THE INDEX OF b-PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH CORNERS

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Abstract. On compact manifolds with corners of arbitrary codimension, we characterize those 'multi-cylindrical end' (or b- type) pseudodifferential operators that are Fredholm on weighted Sobolev spaces and we compute their indices. The index formula contains the usual interior term manufactured from the local symbols of the operator and also contains boundary correction terms corresponding to eta-type invariants of the induced operators on the boundary faces.

1. Introduction

The purpose of this paper is to characterize and give an index formula for b-pseudodifferential operators on compact manifolds with corners (of arbitrary codimension) that are Fredholm on weighted Sobolev space domains. The main application of these results is to study the Fredholm properties of perturbed b-differential operators (e.g. Dirac-type operators) on weighted Sobolev spaces.

The origin of these considerations began with Melrose's interpretation of the Atiyah-Patodi-Singer index formula in the framework of his b-geometry, cf. [1, 2, 24], which we now review. Let X be an even-dimensional compact manifold with boundary. Let g be a Riemannian metric on the interior of X such that in some collar $X \cong [0,1)_\rho \times Y$ of the boundary $Y = \partial X$, the metric takes the form $g = (\frac{d\rho}{\rho})^2 + h_\rho$, where h_ρ is a smoothly varying family of metrics on Y. Such a metric is called an exact b-metric and it gives X the geometric structure of a manifold with an 'asymptotically' cylindrical end. Indeed, the change of variables $t = \log \rho$ turns the interior of the collar $[0,1)_\rho \times Y$ into $(-\infty, 0)_t \times Y$ and the metric $g = \left(\frac{d\rho}{\rho}\right)^2 + h_\rho$ into the cylindrical type metric $g = dt^2 + h_{e^t}$. Note that as $t \to -\infty$, the metric g approaches the product cylindrical metric $dt^2 + h_0$ exponentially. In this sense, X has an asymptotically cylindrical end.

Let $\eth^+ \colon C^\infty(X, E^+) \longrightarrow C^\infty(X, E^-)$ be a Dirac operator associated to g, where E^+ and E^- are the chiral parts of a Clifford bundle over X, and let $\eth_Y$ denote the Dirac operator on Y induced by $\eth^+$. Then (see [24]) the kernel of $\eth_Y$ is exactly the obstruction to $\eth^+$ being Fredholm on its natural domain. More precisely, $\eth_Y$ is invertible if and only if $\eth^+ \colon H^1_b(X, E^+) \longrightarrow L^2_b(X, E^-)$ is Fredholm, where $H^1_b(X, E^+)$ is the natural Sobolev space domain of $\eth^+$, in which case the following

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formula for the index holds:

$$\operatorname{ind} \eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_Y), \tag{1.1}$$

where AS is the Atiyah-Singer density and $\eta(\eth_Y)$ is the eta invariant of $\eth_Y$. The integral ${}^b\!\!\int_X$ AS represents a 'regularized' integral since AS is not integrable on X. The formula (1.1) generalizes the work of Atiyah, Patodi, and Singer in the seminal paper [1] for the product case near the boundary.

On weighted Sobolev spaces it turns out that there are no obstructions to making the Dirac operator Fredholm [24]: Whether or not $\eth_Y$ is invertible, there is an $\varepsilon > 0$ such that for all real numbers α with $0 < |\alpha| < \varepsilon$,

$$\eth^+ \colon \rho^\alpha H^1_b(X, E^+) \longrightarrow \rho^\alpha L^2_b(X, E^-)$$

is Fredholm and if $\operatorname{ind}_\alpha \eth^+$ denotes the index of this map, then

$$\operatorname{ind}_\alpha \eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\left[\eta(\eth_Y) + \operatorname{sgn}\alpha \cdot \dim\ker \eth_Y\right]. \tag{1.2}$$

The special case of $\alpha > 0$ gives the 'extended' L^2-index theorem. Generalizations of the L^2-index theorem have been investigated by many authors, see for instance [7], [38], [3], and for singular manifolds in [5], [36], [9], [4], [8].

