tD_a^2}$ to e^{-tD^2}. We do this by defining another parametrix for $e^{-tD_a^2}$ employing e^{-tD^2} directly. With this in mind, setting $s = \log x$ as usual, we define

$$\tilde{\psi}_1(s) = \rho_{-3/4,-1/2}(s), \quad \tilde{\psi}_2(s) = 1 - \psi_1(s),$$
$$\tilde{\varphi}_1(s) = \rho_{-1,0}(s), \quad \tilde{\varphi}_2(s) = 1 - \rho_{0,1}(s),$$

where the function $\rho_{\alpha,\beta}$ is defined in (5.13). We define

$$\psi_i^a(s) = \tilde{\psi}_i(s/a), \quad \varphi_1^a(s) = \tilde{\varphi}_1(s + 3a/4), \quad \varphi_2^a(s) = \tilde{\varphi}_2(s - a/2). \tag{5.34}$$

Each of these functions extends either by 0 or 1 to define smooth functions on all of X. Henceforth, we assume that $a \geq 5$ so that for $s \geq -1$, we have $\psi_1^a(s), \varphi_1^a(s) = 1$ and $\psi_2^a(s), \varphi_2^a(s) = 0$. We define

$$\tilde{E}_a = \varphi_1^a e^{-tD^2} \psi_1^a + \varphi_2^a H_2^a \psi_2^a,$$

where $H_2^a(t, s, y, s', y') = H(t, s + a, y, s' + a, y')$ with $H(t, s, y, s', y')$ defined in (5.28). Then $(\partial_t + D_a^2)\tilde{E}_a = \tilde{K}_a$, where

$$\tilde{K}_a = [\eth^2, \varphi_1^a] e^{-tD^2} \psi_1^a + [\eth^2, \varphi_2^a] H_2^a \psi_2^a.$$

Note that the Schwartz kernel of $\tilde{K}_a$ is a smooth function on $(\hat{X}_a)^2$ vanishing to infinite order at $t = 0$ and at the boundary hypersurfaces of $(\hat{X}_a)^2$ coming from the boundary $s = -a$ in $\hat{X}_a$. The heat operator of D_a^2 is given by the usual formula

$$e^{-tD_a^2} = \tilde{E}_a + \tilde{E}_a * \tilde{F}_a, \quad \tilde{F}_a = \sum_{j=1}^{\infty} (-1)^j \tilde{K}_j,$$

where $\tilde{K}_1 = \tilde{K}_a$ and $\tilde{K}_j = \tilde{K}_{j-1} * \tilde{K}_a$, with $*$ denoting the convolution of kernels as in (5.16). The estimates in the following lemma provide the last ingredients necessary to complete Step 2 in our program establishing Theorem 5.1. However, as the proof of this lemma is quite long, we shall finish up proving that ${}^b\eta(D) = {}^b\eta(\eth)$ before presenting the proof of this lemma in the appendix to this section.

Lemma 5.10. *Uniformly for $p \in \hat{X}_a$ and for $a \in [5, \infty)$, we have*

$$|\mathrm{tr}(D_a e^{-tD_a^2})(p) - \mathrm{tr}(\psi_1^a D e^{-tD^2})(p)| \leq c_1\, e^{c_1 t} e^{-c_2 a^2/t}\, dg(p), \quad t > 0. \tag{5.35}$$

Here, $c_i > 0$ are independent of $a \in [5, \infty)$. Moreover, for some $c > 0$ independent of $a \in [5, \infty)$, we have

$$\mathrm{Tr}(e^{-tD_a^2}) \leq c\, a\, t^{-n/2}, \qquad t > 0. \tag{5.36}$$

Theorem 5.11. *We have ${}^b\eta(D) = {}^b\eta(\eth)$.*

Proof. We begin by splitting the eta invariant of D_a into two integrals:

$$\eta(D_a) = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2}\, \mathrm{Tr}(D_a e^{-tD_a^2})\, dt + \frac{1}{\sqrt{\pi}} \int_{\sqrt{a}}^{\infty} t^{-1/2}\, \mathrm{Tr}(D_a e^{-tD_a^2})\, dt. \tag{5.37}$$

Consider the first integral in this expression. Using the notation of Lemma 5.10, we can write

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2}\, \mathrm{Tr}(D_a e^{-tD_a^2})\, dt = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \int_{p \in \hat{X}_a} \mathrm{tr}(\psi_1^a D e^{-tD^2})(p)\, dt + \xi(a),$$

where

$$|\xi(a)| \leq \mathrm{vol}(\hat{X}_a)\, c_1 \int_0^{\sqrt{a}} t^{-1/2} e^{c_2 t} e^{-c_3 a^2/t}\, dt \leq c\, a^{3/2}\, e^{c_2 a^{1/2}} e^{-c_3\, a^{3/2}}$$

with $c > 0$ independent of $a \geq 5$. Thus, $\xi(a) \to 0$ as $a \to \infty$. We show that

$$\lim_{a \to \infty} \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \int_{p \in \hat{X}_a} \mathrm{tr}(\psi_1^a D e^{-tD^2})(p)\, dt = {}^b\eta(D). \tag{5.38}$$

Indeed, by Proposition 5.4 and Remark 5.5, we have

$$\mathrm{tr}(De^{-tD^2})(p) = x^{\varepsilon} f(t)\, dg,$$

where $f(t) \in C^0(\hat{X})$ and vanishes exponentially as $t \to \infty$. It follows that the integral defining ${}^b\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \int_{p\in\hat{X}} \operatorname{tr}(De^{-tD^2})(p)\, dt$ is an absolutely convergent integral. This proves the limit (5.38).

We now follow Douglas and Wojciechowski [11, p. 159] and Müller [27, p. 360] applying the "Cheeger-Gromov estimate" to compute the second integral in (5.37). First observe that for any $\mu > 0$, we have

$$\int_{\sqrt{a}}^\infty t^{-1/2}\mu\, e^{-t\mu^2} dt = 2\int_{\mu a^{1/4}}^\infty e^{-t^2} dt \le 2e^{-\sqrt{a}\mu^2}.$$

Let $\{\lambda_j = \lambda_j(a)\}$ denote the non-zero eigenvalues of D_a. Then it follows that

$$\Big|\int_{\sqrt{a}}^\infty t^{-1/2}\operatorname{Tr}(D_a e^{-tD_a^2})\, dt\Big| \le \sum_j \int_{\sqrt{a}}^\infty t^{-1/2}\,|\lambda_j|\, e^{-t\lambda_j^2} dt \le 2\sum_j e^{-\sqrt{a}\,\lambda_j^2}.$$

By statement (4) of Proposition 5.8, for some $C > 0$ independent of a, we have $\lambda_j(a)^2 \ge 2C$. Hence,

$$\sum_j e^{-\sqrt{a}\,\lambda_j^2} \le e^{-\sqrt{a}C}\sum_j e^{-\sqrt{a}\,\lambda_j^2 + (\sqrt{a}/2)\lambda_j^2} \le e^{-\sqrt{a}C}\operatorname{Tr}(e^{-(\sqrt{a}/2)D_a^2}).$$

Since $a \ge 5$, $\operatorname{Tr}(e^{-(\sqrt{a}/2)D_a^2}) \le \operatorname{Tr}(e^{-D_a^2})$. Thus,

$$\Big|\int_{\sqrt{a}}^\infty t^{-1/2}\operatorname{Tr}(D_a e^{-tD_a^2})\, dt\Big| \le e^{-\sqrt{a}C}\operatorname{Tr}(e^{-D_a^2}).$$

By the estimate (5.36) in Lemma 5.10, we have $\operatorname{Tr}(e^{-D_a^2}) \le Ca$ for some $C > 0$ independent of a. Hence, the second integral of (5.37) vanishes as $a \to \infty$. The proof of Theorem 5.11 is now complete. □

Appendix: Proof of Lemma 5.10.

To prove the estimates (5.35) and (5.36) we need precise estimates on the Schwartz kernels of e^{-tD^2} and H_2^a.

Step 1: Estimates for H_2^a. We start with the estimates for H_2^a. In the sequel we shall need the following facts for heat operators on closed manifolds: Given a self-adjoint, elliptic, first-order differential operator P acting on sections of a Hermitian vector bundle F on a closed compact Riemannian manifold Z, we have

$$\begin{aligned} &\|e^{-tP^2}(p,p')\| \le ct^{-n/2}e^{ct}e^{-c'd(p,p')^2/t};\\ &\|Pe^{-tP^2}(p,p')\| \le ct^{-(n+1)/2}e^{-c'd(p,p')^2/t}, \end{aligned} \tag{5.39}$$

where $c, c' > 0$, and where $d(p,p')$ represents the geodesic distance between the points $p, p' \in Z$. The norm $\|\ \|$ is the norm derived from the Hermitian metric on F, and the Riemannian density on Z is used to trivialize the density factor in the Schwartz kernels. The proof of the estimates (5.39) can be found in [11, Prop. 1.1] or [3, Ch. 2]. They also follow from [22, Ch. 7]. Note that a priori, the estimates in (5.39) may be of the sort: $\|e^{-tP^2}(p,p')\| \le C_1 t^{-n/2}e^{C_2 t}e^{-C_3 d(p,p')^2/t}$ and $\|Pe^{-tP^2}(p,p')\| \le C_4 t^{-(n+1)/2}e^{-C_5 d(p,p')^2/t}$. However, choosing the *larger* of C_1 and C_2, we may assume that $C_1 = C_2$; then choosing the *larger* of C_1 and C_4, we may assume that $C_1 = C_4$; then choosing the *smaller* of C_3 and C_5, we may assume that $C_3 = C_5$. In order to save letters for constants, we implicitly follow this practice of saving letters in the sequel. This is the reason constants may be duplicated in the estimates that follow. We now prove that

$$\begin{aligned} \|H_2^a(t,p,p')\| &\le b\,t^{-n/2}e^{-b'd(p,p')^2/t} + b\,t^{-n/2}e^{-b'(s+s'+2a)^2/t};\\ \|\eth H_2^a(t,p,p')\| &\le b\,t^{-(n+1)/2}e^{-b'd(p,p')^2/t} + b\,t^{-(n+1)/2}e^{-b'(s+s'+2a)^2/t}, \end{aligned} \tag{5.40}$$

where the constants $b, b' > 0$ are independent of a, and where $p = (s, y)$ and $p' = (s', y')$ with $s, s' \in [-a, \infty)$ and $y, y' \in Y$, and $d(p, p')^2 = (s - s')^2 + d_Y(y, y')$, where $d_Y(y, y')$ is the geodesic distance between $y, y' \in Y$. Since $H_2^a(t, s, y, s', y') = H_2^0(t, s + a, y, s' + a, y')$, to prove the estimates (5.40), we may assume that $a = 0$. Define

$$H_D = \frac{1}{\sqrt{4\pi t}}\left[e^{-(s-s')^2/4t} - e^{-(s+s')^2/4t}\right]\Pi_0^{\perp}e^{-t\eth_0^2}\Pi_0^{\perp}.$$

Note that $\Pi_0^{\perp}e^{-t\eth_0^2}\Pi_0^{\perp} = e^{-t\eth_0^2}$ on the domain $\Pi_0^{\perp}L^2(Y, E_0)$. Thus, the Schwartz kernel of $\Pi_0^{\perp}e^{-t\eth_0^2}\Pi_0^{\perp}$ is the same as the Schwartz kernel of $e^{-t\eth_0^2}$, and so the estimates (5.39) hold for $\Pi_0^{\perp}e^{-t\eth_0^2}\Pi_0^{\perp}$. It follows that the estimates (5.40) (with $a = 0$) hold for H_D. By the formula (5.28) for H_2^0, we can write

$$H_2^0 = H_D + R_1 + R_2,$$

where

$$R_1 = \frac{e^{-(s+s')^2/4t}}{\sqrt{\pi t}}\sum_j e^{-\lambda_j^2 t}\,\Gamma\varphi_j(y)\otimes\Gamma\varphi_j(y')$$

and

$$R_2 = -\sum_j \lambda_j e^{\lambda_j(s+s')}\mathrm{erfc}\left(\frac{s+s'}{2\sqrt{t}} + \lambda_j\sqrt{t}\right)\Gamma\varphi_j(y)\otimes\Gamma\varphi_j(y').$$

Arguing as in the proof of Proposition 2.21 of [2, Sec. 2], it is straightforward to prove that R_1 satisfies the estimates (5.40) without the first term in each inequality (and with $a = 0$). Since $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\xi^2}d\xi$, we have

$$0 \le \mathrm{erfc}(x) \le \frac{2}{\sqrt{\pi}}e^{-x^2}, \qquad \frac{d}{dx}\mathrm{erfc}(x) = -\frac{2}{\sqrt{\pi}}e^{-x^2},$$

and using these facts about $\mathrm{erfc}(x)$ together with the equality

$$\exp\left(-\Big(\frac{s+s'}{2\sqrt{t}} + \lambda_j\sqrt{t}\Big)^2\right) = e^{-(s+s')^2/4t}\,e^{-\lambda_j(s+s')}\,e^{-t\lambda_j^2},$$

arguments similar to those in [2, Sec. 2] can be used to verify that R_2 satisfies the estimates (5.40) without the first term in each inequality (and with $a = 0$). Thus, the estimates (5.40) are proved.

Step 2: Estimates for e^{-tD^2}. We now prove some estimates on the Schwartz kernel of e^{-tD^2}. Recall that the kernel of e^{-tD^2} is a smooth function on $\hat{X}^2$ away from $x = 0$ where it is a b-operator. Also recall that X has a collar $[0, e]_x \times Y$ near ∂X with $x > e$ off this collar and $\hat{X}$ is just the disjoint union of the two halves of X cut along $x = 1$. We shall prove the following estimates: For $p, p' \in \hat{X}$, both of which are in the region $\{x \ge e\}$, we have

$$\begin{aligned} \|e^{-tD^2}(p,p')\| &\le a_1 t^{-n/2}e^{a_1 t}e^{-a_2 d(p,p')^2/t};\\ \|De^{-tD^2}(p,p')\| &\le a_1 t^{-(n+1)/2}e^{a_1 t}e^{-a_2 d(p,p')^2/t}. \end{aligned} \tag{5.41}$$

For $p, p' \in \hat{X}$, at least one of which is in the region $\{x \leq e\}$, we have

$$\begin{aligned} \|e^{-tD^2}(p,p')\| &\leq a_1 t^{-n/2} e^{a_1 t} e^{-a_2 d(p,p')^2/t} + a_1 t^{-1/2} e^{-a_2 (s(p)-s(p'))^2/t}; \\ \|De^{-tD^2}(p,p')\| &\leq a_1 t^{-(n+1)/2} e^{a_1 t} e^{-a_2 d(p,p')^2/t} + a_1 t^{-1} e^{-a_2 (s(p)-s(p'))^2/t}, \end{aligned} \tag{5.42}$$

where $a_i > 0$ are constants (independent of the variables), $s = \log x$, and $d(p, p')$ is the geodesic distance between p and p' as points in X. Here, $s = \log x$ identifies $(-\infty, 1]_s \times Y$ with the interior of the collar $[0, e]_x \times Y$ and the (b-) Riemannian density dg on X is used to trivialize the density factor in the Schwartz kernels. Note that $dg = ds\, dg_Y$ on the collar of X, where dg_Y is the Riemannian density on Y. The inequalities (5.41) and (5.42) are proved by constructing the heat operator e^{-tD^2} in the usual way, cf. the construction of E_a in (5.30). We remark that e^{-tD^2} has already been constructed since we constructed $e^{-tA_\theta^2}$ around (5.15), and e^{-tD^2} is just the part of $e^{-tA_\theta^2}$ with $\theta = 0$ that maps into $\mathrm{Dom}(D^2)$; see (5.6) and Remark 5.5. However, to achieve the estimates (5.41) and (5.42), we need to construct the heat operator again. To this end, define

$$\begin{aligned} &\psi_1(s) = \rho_{1/4,1/2}(s), \quad \psi_2(s) = 1 - \psi_1(s), \\ &\varphi_1(s) = \rho_{0,1/4}(s), \quad \varphi_2(s) = 1 - \rho_{1/2,3/4}(s), \end{aligned}$$

where the function $\rho_{\alpha,\beta}$ is defined in (5.13). With s interpreted as the variable on the collar $(-\infty, 1]_s \times Y$, each of these functions extends either by 0 or 1 to define smooth functions on all of X. We define

$$E = \varphi_1 H_1 \psi_1 + \varphi_2 H_2 \psi_2 + \varphi_2 H_2^C \psi_2, \tag{5.43}$$

where H_1 is the heat operator for the double of $\eth^2$ on the manifold N doubled across $x = 1$, H_2 is the operator

$$H_2 = \Pi_0^{\perp} \frac{1}{\sqrt{4\pi t}} e^{-(s-s')^2/4t} e^{-t\eth_0^2} \Pi_0^{\perp},$$

and finally, H_2^C is given in (5.29). Here, we understand that $s, s' \in \mathbb{R}$ in the Schwartz kernel of H_2 and $s, s' \in [0, \infty)$ in the Schwartz kernel for H_2^C.

The proof of the estimates (5.41) use the following estimates on the Schwartz kernels of each component of E appearing in (5.43):

$$\begin{aligned} \|H_1(t,p,p')\| &\leq b_1 t^{-n/2} e^{b_1 t} e^{-b_2 d(p,p')^2/t}; \\ \|H_2(t,p,p')\| &\leq b_1 t^{-n/2} e^{b_1 t} e^{-b_2 d(p,p')^2/t}; \\ \|H_2^C(t,p,p')\| &\leq b_1 t^{-1/2} e^{-b_2 (s(p)-s(p'))^2/t}, \end{aligned} \tag{5.44}$$

where $b_i > 0$ and where the variables are described as follows. In the estimate for H_1, $d(p, p')$ represents the geodesic distance between the points p, p' on the double of N. In the estimates for H_2 and H_2^C, $p = (s, y)$ and $p' = (s', y')$, where $s, s' \in \mathbb{R}$ for H_2 and $s, s' \in [0, \infty)$ for H_2^C and $y, y' \in Y$, and $d(p, p')^2 = (s - s')^2 + d_Y(y, y')$, where $d_Y(y, y')$ is the geodesic distance between $y, y' \in Y$. The estimate for H_2^C is immediate from its definition (5.29). The estimates on H_1, a heat operator on a closed manifold, follow from (5.39), and since $\eth_0$ is a an operator on a closed manifold, the estimates on H_2 also follow from (5.39). Note that we can apply (5.39) to H_2 because $\Pi_0^{\perp} e^{-t\eth_0^2} \Pi_0^{\perp} = e^{-t\eth_0^2}$ on the domain $\Pi_0^{\perp} L^2(Y, E_0)$, so the

Schwartz kernel of $\Pi_0^\perp e^{-t\eth_0^2}\Pi_0^\perp$ is the same as the Schwartz kernel of $e^{-t\eth_0^2}$. We have similar estimates for derivatives, which are also straightforward to prove:

$$
\begin{aligned}
\|\eth H_1(t,p,p')\| &\le b_1 t^{-(n+1)/2} e^{-b_2 d(p,p')^2/t};\\
\|\eth H_2(t,p,p')\| &\le b_1 t^{-(n+1)/2} e^{-b_2 d(p,p')^2/t};\\
\|\eth H_2^C(t,p,p')\| &\le b_1 t^{-1} e^{-b_2 (s(p)-s(p'))^2/t}.
\end{aligned}
\tag{5.45}
$$

Now the operator E in (5.43) maps $\Pi_0^\perp \hat{L}_b^2(M,E_0)\oplus L^2(N,E)$ into $\mathrm{Dom}(D^2)$ in a canonical way, and $(\partial_t + D^2)E = K$, where

$$
K = [\eth^2,\varphi_1]H_1\psi_1 + [\eth^2,\varphi_2]H_2\psi_2 + [\eth^2,\varphi_2]H_2^C\psi_2. \tag{5.46}
$$

The Schwartz kernel of K is a smooth function on $\hat{X}^2$ vanishing to infinite order at $t=0$ and at the boundary hypersurfaces of $\hat{X}^2$ coming from the boundary $x=0$ in $\hat{X}$, and

$$
e^{-tD^2} = E + E * F, \quad F = \sum_{j=1}^{\infty}(-1)^j F_j,
$$

where $F_1 = K$ and $F_j = F_{j-1} * K$. First of all, by the estimates (5.44) and (5.45) for H_1, H_2, and H_2^C, it follows that E satisfies estimates of the form (5.41) and (5.42). To complete the proof of the estimates (5.41) and (5.42), it remains to estimate $E * F$. We first give estimates on F. By the estimates (5.44) and (5.45), the formula (5.46) for K implies that

$$
\|K(t,p,p')\| \le d_1 e^{d_1 t} e^{-d_2/t} e^{-d_2(s(p)-s(p'))^2/t}, \qquad p,p' \in \hat{X}, \tag{5.47}
$$

for some constants $d_i > 0$. Here, we used the fact that $[\eth^2,\varphi_1]$ has support in a compact subset of $(0,1/4)_s \times Y$ while ψ_1 has support in $s > 1/4$, and that $[\eth^2,\varphi_2]$ has support in a compact subset of $(1/2,3/4)_s \times Y$ while ψ_2 has support in $s < 1/2$. We claim that

$$
\|F(t,p,p')\| \le d_3 e^{d_3 t} e^{-d_4/t} e^{-d_2(s(p)-s(p'))^2/t}. \tag{5.48}
$$

To see this, we estimate

$$
(K*K)(t,p,p') = \int_0^t \int_{q\in\hat{X}} K(r,p,q)K(t-r,q,p')dr.
$$

The following inequality (cf. [4, Lem. 22.12]) will be useful in what follows: For any real numbers α,β,γ, we have

$$
\frac{(\alpha-\beta)^2}{t} \le \frac{(\alpha-\gamma)^2}{r} + \frac{(\gamma-\beta)^2}{t-r}, \qquad 0<r<t. \tag{5.49}
$$

Since the support of $K(t-r,q,p')$ in the variable q is confined to the interval $[0,1]_s \times Y$, the estimate (5.47) on K and the inequality (5.49) (with $\alpha = s(p)$,

$\beta = s(p')$, and $\gamma = s(q)$) imply that

$$
\begin{aligned}
\|(K*K)(t,p,p')\| &\le \int_0^t \int_{q\in[0,1]_s\times Y} (d_1 e^{d_1 r} e^{-d_2/r} e^{-d_2(s(p)-s(q))^2/r})\times \\
&\qquad (d_1 e^{d_1(t-r)} e^{-d_2/(t-r)} e^{-d_2(s(q)-s(p')^2)/(t-r)}) dg(q)\, dr \\
&\le d_1^2 e^{d_1 t} e^{-d_2(s(p)-s(p')^2)/t} \int_0^t \int_{q\in[0,1]_s\times Y} e^{-d_2/r} e^{-d_2/(t-r)} dg(q)\, dr \qquad (5.50)\\
&\le (\mathrm{vol}(Y) d_1 t)\, (d_1 e^{d_1 t} e^{-d_2/t} e^{-d_2(s(p)-s(p')^2)/t}).
\end{aligned}
$$

Repeating this argument, one can show that

$$
\|F_j(t,p,p')\| \le \frac{(\mathrm{vol}(Y)d_1 t)^{j-1}}{(j-1)!}\, d_1 e^{d_1 t} e^{-d_2/t} e^{-d_2(s(p)-s(p')^2)/t},
$$

where $F_j = K * \cdots * K$ (j convolutions). Since $F(t,p,p') = \sum_j (-1)^j F_j(t,p,p')$, we have

$$
\|F(t,p,p')\| \le d_1 e^{(d_1 \mathrm{vol}(Y) + d_1)t} e^{-d_2/t} e^{-d_2(s(p)-s(p')^2)/t}.
$$

This proves the estimate (5.48). We now estimate $E * F$. First, since E satisfies the estimates (5.41) and (5.42) (as H_1, H_2, and H_2^C satisfy the estimates (5.44) and (5.45)), a straightforward computation using these estimates shows that given any compact subset $\mathcal{K} \subset \hat{X}$ away from the boundary $x = 0$, there are constants $C_1 = C_1(\mathcal{K}), C_2 = C_2(\mathcal{K}) > 0$ such that

$$
\|E(t)u(p)\| \le C_1 e^{a_1 t} e^{-C_2 s(p)^2/t} \|u\|_\infty, \tag{5.51}
$$

for any smooth section u on $\hat{X}$ with support in $\mathcal{K}$, where $\| \,\|_\infty$ is the sup-norm. Let C_1 and C_2 be chosen for $\mathcal{K} = [0,1]_s \times Y$. Since $0 \le s(q) \le 1$, by choosing $C_2 \le d_2$ if necessary, for some $C_3 > 0$, we see that

$$
\int_{q\in[0,1]_s\times Y} e^{-d_2(s(q)-s(p'))^2/(t-r)}\, dg(q) \le C_3 e^{-d_2 s(p')^2/(t-r)} \le C_3 e^{-C_2 s(p')^2/(t-r)}.
$$