The elliptic theory of Dirac operators, and totally characteristic (or b-type) differential operators more generally, can be investigated through appropriate spaces of pseudodifferential operators. Such operators have been studied by many authors; to name a few, Egorov and Schulze [6], Melrose [24], Melrose and Mendoza [26], Plamenevskij [32], Rempel and Schulze [33], and Schulze [35]. The operators we focus on are the b-pseudodifferential operators of Melrose. These operators extend to manifolds with corners of arbitrary codimension [29], [28].

Let $A \in \Psi^m_b(X, E, F)$ be an elliptic b-pseudodifferential operator of order $m \in \mathbb{R}^+$ acting between sections of vector bundles over our manifold with boundary X. The Fredholm condition on such an operator on its natural Sobolev space domain is well-known (cf. Kondrat′ev [16]), $A \colon H^m_b(X, E) \to L^2_b(X, F)$ is Fredholm if and only if its normal operator $N_Y(A)(\tau)$ is invertible for each $\tau \in \mathbb{R}$, where $N_Y(A)(\tau)$ is a family of pseudodifferential operators on the boundary Y depending holomorphically on $\tau \in \mathbb{C}$. Using this result, one can show that there are no obstructions to making an arbitrary elliptic operator $A \in \Psi^m_b(X, E, F)$ Fredholm on weighted Sobolev spaces [24]: If $A_\alpha = \rho^{-\alpha} A \rho^\alpha$, then there is an $\varepsilon > 0$ such that for all $\alpha \in \mathbb{R}$ with $0 < |\alpha| < \varepsilon$, $N_Y(A_\alpha)(\tau)$ is invertible for all $\tau \in \mathbb{R}$, which implies that $A_\alpha \colon H^m_b(X, E) \longrightarrow L^2_b(X, F)$ is Fredholm. That is, for all α with $0 < |\alpha| < \varepsilon$,

$$A \colon \rho^\alpha H^m_b(X, E) \longrightarrow \rho^\alpha L^2_b(X, F) \tag{1.3}$$

is Fredholm. If $\operatorname{ind}_\alpha A$ denotes the index of this map, then Piazza [31] gives the following generalization of the APS formula (1.1):

$$\operatorname{ind}_\alpha A = {}^b\!\!\int_X \omega_{A_\alpha} - \frac{1}{2}\eta(N_Y(A_\alpha)). \tag{1.4}$$

Here ω_{A_α} is the 'analytic' Atiyah-Singer density of A_α manufactured from finitely many homogeneous terms in the local symbol expansions of A_α, and is defined as the constant term in the difference of the small time fiberwise trace asymptotics of the heat operators for $A_\alpha^* A_\alpha$ and $A_\alpha A_\alpha^*$. The term $\eta(N_Y(A_\alpha)(\tau))$ is related to the eta invariant introduced in [25], and is a certain regularization of the integral

$$\eta(N_Y(A_\alpha)) \approx 2 \int_{\mathbb{R}} \mathrm{Tr}(\, D_\tau N_Y(A_\alpha)(\tau)\, N_Y(A_\alpha)(\tau)^{-1})\, đ\tau,$$

where $đ\tau = d\tau/2\pi$. This eta term represents a 'winding number' of $N_Y(A_\alpha)(\tau)$. A key difference between the formulas (1.2) and (1.4) is that each term in (1.2) is *constant* under small variations in α, whereas each term in (1.4) *changes* under small variations in α, but where the difference of the terms is of course constant. We remark that in [19, 21], the index formula (1.4) is extended to b-pseudodifferential operators on compact manifold with corners; Lauter and Moroianu [18] give a cusp version in the spirit of Melrose and Nistor [27]. However, there are some drawbacks to the formulas in [19, 21], [18]. The first is that the operator (1.3) is assumed Fredholm without conditions guaranteeing this, and second the formula (1.4) still has varying terms with the weight α.

A couple related questions arise: Is there a condition characterizing those elliptic b-pseudodifferential operators on a compact manifold with corners (of arbitrary codimension) to be Fredholm on weighted Sobolev spaces and is there a formula with nonvarying terms (corresponding to (1.2)) for the index of such b-pseudodifferential operators on weighted spaces? The answer to the first question is given in Theorem 1.1 below and the second in Theorem 1.2. Based on the manifold with boundary case, it might seem that an elliptic b-pseudodifferential operator on a manifold with corners could always be made Fredholm by considering it on weighted Sobolev spaces. Perhaps surprisingly, this is not true as we now describe.

Let X be a compact manifold with corners and let $H_1, \ldots, H_N$ be any fixed ordering of its boundary hypersurfaces with corresponding defining functions $\rho_1, \ldots, \rho_N$. Then $\rho = \rho_1 \cdots \rho_N$ is a total boundary defining function of X. A codimension k face of X is a nonempty component of the intersection of k hypersurfaces of X; the set of such faces is denoted by $M_k(X)$. A multi-index α is just a set of N real numbers $\alpha = \{\alpha_1, \ldots, \alpha_N\}$, in which case we define $\rho^\alpha = \rho_1^{\alpha_1} \cdots \rho_N^{\alpha_N}$. The notation $\alpha = \{\alpha_H\,;\, H \in M_1(X)\}$ is also common. Manifolds with corners and b-pseudodifferential operators are reviewed in Section 2. Let $A \in \Psi_b^m(X, E, F)$ and $M \in M_k(X)$. Then there is a naturally induced entire family of operators, called the normal operator of A at M, $N_M(A)(\tau) \in \Psi_b^m(M, E, F)$ depending on $\tau \in \mathbb{C}^k$, where E and F here denote the restrictions of E and F, respectively, to M. It is well-known that these families at all codimension one faces of X determine the Fredholm properties of A on (unweighted) Sobolev spaces if A is elliptic (Theorem 2.3). On weighted Sobolev spaces, those normal operators on the codimension two faces are the determining factors.

Theorem 1.1. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$. Then there is an $\varepsilon > 0$ such that for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$A\colon \rho^\alpha H_b^m(X, E) \longrightarrow \rho^\alpha L_b^2(X, F) \tag{1.5}$$

is Fredholm, if and only if, A is elliptic and $N_M(A)(\tau)\colon H_b^m(M, E) \longrightarrow L_b^2(M, F)$ is invertible for each $M \in M_2(X)$ and $\tau \in \mathbb{R}^2$.

Here, $0 < |\alpha| < \varepsilon$ means $0 < |\alpha_H| < \varepsilon$ for each $H \in M_1(X)$. For Dirac-type operators, Theorem 1.1 is quite explicit (see Theorem 3.8). Now that we have characterized such Fredholm operators, what is the formula for the index of a b-pseudodifferential that is Fredholm on weighted Sobolev spaces? The second main result of this paper expresses the formula (1.4) in terms of unperturbed data, that is, each term is *constant* under small variations in α, and we generalize this index formula to operators on manifolds with corners of arbitrary codimension.

Theorem 1.2. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$, and suppose that the conditions of Theorem 1.1 are satisfied and let $\operatorname{ind}_\alpha A$ denote the index of the map (1.5). Then,*

$$\operatorname{ind}_\alpha A = {}^b\!\!\int_X \omega_A - \frac{1}{2}\left\{{}^b\tilde{\eta} + \operatorname{sgn}(\alpha)\cdot \operatorname{rk}(N(A)) + \beta\right\} - \frac{1}{2}{}^b\eta_C.$$