Using this estimate together with the estimate (5.47) on K, the estimate (5.51) (and recalling that the support in the variable q of $K(t-r,q,p')$ is in $[0,1]_s \times Y$), and the estimate (5.49) (with $\alpha = s(p)$, $\beta = s(p')$, and $\gamma = 0$), we obtain

$$
\begin{aligned}
\|(E*K)(t,p,p')\| &= \int_0^t \int_{q\in\hat{X}} E(r,p,q) K(t-r,q,p') dr \\
&\le \int_0^t \int_{q\in[0,1]_s\times Y} C_1\, e^{a_1 r}\, e^{-C_2 s(p)^2/t}\, d_1 e^{d_1(t-r)} \times \\
&\qquad e^{-d_2/(t-r)} e^{-d_2(s(q)-s(p'))^2/(t-r)}\, dg(q)\, dr \\
&\le C_1\, C_3\, d_1\, e^{(a_1+d_1)t} \int_0^t e^{-C_2 s(p)^2/t} e^{-C_2 s(p')^2/(t-r)} e^{-d_2/(t-r)}\, dr \qquad (5.52)\\
&\le C_1\, C_3\, d_1\, t\, e^{(a_1+d_1)t} e^{-d_2/t} e^{-C_2(s(p)-s(p'))^2/t} \\
&\le C_1\, C_3\, d_1 e^{(1+a_1+d_1)t} e^{-d_2/t} e^{-C_2(s(p)-s(p'))^2/t}.
\end{aligned}
$$

Observe that $F = -K + K*K + K*F*K$. Thus,

$$
E*F = -E*K + (E*K)*K + (E*K)*F*K.
$$

Using the estimate (5.52) for $E * K$, (5.47) for K, and (5.48) for F, and then following the arguments used in (5.50) for estimating $K * K$ show that $E * F$ satisfies the estimate

$$\|(E * F)(t,p,p')\| \leq d_5 e^{d_5 t} e^{-d_6/t} e^{-d_6(s(p)-s(p'))^2/t}, \qquad p,p' \in \hat{X}. \tag{5.53}$$

We now prove a similar type of estimate for $DE * F$. Indeed, since E satisfies the estimates (5.41) and (5.42) (as H_1, H_2, and H_2^C satisfy the estimates (5.44) and (5.45)), the same argument used to prove the estimate (5.51) implies that given a compact subset $\mathcal{K} \subset \hat{X}$ away from the boundary $x = 0$, we have

$$\|DE(t)u\|_\infty \leq C_1\, t^{-1/2}\, e^{a_1 t} e^{-C_2 s(p)^2/t} \|u\|_\infty, \tag{5.54}$$

for any smooth section u on $\hat{X}$ with support in $\mathcal{K}$ where the constants are those given in (5.51). Following the argument of (5.52), and using the estimate (5.54), the estimate (5.47) on K, and the fact that $\int_0^t r^{-1/2} dr = t^{1/2} \leq e^t$, give a similar estimate as in (5.52):

$$\|(DE * K)(t,p,p')\| \leq C_1\, C_3\, d_1 e^{(1+a_1+d_1)t} e^{-d_2/t} e^{-C_2'(s(p)-s(p'))^2/t}.$$

Following the argument used to prove (5.53) then shows that

$$\|(DE * F)(t,p,p')\| \leq d_7 e^{d_7 t} e^{-d_8/t} e^{-d_8(s(p)-s(p'))^2/t}, \qquad p,p' \in \hat{X}. \tag{5.55}$$

Since E satisfies the estimates (5.41) and (5.42), the estimates (5.53) and (5.55) imply that e^{-tD^2} and De^{-tD^2} satisfy the estimates (5.41) and (5.42).

Step 3: Finish up the proof of Lemma 5.10. We now prove the main results (5.35) and (5.36) of this lemma. Let us briefly recall our set-up. We have written the heat operator of D_a^2 as $e^{-tD_a^2} = \tilde{E}_a + \tilde{E}_a * \tilde{F}_a$, where

$$\tilde{E}_a = \varphi_1^a e^{-tD^2} \psi_1^a + \varphi_2^a H_2^a \psi_2^a,$$

where φ_i^a, ψ_i^a are defined in (5.34), $H_2^a(t,s,y,s',y') = H(t,s+a,y,s'+a,y')$ with $H(t,s,y,s',y')$ defined in (5.28), and $\tilde{F}_a = \sum_{j=1}^\infty (-1)^j \tilde{K}_j$, where $\tilde{K}_j = \tilde{K}_a * \cdots * \tilde{K}_a$ (j convolutions), with

$$\tilde{K}_a = (\partial_t + D_a^2)\tilde{E}_a = [\eth^2, \varphi_1^a] e^{-tD^2} \psi_1^a + [\eth^2, \varphi_2^a] H_2^a \psi_2^a.$$

We now more or less repeat the arguments used to prove the estimates on e^{-tD^2} to prove the estimates (5.35) and (5.36).

We first prove the estimate (5.36). In view of the estimates (5.40) for H_2^a, and (5.41) and (5.42) for e^{-tD^2}, we have

$$\|\tilde{E}_a(t,p,p)\| \leq a_1'\, t^{-n/2} e^{a_1' t},$$

for some $a_1' > 0$ independent of a. Hence, as $\mathrm{vol}(\hat{X}_a) \leq a_2'\, a$ for some $a_2' > 0$, we have

$$\mathrm{Tr}(\tilde{E}_a) \leq a_1' \,\mathrm{vol}(\hat{X}_a) t^{-n/2} e^{a_1' t} \leq a_1'\, a_2'\, a\, t^{-n/2} e^{a_1' t}. \tag{5.56}$$

We now analyze $\mathrm{Tr}(\tilde{E}_a * \tilde{F}_a)$. To do so, we first estimate $\tilde{K}_a$. By the definitions of $\psi_i^a(s)$ and $\varphi_i^a(s)$ in (5.34), for some $\varepsilon > 0$, $[\eth^2, \varphi_1^a]$ has support in a compact subset of $(-3a/4-1, -3a/4)_s \times Y$ while $\psi_1^a(s)$ has support in $s > (-3/4+\varepsilon)a$, and $[\eth^2, \varphi_2^a]$ has support in a compact subset of $(-a/2, -a/2+1)_s \times Y$ while ψ_2^a has support in

$s < (-1/2 - \varepsilon)a$. Thus, by the estimates (5.40) for H_2^a, and (5.41) and (5.42) for e^{-tD^2}, we have

$$\|\tilde{K}_a(t,p,p')\| \le d_1' e^{d_1' t} e^{-d_2' a^2/t}, \qquad p,p' \in \hat{X}_a, \tag{5.57}$$

for some constants $d_i' > 0$. Using this estimate, we estimate $\tilde{F}_a$ as follows. First, we have

$$(\tilde{K}_a * \tilde{K}_a)(t,p,p') = \int_0^t \int_{q\in\hat{X}} \tilde{K}_a(r,p,q)\tilde{K}_a(t-r,q,p')dr.$$

Since the support of $\tilde{K}_a(t-r,q,p')$ in the variable q is confined to the interval $I_a \times Y$, where $I_a = [-3/4a-1, -3/4a] \cup [-a/2, -a/2+1]$ by (5.57), it follows that

$$\begin{aligned}\|(\tilde{K}_a * \tilde{K}_a)(t,p,p')\| &\le \left(\int_{I_a\times Y} ds\, dg_Y\right)\int_0^t (d_1' e^{d_1' r} e^{-d_2' a^2/r})\,(d_1' e^{d_1'(t-r)} e^{-d_2' a^2/(t-r)})dr \\ &= 2\,\mathrm{vol}(Y)\,(d_1')^2 e^{d_1' t}\cdot e^{-d_2' a^2/t}\, e^{-d_2' a^2/t}\int_0^t dr \\ &\le (2\,\mathrm{vol}(Y) d_1' t)\,(d_1' e^{d_1' t} e^{-d_2'/t}).\end{aligned}$$

Second, we repeat this argument, obtaining the inequality

$$\|\tilde{K}_j(t,p,p')\| \le \frac{(2\,\mathrm{vol}(Y) d_1' t)^{j-1}}{(j-1)!}\, d_1' e^{d_1' t} e^{-d_2' a^2/t},$$

where $\tilde{K}_j = \tilde{K}_a * \cdots * \tilde{K}_a$ (j convolutions). Hence,

$$\|\tilde{F}_a(t,p,p')\| = \|\sum_j (-1)^j \tilde{K}_j(t,p,p')\| \le d_1' e^{(2d_1'\mathrm{vol}(Y)+d_1')t} e^{-d_2' a^2/t},$$

which proves the estimate (5.48). We now estimate $\tilde{E}_a * \tilde{F}_a$. First, the estimates (5.40) for H_2^a, and (5.41) and (5.42) for e^{-tD^2}, imply that for some $C > 0$, we have

$$\|\tilde{E}_a(t)u\|_\infty \le C e^{a_1 t}\|u\|_\infty,$$

for any smooth section u on $\hat{X}_a$, where $\| \ \|_\infty$ is the sup-norm. This estimate, plus the estimate (5.57) on $\tilde{K}_a$, imply that

$$\begin{aligned}\|(\tilde{E}_a * \tilde{K}_a)(t,p,p')\| &\le \int_0^t C\, e^{a_1 r}\, d_1' e^{d_1'(t-r)} e^{-d_2' a^2/(t-r)}\, dr \\ &\le C\, t\, d_1' e^{(a_1+d_1')t} e^{-d_2' a^2/t} \\ &\le C\, d_1' e^{(1+a_1+d_1')t} e^{-d_2' a^2/t}.\end{aligned}$$

Second, the same arguments used to prove the estimate (5.53) for $E * F$ can be used in this situation to prove that

$$\|(\tilde{E}_a * \tilde{F}_a)(t,p,p')\| \le d_5' e^{d_5' t} e^{-d_6' a^2/t}, \qquad p,p' \in \hat{X}_a.$$

As $\mathrm{vol}(\hat{X}_a) \le a_2'\, a$, it follows that

$$\mathrm{Tr}(\tilde{E}_a * \tilde{F}_a) \le d_5' \mathrm{vol}(\hat{X}_a) e^{d_5' t} e^{-d_6' a^2/t} \le a_2'\, d_5'\, a\, e^{d_5' t} e^{-d_6' a^2/t}.$$

This estimate together with (5.56), imply that for some $c > 0$, $\mathrm{Tr}(e^{-tD_a^2}) \le c\, a\, t^{-n/2} e^{ct}$ for all $t > 0$. However, as $\mathrm{Tr}(e^{-tD_a^2}) \le \mathrm{Tr}(e^{-D_a^2})$ for $t \ge 1$, it follows that $\mathrm{Tr}(e^{-tD_a^2}) \le c\, e^c a\, t^{-n/2}$. This proves the trace estimate (5.36).

It remains to prove the estimate (5.35). To see this, observe that $D_a e^{-tD_a^2} = D\tilde{E}_a + D\tilde{E}_a * \tilde{F}_a$. Now

$$D_a\tilde{E}_a = \varphi_1^a D e^{-tD^2}\psi_1^a + \varphi_2^a \eth H_2^a \psi_2^a + [\eth, \varphi_1^a] e^{-tD^2}\psi_1^a + [\eth, \varphi_2^a] H_2^a \psi_2^a, \tag{5.58}$$

so on the diagonal, we have

$$D_a\tilde{E}_a(t,p,p) = \psi_1^a(p)\, De^{-tD^2}(p,p) + \psi_2^a(p)\,\eth H_2^a(t,p,p).$$

Thus,

$$D_a e^{-tD_a^2}(p,p) - \psi_1^a(p)\, De^{-tD^2}(p,p) = \psi_2^a(p)\,\eth H_2^a(t,p,p) + (D_a\tilde{E}_a * \tilde{F}_a)(t,p,p).$$

We now compute $\operatorname{tr}(\eth H_2^a(t,p,p))$. To do so, we use the explicit formula (5.28) for $H(t,s,y,s',y')$ to obtain

$$\begin{aligned}(\eth H)(t,s,y,s,y) = &\sum_j \frac{e^{-\lambda_j^2 t}}{\sqrt{4\pi t}}\Big[\frac{s}{t}e^{-s^2/4t} + \lambda_j(1-e^{-s^2/4t})\Big]\Gamma\varphi_j(y)\otimes\varphi_j(y)\\ &+ \sum_j \frac{e^{-\lambda_j^2 t}}{\sqrt{4\pi t}}\Big[\frac{s}{t}e^{-s^2/4t} + \lambda_j(1-e^{-s^2/4t})\Big]\varphi_j(y)\otimes\Gamma\varphi_j(y)\end{aligned}$$

Since $\Gamma^* = -\Gamma$, we have $\langle\Gamma\varphi_j(y), \varphi_j(y)\rangle = -\langle\varphi_j(y), \Gamma\varphi_j(y)\rangle$. Hence, the pointwise trace $\operatorname{tr}(\eth H_2^a(t,p,p)) = 0$ for all p and therefore,

$$\operatorname{tr}(D_a e^{-tD_a^2}(p,p)) - \operatorname{tr}(\psi_1^a(p) De^{-tD^2}(p,p)) = \operatorname{tr}((D_a\tilde{E}_a * \tilde{F}_a)(t,p,p)).$$

In view of the estimate (5.57) for $\tilde{K}_a(t,p,p')$, and the formula (5.58) for $D_a\tilde{E}_a$, arguments and computations very similar to those used to prove the estimate (5.55) for $DE * F$ can be used to prove that

$$\|(D_a\tilde{E}_a * \tilde{F}_a)(t,p,p')\| \le d_7' e^{d_7' t} e^{-d_8' a^2/t}, \qquad p,p' \in \hat{X}_a.$$

This estimate completes the proof of (5.35).

6. Index theory

6.1. A general index formula. Let X be a compact manifold with corners of arbitrary codimension. The following lemma is the fundamental observation used in all heat operator proofs of the index theorem.

Lemma 6.1. *If $P \in \operatorname{Diff}_b^2(X,E) + \Psi_b^{-\infty}(X,E)$ has a nonnegative scalar principal symbol, is self-adjoint and Fredholm, then $\lim_{t\to\infty} {}^b\operatorname{Tr}(e^{-tP}) = \dim\ker P$. In particular, if $A \in \operatorname{Diff}_b^1(X,E) + \Psi_b^{-\infty}(X,E)$ is Fredholm, then*

$$\operatorname{ind} A = \lim_{t\to\infty}\big[\,{}^b\operatorname{Tr}(e^{-tA^*A}) - {}^b\operatorname{Tr}(e^{-tAA^*})\,\big]. \tag{6.1}$$

Proof. By Proposition C.9 of the appendix, we can write $e^{-tP} = \Pi_0 + R(t)$, where Π_0 is the orthogonal projection onto the null space of P and where, for some $\varepsilon > 0$, $R(t) \in \Psi_b^{-\infty,\varepsilon}(X,E)$ is exponentially decreasing as $t \to \infty$. It follows that

$$\lim_{t\to\infty} {}^b\operatorname{Tr}(e^{-tP}) = \dim\ker P + \lim_{t\to\infty} {}^b\operatorname{Tr}(R(t)) = \dim\ker P.$$

The formula (6.1) follows from the first statement plus the equalities: $\ker A^*A = \ker A$ and $\ker AA^* = \ker A^*$. $\square$

Let $A \in \mathrm{Diff}^1_b(X, E^+, E^-) + \Psi_b^{-\infty}(X, E^+, E^-)$ be Fredholm. Consider the function

$$h(t) = {}^b\mathrm{Tr}(e^{-tA^*A}) - {}^b\mathrm{Tr}(e^{-tAA^*}).$$

By Lemma 6.1, $\mathrm{ind}\, A = \lim_{t\to\infty} h(t)$, hence by the Fundamental Theorem of Calculus, for all $t > 0$,

$$\mathrm{ind}\, A = h(t) + \int_t^\infty h'(s)\, ds. \tag{6.2}$$

We now compute $h'(s)$. Observe that $A^*Ae^{-tA^*A} = A^*e^{-tAA^*}A$, so

$$\begin{aligned} h'(s) &= {}^b\mathrm{Tr}(\, -A^*Ae^{-sA^*A} + AA^*e^{-sAA^*}\,) \\ &= {}^b\mathrm{Tr}(\, AA^*e^{-sAA^*} - A^*e^{-sAA^*}A\,) \\ &= {}^b\mathrm{Tr}(\, [A, A^*e^{-sAA^*}]\,). \end{aligned}$$

According to the trace-defect formula in Theorem 3.7, we have

$$h'(s) = - \sum_{M\in M_k(X),k\geq 1} \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau, \tag{6.3}$$

where $\mathbf{D}^k_\tau = D_{\tau_1} \cdots D_{\tau_k}$ with $D_{\tau_j} = i^{-1}\partial_{\tau_j}$. By Proposition C.9, $e^{-tAA^*} = \Pi_0 + R(t)$, where $R(t) \to 0$ exponentially in the space $\Psi_b^{-\infty,\varepsilon}(X, E^-)$ for some $\varepsilon > 0$. Hence, for any $M \in M_k(X)$, $k \in \mathbb{N}$, $N_M(e^{-tAA^*})(\tau) = N_M(R(t))(\tau)$ is rapidly decreasing in $\Psi_b^{-\infty,\varepsilon}(M, E^-)$ as $t \to \infty$ and as $|\tau| \to \infty$. Hence, we can interchange integrals in the following computation: if $M \in M_k(X)$, $k \in \mathbb{N}$, then

$$\begin{aligned} &\int_t^\infty \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau\, ds \\ &\quad = \int_{\mathbb{R}^k} \int_t^\infty {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, ds\, d\tau \\ &\quad = -\int_{\mathbb{R}^k} \int_t^\infty {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, \partial_s N_M(e^{-sAA^*})(\tau)\,)\, ds\, d\tau \\ &\quad = -\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, [\, N_M(e^{-sAA^*})(\tau)\,]_{s=t}^{s=\infty}\,)\, d\tau \\ &\quad = \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau. \end{aligned}$$

Thus, equations (6.2) and (6.3) imply that for all $t > 0$,

$$\mathrm{ind}\, A = h(t) - \frac{1}{2}\, {}^b\eta_A(t), \tag{6.4}$$

where ${}^b\eta_A(t) = \sum_{M\in M_k(X),k\geq 1} {}^b\eta_M(t)$ with

$$\begin{aligned} {}^b\eta_M(t) &= \frac{2}{(2\pi)^k} \int_t^\infty \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau ds \\ &= \frac{2}{(2\pi)^k} \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau. \end{aligned} \tag{6.5}$$

Definition 6.2. The *b-eta invariant of* A, ${}^b\eta_A$, is the constant term in the expansion of ${}^b\eta_A(t)$ as $t \downarrow 0$.

Remark 6.3. In [16, Lem. 7.4], it is shown that each ${}^b\eta_M(t)$ has an asymptotic expansion in powers of t as $t \downarrow 0$.

Hence, taking the constant term in the expansion of the right hand side of equation (6.4) as $t \downarrow 0$ and using Lemmas C.10 and 4.1 give the following theorem.

Theorem 6.4. *If $A = D + B \in \mathrm{Diff}^1_b(X, E^+, E^-) + \Psi^{-\infty}_b(X, E^+, E^-)$ is Fredholm, then*

$$\operatorname{ind} A = \mathrm{AS}(D) - \frac{1}{2}{}^b\eta_A,$$

where $\mathrm{AS}(D)$ is the constant term in the expansion, as $t \downarrow 0$, of ${}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$, and where ${}^b\eta_A$ is the b-eta invariant of A.

By Lemma C.10, if n is *odd*, then ${}^b\mathrm{Tr}(e^{-tD^*D})$ and ${}^b\mathrm{Tr}(e^{-tDD^*})$ have no constant terms in their expansions as $t \downarrow 0$. Thus, we get the following corollary.

Corollary 6.5. *If $A = D + B \in \mathrm{Diff}^1_b(X, E^+, E^-) + \Psi^{-\infty}_b(X, E^+, E^-)$ is Fredholm and X is odd dimensional, then*

$$\operatorname{ind} A = -\frac{1}{2}{}^b\eta_A,$$

where ${}^b\eta_A$ is the b-eta invariant of A.

Let $\eth$ be a Dirac operator associated to an exact b-metric on an even dimensional compact manifold with corners X of arbitrary codimension and fixed by a $\mathbb{Z}_2$-graded Clifford module E (see Section 1.1).

Lemma 6.6. *Assume that $\eth^+ \in \mathrm{Diff}^1_b(X, E^+, E^-)$ is Fredholm. Then, ${}^b\eta_{\eth^+}(t) = \sum_{H \in M_1(X)} {}^b\eta_H(t)$, where for each $H \in M_1(X)$,*

$${}^b\eta_H(t) = \frac{1}{\sqrt{\pi}} \int_t^\infty s^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-s\eth_H^2}\,)\, ds.$$

Proof. For any $M \in M_k(X)$, $k \in \mathbb{N}$, the normal operator $N_M(\eth^+)(\tau)$ is a first degree polynomial in τ. Thus, $\mathbf{D}^k_\tau N_M(\eth^+)(\tau) = 0$ if $k \geq 2$ and so ${}^b\eta_{\eth^+}(t) = \sum_{H \in M_1(X)} {}^b\eta_H(t)$. Given $H \in M_1(X)$, for $\tau \in \mathbb{R}$, $N_H(\eth^+)(\tau) = \frac{1}{i}\,\sigma_H(i\tau + \eth_H)$ and $N_H(\eth^-)(\tau) = -\frac{1}{i}\,(-i\tau + \eth_H)\sigma_H$, which implies that

$$D_\tau N_H(\eth^+)(\tau)\, N_H(\eth^-)(\tau)\, N_H(e^{-s\eth^+\eth^-})(\tau) = \sigma_H(-i\tau + \eth_H)e^{-s\tau^2}e^{-s\eth_H^2}\sigma_H.$$

Since $\int_{\mathbb{R}} \tau e^{-s\tau^2}\, d\tau = 0$ and $\int_{\mathbb{R}} e^{-s\tau^2}\, d\tau = s^{-1/2} \int_{\mathbb{R}} e^{-\tau^2}\, d\tau = s^{-1/2}\sqrt{\pi}$, we obtain

$$\begin{aligned} {}^b\eta_H(t) &= \frac{1}{\pi}\int_t^\infty \int_{\mathbb{R}} {}^b\mathrm{Tr}(\, D_\tau N_H(\eth^+)(\tau)\, N_H(\eth^-)(\tau)\, N_H(e^{-s\eth^+\eth^-})(\tau)\,)\, d\tau ds \\ &= \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-s\eth_H^2}\,)\, ds. \end{aligned}$$

□

By [22, Th. 8.36], it follows that $s^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-s\eth_H^2}\,)$ is integrable near $s = 0$. Hence, $\lim_{t\downarrow 0} {}^b\eta_{\eth^+}(t) = {}^b\eta_{\eth^+}(0) = \sum_{H \in M_1(X)} {}^b\eta_H(0)$ exists. Moreover, by [22, Ch. 8], we have $\mathrm{AS}(\eth^+) = {}^b\!\int_X \mathrm{AS}$, where AS is the Atiyah-Singer density of E. Thus, for the case of Dirac operators, Theorem 6.4 takes the usual form.

Theorem 6.7. *If $\eth^+$ is Fredholm, then*

$$\operatorname{ind}\eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H\in M_1(X)} {}^b\eta_H,$$

where AS *is the Atiyah-Singer density of E, and where*

$$ {}^b\eta_H = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-t\eth_H^2}\,)\,dt. \tag{6.6}$$

The following result has essentially the same proof as the corresponding result on a manifold with boundary [22, Ch. 9.1]. The details will be left to the reader.