We now give a short description of each term in this formula; Section 3.2 contains the precise descriptions. Just as in the manifold with boundary case, ω_A is the 'analytic' Atiyah-Singer density of A manufactured from the local symbols of A. The term ${}^b\tilde{\eta}$ is the sum over each hypersurface of X of regularized 'b-'eta invariants; these invariants reduce to the usual ones in the Dirac case (Theorem 3.8). The term $\operatorname{sgn}(\alpha)\cdot\operatorname{rk}(N(A))$ is the sum over all the ranks of the poles of the inverses of the normal operators of A at the hypersurfaces with $\pm$ signs at a hypersurface H determined by $\operatorname{sgn}(\alpha_H)$. The term β depends on the principal symbol of A restricted to the hypersurfaces and on the poles of the inverses of the normal operators of A at the hypersurfaces. β is in general nonzero, but vanishes in case A is a b-differential operator modulo a term of lower order. Finally, ${}^b\eta_C$ represents the sum of all the b-eta invariants at all the codimension k faces of X with $k \geq 2$.

Many authors have worked on similar index theorems on manifolds with corners; we only mention a few whose work is most directly related. Already mentioned are the works [19, 21] and the cusp version of Lauter and Moroianu [18]. Müller [30] gives an index formula for Dirac operators on manifolds with corners up to codimension two under the assumptions of Theorem 1.2, which for the case of Dirac operators are just that the induced Dirac operators on the corners are invertible. Without this nondegeneracy assumption, a signature formula was proved in [15] by Hassell, Mazzeo, and Melrose, using the techniques of analytic surgery [14], and Salomonsen [34] gives a similar formula by considering a related problem on a manifold with wedge singularities. Finally, in joint work with Melrose [22], we give an index formula for Dirac operators perturbed by b-pseudodifferential operators of order $-\infty$; in this case all the terms in Theorem 1.2 involve only the Dirac operator and certain Lagrangian subspaces of the null spaces of the corner Dirac operators.

In Section 2 we give a very 'hands-on' description of the b-calculus on manifolds with corners, avoiding the machinery of 'blow-ups'. In Section 3 we prove sufficiency in the Characterization Theorem 1.1. We also state the Index Theorem 3.4 with precise descriptions of each term in the formula and we give various applications of the index formula; for instance, we give conditions under which the β term vanishes and we also apply our formula to Dirac operators. In Section 4 we prove our Index Theorem. Here we rely heavily on the structure and asymptotic properties of the heat kernel proved in the earlier paper [21] to carefully analyze the b-eta invariants. These asymptotic properties are proved using techniques similar to Grubb and Seeley, cf. [37, 13, 12]. Lastly, in Section 5 we prove necessity Theorem 1.1.

Finally, I want to take this opportunity to thank Richard Melrose for his support throughout the years and for many helpful discussions.

2. The b-calculus on manifolds with corners

We review some topics on the b-calculus on manifolds with corners. For notational simplicity we restrict our attention to b-pseudodifferential operators acting on functions, but of course everything can be done with vector bundles too. For more on these topics, see [23], [24], [11], or the appendices of [29] and [28].

2.1. b-pseudodifferential operators.

A compact topological space X is called an n-dimensional compact *manifold with corners* if there is an n-dimensional manifold without boundary $\widetilde{X}$ containing X and smooth functions $\rho_1, \ldots, \rho_N$ on $\widetilde{X}$ such that $X = \{\rho_i \geq 0\,;\, i = 1, \ldots, N\}$, $d\rho_i \neq 0$ on $H_i := \{\rho_i = 0\}$, and such that each boundary hypersurface H_i is connected. We shall fix the boundary defining functions $\{\rho_i\}$ once and for all throughout this paper. Let p be a point in X and suppose that $\rho_{i_1}, \ldots, \rho_{i_\kappa}$ are all the boundary defining functions that vanish at p. Because $d\rho_i \neq 0$ on H_i, if $x = (\rho_{i_1}, \ldots, \rho_{i_\kappa})$, then on a neighborhood of p we have a decomposition

$$X \cong [0,\epsilon)^\kappa_x \times \mathbb{R}^{n-\kappa}, \qquad \text{some } \epsilon > 0, \tag{2.1}$$

whence the name manifold 'with corners'. A *codimension k face*, $k \geq 1$, of X is a nonempty connected component of the intersection of k hypersurfaces of X. The set of such faces is denoted by $M_k(X)$ and the collection of all faces is denoted by $M'(X)$. In particular, a hypersurface is just a codimension one face of X. The largest k where $M_k(X)$ is nonempty is called the *codimension* of X.