Theorem 6.8. *Suppose that* $\ker\eth_M = 0$ *for each* $M \in M_k(X)$ *with* $k \geq 2$. *Then for some* $\delta > 0$, *for all multi-indices* α *with* $0 < |\alpha| < \delta$, *the operator*

$$\eth^+ : \rho^\alpha H^1_b(X, E^+) \longrightarrow \rho^\alpha L^2_b(X, E^-)$$

is Fredholm (see Theorem 2.1), and if we denote its index by $\operatorname{ind}_\alpha\eth^+$, *then*

$$\operatorname{ind}_\alpha\eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H\in M_1(X)} ({}^b\eta_H + \operatorname{sgn}\alpha_H\cdot\dim\ker\eth_H),$$

where AS *is the Atiyah-Singer density of E, and where ${}^b\eta_H$ is given in* (6.6).

6.2. **Non-product type perturbed Dirac operators.** Let $\eth$ be a Dirac operator associated to an exact b-metric on an even dimensional compact manifold with corners X of arbitrary codimension and fixed by a $\mathbb{Z}_2$-graded Clifford module E. We now consider the index of Dirac operators with compatible perturbations as defined in Definition 2.3. We start with the following lemma.

Lemma 6.9. *Let H be a hypersurface of X and let $A(r,\tau) = \eth_H + T(r,\tau)$, where $T(r,\tau)$ is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and bounded as a function with values in $\Psi_b^{-\infty}(H, E_H)$. Assume that $T(r,\tau)$ is self-adjoint and $A(r,\tau)$ is invertible for all $(r,\tau)\in[0,1]\times\mathbb{R}$. Let $B(r,\tau)$ be either*
(1) equal to $A(r,\tau)$, or
(2) be continuous and bounded in $(r,\tau)\in[0,1]\times\mathbb{R}$ with values in $\Psi_b^{-\infty}(H,E_H)$.
Then for all $(r,\tau)\in[0,1]\times\mathbb{R}$, the integral

$$\eta(r,t) = \int_{\mathbb{R}} {}^b\mathrm{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau$$

exists as an absolutely convergent integral and $\eta(r,t)$ decays exponentially as $t\to\infty$ and is $\mathcal{O}(t^{-1/2})$ as $t\to 0$, both uniformly in $r\in[0,1]$.

Proof. It suffices to prove that each of the following integrals

$$\eta_1(r,t) = \int_{|\tau|\geq 1} {}^b\mathrm{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau,$$
$$\eta_2(r,t) = \int_{|\tau|\leq 1} {}^b\mathrm{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau$$

exist and have the required properties. Consider first the analysis of η_1. For this, we need some bounds on the heat operator $e^{-t\tau^2}e^{-tA(r,\tau)^2}$. Since $A(r,\tau) = \eth_H + T(r,\tau)$ is self-adjoint and invertible, the operator $(\eth_H + T(r,\tau))^2$ is positive, so we can write

$$e^{-tA(r,\tau)^2} = \frac{i}{2\pi}\int_\Upsilon e^{-t\lambda}\Big((\eth_H + T(r,\tau))^2 - \lambda\Big)^{-1}\,d\lambda \tag{6.7}$$

where Υ is any counter-clockwise contour in the complex plane around the positive real axis. Since $T(r,\tau)$ is continuous and bounded in $(r,\tau) \in [0,1] \times \mathbb{R}$ as a function with values in $\Psi_b^{-\infty}(H, E_H)$, the explicit resolvent construction in [17] shows that we can write

$$((\eth_H + T(r,\tau))^2 - \lambda)^{-1} = Q(\lambda) + R(r,\tau,\lambda),$$

where $Q(\lambda)$ is a pseudodifferential operator of order -2 living in an appropriate parameter-dependent pseudodifferential calculi, and where $R(r,\tau,\lambda)$ is continuous and bounded in $(r,\tau) \in [0,1] \times \mathbb{R}$ as a function with values in $\Psi_b^{-\infty}(H, E_H)$ and decays in λ to order -1 uniformly as $|\lambda| \to \infty$ in sectors bounded away from the positive real axis. In particular, the contour integral (6.7) implies that for any $\varepsilon > 0$, the heat operator $e^{-tA(r,\tau)^2}$ is of the form $e^{\varepsilon t} \times$ a function that is continuous and bounded in $(r,\tau) \in [0,1] \times \mathbb{R}$ as a function with values in $\Psi_b^{-\infty}(H, E_H)$. Now back to our analysis of η_1. Observe that

$$\int_{|\tau| \geq 1} e^{-t\tau^2} d\tau = \frac{2}{\sqrt{t}} \int_{\sqrt{t}}^{\infty} e^{-\tau^2} d\tau \leq \frac{2C}{\sqrt{t}} \int_{\sqrt{t}}^{\infty} \tau e^{-\tau^2} d\tau = \frac{C}{\sqrt{t}} e^{-t},$$

for some constant C. Hence, for any $0 < \varepsilon < 1$, for some constant C' we have

$$\left| \int_{|\tau| \geq 1} {}^b\mathrm{Tr}(B(r,\tau) e^{-t\tau^2} e^{-tA(r,\tau)^2})\, d\tau \right| \leq \frac{C'}{\sqrt{t}} e^{(\varepsilon - 1)t}.$$

The function on the right decays exponentially as $t \to \infty$ and is $\mathcal{O}(t^{-1/2})$ as $t \downarrow 0$, both uniformly in $r \in [0,1]$.

We now analyze η_2:

$$\eta_2(r,t) = \int_{|\tau| \leq 1} {}^b\mathrm{Tr}(B(r,\tau) e^{-t\tau^2} e^{-tA(r,\tau)^2})\, d\tau. \tag{6.8}$$

Since $A(r,\tau)$ is by assumption continuous in $(r,\tau) \in [0,1] \times \mathbb{R}$ and invertible, it follows that $e^{-tA(r,\tau)^2}$ vanishes exponentially as $t \to \infty$ uniformly for $(r,\tau) \in [0,1] \times [-1,1]$. In particular, the integral (6.8) defining $\eta_2(r,t)$ is absolutely convergent for $t \geq 1$ and vanishes exponentially as $t \to \infty$ uniformly for $r \in [0,1]$. Thus, it remains to analyze the integral (6.8) for t over the bounded interval $[0,1]$. If $B(r,\tau)$ is continuous and bounded in $(r,\tau) \in [0,1] \times [-1,1]$ as a function with values in $\Psi_b^{-\infty}(H, E_H)$, then the integrand of $\eta_2(r,t)$ involves the trace of an operator of order $-\infty$; this trace is certainly a continuous function of $(r,\tau) \in [0,1] \times [-1,1]$ and $t \in [0,1]$. Suppose now that $B(r,\tau) = A(r,\tau) = \eth_H + T(r,\tau)$. Since $A(r,\tau)^2 = \eth_H^2 + R(r,\tau)$ where $R(r,\tau)$ is continuous in $(r,\tau) \in [0,1] \times [-1,1]$ with values in $\Psi_b^{-\infty}(H, E_H)$, according to Lemma 4.1, for some $S(r,\tau,t)$ that is continuous in $(r,\tau,t) \in [0,1] \times [-1,1] \times [0,\infty)$ with values in $\Psi_b^{-\infty}(H, E_H)$, we can write

$$e^{-tA(r,\tau)^2} = e^{-t\eth_H^2} + tS(r,\tau,t).$$

Hence,

$${}^b\mathrm{Tr}(A(r,\tau) e^{-tA(r,\tau)^2}) = {}^b\mathrm{Tr}(\eth_H e^{-t\eth_H^2}) + t\tilde{S}(r,t,\tau),$$

where $\tilde{S}(r,\tau,t)$ is continuous in $(r,\tau,t) \in [0,1] \times [-1,1] \times [0,\infty)$. Since ${}^b\mathrm{Tr}(\eth_H e^{-t\eth_H^2}) = \mathcal{O}(\sqrt{t})$ near $t = 0$ [22, Th. 8.36], it follows that the integral (6.8) decays exponentially as $t \to \infty$ and is $\mathcal{O}(t^{-1/2})$ as $t \downarrow 0$, both uniformly in $r \in [0,1]$. Our proof is now complete. $\square$

Proposition 6.10. *Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be a Clifford compatible perturbation. In other words, for each $H \in M_1(X)$, we can write $N_H(R) = \frac{1}{i}\sigma_H R_H(\tau)$, where $R_H(\tau) \in \Psi_b^{-\infty}(H, E_H)$ is self-adjoint and even in $\tau \in \mathbb{R}$. Assume that $\eth^+ + R$ is Fredholm. Then the index of the operator $\eth^+ + R$ is given by*

$$\operatorname{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H \in M_1(X)} {}^b\eta_H(R), \tag{6.9}$$

where AS *is the Atiyah-Singer density of E and*

$$ {}^b\eta_H(R) = \frac{1}{\pi}\int_0^\infty\!\!\int_{\mathbb{R}} {}^b\mathrm{Tr}(LB_H\, e^{-t\tau^2 - B_H^2}\,)\, d\tau dt, \tag{6.10}$$

where $L = 2t^{1/2}\partial_t - t^{-1/2}\tau\partial_\tau$ and $B_H = t^{1/2}(\eth_H + R_H(\tau))$.

Proof. If $A = \eth^+ + R$, then by Theorem 6.4, $\operatorname{ind}(\eth^+ + R) = \mathrm{AS}(\eth^+) - \frac{1}{2}{}^b\eta_A$. As in Theorem 6.7, we have $\mathrm{AS}(\eth^+) = {}^b\!\int_X \mathrm{AS}$. By definition, ${}^b\eta_A$ is the constant term as $t \downarrow 0$ of ${}^b\eta_A(t) = \sum_{M \in M_k(X), k \geq 1} {}^b\eta_M(t; R)$, where

$$ {}^b\eta_M(t;R) = \frac{2}{(2\pi)^k}\int_t^\infty\!\!\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau ds.$$

Thus, our proof is complete once we show that ${}^b\eta_M(t;R) = 0$ for each $M \in M_k(X)$ with $k \geq 2$ and that $\lim_{t\downarrow 0} {}^b\eta_H(t;R)$ is given in (6.10).

We first show that ${}^b\eta_M(t;R) = 0$ for each $M \in M_k(X)$ with $k \geq 2$. Let $M \in M_k(X)$ with $k \geq 2$. Since (cf. (1.5))

$$N_M(A)(\tau) = \sigma_1\tau_1 + \cdots + \sigma_k\tau_k + B_M + N_M(R)(\tau), \tag{6.11}$$

and since B_M and $N_M(R)(\tau)$ both anti-commute with the Clifford action (cf. (2.7)), we obtain

$$N_M(AA^*)(\tau) = \tau_1^2 + \cdots + \tau_k^2 + (B_M + N_M(R)(\tau))(B_M + N_M(R)(\tau))^*.$$

Thus, $N_M(e^{-sAA^*})(\tau) = e^{-sN_M(AA^*)(\tau)}$ is an even function in each of the variables $\tau_i \in \mathbb{R}$ and $\mathbf{D}_\tau^k N_M(A)(\tau) = D_{\tau_1}\cdots D_{\tau_k} N_M(R)(\tau)$ is an odd function in each of the variables τ_i. It follows that the function $\mathbf{D}_\tau^k N_M(A)(\tau) N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)$ is odd in at least one of the variables τ_i. Since the integral of an odd function over $\mathbb{R}$ is zero, we have ${}^b\eta_M(t;R) = 0$ for $M \in M_k(X)$ with $k \geq 2$.

Hence, ${}^b\eta_A(t) = \sum_{H \in M_1(X)} {}^b\eta_H(t;R)$. By (6.5), ${}^b\eta_H(t;R) = \frac{1}{\pi}\int_t^\infty\!\int_{\mathbb{R}} \eta(s,\tau)\, d\tau ds$, where

$$\eta(s,\tau) = {}^b\mathrm{Tr}(\, D_\tau N_H(A)(\tau)\, N_H(A^*)(\tau)\, N_H(e^{-sAA^*})(\tau)\,).$$

For $\tau \in \mathbb{R}$, we have $N_H(A)(\tau) = \frac{1}{i}\,\sigma_H(i\tau + \eth_H + R_H(\tau))$ and $N_H(A^*)(\tau) = -\frac{1}{i}\,(-i\tau + \eth_H + R_H(\tau))\sigma_H$. Hence, $D_\tau N_H(A)(\tau) = \frac{1}{i}\,\sigma_H(1 + D_\tau R_H(\tau))$ and $N_H(AA^*)(\tau) = \sigma_H(\tau^2 + (\eth_H + R_H(\tau))^2)\sigma_H$, thus

$$\begin{aligned}\eta(s,\tau) &= {}^b\mathrm{Tr}((1 + D_\tau R_H(\tau))(-i\tau + \eth_H + R_H(\tau))e^{-s\tau^2}e^{-s(\eth_H + R_H(\tau))^2}) \\ &= {}^b\mathrm{Tr}((\eth_H + R_H(\tau))e^{-s\tau^2}e^{-s(\eth_H+R_H(\tau))^2}) \\ &\qquad - {}^b\mathrm{Tr}(\tau\partial_\tau R_H(\tau)e^{-s\tau^2}e^{-s(\eth_H+R_H(\tau))^2}) \;+ \zeta(s,\tau),\end{aligned}$$

where $\zeta(s,\tau)$ is odd in τ. Since the integral of an odd function over $\mathbb{R}$ is zero and since $LB_H = \eth_H + R_H(\tau) - \tau\partial_\tau R_H(\tau)$, we have

$$ {}^b\eta_H(t;R) = \frac{1}{\pi}\int_t^\infty\!\!\int_{\mathbb{R}} \eta(s,\tau)\, d\tau ds = \frac{1}{\pi}\int_t^\infty\Big[\int_{\mathbb{R}} {}^b\mathrm{Tr}(LB_H e^{-s\tau^2}e^{-B_H^2})\, d\tau\Big] ds.$$

The previous lemma implies that the integral in the brackets on the right is an absolutely convergent integral and decays exponentially as $s \to \infty$ and is $\mathcal{O}(s^{-1/2})$ as $s \to 0$. In particular, ${}^b\eta_H(t;R)$ is continuous at $t=0$, which completes the proof of the proposition. □

Assume now that X is of *codimension two.* Then for perturbations considered in Section 2.3, the previous proposition simplifies as follows.

Theorem 6.11. *On a codimension two manifold with corners, suppose that the index of the positive parts of each induced Dirac operator on the codimension two faces is zero. Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be a Clifford compatible perturbation such that for each $M \in M_2(X)$, $N_M(R)(\tau)$ has the form given in* (2.12)*. This holds, for example, for the operator in Proposition 2.8. If $\eth^+ + R$ is Fredholm, then*

$$\operatorname{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H\in M_1(X)} {}^b\eta(\eth_H + R_H), \tag{6.12}$$

where AS *is the Atiyah-Singer density of E, $\eth_H$ is the induced Dirac operator on H, $R_H = R_H(0)$ with $R_H(\tau) = i\sigma_H N_H(R)(\tau)$, and where ${}^b\eta(\eth_H + R_H)$ is the b-eta invariant*

$$ {}^b\eta(\eth_H + R_H) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(\, (\eth_H + R_H)e^{-t(\eth_H+R_H)^2}\,)\, dt$$

introduced in (4.1).

Proof. The formula (6.12) follows from Proposition 6.10 but with ${}^b\eta(\eth_H + R_H)$ replaced with ${}^b\eta_H(R)$ in (6.10). Thus, we just need to prove that ${}^b\eta_H(R) = {}^b\eta(\eth_H + R_H)$. Fixing $H \in M_1(X)$, for $r \in [0,1]$, we define $B_H(r) = t^{1/2}(\eth_H + R_H(r\tau))$ and ${}^b\eta_H(r) = \frac{1}{\pi}\int_0^\infty \eta(t,r)dt$, where omitting the variable r in $B_H(r)$,

$$\eta(t,r) = \int_{\mathbb{R}} {}^b\mathrm{Tr}(LB_H\, e^{-t\tau^2 - B_H^2}\,)\, d\tau$$

with $L = 2t^{1/2}\partial_t - t^{-1/2}\tau\partial_\tau$ (thus, $LB_H = \eth_H + R_H(r\tau) - (\tau\partial_\tau R_H)(r\tau)$). By Lemma 6.9, ${}^b\eta_H(r)$ is continuous as a function of $r \in [0,1]$. Moreover, ${}^b\eta_H(1) = {}^b\eta_H(R)$ and since $(\tau\partial_\tau R_H)(0) = 0$ and $\int_{\mathbb{R}} e^{-t\tau^2}d\tau = \sqrt{\pi/t}$, we have ${}^b\eta_H(0) = {}^b\eta(\eth_H + R_H)$. We shall prove that ${}^b\eta_H(r)$ is in fact constant, which implies that ${}^b\eta_H(R) = {}^b\eta_H(1) = {}^b\eta_H(0) = {}^b\eta(\eth_H + R_H)$ and completes our proof.

The same arguments, which are based on Duhamel's principle, used in the proof of Proposition 13 of [26] show that

$$\begin{aligned}\frac{d}{dr}\eta(t,r) &= \frac{d}{dt}\Big\{\int_{\mathbb{R}} {}^b\mathrm{Tr}(2t^{1/2}\dot{B}_H\, e^{-t\tau^2 - B_H^2}\,)\, d\tau\Big\}\\ &+ \int_{\mathbb{R}}\int_0^t e^{-t\tau^2}\, {}^b\mathrm{Tr}(\, [\dot{B}_H\, e^{-(1-u)B_H^2}, LB_H \cdot B_H e^{-uB_H^2}]\,)\, du d\tau\\ &+ \int_{\mathbb{R}}\int_0^t e^{-t\tau^2}\, {}^b\mathrm{Tr}(\, [\dot{B}_H \cdot B_H\, e^{-(1-u)B_H^2}, LB_H e^{-uB_H^2}]\,)\, du d\tau,\end{aligned}$$

where $\dot{B}_H = \frac{d}{dr}B_H$. Analyzing the b-traces as in the proofs of Lemmas 4.5 and 4.6 one can show that the second and third terms on the right vanish. Thus,

$$\frac{d}{dr}\eta(t,r) = \frac{d}{dt}\Big\{\int_{\mathbb{R}} {}^b\mathrm{Tr}(2t^{1/2}\dot{B}_H\, e^{-t\tau^2 - B_H^2}\,)\, d\tau\Big\},$$

and so given $a, \varepsilon > 0$, we have

$$\int_\varepsilon^a \frac{d}{dr}\eta(t,r)dt = \int_{\mathbb{R}} {}^b\mathrm{Tr}(2ar\tau\partial_\tau R_H(r\tau)\, e^{-a\tau^2 - a(\eth_H + R_H(r\tau))^2}\,)\, d\tau - \int_{\mathbb{R}} {}^b\mathrm{Tr}(2\varepsilon r\tau\partial_\tau R_H(r\tau)\, e^{-\varepsilon\tau^2 - \varepsilon(\eth_H + R_H(r\tau))^2}\,)\, d\tau. \tag{6.13}$$

Lemma 6.9 shows that each term on the right vanishes as $a \to \infty$ and as $\varepsilon \downarrow 0$, respectively. Taking $\varepsilon \downarrow 0$ and $a \to \infty$ in (6.13) then gives $\frac{d}{dr}\, {}^b\eta_H(r) = 0$. □

6.3. **Dirac operators of product type.** We now present a generalization of Theorem 0.1 found in the introduction. Let X be an even dimensional compact manifold with corners of codimension two equipped with an exact b-metric. Assume that $\eth$ is of product type near the corners in the sense that given $M \in M_2(X)$, for some product decomposition $X \cong [0,1)_{x_1} \times [0,1)_{x_2} \times M$ near M, we can write

$$\eth = \sigma_1 x_1 D_{x_1} + \sigma_2 x_2 D_{x_2} + B_M,$$

where, cf. (1.5), $B_M \in \mathrm{Diff}^1_b(M, E|_M)$ is self-adjoint and is odd with respect to the $\mathbb{Z}_2$-grading of E, $\sigma_j = \sigma(\frac{dx_j}{x_j})|_M$, and where x_j represents an appropriate boundary defining function in the definition of the exact b-metric (1.1).

We assume that $\mathrm{ind}\, \eth^+_M = 0$ for each $M \in M_2(X)$. For each $M \in M_2(X)$, let $T_M : \ker \eth_M \longrightarrow \ker \eth_M$ be a unitary self-adjoint isomorphism that is odd with respect to the $\mathbb{Z}_2$-grading on E_M. Recall that the hypersurfaces are in a fixed order $\{H_j\}$. For each j, let $V_j = \oplus_k \ker \eth_{jk}$, where $\eth_{jk}$ is the induced Dirac operator on $H_j \cap H_k$ (provided this intersection is not empty), see Section 1.2 for a discussion of induced operators. As defined in Section 1.2, the $\mathbb{Z}_2$-grading on $\ker \eth_{jk}$ is given by $\omega_{jk} = \mathrm{sgn}(k-j)\, i\sigma_k\sigma_j$. Then $V_j = V_j^+ \oplus V_j^-$ is $\mathbb{Z}_2$-graded with grading defined by $\omega_j = \bigoplus_k \mathrm{sgn}(k-j)\omega_{jk}$. We define an operator

$$T_j : V_j \longrightarrow V_j$$

by $T_j = [T_{jk}]$, where $T_{jk} : \ker \eth_{jk} \longrightarrow \ker \eth_{jk}$ is given by

$$T_{jk} = \begin{cases} \sum T_M, & \text{if } j < k; \\ i\omega_{jk} \sum T_M, & \text{if } j > k, \end{cases}$$

where the sums are over those $M \in M_2(X)$ with $M \subset M_{jk}$. Then T_j is odd with respect to the $\mathbb{Z}_2$-grading of V_j, and so T_j induces maps $T_j^\pm : V_j^\pm \longrightarrow V_j^\mp$. The reason we defined T_j and ω_j as we did stems from the discrepancies found between equations (1.8) and (1.9). Let $C_j : V_j \longrightarrow V_j$ be the unitary isomorphism (4.15) corresponding to the scattering Lagrangian for $\eth_{H_j}$, let $C = \oplus_j C_j$, and let $T = \oplus_j T_j$. Then $C, T \in \mathcal{L}(V)$, where $V = \oplus_j V_j$ is the vector space with $\mathbb{Z}_2$-grading defined by $\omega = \oplus_j \omega_j$. Finally, let Λ_T and Λ_C be the $+1$ eigenspaces of the matrices T and C, respectively.

Theorem 6.12. *Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be a Clifford compatible perturbation constructed from the T_M's as shown in Proposition 2.8. Then with the notation described above, we have*

$$\mathrm{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H \in M_1(X)} \Big\{ {}^b\eta(\eth_H) + \dim\ker \eth_H \Big\} - \frac{1}{2}\Big\{ \dim(\Lambda_T \cap \Lambda_C) + m(\Lambda_T, \Lambda_C) \Big\}, \tag{6.14}$$

where $m(\Lambda_T, \Lambda_C)$ *is defined in* (4.20).