A *b-measure* is a density on X of the form $\rho^{-1} \times$ a smooth nonvanishing density on X, where $\rho = \rho_1 \cdots \rho_N$ is the product of all the boundary defining functions for X, called a *total boundary defining function*. Henceforth we fix a b-measure $\mathfrak{m}$.

We now describe the small calculus. Let $\dot{C}^\infty(X)$ denote the space of smooth functions on X that vanish to infinite order at the boundary of X. We first define the space $\Psi_b^{-\infty}(X)$ as operators R on $\dot{C}^\infty(X)$ described in local coordinates as follows. Let $\mathcal{U}$ and $\mathcal{U}'$ be coordinate patches on X of the form given in (2.1). (We allow $\kappa = 0$ in (2.1); this just means that the coordinate patch is located in the interior of X.) Let $x = (x_1, \ldots, x_\ell)$ denote those boundary defining functions, if any, that are common to both coordinate patches $\mathcal{U}$ and $\mathcal{U}'$, so that

$$\mathcal{U} = [0,\epsilon)^\ell_x \times \mathcal{V}, \quad \mathcal{U}' = [0,\epsilon)^\ell_x \times \mathcal{V}', \tag{2.2}$$

where $\mathcal{V}$ and $\mathcal{V}'$ are also of the form (2.1) and where ℓ may be zero. Let y denote the coordinates on $\mathcal{V}$ and y' the coordinates on $\mathcal{V}'$. Then given any open set $\mathcal{W}$ with compact closure in $\mathcal{U}'$, for any $u \in \dot{C}^\infty(X)$ having support in $\mathcal{W}$, the restriction of $R\,u$ to $\mathcal{U}$ is of the form

$$R\,u = \int_{\mathcal{U}'} k\Big(x, \frac{x}{x'}, y, y'\Big)\, u(x', y')\, \mathfrak{m}(x', y'), \tag{2.3}$$

where $x/x' = (x_1/x'_1, \ldots, x_\ell/x'_\ell)$ and where $k(x, z, y, y')$ has the following regularity properties: It is smooth in all variables, vanishes to infinite order with all derivatives at any $z_i = 0$ or as $z_i \to \infty$, and vanishes to infinite order at any $y_i = 0$ or $y'_i = 0$ if these sets represent boundary hypersurfaces of X.

We now consider the general case. An element $A \in \Psi_b^m(X)$, $m \in \mathbb{R}$, is an operator on $\dot{C}^\infty(X)$ described in local coordinates as follows. Let $\mathcal{U}$ and $\mathcal{U}'$ be coordinate patches of the form (2.2) and let $\mathcal{W}$ be an open set with compact closure in $\mathcal{U}'$. If $\mathcal{U}$ and $\mathcal{U}'$ are disjoint, then given any $u \in \dot{C}^\infty(X)$ having support in $\mathcal{W}$, the restriction of Au to $\mathcal{U}$ is given by an operator $R \in \Psi_b^{-\infty}(X)$ as in (2.3). Suppose now that $\mathcal{U} = \mathcal{U}'$. In this case, $\mathcal{U} = [0,\epsilon)_x^\ell \times \mathbb{R}_y^{n-\ell}$. Then there is a function $a(x,y,\xi)$, smooth in $(x,y) \in \mathcal{U}$ and a classical symbol of order m in ξ, such that given any $u \in \dot{C}^\infty(X)$ having support in $\mathcal{W}$, we have