Proof. By Theorem 6.11, we have

$$\operatorname{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_j {}^b\eta(\eth_j + R_j),$$

where $R_j = R_j(0)$ with $R_j(\tau) = i\sigma_{H_j} N_{H_j}(R)(\tau)$. Hence, (6.14) is proved once we show that for each j,

$$(6.15) \qquad {}^b\eta(\eth_j + R_j) = {}^b\eta(\eth_j) + \dim\ker\eth_j + \dim(\Lambda_{T_j} \cap \Lambda_{C_j}) + m(\Lambda_{T_j}, \Lambda_{C_j}).$$

To see this, assume for simplicity that H_j intersects only one $H_k \in M_1(X)$; the general case is no harder, only notationally cumbersome. In this case, H_j is a manifold with connected boundary given by $\partial H_j = H_j \cap H_k$. From the proof of Proposition 2.8, we can write $R_j = S_j + K_j$, where S_j and K_j have the following properties. By the discussion of induced Dirac operators in Section 1.2, see especially (1.8) and (1.9), and by the formulas (2.11) and (2.12), it follows that on a product decomposition near ∂H_j, we can write

$$\eth_j + S_j = \Gamma[x\partial_x + \eth_0] + S_j,$$

where $\Gamma = i\omega_j$, $S_j = -\Gamma Q^2 T_{jk}$ with Q an operator of the form described in (2.9), and where $\eth_0$ is a Dirac operator on ∂H_j (which is given by $\eth_0 = \eth_{jk}$ if $j < k$; or $\eth_0 = i\omega_{jk}\,\eth_{jk}$ if $j > k$). The operator $K_j \in \Psi^{-\infty,\varnothing}(H_j, E_{H_j})$ was chosen such that if $R_j(r) = S_j + K_j + r(\Pi_j - K_j)$ for $r \in [0,1]$, where Π_j is the orthogonal projection onto the null space of $\eth_j + S_j$, then $\eth_j + R_j(r)$ is invertible for all $r \in [0,1]$. By the variation formula in Theorem 4.7, we have

$$(6.16) \quad {}^b\eta(\eth_j + R_j) = {}^b\eta(\eth_j + R_j(r))|_{r=0} = {}^b\eta(\eth_j + R_j(r))|_{r=1} = {}^b\eta(\eth_j + S_j + \Pi_j)$$

since the corner unitary isomorphisms are constant in r. Now let $A(r) = \eth_j + S_j + r\Pi_j$ for $r \in [0,1]$. Then by Lemma 4.6, it follows that

$$\begin{aligned} {}^b\eta(\eth_j + S_j + \Pi_j) - {}^b\eta(\eth_j + S_j) &= \lim_{t\to\infty}\Big\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_0^1 {}^b\mathrm{Tr}(\dot A(r)e^{-tA(r)^2})\,dr\Big\}. \\ &= \lim_{t\to\infty}\Big\{\frac{2t^{1/2}}{\sqrt{\pi}}\int_0^1 {}^b\mathrm{Tr}(\Pi_j e^{-tA(r)^2})\,dr\Big\}. \end{aligned}$$

Since $\Pi_j e^{-tA(r)^2} = e^{-tr^2}\Pi_j$ and $\lim_{t\to\infty}(2t^{1/2}/\sqrt{\pi})\int_0^1 e^{-tr^2}\,dr = 1$, in view of (6.16), we obtain

$${}^b\eta(\eth_j + R_j) - {}^b\eta(\eth_j + S_j) = \dim\ker(\eth_j + S_j).$$

Now (6.15) follows from the fact that ${}^b\eta(\eth_j + S_j) = {}^b\eta(\eth_j) + m(\Lambda_{T_j}, \Lambda_{C_j})$ by Theorem 5.1 and that $\dim\ker(\eth_j + S_j) = \dim\ker\eth_j + \dim(\Lambda_{T_j} \cap \Lambda_{C_j})$ by Theorem 4.13. □

For weighted Sobolev spaces, we have the following theorem involving the operator constructed in Lemma 2.7.

Theorem 6.13. *Let* $S \in \Psi_b^{-\infty}(X, E^+, E^-)$ *be defined from the* T_M*'s near the corners as shown in Lemma 2.7 using the same product decompositions near the corners (see* (2.10) *and* (2.11)*) that make* $\eth$ *of product type. Then for some* $\delta > 0$*, for all multi-indices* α *with* $0 < |\alpha| < \delta$*, the operator*

$$\eth^+ + S : \rho^\alpha H_b^1(X, E^+) \longrightarrow \rho^\alpha L_b^2(X, E^-)$$

is Fredholm (see Theorem 2.1 and Lemma 2.7), and if we denote its index by $\mathrm{ind}_\alpha(\eth^+ + S)$*, then*

$$\begin{aligned}\mathrm{ind}_\alpha(\eth^+ + S) = &{}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_j {}^b\eta(\eth_j) - \frac{1}{2} m(\Lambda_T, \Lambda_C) \\ &- \frac{1}{2}\sum_j \operatorname{sgn}\alpha_j \Big\{ \dim\ker\eth_j + \dim(\Lambda_{T_j} \cap \Lambda_{C_j}) \Big\},\end{aligned} \tag{6.17}$$

where $m(\Lambda_T, \Lambda_C)$ *is defined in* (4.20).

Proof. The proof of this result is similar to that of Theorem 6.12, but following the proof of [22, Ch. 9.1]. □

The following interpretation of the "corner term" $m(\Lambda_T, \Lambda_C)$ is taken from [13], cf. also [18]. This interpretation reflects the geometry and Clifford structure of the manifold quite elegantly. We define a graph $\mathcal{G}$ and a Dirac operator $\eth_{\mathcal{G}}$ on the graph such that the eta invariant of $\eth_{\mathcal{G}}$ is the corner term.

The graph $\mathcal{G}$ is defined as follows. Each boundary hypersurface H_j of X represents a vertex v_j of the graph $\mathcal{G}$. Each intersection $M_{jk} = H_j \cap H_k$ represents an edge e_{jk} of $\mathcal{G}$ (of course, provided that M_{jk} is not empty). Although $M_{jk} = M_{kj}$ as sets, the edges e_{jk} and e_{kj} are to be considered distinct edges joining v_j and v_k. We do this because the Clifford structures and Dirac operators on the corners depend on the ordering of the boundary hypersurfaces, see (1.8) and (1.9). To put a manifold structure on this graph, we identify e_{jk} with the interval $[-1,1]_s$, where the vertex v_j corresponds to $s = -1$ and the vertex v_k to $s = +1$. We consider $V = \oplus_j V_j = \oplus_{jk} \ker\eth_{jk}$ as a "vector bundle" over $\mathcal{G}$ with fiber $\ker\eth_{jk}$ over the edge e_{jk}. Thus, a section of this vector bundle consists of a collection of smooth sections $\oplus_{jk} s_{jk}$, where $s_{jk} : e_{jk} = [-1,1] \longrightarrow \ker\eth_{jk}$.

We define a Dirac operator $\eth_{\mathcal{G}}$ acting on sections of $\mathcal{G}$ by $\eth_{\mathcal{G}} = \oplus_{jk}\Gamma_{jk}\, d/ds$, where if ω_{jk} is the induced $\mathbb{Z}_2$ grading on $E_{M_{jk}}$, then we define $\Gamma_{jk} = i\omega_{jk}$ if $j < k$, or $\Gamma_{jk} = -i\omega_{jk}$ if $j > k$. The reason why we define Γ_{jk} in this way stems from the discrepancies found between (1.8) and (1.9). We now describe the domain of $\eth_{\mathcal{G}}$. Let $\oplus_{jk} s_{jk}$ be a section of V. Observe that given a vertex v_j of $\mathcal{G}$, we have $\oplus_k s_{jk}(v_j) \in V_j$. Similarly, $\oplus_j s_{jk}(v_k) \in V_k$. Let $\Lambda_{T_j} \subset V_j$ denote the Lagrangian subspace associated to T_j, and let $\Lambda_{C_j} \subset V_j$ denote the Lagrangian subspace associated to C_j. Then the domain of $\eth_{\mathcal{G}}$ consists of those sections $\oplus_{jk} s_{jk}$ such that $\oplus_k s_{jk}(v_j) \in \Lambda_{T_j}$ and $\oplus_j s_{jk}(v_k) \in \Lambda_{C_k}$. The eta invariant of $\eth_{\mathcal{G}}$ can be interpreted exactly as the corner term, see for instance, [18] or [13], cf. also [7, Sec. 6] or [14] for related results:

$$\eta(\eth_{\mathcal{G}}) = m(\Lambda_T, \Lambda_C).$$

Thus, the index formula (6.14) can be written in the form

$$\begin{aligned}\mathrm{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H \in M_1(X)} &\Big\{ {}^b\eta(\eth_H) + \dim\ker\eth_H \Big\} \\ &- \frac{1}{2}\Big\{ \dim(\Lambda_T \cap \Lambda_C) + \eta(\eth_{\mathcal{G}}) \Big\}.\end{aligned}$$

Assume now that the only two boundary hypersurfaces of X that intersect are H_1 and H_2 and let $M = H_1 \cap H_2$, which is a disjoint union of codimension two boundary faces of X. Let C_1 and C_2 be the scattering matrices for $\eth_{H_1}$ and $\eth_{H_2}$ respectively, and let $\Lambda_{C_1}, \Lambda_{C_2} \subset \ker \eth_M$ be the $+1$ eigenspaces of C_1 and C_2 respectively. Recall that $\Gamma = i\omega$ with $\omega = i\sigma_2\sigma_1$ the induced $\mathbb{Z}_2$-grading on E_M. Here, $\sigma_i = \sigma(dx_i/x_i)|_M$ where x_i is the boundary defining function for H_i. Also recall that m is defined on pairs (Λ_T, Λ_S) of Lagrangian subspaces of $\ker \eth_M$ by (see (4.20))

$$m(\Lambda_T, \Lambda_S) = -\frac{1}{i\pi} \sum_{\substack{e^{i\theta} \in \operatorname{spec}(-T^- S^+) \\ \theta \in (-\pi, \pi)}} i\theta, \tag{6.18}$$

where T^- and S^+ are the restrictions of T and S to the -1 and $+1$ eigenspaces of ω, respectively. Two straightforward properties that $m(\Lambda_T, \Lambda_S)$ satisfies are

$$m(\Lambda_T, \Lambda_S) = -m(\Lambda_S, \Lambda_T), \qquad m(\Lambda_T, \Lambda_S) = -\tilde{m}(\Lambda_T, \Lambda_S), \tag{6.19}$$

where $\tilde{m}(\Lambda_T, \Lambda_S)$ is the function (6.18), but using the opposite $\mathbb{Z}_2$-grading given by $-\omega$. In other words, the T^- and S^+ in (6.18) are defined as the restrictions of T and S to the -1 and $+1$ eigenspaces of $\tilde{\omega} = -\omega$. Theorem 0.1 in the introduction is a consequence of the following result.

Corollary 6.14. *Let $R \in \Psi_b^{-\infty}(X, E^+, E^-)$ be any compatible perturbation constructed as shown in Proposition 2.8 from a unitary self-adjoint isomorphism $T : \ker \eth_M \longrightarrow \ker \eth_M$ that is odd with respect to the $\mathbb{Z}_2$ grading ω on E_M and is diagonal with respect to the decomposition $\ker \eth_M = \oplus_{F \in M_2(X)} \ker \eth_F$. Then*

$$\begin{aligned} \operatorname{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2} \sum_{H \in M_1(X)} \Big\{ {}^b\eta(\eth_H) + \dim \ker \eth_H \Big\} \\ - \frac{1}{2} \Big\{ \dim(\Lambda_T \cap \Lambda_{C_1}) + \dim(\Lambda_{\Gamma T} \cap \Lambda_{C_2}) + m(\Lambda_T, \Lambda_{C_1}) + m(\Lambda_{C_2}, \Lambda_{\Gamma T}) \Big\}. \end{aligned} \tag{6.20}$$

Proof. The index of $\eth^+ + R$ is given in (6.14). In this specific case of only two hypersurfaces intersecting, $V_1 = \ker \eth_M = V_2$ and $\omega \oplus \tilde{\omega}$ is the $\mathbb{Z}_2$-grading on $V = V_1 \oplus V_2$. Moreover, by definition of T_j, we have $T_1 = T$ and $T_2 = \Gamma T$. Hence, the identities (6.19) imply that twice the second line of (6.14) is given by

$$\begin{aligned} &\dim(\Lambda_{T_1} \cap \Lambda_{C_1}) + m(\Lambda_{T_1}, \Lambda_{C_1}) + \dim(\Lambda_{T_2} \cap \Lambda_{C_2}) + \tilde{m}(\Lambda_{T_2}, \Lambda_{C_2}) \\ &\quad = \dim(\Lambda_T \cap \Lambda_{C_1}) + \dim(\Lambda_{\Gamma T} \cap \Lambda_{C_2}) + m(\Lambda_T, \Lambda_{C_1}) + m(\Lambda_{C_2}, \Lambda_{\Gamma T}). \end{aligned}$$

□

Corollary 6.15. *Let τ be the triple Maslov index, defined on a triple $(\Lambda_A, \Lambda_B, \Lambda_C)$ of Lagrangian subspaces of $\ker \eth_M$ by*

$$\tau(\Lambda_A, \Lambda_B, \Lambda_C) = m(\Lambda_A, \Lambda_B) + m(\Lambda_B, \Lambda_C) + m(\Lambda_C, \Lambda_A). \tag{6.21}$$

Under the same assumptions as in Corollary 6.14, in terms of the triple Maslov index, the index formula (6.20) takes the form

$$\begin{aligned} \operatorname{ind}(\eth^+ + R) = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2} \sum_{H \in M_1(X)} \Big\{ {}^b\eta(\eth_H) + \dim \ker \eth_H \Big\} \\ - \frac{1}{2} \Big\{ \dim(\Lambda_T \cap \Lambda_{C_1}) + \dim(\Lambda_T \cap (\mathrm{Id} - \Gamma)\Lambda_{C_2}) \\ + \tau(\Lambda_{C_1}, (\mathrm{Id} - \Gamma)\Lambda_{C_2}, \Lambda_T) - m(\Lambda_{C_1}, (\mathrm{Id} - \Gamma)\Lambda_{C_2}) \Big\}. \end{aligned} \tag{6.22}$$

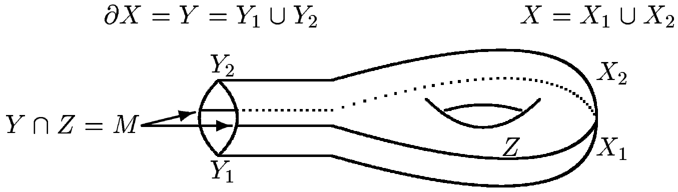

FIGURE 1. The submanifold Z cuts X as well as Y into two pieces.

Proof. The following facts are straightforward to verify:

(1) $\mathrm{Id} + \Gamma : \Lambda_{\Gamma T} \cap \Lambda_{C_2} \longrightarrow \Lambda_T \cap \Lambda_{-\Gamma C_2}$ is an isomorphism;

(2) $m(\Lambda_{C_2}, \Lambda_{\Gamma T}) = m(\Lambda_{-\Gamma C_2}, \Lambda_T) = -m(\Lambda_T, \Lambda_{-\Gamma C_2})$

(3) $\Lambda_{-\Gamma C_2} = (\mathrm{Id} - \Gamma)\Lambda_{C_2}$.

Using these facts, it is straightforward to verify (6.22) using the definition of τ and the formula (6.20). □

7. A splitting formula for the eta invariant

We give an application of Theorem 6.12 to prove a splitting formula for eta invariants, see Theorems 7.1 and 7.3. Such formulas are well-known, see [6], [10], [33], [12], [19], [28] [5], and [34]. In Theorem 7.1, we present a mod $\mathbb{Z}$ version, and in Theorem 7.3, we identify the integer part in terms of index theoretic objects. We follow [28, Sec. 8].

7.1. **The set-up.** We begin by setting up our problem. Let $E \to X$ be a $\mathbb{Z}_2$-graded Hermitian Clifford module over an even dimensional compact Riemannian manifold with boundary. Let $Z \subset X$ be a hypersurface that divides X into two pieces, X_1 and X_2, and which intersects the boundary $Y = \partial X$ transversally and divides it into two pieces, Y_1 and Y_2, where $Y_1 = \partial X_1 \cap Y$ and $Y_2 = \partial X_2 \cap Y$. See Figure 1. Note that X_1 and X_2 are manifolds with corners of codimension two. For simplicity, we assume that $M = Y \cap Z$ is *connected* (contrary to the picture shown in Figure 1). We derive a splitting formula for the eta invariant of a Dirac operator on Y in terms of eta invariants of the Dirac operator restricted to each of the two components Y_1 and Y_2. Although we are assuming that Y bounds, the splitting formula that we derive also holds for twisted Dirac operators on spin manifolds that don't necessarily bound, see [28, Sec. 8] for the details.

Let D be a generalized Dirac operator on X associated to a Clifford compatible connection on E and metric g. We assume that Y and Z have collar neighborhoods over which all the geometric structures are of product type. Thus, for instance, if g denotes the metric on X, then we assume that $X \cong [0,1)_x \times Y$ near Y where $g = dx^2 + h_Y$, where h_Y is a metric on Y, and we assume that $X \cong (-1,1)_y \times Z$ near Z where $g = dy^2 + h_Z$, where h_Z is a metric on Z. *We assume that ∂_y points into X_2*. Over each of these product neighborhoods, E and the connection on E are also products. On the the collar $[0,1)_x \times Y$ of X, we have

$$D = \frac{1}{i}\sigma(dx)[\partial_x + \eth],$$

where $\sigma(dx)$ is Clifford multiplication by dx, and where $\eth$ is the induced Dirac operator on Y and on the collar $[0,1)_x \times (-1,1)_y \times M$, we can write

$$D = \frac{1}{i}\sigma(dx)\partial_x + \frac{1}{i}\sigma(dy)\partial_y + B_M,$$

where B_M is a Dirac operator over M. This decomposition of D plus the decomposition $D = \frac{1}{i}\sigma(dx)[\partial_x + \eth]$ imply that over the collar $(-1,1)_y \times M$ in Y, we have

$$\eth = \Gamma[\partial_y + \eth_M], \tag{7.1}$$

where

$$\eth_M = i\sigma(dy)B_M, \quad \Gamma = i\omega \text{ with } \omega = i\sigma(dy)\sigma(dx). \tag{7.2}$$

The endomorphism ω defines a $\mathbb{Z}_2$-grading on $\ker \eth_M$.

7.2. **The splitting formula.** The ξ-invariant of $\eth$ is defined by

$$\xi(\eth) = \frac{1}{2}[\eta(\eth) + \dim\ker\eth].$$

Let $\eth_1$ and $\eth_2$ denote the restrictions of $\eth$ to Y_1 and Y_2, respectively, and denote by Λ_1 and Λ_2 their corresponding scattering Lagrangians. Let Π_- be the negative spectral projection of $\eth_M$ and let $(\eth_1, \Pi_-^{\Lambda_1})$ denote the operator $\eth_1$ with "augmented" APS boundary condition fixed by the projection $\Pi_-^{\Lambda_1} = \Pi_- + \Pi_{C_1}$, where Π_{C_1} is the orthogonal projection onto Λ_1. Thus, $\eth_1$ has domain defined by

$$\operatorname{Dom}(\eth_1, \Pi_-^{\Lambda_1}) = \{u \in H^1(Y_1, E|_Y)\,;\, \Pi_-^{\Lambda_1} u|_{y=0} = (\Pi_- + \Pi_{C_1})u|_{y=0} = 0\}.$$

Likewise, $(\eth_2, \Pi_+^{\Lambda_2})$ has a similar meaning, but here, Π_+ is the positive spectral projection of $\eth_M$. The following theorem is a corollary of Theorem 7.3 to be proved shortly.

Theorem 7.1. *Under the assumptions described above, we have*

$$\xi(\eth) = \xi(\eth_1, \Pi_-^{\Lambda_1}) + \xi(\eth_2, \Pi_+^{\Lambda_2}) + \frac{1}{2}\Big\{m(\Lambda_1, \Lambda_2) + \dim(\Lambda_1 \cap \Lambda_2)\Big\} \mod \mathbb{Z}. \tag{7.3}$$

Here, $\xi(\eth_1, \Pi_-^{\Lambda_1})$ and $\xi(\eth_2, \Pi_+^{\Lambda_2})$ are the ξ-invariants of $(\eth_1, \Pi_-^{\Lambda_1})$ and $(\eth_2, \Pi_+^{\Lambda_2})$ respectively, and the function m is defined by (6.18) using the $\mathbb{Z}_2$-grading ω fixed in (7.2). As a direct corollary, we have

$$\eta(\eth) = \eta(\eth_1, \Pi_-^{\Lambda_1}) + \eta(\eth_2, \Pi_+^{\Lambda_2}) + m(\Lambda_1, \Lambda_2) \mod \mathbb{Z}.$$

We prove Theorem 7.1 as follows. First of all, the APS index theorem [2] gives

$$\operatorname{ind}(D^+, \Pi_{\geq 0}) = \int_X \mathrm{AS} - \xi(\eth), \tag{7.4}$$

where $\Pi_{\geq 0}$ is the nonnegative spectral projection of $\eth$. This formula accounts for the term $\xi(\eth)$ in (7.3). The integer difference in (7.3) will be written, see Theorem 7.3, as a combination involving Maslov type indexes of certain Lagrangian subspaces of $\ker \eth_M$ and in terms of index theoretic objects involving certain operators on X_1 and X_2, $(D^+, \Pi_{\geq 0})$, and $\eth_Z$, where $\eth_Z$ is the induced Dirac operator on Z. Here, in a collar $(-1,1)_y \times Z$ of X near Z, we can write

$$D = \frac{1}{i}\sigma(dy)[\partial_y + \eth_Z], \tag{7.5}$$

where $\eth_Z$ is the induced Dirac operator on Z.

The operators on X_1 and X_2 are defined as follows. Consider first X_2. We begin by attaching infinite cylinders to the boundary of X_2 to make it into a manifold with cylindrical ends. Thus, we attach $(-\infty,0]_x\times Y_2$ to Y_2, $(-\infty,0]_y\times Z$ to Z, and $(-\infty,0]_x\times(-\infty,0]_y\times M$ to M. Note that all the geometric structures extend in a canonical fashion to this new manifold with cylindrical ends since all the structures were of product type near ∂X_2. Next, we compactify this manifold into a compact manifold with corners diffeomorphic to X_2, and make metric and operators into corresponding b-objects, by introducing the change of variables $x'=e^x$ and $y'=e^y$ (so that $x',y'\to 0$ as $x,y\to-\infty$). Abusing notation, we denote this new manifold by X_2, and we use the *same* notation for all the structures inherited on (the new) X_2 that were on the old X_2. For example, the boundaries of (the new) X_2 are still denoted by Y_2 and Z, the induced Dirac operator on Y_2 is still denoted by $\eth_2$, ... etc. We are now in the setting where we can apply Corollary 6.14.

Fix any unitary, self-adjoint isomorphism $T:\ker\eth_M\longrightarrow\ker\eth_M$ that is odd with respect to the $\mathbb{Z}_2$-grading of $\ker\eth_M$ defined by ω. We can choose such a map by Corollary 2.15. Let D_2 be the Dirac operator D restricted to X_2, $H_1=Y_2$ and $H_2=Z$, and let $R_2\in\Psi_b^{-\infty}(X_2,E^+,E^-)$ be any compatible perturbation constructed as shown in Proposition 2.8 from T. Let Λ_Z denote the scattering Lagrangian for $\eth_Z$. Then by Corollary 6.14, we have

$$\begin{aligned}\operatorname{ind}(D_2^++R_2)&=\int_{X_2}\mathrm{AS}_2-\xi(\eth_2,\Pi_+^{\Lambda_2})-\xi(\eth_Z,\Pi_+^{\Lambda_Z})\\&\quad-\frac{1}{2}\Big\{\dim(\Lambda_T\cap\Lambda_2)+\dim(\Lambda_{\Gamma T}\cap\Lambda_Z)+m(\Lambda_T,\Lambda_2)+m(\Lambda_Z,\Lambda_{\Gamma T})\Big\},\end{aligned}\tag{7.6}$$

where we used the fact that ${}^b\eta(\eth_2)=\eta(\eth_2,\Pi_+^{\Lambda_2})$ from [27], and that $\ker_{L_b^2}\eth_2=\ker(\eth_2,\Pi_+^{\Lambda_2})$ from [2]. Similar statements hold for $\xi(\eth_Z,\Pi_+^{\Lambda_Z})$. We also used the fact that since all structures are constant near ∂X_2, the density AS_2 vanishes on the product decompositions near ∂X_2, and hence ${}^b\!\int_{X_2}\mathrm{AS}_2=\int_{X_2}\mathrm{AS}_2$. Since $H_1=Y_2$ and $H_2=Z$, the function m in (7.6) is defined by (6.18) using the $\mathbb{Z}_2$-grading ω fixed in (7.2). The map Γ in (7.6) is also given in (7.2).

We now consider X_1. Here, we make the change of variables $\tilde y=-y$ so that in a neighborhood of Z in X_1, we have $X_1\cong[0,1)_{\tilde y}\times Z$. As we did for X_2, we make X_1 into a compact manifold with corners with the corresponding b-objects. Let $H_1=Y_1$ and $H_2=Z$. Let D_1 be the Dirac operator D restricted to X_1 and let $R_1\in\Psi_b^{-\infty}(X_1,E^+,E^-)$ be any compatible perturbation constructed as shown in Proposition 2.8 from T. Because of the change of variables $\tilde y=-y$, by (7.1) and (7.5), the induced Dirac operator of D_2 on Z is $-\eth_Z$ and the induced Dirac operator of $\eth_2$ on M is $-\eth_M$; moreover, the induced $\mathbb{Z}_2$-grading on E_M is now $\tilde\omega=-\omega$. Hence, by similar reasons as we gave to write (7.6), we have

$$\begin{aligned}\operatorname{ind}(D_1^++R_1)&=\int_{X_1}\mathrm{AS}_1-\xi(\eth_1,\Pi_-^{\Lambda_2})-\xi(-\eth_Z,\Pi_+^{\Lambda_Z})\\&\quad-\frac{1}{2}\Big\{\dim(\Lambda_T\cap\Lambda_1)+\dim(\Lambda_{-\Gamma T}\cap\Lambda_Z)+\tilde m(\Lambda_T,\Lambda_1)+\tilde m(\Lambda_Z,\Lambda_{-\Gamma T})\Big\},\end{aligned}\tag{7.7}$$

where $\tilde m$ is defined using the $\mathbb{Z}_2$-grading $\tilde\omega=-\omega$. Note that $\eta(-\eth_Z,\Pi_+^{\Lambda_Z})=-\eta(\eth_Z,\Pi_+^{\Lambda_Z})$ and $\int_X\mathrm{AS}=\int_{X_1}\mathrm{AS}_1+\int_{X_2}\mathrm{AS}_2$ (as these densities are defined locally). Also, by (6.19), we have $\tilde m(\Lambda_T,\Lambda_1)=m(\Lambda_1,\Lambda_T)$ and $\tilde m(\Lambda_Z,\Lambda_{-\Gamma T})=m(\Lambda_{-\Gamma T},\Lambda_Z)$. Hence, adding (7.6) and (7.7) and then subtracting (7.4) yields, after some simple algebraic manipulations,

$$\begin{aligned}\xi(\eth) = \xi(\eth_1, \Pi_-^{\Lambda_1}) + \xi(\eth_2, \Pi_+^{\Lambda_2}) + \frac{1}{2}\Big\{m(\Lambda_1, \Lambda_2) + \dim(\Lambda_1 \cap \Lambda_2)\Big\} \\ + \dim\ker(\eth_Z, \Pi_+^{\Lambda_Z}) - \operatorname{ind}(D^+, \Pi_{\geq 0}) + \operatorname{ind}(D_1^+ + R_1) + \operatorname{ind}(D_2^+ + R_2) + J,\end{aligned}$$

where J is given by

$$\begin{aligned}J = \frac{1}{2}\Big\{\tau(\Lambda_T,\, \Lambda_1, \Lambda_2) - \dim(\Lambda_T \cap \Lambda_1) - \dim(\Lambda_T \cap \Lambda_2) - \dim(\Lambda_1 \cap \Lambda_2) \\ + \tau(\Lambda_Z, \Lambda_{-\Gamma T}, \Lambda_{\Gamma T}) - \dim(\Lambda_Z \cap \Lambda_{-\Gamma T}) - \dim(\Lambda_Z \cap \Lambda_{\Gamma T})\Big\}\end{aligned} \tag{7.8}$$

with τ is the triple Maslov index defined in (6.21). Note that J depends on T, Λ_Z, Λ_1, and Λ_2. The following lemma finishes the proof of Theorem 7.1.

Lemma 7.2. *We have* $J \in \mathbb{Z}$.

Proof. By [15, Prop. 1.9.3], we have

$$\tau(\Lambda_T, \Lambda_1, \Lambda_2) - \dim(\Lambda_T \cap \Lambda_1) - \dim(\Lambda_T \cap \Lambda_2) - \dim(\Lambda_1 \cap \Lambda_2) - \dim\ker\eth_M^+ \in 2\mathbb{Z}.$$

Similarly, as $(-\Gamma T)(\Gamma T) = -\mathrm{Id}$ has no $+1$ eigenvalue, by Lemma 4.9, we have $\Lambda_{-\Gamma T} \cap \Lambda_{\Gamma T} = \{0\}$, hence, again by [15, Prop. 1.9.3], we have

$$\tau(\Lambda_Z, \Lambda_{-\Gamma T}, \Lambda_{\Gamma T}) - \dim(\Lambda_Z \cap \Lambda_{-\Gamma T}) - \dim(\Lambda_Z \cap \Lambda_{\Gamma T}) - \dim\ker\eth_M^+ \in 2\mathbb{Z}.$$

It follows that $J \in \mathbb{Z}$. □

We summarize our results in the following theorem.

Theorem 7.3. *Under the assumptions described above, we have*

$$\begin{aligned}\xi(\eth) = \xi(\eth_1, \Pi_-^{\Lambda_1}) + \xi(\eth_2, \Pi_+^{\Lambda_2}) + \frac{1}{2}\Big\{m(\Lambda_1, \Lambda_2) + \dim(\Lambda_1 \cap \Lambda_2)\Big\} \\ + \dim\ker(\eth_Z, \Pi_+^{\Lambda_Z}) - \operatorname{ind}(D^+, \Pi_{\geq 0}) + \mathcal{E},\end{aligned}$$

where the error term $\mathcal{E} \in \mathbb{Z}$ *is given by*

$$\mathcal{E} = \operatorname{ind}(D_1^+ + R_1) + \operatorname{ind}(D_2^+ + R_2) + J,$$

where $J \in \mathbb{Z}$ *is defined by* (7.8).

By Proposition 4.10, we may choose T so that the intersections in (7.8) are all trivial, in which case, $J = (1/2)\{\tau(\Lambda_T, \Lambda_1, \Lambda_2) + \tau(\Lambda_Z, \Lambda_{-\Gamma T}, \Lambda_{\Gamma T})\}$.

Appendix A. The b-calculus

In this appendix, we fix the notations used in the main part of the paper. For more detailed accounts of the category of b-objects; specifically, discussions about b-vector fields, conormal functions, b-tangent and cotangent bundles, blow-ups, and b-differential and b-pseudodifferential operators, see [25], [21], [30], or [17].

A.1. **The small calculus.** By a manifold with corners X of dimension n, we mean a Hausdorff, paracompact topological space with local models of the form $\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}$, where k can run anywhere between 0 and n. We further assume that there are only finitely many boundary hypersurfaces, $\{H_1,\dots,H_N\}$, each one of which is embedded. The largest k that appears in a local model is the codimension of X. The set of boundary faces of codimension $k \in \mathbb{N}_0$ is denoted by $M_k(X)$, the set of all boundary faces is denoted by $M(X)$, and the set of all proper faces, the faces except X itself, is denoted by $M'(X)$.

If $\alpha \in \mathbb{R}$, the b-alpha density bundle, Ω_b^α, is the line bundle with local basis of the form $|\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}dy_1\cdots dy_{n-k}|^\alpha$ on a model $\mathbb{R}^{n,k} = [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$.

The space of smooth functions on X is denoted by $C^\infty(X)$ and the subspace of functions that vanish to infinite order at ∂X is denoted by $\dot{C}^\infty(X)$. The space of 0-th order symbols, $S^0(X)$, consists of those functions u on X such that Pu is a bounded function for any b-differential operator P.

Henceforth, X will always be compact. Given vector bundles E and F over X, we denote the space of classical b-pseudodifferential operators of order $m \in \mathbb{R}$, mapping sections of E to sections of F, by $\Psi_b^m(X,E,F)$. If $m \in \mathbb{N}_0$, we denote the subspace of b-differential operators by $\mathrm{Diff}_b^m(X,E,F)$. For symmetry reasons, we usually assume that $E = F = \Omega_b^{\frac12}$. Then $\Psi_b^m(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}$, is the space of operators on $C^\infty(X,\Omega_b^{\frac12})$ which have Schwartz kernels that are distributions on the b-stretched product X_b^2, the blown-up manifold $X_b^2 = [X^2;\mathcal{B}]$, where $\mathcal{B} = \{H\times H\,;\, H \in M_1(X)\}$. These kernels are smooth up to *ff*, the hypersurfaces in X_b^2 coming from the blow-up of $\mathcal{B}$; vanish to infinite order at *lb* and *rb*, the hypersurfaces coming from left and right boundary hypersurfaces of X^2; and are conormal of order m to Δ_b, the "lifted diagonal" coming from the diagonal of X^2. We refer the reader to [25, Appendix] for a discussion of this view point of the kernels.

For each $m \in \mathbb{R}$, there is a principal symbol map ${}^b\sigma_m$ from $\Psi_b^m(X,\Omega_b^{\frac12})$ onto $C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$, where $C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$ is the space of homogeneous functions of degree m on the b-cotangent bundle ${}^bT^*X$ minus the zero section. This map gives a short exact sequence

$$0 \hookrightarrow \Psi_b^{m-1}(X,\Omega_b^{\frac12}) \hookrightarrow \Psi_b^m(X,\Omega_b^{\frac12}) \xrightarrow{{}^b\sigma_m} C^\infty_{\mathrm{hom}(m)}({}^bT^*X) \to 0,$$

which preserves compositions and adjoints. If $A \in \Psi_b^m(X,\Omega_b^{\frac12})$ and ${}^b\sigma_m(A)$ is invertible, then A is called elliptic. Given such an elliptic operator, there exists a $B \in \Psi_b^{-m}(X,\Omega_b^{\frac12})$ such that $AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi_b^{-\infty}(X,\Omega_b^{\frac12})$.

We denote the space of L^2-integrable b-half densities by $L_b^2(X,\Omega_b^{\frac12})$ and we denote the b-Sobolev space of order $m \in \mathbb{R}$, by $H_b^m(X,\Omega_b^{\frac12})$. Then given $A \in \Psi_b^m(X,\Omega_b^{\frac12})$, A defines a continuous map from $H_b^s(X,\Omega_b^{\frac12})$ into $H_b^{s-m}(X,\Omega_b^{\frac12})$ for any $s \in \mathbb{R}$. In particular, elements of $\Psi_b^0(X,\Omega_b^{\frac12})$ define bounded operators on $L_b^2(X,\Omega_b^{\frac12})$.

In order to find parametrices for Fredholm b-pseudodifferential operators, we need to enlarge the calculus to include operators with kernels having more general conormal behaviors on X_b^2.

A.2. **Calculus with bounds.** A *multi-index* α on any manifold with corners is an assignment of a real number $\alpha(H)$ to each hypersurface H of the manifold. We identify a number $a \in \mathbb{R}$ with the multi-index that assigns to every hypersurface the number a. Given any multi-index α on X^2, we define

$$\Psi^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}) = \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} H_b^\infty(X^2,\Omega_b^{\frac{1}{2}}). \tag{A.1}$$

Now let α be a multi-index for X_b^2 with $\alpha|_{ff} \geq 0$. If ρ_{lb} and ρ_{rb} are total boundary defining functions for lb and rb respectively, we define

$$\Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}) = \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} \bigcup_{\varepsilon>0} \rho_{lb}^{\varepsilon}\rho_{rb}^{\varepsilon} S_{ff}^{0,\alpha|_{ff}}(X_b^2,\Omega_b^{\frac{1}{2}}), \tag{A.2}$$

where $S_{ff}^{0,\alpha|_{ff}}(X_b^2,\Omega_b^{\frac{1}{2}})$ is the subspace of $S^0(X_b^2,\Omega_b^{\frac{1}{2}})$, the space of b-half density symbols u of order 0 on X_b^2, such that given any hypersurface H of the front face of X_b^2, we can write $u = v + \rho_H^{\alpha(H)} w$, where v is smooth up to H, where ρ_H is a boundary defining function for H, and where $w \in S^0(X_b^2,\Omega_b^{\frac{1}{2}})$ is continuous, with all b-derivatives, up to H.

For any $m \in \mathbb{R}$, we define

$$\Psi_b^{m,\alpha}(X,\Omega_b^{\frac{1}{2}}) = \Psi_b^{m}(X,\Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}).$$

These spaces $\Psi_b^{m,*}(X,\Omega_b^{\frac{1}{2}})$ form the *calculus with bounds*.

If ρ_{ff} is a boundary defining function for *ff*, we have the following composition property: Provided that $\alpha|_{rb} + \alpha'|_{lb} \geq 0$ and $\alpha|_{lb} + \alpha'|_{rb} \geq \gamma + \gamma' + \alpha''|_{ff}$ where $\alpha''|_{ff} = \min\{\alpha|_{ff} + \alpha'|_{ff}\}$,

$$\rho_{ff}^{\gamma}\Psi_b^{m,\alpha}(X,\Omega_b^{\frac{1}{2}}) \circ \rho_{ff}^{\gamma'}\Psi_b^{m',\alpha'}(X,\Omega_b^{\frac{1}{2}}) \subset \rho_{ff}^{\gamma+\gamma'}\Psi_b^{m+m',\alpha''}(X,\Omega_b^{\frac{1}{2}}), \tag{A.3}$$

where $\alpha''|_{lb} = \min\{\alpha|_{lb}, \alpha'|_{lb} + \gamma\}$, $\alpha''|_{rb} = \min\{\alpha|_{rb} + \gamma',\ \alpha'|_{rb}\}$.

As a corollary of (A.3), we get the following result: For any $\alpha \geq 0$,

$$\Psi_b^{m,\alpha}(X,\Omega_b^{\frac{1}{2}}) \circ \Psi_b^{m',\alpha}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi_b^{m+m',\alpha}(X,\Omega_b^{\frac{1}{2}}).$$

A.3. **The normal operator.** Let $S \subset \mathbb{C}^k$ be a *horizontal k-strip*; that is, a subset $S \subset \mathbb{C}^k$ of the form $S = \{\tau \in \mathbb{C}^k\,;\, a < \operatorname{Im}\tau < b\}$ for some $a, b \in [-\infty,\infty]^k$. (Here, for any $v, w \in [-\infty,\infty]^k$, we define $v < w$ if $v_i < w_i$ for each i.) For each $m \in \mathbb{R}$, we define $S_h^m(S \times \mathbb{R}^p)$ as those $a(\tau,\xi) \in C^\infty(S \times \mathbb{R}^p)$ that are holomorphic in τ such that for all α, β and $a < a' < b' < b$, there is a $C > 0$ such that

$$|\partial_\tau^\alpha \partial_\xi^\beta a(\tau,\xi)| \leq C(1 + |\tau| + |\xi|)^{m-|\alpha|-|\beta|}$$

for all $\xi \in \mathbb{R}^p$ and $a' \leq \operatorname{Im}\,\tau \leq b'$. If $p = 0$, we denote $S_h^m(S \times \mathbb{R}^0)$ by $S_h^m(S)$. If $\mathcal{F}$ is a Frechét space, then $S_h^m(S,\mathcal{F})$ is defined (we shall need this generality below).

If $m \in \mathbb{R}$, then we define the subspace of holomorphic operators

$$\Psi_{b,S}^m(X,\Omega_b^{\frac{1}{2}}) \subset \mathcal{H}ol(S, \Psi_b^m(X,\Omega_b^{\frac{1}{2}})\,) \tag{A.4}$$

as those operators $A \in \mathcal{H}ol(S, \Psi_b^m(X,\Omega_b^{\frac{1}{2}})\,)$ satisfying the following conditions:

1. For any $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, $\varphi\, A(\tau) \in S_h^{-\infty}(S, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})\,)$.
2. Given any coordinate patch $\mathbb{R}_t^{n,k} \times \mathbb{R}_z^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function φ on the coordinate patch, we can write

$$\varphi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(t,\tau,\xi)\, d\xi \otimes \nu, \quad \nu \in C^\infty(X_b^2,\Omega_b^{\frac{1}{2}}),$$

where $a(t,\tau,\xi) \in C_c^\infty(\mathbb{R}_t^{n,k}; S_h^m(S\times\mathbb{R}^n))$.

For the rest of this appendix, we assume that each boundary hypersurface H_i of X has a *fixed* boundary defining function ρ_i. Let $M \in M_k(X)$ be defined by $x_1,\dots,x_k$, where each x_i is one of the fixed boundary defining functions. Then the *normal operator of* $A \in \Psi_b^m(X,\Omega_b^{\frac12})$ *at* M is the holomorphic family

$$\mathbb{C}^k \ni \tau \mapsto N_M(A)(\tau) = (x^{-i\tau}Ax^{i\tau})|_M \in \Psi_b^m(M,\Omega_b^{\frac12}),\ x^{\pm i\tau} = x_1^{\pm i\tau_1}\cdots x_k^{\pm i\tau_k}.$$

Then $N_M : \Psi_b^m(X,\Omega_b^{\frac12}) \longrightarrow \Psi_{b,\mathbb{C}^k}^m(M,\Omega_b^{\frac12})$, where $\Psi_{b,\mathbb{C}^k}^m(M,\Omega_b^{\frac12})$ is the space given in (A.4) with $S=\mathbb{C}^k$.

The normal operator extends to the calculus with bounds as follows. If $S\subset\mathbb{C}^k$ is a horizontal k-strip, and α is a multi-index on X_b^2 with $\alpha|_{\mathit{ff}}\ge 0$, we define

$$\Psi_{b,S}^{-\infty,\alpha}(X,\Omega_b^{\frac12}) = \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}}\bigcup_{\varepsilon>0}\rho_{lb}^{\varepsilon}\rho_{rb}^{\varepsilon}S_h^{-\infty}(S,\ S_{\mathit{ff}}^{0,\alpha|_{\mathit{ff}}}(X_b^2,\Omega_b^{\frac12}))$$

and for each $m\in\mathbb{R}$, we define

$$\Psi_{b,S}^{m,\alpha}(X,\Omega_b^{\frac12}) = \Psi_{b,\mathbb{C}^k}^m(X,\Omega_b^{\frac12}) + \Psi_{b,S}^{-\infty,\alpha}(X,\Omega_b^{\frac12}).$$

If $M\in M_k(X)$ is defined by $\rho_{i_1},\dots,\rho_{i_k}$, then

$$N_M : \Psi_b^{m,\alpha}(X,\Omega_b^{\frac12}) \longrightarrow \bigcup_{\varepsilon>0}\Psi_{b,S_\varepsilon}^{m,\alpha_M}(M,\Omega_b^{\frac12}),$$

where S_ε is the strip: $-(\alpha|_{lb})_{i_j}-\varepsilon < \operatorname{Im}\tau_j < (\alpha|_{rb})_{i_j}+\varepsilon$, and where α_M is the multi-index induced on $M_b^2\subset X_b^2$ by α. Moreover, N_M is an algebra homomorphism preserving adjoints, and is surjective, with null space the subspace of $\Psi_b^{m,\alpha}(X,\Omega_b^{\frac12})$ that vanishes at $\mathit{ff}(M)$ of X_b^2.

Appendix B. Fredholm properties of b-pseudodifferential operators

The goal of this section is to prove the following results. Let $\rho=\rho_1\cdots\rho_N$ be a total boundary defining function for X.

Theorem B.1. *Let* $A\in\Psi_b^0(X,\Omega_b^{\frac12})$. *Then the following are equivalent:*

(1) A *is compact on* $L_b^2(X,\Omega_b^{\frac12})$.
(2) $A\in\rho\Psi_b^{-1}(X,\Omega_b^{\frac12})$.
(3) ${}^b\sigma_0(A)=0$ *and for each* $H\in M_1(X)$, $N_H(A)(\tau)=0$.

Theorem B.2. *Let* $A\in\Psi_b^m(X,\Omega_b^{\frac12})$, $m\in\mathbb{R}$. *Then the following are equivalent:*

(1) $A: H_b^m(X,\Omega_b^{\frac12}) \longrightarrow L_b^2(X,\Omega_b^{\frac12})$ *is a Fredholm.*
(2) A *is elliptic and for each* $M\in M'(X)$, $N_M(A)(\tau)$ *is invertible for all real parameters.*
(3) A *is elliptic and for each* $H\in M_1(X)$, $N_H(A)(\tau)$ *is invertible for all* $\tau\in\mathbb{R}$.

B.1. **Sufficiency of characterization.** The following lemma proves the sufficiency part of Theorem B.1.

Lemma B.3. *If* $A\in\rho\Psi_b^{-1}(X,\Omega_b^{\frac12})$, *then* A *is compact on* $L_b^2(X,\Omega_b^{\frac12})$.

Proof. Since A vanishes to order ρ at ∂X, there is a sequence of functions $\varphi_j \in C^\infty(X)$ with support sufficiently close to ∂X such that $\varphi_j = 1$ near ∂X and if $B_j = A - (1-\varphi_j)A(1-\varphi_j)$, then $\|B_j\|_{L^2_b} \leq 1/j$.

Let $\{u_j\}$ be a bounded sequence in $L^2_b(X, \Omega_b^{\frac{1}{2}})$. Then, since $(1-\varphi_j) = 0$ near ∂X, $(1-\varphi_j)A(1-\varphi_j)$ is a pseudodifferential operator of order -1 on any compact manifold $\widetilde{X}$ without boundary such that $X \hookrightarrow \widetilde{X}$. As operators of order -1 on compact manifolds without boundary are compact on L^2, it follows that there exists a subsequence u_{j_k} such that $(1-\varphi_j)A(1-\varphi_j)u_{j_k}$ converges as $k \to \infty$. Since for any j and k, $Au_{j_k} = (1-\varphi_j)A(1-\varphi_j)u_{j_k} + B_j u_{j_k}$ and $\|B_j\|_{L^2_b} \leq 1/j$, it follows that the sequence Au_{j_j} converges in L^2_b as $j \to \infty$. □

In order to prove sufficiency in Theorem B.2, we need three lemmas. We begin with some definitions. Let $S = \{\tau \in \mathbb{C}^k\,;\, -\infty \leq a < \operatorname{Im}\tau < b \leq \infty\}$ be a horizontal k-strip and α be a multi-index on X^2. We define

$$\Psi_S^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}) = S_h^{-\infty}(S; \Psi^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})).$$

Let $S \ni \tau \mapsto K(\tau) \in \Psi^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})$ be meromorphic. Then we regard $K(\tau)$ as being meromorphic in $\Psi_S^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})$ if for any $a < a' < b' < b$, $K(\tau)$ has only finitely many poles with $a' \leq \operatorname{Im}\tau \leq b'$, and for $|\Re\tau|$ sufficiently large, $K(\tau)$ satisfies the same estimates as an operator in $S_h^{-\infty}(a' < \operatorname{Im}\tau < b'; \Psi^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}))$.

Lemma B.4. *Let S be a horizontal k-strip and $K(\tau) \in \Psi_S^{-\infty,\varepsilon}(X, \Omega_b^{\frac{1}{2}})$, $\varepsilon > 0$. Then,* $\mathrm{Id} - K(\tau)$ *is invertible on $L^2_b(X, \Omega_b^{\frac{1}{2}})$ with inverse of the form* $\mathrm{Id} + K'(\tau)$, *where $K'(\tau) \in \Psi_S^{-\infty,\varepsilon}(X, \Omega_b^{\frac{1}{2}})$ is meromorphic with finite rank singularities.*

Proof. This is just analytic Fredholm theory [22, Section 5.3]. □

Lemma B.5. *Let $A \in \Psi_b^{m,\alpha}(X, \Omega_b^{\frac{1}{2}})$ and $B \in \Psi_b^{m',\beta}(X, \Omega_b^{\frac{1}{2}})$ where the composition AB defined as in* (A.3). *Suppose that for some $H_1, \ldots, H_q \in M_1(X)$, $B|_{\mathit{ff}(H_i)} = 0$ for $i = 1, \ldots, q$. Then, $(AB)|_{\mathit{ff}(H_i)} = 0$ for $i = 1, \ldots, q$.*

Proof. Let $0 \leq i \leq q$. Then, as $N_{H_i}(A \circ B)(\tau) = N_{H_i}(A)(\tau) \circ N_{H_i}(B)(\tau)$ and $B|_{\mathit{ff}(H_i)} = 0$, we have $N_{H_i}(A \circ B)(\tau) = 0$, so $(A \circ B)|_{\mathit{ff}(H_i)} = 0$. □

Lemma B.6. *Let $A_t \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}}), m \in \mathbb{R}$ be elliptic and depend continuously on a parameter $t \in \mathcal{T}$ where $\mathcal{T}$ is a compact topological space. Then given any $r > 0$, there is an $r' > 0$ such that $N_M(A_t)(\tau) : H_b^m(M, \Omega_b^{\frac{1}{2}}) \longrightarrow L^2_b(M, \Omega_b^{\frac{1}{2}})$ is invertible for each $M \in M'(X)$ and for all $t \in \mathcal{T}$, $|\operatorname{Im}\tau| \leq r$, and $|\Re\tau| > r'$. If for each $t \in \mathcal{T}$, $N_M(A_t)(\tau)$ is invertible for all real parameters, then there is a $\theta > 0$ such that that $N_M(A_t)(\tau)$ is invertible for all $|\operatorname{Im}\tau| \leq \theta$ and $t \in \mathcal{T}$.*

Proof. Since A_t is elliptic, we can write $A_t B_t = \mathrm{Id} - R_t$, where $B_t \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$ and $R_t \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ depend continuously on t. Therefore, given $M \in M_k(X)$ with $k \geq 1$, we have $N_M(A_t)(\tau)\, N_M(B_t)(\tau) = \mathrm{Id} - N_M(R_t)(\tau)$. Let $r > 0$. Then, as $N_M(R_t)(\tau) \in S_h^{-\infty}(\mathbb{C}^k; \Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}}))$ and since elements of $\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}})$ are bounded operators on $L^2_b(M, \Omega_b^{\frac{1}{2}})$, $(\mathrm{Id} - N_M(R_t)(\tau))^{-1}$ exists as a bounded operator on $L^2_b(M, \Omega_b^{\frac{1}{2}})$ for all $|\operatorname{Im}\tau| \leq r$ and $|\Re\tau| \geq r'_M$ for some $r'_M > 0$. Setting $r' =$

$\min_{M\in M'(X)}\{r'_M\}$, it follows that for each $k\in\mathbb{N}$ and $M\in M_k(X)$, $N_M(A_t)(\tau):H^m_b(M,\Omega_b^{\frac12})\longrightarrow L^2_b(M,\Omega_b^{\frac12})$ is invertible for all $|\operatorname{Im}\tau|\le r$ and $|\Re\tau|\ge r'$.

The second statement of this lemma follows from the first statement and the fact that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then the space of isomorphisms of $\mathcal{H}_1$ onto $\mathcal{H}_2$ is an open subset of the space of bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_2$. □

The following lemma proves sufficiency in Theorem B.2.

Lemma B.7. *Let $A\in\Psi^m_b(X,\Omega_b^{\frac12})$, $m\in\mathbb{R}$ be elliptic and suppose that for some $\theta>0$, for each $M\in M'(X)$, $N_M(A)(\tau):H^m_b(M,\Omega_b^{\frac12})\longrightarrow L^2_b(M,\Omega_b^{\frac12})$ is invertible for all $|\operatorname{Im}\tau|\le\theta$. Then for each $\varepsilon<\theta$, there exists a $B\in\Psi_b^{-m,\varepsilon}(X,\Omega_b^{\frac12})$ such that*

(B.1) $$AB=\operatorname{Id}-K_1;\ \ BA=\operatorname{Id}-K_2,\ \ \textit{where } K_1,\ K_2\in\Psi^{-\infty,\varepsilon}(X,\Omega_b^{\frac12}).$$

In particular, $A:H^m_b(X,\Omega_b^{\frac12})\longrightarrow L^2_b(X,\Omega_b^{\frac12})$ is Fredholm.

Proof. For induction purposes, we will understand $N_M(A)(\tau)\equiv A$ if $M\in M_k(X)$ where $k=0$. Also, for each $M\in M(X)$ and $Q\in\Psi^*_b(X,\Omega_b^{\frac12})$, we will denote $N_M(Q)$ by Q_M. If $\operatorname{codim}X=n'$, then we shall prove the following statement by induction on $k=n',n'-1,\dots,0$: Let $M\in M_p(X)$ with $p\ge k$, let $0<\varepsilon<\theta$, and let S_ε be the strip $S_\varepsilon=\{\tau\in\mathbb{C}^p\,;\,-\varepsilon<\operatorname{Im}\tau<\varepsilon\}$. Then there exists a $B(\tau)\in\Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M,\Omega_b^{\frac12})$ and an $R(\tau)\in\Psi^{-\infty,\varepsilon}_{S_\varepsilon}(M,\Omega_b^{\frac12})$, such that

1. $A_M(\tau)B(\tau)=\operatorname{Id}-R(\tau)$;
2. if $p>0$, then $A_M(\tau)^{-1}\in\Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M,\Omega_b^{\frac12})$.

Once this induction statement is proved, our proposition is proved. Indeed, setting $k=0$ into the above statement implies that if $\varepsilon<\theta$ is given, then there exists a $B\in\Psi_b^{-m,\varepsilon}(X,\Omega_b^{\frac12})$ and a $K_1\in\Psi^{-\infty}(X,\Omega_b^{\frac12})$ such that $AB=\operatorname{Id}-K_1$. Since for each $M\in M'(X)$, $N_M(A^*)(\tau)=N_M(A)(\overline{\tau})^*$, A^* satisfies the same hypothesis as A. Hence, there exists a $B'\in\Psi_b^{-m,\varepsilon}(X,\Omega_b^{\frac12})$ and a $K'\in\Psi^{-\infty,\varepsilon}(X,\Omega_b^{\frac12})$ such that $A^*B'=\operatorname{Id}-K'$. Thus, if $B''=(B')^*$ and $K''=(K')^*$, then $B''A=\operatorname{Id}-K''$. One can check that $B''=B$ modulo $\Psi^{-\infty,\varepsilon}(X,\Omega_b^{\frac12})$. Thus, B satisfies (B.1).

We now prove our induction statement. For our base case, $M\in M_{n'}(X)$ is a manifold without boundary. Since A is elliptic, we can choose a $B\in\Psi_b^{-m}(X,\Omega_b^{\frac12})$ such that $\operatorname{Id}-AB=R\in\Psi_b^{-\infty}(X,\Omega_b^{\frac12})$. Taking normal operators, we get $A_M(\tau)B_M(\tau)=\operatorname{Id}-R_M(\tau)$, where $B_M(\tau)\in\Psi^{-m}_{\mathbb{C}^{n'}}(M,\Omega_b^{\frac12})$ and $R_M(\tau)\in\Psi^{-\infty}_{\mathbb{C}^{n'}}(M,\Omega_b^{\frac12})$. Hence, by Lemma B.4, $(\operatorname{Id}-R_M(\tau))^{-1}=\operatorname{Id}-S_M(\tau)$, where $S_M(\tau)\in\Psi^{-\infty}_{\mathbb{C}^{n'}}(M,\Omega_b^{\frac12})$ is meromorphic with finite rank singularities. Hence, $A_M(\tau)^{-1}=B_M(\tau)-K_M(\tau)$, where $K_M(\tau)=B_M(\tau)\circ S_M(\tau)\in\Psi^{-\infty}_{\mathbb{C}^{n'}}(M,\Omega_b^{\frac12})$ is meromorphic with finite rank singularities. By assumption, $A_M(\tau)^{-1}$ exists for all $|\operatorname{Im}\tau|<\theta$. Thus, $K_M(\tau)$ has no poles on $|\operatorname{Im}\tau|<\theta$ and so $K_M(\tau)$ is holomorphic on the n'-strip S_θ. Thus, $A_M(\tau)^{-1}\in\Psi^{-m}_{S_\theta}(M,\Omega_b^{\frac12})$.

Now let $1\le k\le n'-1$ and assume that (1) and (2) hold for all ℓ with $\ell\ge k+1$; we will prove it is true for k, so let $M\in M_k(X)$. As in the $k=n'$ case, we can choose $B(\tau) \in \Psi^{-m}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})$ and $R(\tau) \in \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})$ such that

$$A_M(\tau)B(\tau) = \mathrm{Id} - R(\tau). \tag{B.2}$$

Let $M_1(M) = \{F_1, \ldots, F_q\} \subset M_{k+1}(X)$. Then, by the induction hypothesis, for any $\varepsilon < \theta$, $A_{F_i}(\sigma)^{-1} \in \Psi^{-m,\varepsilon}_{b,S_\varepsilon}(F_i, \Omega_b^{\frac{1}{2}})$, $i = 1, \ldots, q$. For $1 \le \ell \le q$ and any $\varepsilon > 0$, define

$$\Psi^{-m,\varepsilon}_{b,S_\varepsilon,\ell}(M, \Omega_b^{\frac{1}{2}}) = \{S(\tau) \in \Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M, \Omega_b^{\frac{1}{2}})\,;\; S|_{\mathit{ff}(F_i)} = 0,\; i = 1, \ldots, \ell\}$$

and for $\ell = 0$, define $\Psi^{-m,\varepsilon}_{b,S_\varepsilon,0}(M, \Omega_b^{\frac{1}{2}}) = \Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M, \Omega_b^{\frac{1}{2}})$. We shall prove the following statement by induction on ℓ: For each $\ell = 0, \ldots, q$ and $\varepsilon < \theta$, there exists a $B_\ell(\tau) \in \Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M, \Omega_b^{\frac{1}{2}})$ and an $R_\ell(\tau) \in \Psi^{-\infty,\varepsilon}_{b,S_\varepsilon,\ell}(M, \Omega_b^{\frac{1}{2}})$ such that

$$A_M(\tau)B_\ell(\tau) = \mathrm{Id} - R_\ell(\tau). \tag{B.3}$$

The $\ell = 0$ case is given in (B.2). Thus, assume that (B.3) holds for $\ell \le q - 1$; we will prove it holds for $\ell + 1$. Let $\varepsilon < \delta < \theta$. Then by induction hypothesis, there exists a $B_\ell(\tau) \in \Psi^{-m,\delta}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$ and an $R_\ell(\tau) \in \Psi^{-\infty,\delta}_{b,S_\delta,\ell}(M, \Omega_b^{\frac{1}{2}})$ such that $A_M(\tau)B_\ell(\tau) = \mathrm{Id} - R_\ell(\tau)$. Write $\sigma = (\tau, \tau') \in \mathbb{C}^k_\tau \times \mathbb{C}_{\tau'}$. Then, since $A_{F_{\ell+1}}(\sigma)^{-1} \in \Psi^{-m,\delta}_{b,S_\delta}(F_{\ell+1}, \Omega_b^{\frac{1}{2}})$ and $(R_\ell)_{F_{\ell+1}}(\sigma) = N_{F_{\ell+1}}(\,R_\ell(\tau)\,)(\tau') \in \Psi^{-\infty,\delta}_{b,S_\delta}(F_{\ell+1}, \Omega_b^{\frac{1}{2}})$, we have $A_{F_{\ell+1}}(\sigma)^{-1} \circ (R_\ell)_{F_{\ell+1}}(\sigma) \in \Psi^{-\infty,\delta}_{b,S_\delta}(F_{\ell+1}, \Omega_b^{\frac{1}{2}})$. Since $R_\ell(\tau)|_{\mathit{ff}(F_i)} = 0$ for $i = 1, \ldots, \ell$, by Lemma B.5, $(\,A_{F_{\ell+1}}(\sigma)^{-1} \circ (R_\ell)_{F_{\ell+1}}(\sigma)\,)|_{\mathit{ff}(F_{\ell+1} \cap F_i)} = 0$ for $i = 1, \ldots, \ell$. Hence, we can choose a $C_\ell(\tau) \in \Psi^{-\infty,\varepsilon}_{b,S_\varepsilon,\ell}(M, \Omega_b^{\frac{1}{2}})$ with $(C_\ell)_{F_{\ell+1}}(\sigma) = A_{F_{\ell+1}}(\sigma)^{-1} \circ (R_\ell)_{F_{\ell+1}}(\sigma)$. Defining $B_{\ell+1}(\tau) = B_\ell(\tau) + C_\ell(\tau)$, one can check that $A_M(\tau) \circ B_{\ell+1}(\tau) = \mathrm{Id} - R_{\ell+1}(\tau)$, where $R_{\ell+1}(\tau) \in \Psi^{-\infty,\varepsilon}_{S_\varepsilon,\ell+1}(M, \Omega_b^{\frac{1}{2}})$. Our induction step is thus finished and so (B.3) holds for each $\ell = 0, \ldots, q$.

Let $\varepsilon < \delta < \theta$. Setting $\ell = q$ in (B.3), we conclude that there exists a $B(\tau) \in \Psi^{-m,\delta}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$ and an $R(\tau) \in \Psi^{-\infty,\delta}_{b,S_\delta,q}(M, \Omega_b^{\frac{1}{2}})$ such that

$$A_M(\tau) \circ B(\tau) = \mathrm{Id} - R(\tau). \tag{B.4}$$

Since $R(\tau)|_{\mathit{ff}(F_i)} = 0$, $i = 1, \ldots, q$, we have $R(\tau) \in \rho_{\mathit{ff}}^{\min\{1,\delta\}} \Psi^{-\infty,(\delta,\delta,0)}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$, where $(\delta, \delta, 0)$ is the multi-index on X_b^2 that assigns δ to *lb* and *rb* and 0 to *ff*. Choose $p \in \mathbb{N}$ and choose $0 < \delta' \le \min\{1, \delta\}$ such that $p\,\delta' = 2\,\delta$. Then $R(\tau) \in \rho_{\mathit{ff}}^{\delta'} \Psi^{-\infty,(\delta,\delta,0)}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$ and by (A.3), $R(\tau)^p \in \rho_{\mathit{ff}}^{2\delta} \Psi^{-\infty,(\delta,\delta,0)}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$. Since

$$\rho_{\mathit{ff}}^{2\delta} \Psi_b^{-\infty,(\delta,\delta,0)}(M, \Omega_b^{\frac{1}{2}}) \subset \rho_{lb}^{\delta}\rho_{rb}^{\delta} S^0(M^2, \Omega_b^{\frac{1}{2}}) \subset \rho_{lb}^{\varepsilon}\rho_{rb}^{\varepsilon} H_b^{\infty}(M^2, \Omega_b^{\frac{1}{2}}),$$

$R(\tau)^p \in \Psi^{-\infty,\varepsilon}_{S_\delta}(M, \Omega_b^{\frac{1}{2}})$ and since $(\mathrm{Id} - R(\tau))(\,\sum_{\ell=0}^{p-1} R(\tau)^\ell\,) = \mathrm{Id} - R(\tau)^p$, multiplying both sides of (B.4) by $\sum_{\ell=0}^{p-1} R(\tau)^\ell$ we obtain

$$A_M(\tau)B_M(\tau) = \mathrm{Id} - R_M(\tau), \tag{B.5}$$

where $B_M(\tau) = B(\tau)(\,\sum_{\ell=0}^{p-1} R(\tau)^\ell\,) \in \Psi^{-m,\delta}_{b,S_\delta}(M, \Omega_b^{\frac{1}{2}})$ and $R_M(\tau) = R(\tau)^p \in \Psi^{-\infty,\varepsilon}_{S_\varepsilon}(M, \Omega_b^{\frac{1}{2}})$. Thus, (B.5) proves (1) for k. If $k > 0$, one can use Lemma B.4, as we did in the $k = n'$ case, to deduce that $A_M(\tau)^{-1} \in \Psi^{-m,\varepsilon}_{b,S_\varepsilon}(M, \Omega_b^{\frac{1}{2}})$. □

The following two results are proved using Lemma B.7 exactly like the corresponding results are proved in the closed manifold case.

Theorem B.8. *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}}), m \in \mathbb{R}$ be elliptic and assume that for some $\theta > 0$, for each $M \in M'(X)$, $N_M(A)(\tau) : H_b^m(M, \Omega_b^{\frac{1}{2}}) \longrightarrow L_b^2(M, \Omega_b^{\frac{1}{2}})$ is invertible for all $|\operatorname{Im}\tau| \le \theta$. Then for each $s \in \mathbb{R}$, $A : H_b^s(X, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^{s-m}(X, \Omega_b^{\frac{1}{2}})$ is Fredholm, and for all $\varepsilon < \theta$, $\ker A \subset \rho^\varepsilon H_b^\infty(X, \Omega_b^{\frac{1}{2}})$ and $\operatorname{coker} A \cong \ker A^* \subset \rho^\varepsilon H_b^\infty(X, \Omega_b^{\frac{1}{2}})$. In particular, the index,*

$$\operatorname{ind}\left(A : H_b^s(X, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^{s-m}(X, \Omega_b^{\frac{1}{2}}) \right) \in \mathbb{Z},$$

is defined independent of $s \in \mathbb{R}$.

Theorem B.9 (Analytic Fredholm Theory)**.** *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be elliptic and formally self-adjoint and suppose that A is Fredholm. Then there exists an open subset $\mathcal{U} \subset \mathbb{C}$ containing zero such that given any open, relatively compact subset $\mathcal{U}' \subset \mathbb{C} \setminus \mathbb{R}$, there exists an $\varepsilon > 0$ such that*

$$\mathcal{U} \cup \mathcal{U}' \ni \lambda \mapsto (A - \lambda)^{-1} \in \Psi_b^{-m,\varepsilon}(X, \Omega_b^{\frac{1}{2}})$$

is meromorphic having only simple poles, all in a discrete subset $\{\lambda_j\} \subset \mathbb{R}$, which are (minus) the self-adjoint projections onto $\ker(A - \lambda_j)$ at λ_j. If A is also positive, then $\mathcal{U}'$ can be chosen to be a subset of $\mathbb{C} \setminus [0, \infty)$; and the same result holds, but with $\{\lambda_j\} \subset [0, \infty)$.

We shall need the following result in the next section.

Corollary B.10. *For any $m \in \mathbb{R}$, there is an elliptic operator $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ such that for any $s \in \mathbb{R}$, $A : H_b^s(X, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^{s-m}(X, \Omega_b^{\frac{1}{2}})$ is an isomorphism and such that for each $M \in M'(X)$, $N_M(A)(\tau) : H_b^s(M, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^{s-m}(M, \Omega_b^{\frac{1}{2}})$ is an isomorphism for all real parameters.*

Proof. Let $B \in \Psi_b^{m/2}(X, \Omega_b^{\frac{1}{2}})$ and $B' \in \Psi_b^{-m/2}(X, \Omega_b^{\frac{1}{2}})$ be such that $B'B = \operatorname{Id} - R$, where $R \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. One can check that the operator $A = B^*B + R^*R \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ satisfies the conditions of the corollary. □

B.2. **Necessity of characterization.** Let $\varphi \in C_c^\infty([0,1)_x)$ with $0 \le \varphi \le 1$ and $\varphi(x) = 1$ for $0 \le x \le 1/2$. Let $k \in \mathbb{N}$ and define φ_ε on $[0,1)_x^k = [0,1)_{x_1} \times \cdots \times [0,1)_{x_k}$ by

$$\varphi_\varepsilon(x) = \varepsilon^{k/2} x^{\varepsilon/2} \tilde{\varphi}(\frac{x}{\varepsilon}) \, |\frac{dx}{x}|^{1/2}, \tag{B.6}$$

where $\tilde{\varphi}(x) = \varphi(x_1) \cdots \varphi(x_k)$ and $x^{\varepsilon/2} = x_1^{\varepsilon/2} \cdots x_k^{\varepsilon/2}$. If $M \in M_k(X)$, then near M, $X \cong [0,1)_x^k \times M_y$, where $x = (x_1, \ldots, x_k)$ with each x_i one of the fixed boundary defining functions. With φ_ε defined by (B.6), observe that if $u \in H_b^m(M, \Omega_b^{\frac{1}{2}})$, then we can consider $\varphi_\varepsilon u$ as an element of $H_b^m(X, \Omega_b^{\frac{1}{2}})$ using the product decomposition $X \cong [0,1)_x^k \times M_y$.

Simple calculations prove that the functions φ_ε have the following properties: If $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$, $u \in L_b^2(M, \Omega_b^{\frac{1}{2}})$, and $v \in H_b^m(M, \Omega_b^{\frac{1}{2}})$, then

(B.7)

(1) $\lim_{\varepsilon \downarrow 0} (x^{-i\tau} A x^{i\tau} \varphi_\varepsilon v, \varphi_\varepsilon u) = (N_M(A)(\tau) v, u)$ for all $\tau \in \mathbb{C}^k$;

(2) $x^{i\tau} \varphi_\varepsilon u \to 0$ in $L_b^2(X, \Omega_b^{\frac{1}{2}})$ for all $\tau \in \mathbb{R}^k$

(3) $\lim_{\varepsilon \downarrow 0} \| x^{i\tau} \varphi_\varepsilon u \|_{L_b^2} \| = \| u \|_{L_b^2}$ for all $\tau \in \mathbb{C}^k$.

For our first application of these functions, we finish with the necessity part of Theorem B.1. Assume that A is compact on L^2_b; we show that $A \in \rho\Psi_b^{-1}(X, \Omega_b^{\frac{1}{2}})$. By the usual oscillatory testing argument for pseudodifferential operators on closed manifolds, we know that $A \in \Psi_b^{-1}(X, \Omega_b^{\frac{1}{2}})$. To see that A vanishes at ∂X, it suffices to show that $N_H(A)(\tau)u = 0$ for all $H \in M_1(X)$, $u \in L^2_b(H, \Omega_b^{\frac{1}{2}})$, and $\tau \in \mathbb{R}$. Since $N_H(A)(\tau)$ is a holomorphic family, this implies that $N_H(A)(\tau) = 0$ for all $\tau \in \mathbb{C}$. Thus, let $u \in L^2_b(H, \Omega_b^{\frac{1}{2}})$ and $\tau \in \mathbb{R}$ and write $X \cong [0,1)_x \times H$ near H, where x is the fixed boundary defining function for H. With φ_ε defined as in (B.6) for $k = 1$, by (3) of (B.7), $x^{i\tau}\varphi_\varepsilon u$ is a bounded family and thus, as A is compact on L^2_b, $A(x^{i\tau}\varphi_\varepsilon u)$ converges in L^2_b. Then (2) of (B.7) implies that $A(x^{i\tau}\varphi_\varepsilon u)$ must converge to 0, therefore (1) of (B.7) implies that

$$0 = \lim_{\varepsilon \downarrow 0} \|A(x^{i\tau}\varphi_\varepsilon u)\|_{L^2_b} = \lim_{\varepsilon \downarrow 0} (\, A^*A(x^{i\tau}\varphi_\varepsilon u), x^{i\tau}\varphi_\varepsilon u \,) = (\, N_H(A^*A)(\tau)u, u \,)$$
$$= \|N_H(A)(\tau)u\|_{L^2_b}.$$

The next three lemmas constitute the necessity proof of Theorem B.2.

Lemma B.11. *Let $A \in \Psi_b^0(X, \Omega_b^{\frac{1}{2}})$ be Fredholm on $L^2_b(X, \Omega_b^{\frac{1}{2}})$. Then for each $M \in M'(X)$, $\ker N_M(A)(\tau) = \{0\}$ for all real parameters.*

Proof. Suppose on the contrary that A is Fredholm and that for some $k \in \mathbb{N}$ and $M \in M_k(X)$, there exists a $\tau \in \mathbb{R}^k$ and a non-zero element $u \in L^2_b(M, \Omega_b^{\frac{1}{2}})$ such that $N_M(A)(\tau)u = 0$. Writing $X \cong [0,1)^k_x \times M$ near M and defining φ_ε as in (B.6), property (1) of (B.7) implies that

$$\lim_{\varepsilon \downarrow 0} \|A(x^{i\tau}\varphi_\varepsilon u)\|^2_{L^2_b} = \lim_{\varepsilon \downarrow 0} (\, A(x^{i\tau}\varphi_\varepsilon u), A(x^{i\tau}\varphi_\varepsilon u) \,)$$
$$= \lim_{\varepsilon \downarrow 0} (\, x^{-i\tau} A^*A(x^{i\tau}\varphi_\varepsilon u), \varphi_\varepsilon u \,) = (\, N_M(A^*)(\tau) N_M(A)(\tau)u, u \,) = 0.$$

Thus, $A(x^{i\tau}\varphi_\varepsilon u) \to 0$ in $L^2_b(X, \Omega_b^{\frac{1}{2}})$. Since $x^{i\tau}\varphi_\varepsilon u \to 0$ weakly in $L^2_b(X, \Omega_b^{\frac{1}{2}})$ and A is invertible up to a compact operator (as A is Fredholm) it follows that $x^{i\tau}\varphi_\varepsilon u \to 0$ strongly in $L^2_b(X, \Omega_b^{\frac{1}{2}})$. But this is impossible by property (3) in (B.7). □

Lemma B.12. *Let $A \in \Psi_b^0(X, \Omega_b^{\frac{1}{2}})$ be elliptic and suppose that for some proper face $M_0 \in M_{k_0}(X)$ and some $\tau_0 \in \mathbb{R}^{k_0}$, $N_{M_0}(A)(\tau_0)$ is not invertible. Then there exists some proper face $M \in M_k(X)$ and $\tau \in \mathbb{R}^k$ such that $\ker N_M(A)(\tau) \neq \{0\}$.*

Proof. Let $n' = \operatorname{codim} X$. Define

$$k = \max \big\{ 1 \leq \ell \leq n' \,;\ \text{there exists an } M \in M_\ell(X) \text{ and a } \tau \in \mathbb{R}^\ell$$
$$\text{such that } N_M(A)(\tau)^{-1} \text{ does not exist on } L^2_b(M, \Omega_b^{\frac{1}{2}}) \big\}.$$

The definition of k implies that there exists an $M \in M_k(X)$ and a $\tau \in \mathbb{R}^k$ such that $N_M(A)(\tau)^{-1}$ does not exist and also for every $F \in M_\ell(X)$ with $\ell \geq k+1$, $N_F(A)(\lambda)^{-1}$ exists for all $\lambda \in \mathbb{R}^\ell$. Thus, Lemma B.7 implies that $N_M(A)(\lambda)$ is a continuous family of Fredholm operators on L^2_b for $\lambda \in \mathbb{R}^k$. By Lemma B.6, $N_M(A)(\lambda)$ is invertible for $|\lambda|$ large. Since the index of any continuous family of of Fredholm operators is constant, we have $\operatorname{ind} N_M(A)(\lambda) = 0$ for all $\lambda \in \mathbb{R}^k$. In particular, for $\lambda = \tau$, $N_M(A)(\tau)$ is Fredholm and $\operatorname{ind} N_M(A)(\tau) = 0$. Since $N_M(A)(\tau)^{-1}$ does not exist, we must have $\ker(\, N_M(A)(\tau) \,) \neq \{0\}$. □

Lemma B.13. *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$ and suppose that A is Fredholm as an operator from $H_b^m(X, \Omega_b^{\frac{1}{2}})$ into $L_b^2(X, \Omega_b^{\frac{1}{2}})$. Then for each $M \in M'(X)$, $N_M(A)(\tau) : H_b^m(M, \Omega_b^{\frac{1}{2}}) \longrightarrow L_b^2(M, \Omega_b^{\frac{1}{2}})$ is invertible for all real parameters.*

Proof. We first reduce to the case $m = 0$. By Corollary B.10, there exists an elliptic operator $B \in \Psi_b^{-m}(X, \Omega_b^{\frac{1}{2}})$ such that $B : L_b^2(X, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^m(X, \Omega_b^{\frac{1}{2}})$ is an isomorphism and such that $N_M(B)(\tau) : L_b^2(M, \Omega_b^{\frac{1}{2}}) \longrightarrow H_b^m(M, \Omega_b^{\frac{1}{2}})$ is invertible for all $\tau \in \mathbb{R}^k$ and $M \in M_k(X)$. It follows that $A \circ B \in \Psi_b^0(X, \Omega_b^{\frac{1}{2}})$ is such that $AB : L_b^2(X, \Omega_b^{\frac{1}{2}}) \longrightarrow L_b^2(X, \Omega_b^{\frac{1}{2}})$ is Fredholm. Since $N_M(AB) = N_M(A)N_M(B)$ and $N_M(B)$ is invertible for all real parameters, to show that $N_M(A)$ is invertible for all real parameters, we just have to show that $N_M(AB)$ is invertible for all real parameters.

Thus, it remains to prove the following statement: If $A \in \Psi_b^0(X, \Omega_b^{\frac{1}{2}})$ is Fredholm on $L_b^2(X, \Omega_b^{\frac{1}{2}})$, then all the normal operators of A are invertible for all real parameters. But if all the normal operators of A are not invertible for all real parameters, then Lemma B.12 implies that there exists an $M \in M_k(X)$ for some $k \in \mathbb{N}$ and a $\tau \in \mathbb{R}^k$ such that $\ker N_M(A)(\tau) \neq \{0\}$. Then Lemma B.11 implies that A cannot be Fredholm on $L_b^2(X, \Omega_b^{\frac{1}{2}})$. □

Appendix C. Heat Calculus

C.1. **The heat space.** Let X be an arbitrary codimension manifold with corners. In this section, we construct the heat kernel for an element $A \in \mathrm{Diff}_b^2(X, \Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. Let $[0, \infty)_s$ be the half-line with variable s. We define

$$X_{b,H}^2 = [\,[0, \infty)_s \times X_b^2; \{0\} \times \Delta_b\,].$$

We refer to [19] and [12] for the notation and definitions of blow-up.

We define $tf = \beta^{-1}(\{0\} \times \Delta_b)$ and $tb = \overline{\beta^{-1}(\{0\} \times X_b^2 \setminus \{0\} \times \Delta_b)}$, and we call tf the 'temporal face' and tb the 'temporal boundary' respectively. To avoid new notation, we continue to denote $\beta^{-1}(\mathit{ff}(X_b^2))$ by $\mathit{ff}(X_b^2)$.

If $\mathcal{U} = \mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ is a coordinate patch on X_b^2 with $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$, then

$$[0, \infty)_s \times X_b^2 \cong [0, \infty)_s \times \mathbb{R}_y^{n,k} \times \mathbb{R}_z^n, \text{ with } \{0\} \times \Delta_b \cong \{0\} \times \mathbb{R}_y^{n,k} \times \{0\}.$$

Hence, $X_{b,H}^2 \cong H^n \times \mathbb{R}_y^{n,k}$, where $H^n = [\,[0, \infty)_s \times \mathbb{R}_z^n; \{0\} \times \{0\}]$. Now by definition of blow-up, $H^n \equiv [0, \infty)_\rho \times \mathbb{S}^{n,1}_{(\omega_0, \omega')}$, where

$$\left.\begin{array}{l} \rho = (|z|^2 + s^2)^{1/2} \\ \omega_0 = s/(|z|^2 + s^2)^{1/2} \\ \omega' = z/(|z|^2 + s^2)^{1/2} \end{array}\right\} \Longleftrightarrow \left\{ \begin{array}{l} s = \rho\, \omega_0 \\ z = \rho\, \omega'. \end{array}\right.$$

A short computation proves the following lemma [17, Lem. 6.1].

Lemma C.1. *If $0 < \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, then*

$$\beta^*\Big(\nu \left|\frac{ds}{s}\right|^{\frac{1}{2}}\Big) = \rho_{tf}^{\frac{n}{2}}\, \mu, \quad \text{where } 0 < \mu \in C^\infty(X_{b,H}^2, \Omega_b^{\frac{1}{2}}).$$

We now consider some special subspaces of $C^\infty(X^2_{b,H})$. If $X=\mathbb{R}^n$, we define $C^\infty_{\rm evn}(X^2_{b,H})$ to be the subspace of C^∞ function f on $X^2_{b,H}=H^n\times\mathbb{R}^n$ such that when written in Taylor series at tf, it has the form

$$f\sim\sum_{j=0}^\infty r^{2j}f'_j(\omega_0,\omega',x')+\sum_{j=0}^\infty r^{2j+1}f''_j(\omega_0,\omega',x'),$$

where for each j, $f'_j(\omega_0,-\omega',x')=f'_j(\omega_0,\omega',x')$; $f''_j(\omega_0,-\omega',x')=-f''_j(\omega_0,\omega',x')$. We define $C^\infty_{\rm odd}(X^2_{b,H})$ to be the subspace of C^∞ functions having the opposite parity: $f'_j(\omega_0,-\omega',x')=-f'_j(\omega_0,\omega',x')$; $f''_j(\omega_0,-\omega',x')=f''_j(\omega_0,\omega',x')$. One can check that the definition of these even and odd spaces are in fact independent of the coordinates chosen and hence are defined for any manifold with corners X.

C.2. **The heat calculus.** To serve as motivation for the general case, we consider the heat kernel for the model case of the Laplacian on $\mathbb{R}^n$:

$$h=e^{-t\Delta}\,|dt|^{\frac12}=\frac{1}{(4\pi t)^{\frac n2}}\,e^{-\frac{|z|^2}{4t}}\,|dtdzdx'|^{\frac12},\quad z=x-x'.$$

If we set $t=s^2$, then up to factors, $h=s^{-n+1}\,e^{-\frac{|z|^2}{4s^2}}\,|\frac{ds}{s}dzdx'|^{\frac12}$. Hence, lifting h to $X^2_{b,H}=H^n\times\mathbb{R}^n$, where $X=\mathbb{R}^n$ and using Lemma C.1, we obtain

$$h=\rho^{-n+1}\omega_0^{-n+1}\,e^{-\frac{|\omega'|^2}{4\omega_0^2}}\,\rho^{\frac n2}\mu=\rho^{-\frac n2+1}\omega_0^{-n+1}\,e^{-\frac{|\omega'|^2}{4\omega_0^2}}\,\mu,\quad \mu\in C^\infty(X^2_{b,H},\Omega_b^{\frac12}).$$

Observe that $h\equiv0$ at $tb=\{\omega_0=0\}$. For any manifold with corners Y and subset $\mathcal{C}\subset M_1(Y)$, the space $C^\infty_{\mathcal{C}}(Y)$ consists of those C^∞ functions which vanish in Taylor series at all boundary faces $H\notin\mathcal{C}$. Thus, $h\in\rho^{-\frac n2+1}C^\infty_{tf,\rm evn}(X^2_{b,H},\Omega_b^{\frac12})$. If $A\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$, then by Duhamel's principle, $e^{-t(\Delta+A)}=e^{-t\Delta}+A(t)$, where

$$A(t)=\int_0^t e^{-(t-u)(\Delta+A)}Ae^{-u\Delta}\,du\in t\,C^\infty([0,\infty)_t\times X^2,\Omega_b^{\frac12}(X^2)).$$

Hence,

$$\begin{aligned}e^{-t(\Delta+A)}\,|dt|^{\frac12}&\in\rho^{-\frac n2+1}C^\infty_{tf,\rm evn}(X^2_{b,H},\Omega_b^{\frac12})+t\,C^\infty([0,\infty)_t\times X^2,|dt|^{\frac12}\otimes\Omega_b^{\frac12}(X^2))\\&=\rho^{-\frac n2+1}C^\infty_{tf,\rm evn}(X^2_{b,H},\Omega_b^{\frac12})+s^3C^\infty_{\rm evn}([0,\infty)_s\times X^2,\Omega_b^{\frac12}),\end{aligned}$$

where for any manifold with corners X, we define $C^\infty_{\rm evn}([0,\infty)_s\times X^2_b,\Omega_b^{\frac12})$ and $C^\infty_{\rm odd}([0,\infty)_s\times X^2_b,\Omega_b^{\frac12})$ to be the restrictions to $[0,\infty)_s$ of the smooth functions in $C^\infty((-\infty,\infty)_s\times X^2_b,\Omega_b^{\frac12})$ that are even and odd in s, respectively.

We now define the heat spaces in the general case. Let $\mathcal{A}=\{\mathit{ff}(X^2_b),\,tf\}\subset M(X^2_{b,H})$, and $\mathcal{B}=\{\{s=0\},\,\mathit{ff}(X^2_b)\}\subset M([0,\infty)_s\times X^2_b)$. Then for each $k\in\mathbb{Z}$, we define

$$\Psi^k_H(X,\Omega_b^{\frac12})=\rho_{tf}^{-\frac n2-k-1}C^\infty_{\mathcal{A}}(X^2_{b,H},\Omega_b^{\frac12})+s^{-k+1}C^\infty_{\mathcal{B}}([0,\infty)_s\times X^2_b,\Omega_b^{\frac12}).\tag{C.1}$$

We define the even and odd heat spaces as follows: If $k\in\mathbb{Z}$ is even,

$$\Psi^k_{H,\rm evn}(X,\Omega_b^{\frac12})=\rho_{tf}^{-\frac n2-k-1}C^\infty_{\mathcal{A},\rm evn}(X^2_{b,H},\Omega_b^{\frac12})+s^{-k+1}C^\infty_{\mathcal{B},\rm evn}([0,\infty)_s\times X^2_b,\Omega_b^{\frac12});$$
$$\Psi^k_{H,\rm odd}(X,\Omega_b^{\frac12})=\rho_{tf}^{-\frac n2-k-1}C^\infty_{\mathcal{A},\rm odd}(X^2_{b,H},\Omega_b^{\frac12})+s^{-k+1}C^\infty_{\mathcal{B},\rm odd}([0,\infty)_s\times X^2_b,\Omega_b^{\frac12});$$

and if $k \in \mathbb{Z}$ is odd,

$$\Psi^k_{H,\mathrm{evn}}(X,\Omega_b^{\frac12}) = \rho_{tf}^{-\frac{n}{2}-k-1} C^\infty_{\mathcal{A},\mathrm{odd}}(X^2_{b,H},\Omega_b^{\frac12}) + s^{-k+1} C^\infty_{\mathcal{B},\mathrm{odd}}([0,\infty)_s \times X^2_b,\Omega_b^{\frac12});$$
$$\Psi^k_{H,\mathrm{odd}}(X,\Omega_b^{\frac12}) = \rho_{tf}^{-\frac{n}{2}-k-1} C^\infty_{\mathcal{A},\mathrm{evn}}(X^2_{b,H},\Omega_b^{\frac12}) + s^{-k+1} C^\infty_{\mathcal{B},\mathrm{evn}}([0,\infty)_s \times X^2_b,\Omega_b^{\frac12}).$$

We shall prove the following theorem.

Theorem C.2. *If $A \in \mathrm{Diff}^2_b(X,\Omega_b^{\frac12}) + \Psi_b^{-\infty}(X,\Omega_b^{\frac12})$ has a nonnegative principal symbol and is elliptic, then there exists a unique operator $H \in \Psi^{-2}_{H,\mathrm{evn}}(X,\Omega_b^{\frac12})$ such that*

$$(\partial_t + A)H = 0 \qquad \text{for } t > 0, \quad H|_{t=0} = \mathrm{Id}\,|dt|^{\frac12}.$$

This unique operator H is denoted by $e^{-tA}\,|dt|^{\frac12}$.

To prove this theorem, we need to discuss various properties of the heat calculus. If $t = s^2$, then by Lemma C.1, it follows that for $Q \in \rho_{tf}^{-\frac{n}{2}-k-1} C^\infty_{\mathcal{A}}(X^2_{b,H},\Omega_b^{\frac12})$,

$$t^{\frac{n+k+2}{2}} Q \in C^\infty_{\mathcal{A}}(X^2_{b,H},\beta^*(|dt|^{\frac12}\Omega_b^{\frac12}(X^2_b))\,).$$

Since $\Omega_b^{\frac12}(X^2_b)|_{\Delta_b} \equiv \Omega_{\mathrm{fibre}}({}^bTX)$, we have $t^{\frac{n+k+2}{2}} Q|_{tf} \in C^\infty_{\mathit{ff}(X^2_b)}(\mathit{tf}(X^2_{b,H}),\Omega_{\mathrm{fibre}})$ where we omit the $|dt|^{\frac12}$ factor. We denote this function by $N_k(Q)$. Note that since $N\Delta_b = {}^bTX$, by the definition of blow-up,

$$\mathit{tf}(X^2_{b,H}) = (N^+([0,\infty)_s \times \Delta_b)\setminus\{0\})/\mathbb{R}^+ = ([0,\infty)_s \times {}^bTX \setminus \{0\})/\mathbb{R}^+,$$

which is just the radial compactification of ${}^bTX$. Hence, $N_k(Q) \in \mathcal{S}({}^bTX,\Omega_{\mathrm{fibre}})$, where the right hand side is the space of smooth functions on ${}^bTX$ vanishing rapidly at infinity with all derivatives. The *normal operator*, $\widetilde{N}_k$, is the map

$$\widetilde{N}_k : \Psi^k_H(X,\Omega_b^{\frac12}) \ni Q + B \longmapsto N_k(Q) \oplus s^{k-1}\,B|_{s=0} \in \mathcal{S}({}^bTX,\Omega_{\mathrm{fibre}}) \oplus C^\infty_{\mathit{ff}}(X^2_b,\Omega_b^{\frac12}),$$

where $Q \in \rho_{tf}^{-\frac{n}{2}-k-1} C^\infty_{\mathcal{A}}(X^2_{b,H},\Omega_b^{\frac12})$ and $B \in s^{-k+1} C^\infty_{\mathcal{B}}([0,\infty)_s \times X^2_b,\Omega_b^{\frac12})$. Note that there are short exact sequences

$$0 \to \Psi^{k-1}_H(X,\Omega_b^{\frac12}) \to \Psi^k_H(X,\Omega_b^{\frac12}) \xrightarrow{\widetilde{N}_k} \mathcal{S}({}^bTX,\Omega_{\mathrm{fibre}}) \oplus C^\infty_{\mathit{ff}}(X^2_b,\Omega_b^{\frac12}) \to 0$$

and

$$0 \to \Psi^{k-1}_{H,\mathrm{evn}}(X,\Omega_b^{\frac12}) \to \Psi^k_{H,\mathrm{evn}}(X,\Omega_b^{\frac12}) \xrightarrow{\widetilde{N}_k} \mathcal{S}_{\mathrm{evn}}({}^bTX,\Omega_{\mathrm{fibre}}) \oplus C^\infty_{\mathit{ff}}(X^2_b,\Omega_b^{\frac12}) \to 0,$$

where $\mathcal{S}_{\mathrm{evn}}({}^bTX,\Omega_{\mathrm{fibre}}) \subset \mathcal{S}({}^bTX,\Omega_{\mathrm{fibre}})$ are those functions that are invariant under reflection through 0 of each fibre.

Lemma C.3. *If $Q \in \Psi^k_H(X,\Omega_b^{\frac12})$, $k \in \mathbb{Z}$, then*

$$Q : \dot{C}^\infty(X,\Omega_b^{\frac12}) \longrightarrow s^{-k-1} C^\infty_{\{s=0\}}([0,\infty)_s \times X,\Omega_b^{\frac12}); \tag{C.2}$$

and $Q \in \Psi^k_{H,\mathrm{evn}}(X,\Omega_b^{\frac12})$ if and only if

$$Q : \dot{C}^\infty(X,\Omega_b^{\frac12}) \longrightarrow s^{-k-1} C^\infty_{\mathcal{B},\mathrm{evn}}([0,\infty)_s \times X,\Omega_b^{\frac12}), \text{ if } k \text{ is even}; \tag{C.3}$$

$$Q : \dot{C}^\infty(X,\Omega_b^{\frac12}) \longrightarrow s^{-k-1} C^\infty_{\mathcal{B},\mathrm{odd}}([0,\infty)_s \times X,\Omega_b^{\frac12}), \text{ if } k \text{ is odd}. \tag{C.4}$$

Moreover, if $Q \in \Psi_H^k(X, \Omega_b^{\frac12})$ *where* $k \le -2$*, then setting* $t = s^2$ *and dropping the* $|dt|^{\frac12}$ *factor, restriction to* $t = 0$ *is well-defined:*

$$Q|_{t=0} : \dot{C}^\infty(X, \Omega_b^{\frac12}) \ni \varphi \mapsto Q\varphi|_{t=0} \in \dot{C}^\infty(X, \Omega_b^{\frac12}).$$

When $k < -2$*,* $Q|_{t=0} = 0$*, and when* $k = -2$*,* $Q|_{t=0}$ *is multiplication by the function* $\int_{\text{fibre}} N_{-2}(Q)$*, the fibre-wise integral of* $N_{-2}(Q) \in \mathcal{S}({}^bTX, \Omega_{\text{fibre}})$*.*

Proof. We may assume that $Q \in \rho_{tf}^{-\frac{n}{2}-k-1} C_{\mathcal{A}}^\infty(X_{b,H}^2, \Omega_b^{\frac12})$ since the other term of (C.1) almost by definition has the properties of the Lemma. Let $\pi_{L,b}$, respectively $\pi_{R,b}$, be the composition of the blow-down map from X_b^2 onto X^2 and the projection of X^2 onto the left, respectively right, factor of X. Let $\pi_{L,H}$ be the composition of the maps

$$X_{b,H}^2 \xrightarrow{\beta} [0,\infty)_s \times X_b^2 \xrightarrow{\text{Id}\times\pi_{L,b}} [0,\infty)_s \times X$$

and $\pi_{R,H}$ to be the composition of the maps

$$X_{b,H}^2 \xrightarrow{\beta} [0,\infty)_s \times X_b^2 \xrightarrow{\pi_2} X_b^2 \xrightarrow{\pi_{R,b}} X,$$

where π_2 is projection onto the second factor. If $\varphi \in \dot{C}^\infty(X, \Omega_b^{\frac12})$ and $0 < \nu \in C^\infty(X, \Omega_b^{\frac12})$, then it follows that

$$\left|\frac{ds}{s}\right|^{\frac12} \nu\, Q\varphi = (\pi_{L,H})_* (\pi_{L,H}^* (|\frac{ds}{s}|^{\frac12} \nu)\, Q \cdot \pi_{R,H}^* \varphi).$$

Observe that $\pi_{L,H}^*(|\frac{ds}{s}|^{\frac12}\nu)\, Q \cdot \pi_{R,H}^*\varphi \in \rho_{tf}^{-k-1} C_{tf}^\infty(X_{b,H}^2, \Omega_b)$, where we used Lemma C.1 to work out the density factor. Hence, (C.2)) follows by applying the push-forward results of [21]. For the proofs of (C.3) and (C.4), we refer the reader to [22, Lem. 7.11]. The proof that the restriction to $t = 0$ when $k = -2$ is given by $\int_{\text{fibre}} N_{-2}(Q)$ can be found in [22, p. 264]. □

Lemma C.4. *If* $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac12})$ *and* $Q \in \Psi_H^k(X, \Omega_b^{\frac12})$*, then*

(C.5) $$A \circ Q,\ Q \circ A \in s^{-k-1}\, C_{\mathcal{B}}^\infty([0,\infty)_s \times X_b^2, \Omega_b^{\frac12}).$$

If $Q \in \Psi_{H,\text{evn}}^k(X, \Omega_b^{\frac12})$*, then*

(C.6) $$\begin{aligned} &A \circ Q,\ Q \circ A \in s^{-k-1}\, C_{\mathcal{B},\text{evn}}^\infty([0,\infty)_s \times X_b^2, \Omega_b^{\frac12}), \text{ if } k \text{ is even;} \\ &A \circ Q,\ Q \circ A \in s^{-k-1}\, C_{\mathcal{B},\text{odd}}^\infty([0,\infty)_s \times X_b^2, \Omega_b^{\frac12}), \text{ if } k \text{ is odd.} \end{aligned}$$

Proof. We may assume that $Q \in \rho_{tf}^{-\frac{n}{2}-k-1} C_{\mathcal{A}}^\infty(X_{b,H}^2, \Omega_b^{\frac12})$ since the other term of (C.1) almost by definition, has the property that it, composed with an operator of order $-\infty$, satisfies the lemma. Let $Z = [\,[0,\infty)_s \times X_b^3; \{0\} \times \pi_{S,b}^* \Delta_b\,]$ and denote by β_Z its blow-down map onto $[0,\infty)_s \times X_b^3$. Let $\pi_{F,H}$, $\pi_{S,H}$, and $\pi_{C,H}$ be the following compositions:

$$\pi_{F,H} : Z \xrightarrow{\beta_Z} [0,\infty)_s \times X_b^3 \xrightarrow{\pi_2} X_b^3 \xrightarrow{\pi_{F,b}} X_b^2;$$

$$\pi_{S,H} : Z \xrightarrow{\beta_Z} [0,\infty)_s \times X_b^3 \xrightarrow{\text{Id}\times\pi_{S,b}} [0,\infty)_s \times X_b^2;$$

$$\pi_{C,H} : Z \xrightarrow{\beta_Z} [0,\infty)_s \times X_b^3 \xrightarrow{\text{Id}\times\pi_{C,b}} [0,\infty)_s \times X_b^2,$$

where $\pi_{F,b}$, $\pi_{S,b}$, and $\pi_{C,b}$ are the unique b-fibrations that give a commutative diagram [20]

$$\begin{array}{ccc} X_b^3 & \xrightarrow{\pi_{O,b}} & X_b^2 \\ \downarrow & & \downarrow \\ X^3 & \xrightarrow{\pi_O} & X^2, \end{array} \tag{C.7}$$

where the vertical arrows represent blow-down maps and where $O = F$, S, or C. If $0 < \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, then it follows that

$$\Big|\frac{ds}{s}\Big|^{\frac{1}{2}} \nu\, A \circ Q = (\pi_{C,H})_* \Big(\pi_{C,H}^* \Big(\Big|\frac{ds}{s}\Big|^{\frac{1}{2}} \nu \Big) \cdot \pi_{F,H}^* A \cdot \pi_{S,H}^* Q \Big).$$

Observe that $\pi_{C,H}^*(|\frac{ds}{s}|^{\frac{1}{2}}\nu) \cdot \pi_{F,H}^* A \cdot \pi_{S,H}^* Q \in \rho_{\mathit{ff}(Z)}^{-k-1} C^\infty_{\mathit{ff}(X_b^3),\mathit{ff}(Z)}(Z, \Omega_b)$, where we used Lemma C.1 to work out the density factor. Hence, by the push-forward results of [21], we have $A \circ Q \in s^{-k-1}\, C^\infty_{\mathcal{B}}([0,\infty)_s \times X_b^2, \Omega_b^{\frac{1}{2}})$. For the proof of (C.6), we refer the reader to [22, Lem. 7.11]. Since $Q \circ A = (A^* \circ Q^*)^*$, the operator $Q \circ A$ also satisfies the properties of the lemma. □

The proof of the following lemma can be found in [22, Lem. 7.14].

Lemma C.5. *For any $P \in \mathrm{Diff}_b^2(X, \Omega_b^{\frac{1}{2}})$ and $Q \in \Psi_H^k(X, \Omega_b^{\frac{1}{2}})$, $k \in -\mathbb{Z}$, then*

$$(\partial_t + P)Q \in \Psi_H^{k+2}(X, \Omega_b^{\frac{1}{2}})$$

and moreover, if Q is in the even calculus, then so is $(\partial_t + P)Q$.

The following lemma is the last ingredient we need to prove Theorem C.2.

Lemma C.6. *If $A \in \mathrm{Diff}_b^2(X, \Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ is elliptic with a nonnegative principal symbol, there exists a $Q \in \Psi_{H,\mathrm{evn}}^{-2}(X, \Omega_b^{\frac{1}{2}})$ such that when dropping the factor $|dt|^{\frac{1}{2}}$, $Q|_{t=0} = \mathrm{Id}$ and*

$$(\partial_t + A)Q = R \in \dot{C}^\infty(\,[0,\infty)_t; \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,).$$

Proof. Let $A = P + B$, where $P \in \mathrm{Diff}_b^2(X, \Omega_b^{\frac{1}{2}})$ is elliptic with a nonnegative principal symbol and $B \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. Our first step is to find a

$$Q_0 \in \Psi_{H,\mathrm{evn}}^{-2}(X, \Omega_b^{\frac{1}{2}}) \text{ with } Q_0|_{t=0} = \mathrm{Id} \text{ and } (\partial_t + A)Q_0 \in \Psi_{H,\mathrm{evn}}^{-1}(X, \Omega_b^{\frac{1}{2}}). \tag{C.8}$$

As in [22, Lem. 7.16] we can find a $G_0 \in \Psi_{H,\mathrm{evn}}^{-2}(X, \Omega_b^{\frac{1}{2}})$ such that $(G_0)_0 = \mathrm{Id}$ and $(\partial_t + P)G_0 \in \Psi_{H,\mathrm{evn}}^{-1}(X, \Omega_b^{\frac{1}{2}})$. If $Q_0 = G_0 - tB|dt|^{\frac{1}{2}} \in \Psi_{H,\mathrm{evn}}^{-2}(X, \Omega_b^{\frac{1}{2}})$, then $(\partial_t + A)Q_0 = (\partial_t + P)G_0 + BG_0 - B|dt|^{\frac{1}{2}} - tAB|dt|^{\frac{1}{2}}$. The first term on the right is in $\Psi_{H,\mathrm{evn}}^{-1}(X, \Omega_b^{\frac{1}{2}})$ and so is the last term. By Lemma C.4, $BG_0 \in C^\infty([0,\infty)_t \times X_b^2, |dt|^{\frac{1}{2}} \otimes \Omega_b^{\frac{1}{2}}(X_b^2))$. Since $G_0|_{t=0} = \mathrm{Id}$, it follows that $BG_0 - B|dt|^{\frac{1}{2}} \in tC^\infty([0,\infty)_t \times X_b^2, |dt|^{\frac{1}{2}} \otimes \Omega_b^{\frac{1}{2}}(X_b^2)) \subset \Psi_{H,\mathrm{evn}}^{-2}(X, \Omega_b^{\frac{1}{2}})$. This proves (C.8).

Suppose that $Q_j \in \Psi_{H,\mathrm{evn}}^{-2-j}(X, \Omega_b^{\frac{1}{2}})$, $1 \le j \le k-1$ have been found such that

$$(\partial_t + A)\Big(\sum_{j=0}^{k-1} Q_j\Big) = Q + R \in \Psi_{H,\mathrm{evn}}^{-k}(X, \Omega_b^{\frac{1}{2}}),$$

where $Q \in \rho_{tf}^{-\frac{n}{2}+k-1} C^{\infty}_{\mathcal{A}}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$ and $R \in s^{k+1} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$ are even or odd depending on whether k is even or odd. We will find a $Q_k \in \Psi^{-2-k}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$ such that $(\partial_t + A)(\sum_{j=1}^k Q_j) \in \Psi^{-k-1}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$. As in [22, Lem. 7.16] we can find a $G_k \in \Psi^{-2-k}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$ such that

$$Q + (\partial_t + P)G_k \in \Psi^{-k-1}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}}). \tag{C.9}$$

Let $F \in s^{k+3} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$ be even or odd depending on whether k is even or odd. Then observe that

$$(\partial_t + A)\Big(\sum_{j=0}^{k-1} Q_j + G_k + F\Big) = Q + R + (\partial_t + P)G_k + BG_k + \partial_t F + AF.$$

By (C.9), we have $Q + (\partial_t + P)G_k \in \Psi^{-k-1}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$. Also, note that $AF \in s^{k+3} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$ and is even or odd depending on whether k is even or odd, and by Lemma C.4, $BG_k \in s^{k+1} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$ and is even or odd depending on whether k is even or odd. Hence,

$$(\partial_t + A)\Big(\sum_{j=0}^{k-1} Q_j + G_k + F\Big) \in \Psi^{-k-1}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}}) \text{ if and only if} \tag{C.10}$$
$$R + BG_k + \partial_t F \in s^{k+2} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}}) \text{ and is even or odd with } k.$$

Now, since $R,\ BG_k \in s^{k+1} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$, we can write $R + BG_k = s^{k+1}T(s)$, where $T(s) \in C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$. We define $Q_k = G_k + F \in \Psi^{-2-k}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$, where $F = -\frac{2}{k+3} s^{k+3} T(0)$. Since $\partial_t = \frac{1}{2s}\partial_s$, one can check that $R + BG_k + \partial_t S \in s^{k+2} C^{\infty}_{\mathcal{B}}([0,\infty)_s \times X^2_b, \Omega_b^{\frac{1}{2}})$ and is even or odd with k, and so by (C.10), it follows that $(\partial_t + A)Q_k \in \Psi^{-k-1}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$. The induction step is proved and via a standard asymptotic summation argument, the lemma is proved. □

Proof of Theorem C.2: For each $M \in M(X)$ and $B \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$, we denote $N_M(B)$ by B_M. If $n' = \operatorname{codim} X$, then we shall prove the following statement by induction on $k = n',\ n'-1,\ n'-2,\ \ldots, 2,\ 1,\ 0$:

$$\begin{aligned}&\text{If } M \in M_k(X), \text{ then the heat kernel } H_M \text{ for } A_M\\ &\text{exists and } H_M \in S^0_h(\mathbb{C}^k; \Psi^{-2}_{H,\text{evn}}(M, \Omega_b^{\frac{1}{2}})).\end{aligned} \tag{C.11}$$

Setting $k = 0$ proves our theorem. If $M \in M_{n'}(X)$, then by Lemma C.6, there is a $Q \in \Psi^{-2}_{H,\text{evn}}(X, \Omega_b^{\frac{1}{2}})$ such that $(\partial_t + A)Q = R \in \dot{C}^{\infty}([0,\infty)_t; \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}))$ and $Q|_{t=0} = \text{Id}$. Hence, $(\partial_t + A_M)Q_M = R_M \in \dot{C}^{\infty}([0,\infty)_t; \Psi^{-\infty}_{\mathbb{C}^k}(M, \Omega^{\frac{1}{2}}))$. Since R_M is a smoothing operator, one can follow the argument of [22, Prop. 7.17] to show that H_M exists. One can check that $Q_M \in S^0_h(\mathbb{C}^k; \Psi^{-2}_{H,\text{evn}}(M, \Omega^{\frac{1}{2}}))$ and hence, this must also be true of H_M. Now assume that (C.11) is true for $k+1$; we will prove it for k. Fix $M \in M_k(X)$ and let ρ_M be a total boundary defining function for M.

First, we prove that given $S \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$, there exists an $R \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that

$$S + (\partial_t + A_M)R \in \rho_M \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})). \tag{C.12}$$

Indeed, let $M_1(M) = \{F_1, \ldots, F_\ell\}$ with corresponding boundary defining functions $\{x_1, \ldots, x_\ell\}$. Then to prove (C.12), we first claim that for each $j = 1, \ldots, \ell$, there exists an $R_j \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that

$$S + (\partial_t + A_M)R_j = T_j \in x_1 \cdots x_j \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})). \tag{C.13}$$

To see this, we use induction on j. Assume that $j = 1$. Since $F_1 \in M_{k+1}(X)$, by the induction hypothesis for (C.11), the heat kernel for $N_{F_1}(A_M) = A_{F_1}$ exists and is an element of $S^0_h(\mathbb{C}^{k+1}; \Psi^{-2}_{H,\mathrm{evn}}(F_1, \Omega_b^{\frac{1}{2}}))$. Hence, by Duhamel's principle we can find an $S_1 \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^{k+1}}(F_1, \Omega_b^{\frac{1}{2}}))$ such that

$$N_{F_1}(S) + (\partial_t + N_{F_1}(A_M))S_1 = 0.$$

Thus, choosing $R_1 \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that $N_{F_1}(R_1) = S_1$ proves the $j = 1$ case of (C.13). Assume (C.13) is true for j, we prove it is true for $j+1$. Indeed, by using a similar argument as we did in the $j = 1$ case, we can choose an $S_{j+1} \in x_1 \cdots x_j \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that

$$N_{F_{j+1}}(T_j) + (\partial_t + N_{F_{j+1}}(A_M))N_{F_{j+1}}(S_{j+1}) = 0.$$

So, with $R_{j+1} = R_j + S_{j+1}$, (C.13) holds for $j+1$. Setting $j = \ell$ proves (C.12).

Second, we prove there is a $G \in S^0_h(\mathbb{C}^k; \Psi^{-2}_{H,\mathrm{evn}}(M, \Omega_b^{\frac{1}{2}}))$ such that $G|_{t=0} = \mathrm{Id}$ and

$$(\partial_t + A_M)G = R \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})). \tag{C.14}$$

By Lemma C.6, there is a $Q \in \Psi^{-2}_{H,\mathrm{evn}}(X, \Omega_b^{\frac{1}{2}}))$ such that $Q|_{t=0} = \mathrm{Id}$ and $(\partial_t + A)Q = S \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_b(X, \Omega_b^{\frac{1}{2}}))$. Taking normal operators of this equation, we obtain $(\partial_t + A_M)Q_M = R_M \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$. By (C.12), for some $R_1 \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$, we have

$$(\partial_t + A_M)(Q_M + R_1) = \\ R_M + (\partial_t + A)R_1 = \rho_M T_1 \in \rho_M \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})).$$

Assume by induction that there are $R_1, \ldots, R_\ell \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that

$$(\partial_t + A_M)\Big(Q_M + \sum_{j=1}^{\ell} \rho_M^{j-1} R_j\Big) = \rho_M^\ell T_\ell \in \rho_M^\ell \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})).$$

By (C.12), there is an $R_{\ell+1} \in \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ such that

$$T_\ell + (\partial_t + \rho_M^{-\ell} A_M \rho_M^\ell)R_{\ell+1} = \rho_M T_{\ell+1} \in \rho_M \dot{C}^\infty([0,\infty)_t; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}})).$$

Thus,

$$(\partial_t + A_M)\Big(Q_M + \sum_{j=1}^{\ell+1} \rho_M^{j-1} R_j\Big) = \rho_M^\ell T_\ell + (\partial_t + A_M)\rho_M^\ell R_{\ell+1} =$$

$$\rho_M^\ell(\rho_M T_{\ell+1}) \in \rho_M^{\ell+1} \dot{C}^\infty(\,[0,\infty)_t; \Psi_{b,\mathbb{C}^k}^{-\infty}(M, \Omega_b^{\frac{1}{2}})\,).$$

Setting G to be the asymptotically sum of $Q_M + \sum_{j=1}^\infty \rho_M^{j-1} R_j$ proves (C.14).

Since R in (C.14) is a smoothing operator, one can now follow the argument found in [22, Prop. 7.17] to show the existence of H_M. Thus, (C.11) is proved and setting $k = 0$ proves our theorem. Uniqueness is proved in [22, p. 271].

C.3. **Asymptotics of the heat kernel.** A sector $\Lambda \subset \mathbb{C}$ is a closed angle of $\mathbb{C}$. Given a Frechét space $\mathcal{F}$, the space $S^k(\Lambda, \mathcal{F})$ denotes the space of $\mathcal{F}$-valued symbols of order k on Λ.

Lemma C.7. *Let $A \in \mathrm{Diff}_b^2(X, \Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ have a nonnegative principal symbol and be elliptic, and let Λ be a sector of $\mathbb{C}$ such that for some $0 < \delta < \pi/2$, $|\arg(\lambda)| \geq \delta$ for all $\lambda \in \Lambda$ Then given any $a > 0$, there is an $r > 0$ such that $(A - \lambda)^{-1} \in S^0(\Lambda; \Psi_b^{-2,a}(X, \Omega_b^{\frac{1}{2}})\,)$ for $\lambda \in \Lambda$ where $|\lambda| \geq r$.*

Proof. For each $M \in M(X)$ and $B \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, we will denote $N_M(B)$ by B_M. If $n' = \operatorname{codim} X$, then we prove the following statement by induction on $k = n'$, $n' - 1$, $n' - 2$, $\ldots, 2, 1, 0$:

$$\begin{aligned}&\text{If } M \in M_k(X), \text{ then given any } a > 0, \text{ andhorizontal strip } S \subset \mathbb{C}^k,\\ &\text{there is an } r > 0 \text{ such that } (A_M - \lambda)^{-1} \in S^0(\Lambda; \Psi_{b,S}^{-2,a}(M, \Omega_b^{\frac{1}{2}})\,)\\ &\text{for } \lambda \in \Lambda \text{ where } |\lambda| \geq r.\end{aligned} \tag{C.15}$$

Setting $k = 0$ proves our lemma. Assume that $M \in M_{n'}(X)$. By the argument found in [22, p. 284–86], there is a $G \in S^0(\Lambda; \Psi_b^{-2}(X, \Omega_b^{\frac{1}{2}})\,)$ and an $R \in S^{-\infty}(\Lambda; \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,)$ such that $(A - \lambda)G = \mathrm{Id} - R$. Hence, $(A_M - \lambda)G_M = \mathrm{Id} - R_M$, where $G_M \in S^0(\Lambda; \Psi_{\mathbb{C}^k}^{-2}(M, \Omega^{\frac{1}{2}})\,)$ and $R_M \in S^{-\infty}(\Lambda; \Psi_{\mathbb{C}^k}^{-\infty}(M, \Omega^{\frac{1}{2}})\,)$. Since R_M is a smoothing operator, one can use the argument found in [22, p. 284] to prove (C.15) for $k = n'$. Assume that (C.15) holds for $k + 1$; we prove it for k. Fix a horizontal strip $S \subset \mathbb{C}^k$ and fix $M \in M_k(X)$. Let ρ_M be a total boundary defining function for M.

We first show that given $a > 0$, there is a $Q \in S^0(\Lambda; \Psi_{b,S}^{-2,a}(M, \Omega_b^{\frac{1}{2}})\,)$ and $T \in S^{-\infty}(\Lambda; \Psi_{b,S}^{-\infty,a}(M, \Omega_b^{\frac{1}{2}})\,)$ such that for some $r > 0$,

$$(A_M - \lambda)Q = \mathrm{Id} - \rho_M T \qquad \text{for all } \lambda \in \Lambda \text{ with } |\lambda| \geq r. \tag{C.16}$$

Indeed, let $M_1(M) = \{F_1, \ldots, F_\ell\}$ with corresponding boundary defining functions $\{x_1, \ldots, x_\ell\}$. Then to prove (C.16), we first claim that given $a > 0$, for each $j = 1, \ldots, \ell$, there is a $Q_j \in S^0(\Lambda; \Psi_{b,S}^{-2,a}(M, \Omega_b^{\frac{1}{2}})\,)$ and an $r_j > 0$ such that for all $\lambda \in \Lambda$, $|\lambda| \geq r_j$,

$$(A_M - \lambda)Q_j = \mathrm{Id} - T_j, \quad T_j \in x_1 \cdots x_j S^{-\infty}(\Lambda; \Psi_{b,S}^{-\infty,a}(M, \Omega_b^{\frac{1}{2}})\,). \tag{C.17}$$

We use induction on j. Assume that $j = 1$. By the argument found in [22, p. 284-86], there is a $G \in S^0(\Lambda; \Psi_b^{-2}(X, \Omega_b^{\frac{1}{2}})\,)$ and $T \in S^{-\infty}(\Lambda; \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,)$

such that $(A-\lambda)G = \mathrm{Id} - T$. Hence, $(A_M - \lambda)G_M = \mathrm{Id} - T_M$, where $G_M \in S^0(\Lambda; \Psi^{-2}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$ and $T_M \in S^{-\infty}(\Lambda; \Psi^{-\infty}_{b,\mathbb{C}^k}(M, \Omega_b^{\frac{1}{2}}))$. Let $S_a = (a-2, a+2) \times S$. Then, as $F_1 \in M_{k+1}(X)$ and $N_{F_1}(A)_M = A_{F_1}$, by the induction hypothesis for (C.15), for some $r_1 > 0$, $(N_{F_1}(A_M) - \lambda)^{-1} \in S^0(\Lambda; \Psi^{-2,a+1}_{b,S_a}(F_1, \Omega_b^{\frac{1}{2}}))$ for all $\lambda \in \Lambda$ with $|\lambda| \geq r_1$. Hence, we can choose $S_1 \in S^{-\infty}(\Lambda; \Psi^{-\infty,a+1}_{b,S}(M, \Omega_b^{\frac{1}{2}}))$ such that $N_{F_1}(S_1) = (N_{F_1}(A_M) - \lambda)^{-1} N_{F_1}(T_M)$. Set $Q_1 = G_M + S_1$. Then $(A_M - \lambda)Q_1 = \mathrm{Id} - (T_M - (A_M - \lambda)S_1)$. Observe that $N_{F_1}(T_M - (A_M - \lambda)S_1) = 0$ and so (C.17) is proved for $j = 1$. Assume that (C.17) holds for j; we'll prove it is true for $j+1$. Indeed, by using a similar argument as we did when $j = 1$, for some $r_{j+1} \geq 0$, we can choose an $S_{j+1} \in x_1 \cdots x_j S^{-\infty}(\Lambda; \Psi^{-\infty,a+1}_{b,S}(M, \Omega_b^{\frac{1}{2}}))$ such that $N_{F_{j+1}}(S_{j+1}) = (N_{F_{j+1}}(A_M) - \lambda)^{-1} N_{F_{j+1}}(T_j)$ for all $\lambda \in \Lambda$, $|\lambda| \geq r_{j+1}$. Set $Q_{j+1} = Q_j + S_{j+1}$. Then it follows that (C.17) holds for $j+1$. Now setting $j = \ell$ in (C.17) proves (C.16).

Let $a > 0$ and choose $p \in \mathbb{N}$ such that $p > a$. Then by (C.16), there is a $Q \in S^0(\Lambda; \Psi^{-2,2p+1}_{b,S}(M, \Omega_b^{\frac{1}{2}}))$ and $T \in S^{-\infty}(\Lambda; \Psi^{-\infty,2p+1}_{b,S}(M, \Omega_b^{\frac{1}{2}}))$ such that for some $r > 0$, (C.16) holds. Then by the composition property (A.3), it follows that $S' = (\rho_M T)^{2p+1} \in \rho_M^{2p+1} S^{-\infty}(\Lambda; \Psi^{-\infty,2p+1}_{b,S}(M, \Omega_b^{\frac{1}{2}})) \subset S^{-\infty}(\Lambda; \Psi^{-\infty,p}_{S}(M, \Omega_b^{\frac{1}{2}}))$. Hence, if $S = \sum_{j=1}^{2p} (\rho_M T)^j \in \rho_M S^{-\infty}(\Lambda; \Psi^{-\infty,2p+1}_{b,S}(M, \Omega_b^{\frac{1}{2}}))$, then

$$(A_M - \lambda)(Q \circ (\mathrm{Id} + S)) = \mathrm{Id} - S'.$$

Now one can use the argument found in [22, p. 284] to invert $\mathrm{Id} - S'$ and prove (C.15) for k. □

Remark C.8. Using [22, p. 286], this same proof can be used to show that $(A - \lambda)^{-1} \in S^{-1}(\Lambda; \Psi^{0,a}_b(X, \Omega_b^{\frac{1}{2}}))$ for $\lambda \in \Lambda$ where $|\lambda| \geq r$.

Proposition C.9. *If $A \in \mathrm{Diff}^2_b(X, \Omega_b^{\frac{1}{2}}) + \Psi^{-\infty}_b(X, \Omega_b^{\frac{1}{2}})$ has a nonnegative principal symbol, is self-adjoint and Fredholm, then*

$$e^{-tA} = \sum_{\text{finite}} e^{-t\lambda_j} \Pi_j + R(t),$$

where the Π_j are finite rank projections onto the eigenspaces of A less than some positive real number and there exists an $\varepsilon > 0$ such that as $t \to \infty$, $R(t) \to 0$ exponentially, with all t derivatives, with values in $\Psi^{-\infty,\varepsilon}_b(X, \Omega_b^{\frac{1}{2}})$.

Proof. We know that $e^{-tA} = \frac{i}{2\pi} \int_\Upsilon e^{-t\lambda} (A - \lambda)^{-1} \, d\lambda$, where Υ is a contour of the form $\Upsilon = \Upsilon_a = a + \{\lambda \in \mathbb{C}\,;\, \arg(\lambda) = \pm\pi/4\}$, where $a < 0$. Hence by Theorem B.9 and Lemma C.7, by shifting the contour Υ to $\Upsilon' = \Upsilon_{a'}$, where $a' > 0$, we can write

$$e^{-tA} = \sum_{\text{finite}} e^{-t\lambda_j} \Pi_j + R(t),$$

where the Π_j are the finite rank projections onto the eigenspaces of A with eigenvalues less than a' and $R(t) = \frac{i}{2\pi} \int_{\Upsilon'} e^{-t\lambda} (A - \lambda)^{-1} \, d\lambda$ where for some $\varepsilon > 0$, $(A - \lambda)^{-1}$ is uniformly an element of $\Psi^{-2,\varepsilon}_b(X, \Omega_b^{\frac{1}{2}})$ for all $\lambda \in \Upsilon'$. Observe that for any j, integration by parts gives

$$R(t) = \frac{i}{2\pi} \cdot j! \cdot t^{-j} \int_{\Upsilon'} e^{-t\lambda} (A - \lambda)^{-j-1} \, d\lambda.$$

Since $(A-\lambda)^{-j-1}$ is uniformly an element of $\Psi_b^{-2(j+1),\varepsilon}(X,\Omega_b^{\frac{1}{2}})$ for all $\lambda \in \Upsilon'$, and j is arbitrary, it follows that $R(t) \in \Psi_b^{-\infty,\varepsilon}(X,\Omega_b^{\frac{1}{2}})$. □

For $s > 0$, observe that $X^2_{b,H} \equiv (0,\infty)_s \times X^2_b$. Let $Q \in \Psi^k_H(X,\Omega_b^{\frac{1}{2}})$. Then for $s > 0$, we denote the restriction of Q to $[0,\infty)_s \times \Delta_b$ by $Q|_{\Delta_b}$. The proof of the following lemma is a consequence of the definition of the heat spaces, the fact that $\Delta_b \cong X$, and Theorem C.2.

Lemma C.10. *Let $Q \in \Psi^k_H(X,\Omega_b^{\frac{1}{2}})$. Then as $t \downarrow 0$, $Q|_{\Delta_b} \sim \sum_{j=0}^\infty t^{\frac{j-n-k-2}{2}} \gamma_j(x)$ where $\gamma_j \in C^\infty(X,\Omega_b)$. In particular, if $A \in \mathrm{Diff}^2_b(X,\Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$ has a nonnegative principal symbol and is elliptic, then*

$$e^{-tA}|_{\Delta_b} \sim \sum_{j=0}^\infty t^{\frac{j-n}{2}} \gamma_j(x), \quad \textit{where } \gamma_j \in C^\infty(X,\Omega_b).$$

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