$$Au = \int_{\mathbb{R}^n} x^{i\xi'} e^{iy\cdot\xi''} a(x,y,\xi)\, \widetilde{u}(\xi)\, đ\xi, \tag{2.4}$$

where $\xi' = (\xi_1,\ldots,\xi_\ell)$, $\xi'' = (\xi_{\ell+1},\ldots,\xi_n)$, $x^{i\xi'} = x^{i\xi_1}\cdots x^{i\xi_\ell}$, and $\widetilde{u}(\xi)$ is the Mellin transform in x and the Fourier transform in y of u,

$$\widetilde{u}(\xi) = \int_{\mathcal{U}} x^{-i\xi'} e^{-iy\cdot\xi''} u(x,y) \frac{dx}{x} dy. \tag{2.5}$$

For technical purposes we also need to assume that $a(x,y,\xi)$ with all its derivatives extends to be an entire function of ξ', and for $|\operatorname{Im}\xi'|$ bounded by any fixed number, is a classical symbol of order m in ξ as $|\operatorname{Re}\xi'|, |\xi''| \to \infty$. The space $\Psi_b^m(X)$ is called the *small calculus of b-pseudodifferential operators* of order m. It also turns out that any $A \in \Psi_b^m(X)$ defines a continuous map on $C^\infty(X)$. Authors often refer to b-operators in terms of their Schwartz kernels. Observe that combining (2.4) and (2.5), we see that the Schwartz kernel of A on the product $X \times X$ near the diagonal is of the form

$$K_A = \int \left(\frac{x}{x'}\right)^{i\xi'} e^{i(y-y')\cdot\xi''} a(x,y,\xi)\, đ\xi \cdot \frac{dx'}{x'} dy', \tag{2.6}$$

where (x,y) are coordinates on the left factor of X and (x',y') are the same coordinates on the right factor of X. Introducing 'logarithmic coordinates' $w = (\log x_1,\ldots,\log x_\ell, y)$, we can write this as

$$K_A = \int e^{i(w-w')\cdot\xi}\, a(x,y,\xi)\, đ\xi \cdot dw', \tag{2.7}$$

which looks like the Schwartz kernel of a 'usual' pseudodifferential operator.

The space of b-pseudodifferential operators share many properties with the usual pseudodifferential operators. For example, $\operatorname{Diff}_b^m(X) \subset \Psi_b^m(X)$, where $\operatorname{Diff}_b^m(X)$ is the space of totally characteristic differential operators, and the space of b-pseudodifferential is a symbolically filtered $*$-algebra of operators. Thus, it is closed under taking adjoints and compositions, and there is a principal symbol map preserving these operations obtained by taking the leading homogeneous term of each symbol $a(x,y,\xi)$ in the local representation (2.6). The principal symbol turns out to be a function on the b-cotangent bundle ${}^bT^*X$ of X minus the zero section [24]. An operator A is said to be *elliptic* if its principal symbol is invertible. The space $L_b^2(X)$ consists of those functions on X that are square integrable with respect to $\mathfrak{m}$; this space is defined independent of the choice of b-measure. For any $m \in \mathbb{R}$, the Sobolev space $H_b^m(X)$ consists of those distributions u on X such that $Au \in L_b^2(X)$ for all $A \in \Psi_b^m(X)$. We also define $H_b^\infty(X) = \bigcap_{m\in\mathbb{R}} H_b^m(X)$. Any $A \in \Psi_b^m(X)$ defines a linear map

$$A\colon H_b^m(X) \longrightarrow L_b^2(X),$$

which is continuous with respect to the 'obvious' topologies [24]. We note that by *convention*, the words 'Fredholm' or 'invertible' mean with respect to their natural, that is *unweighted*, Sobolev space domains unless explicitly stated otherwise.

Unfortunately, $\Psi_b^\bullet(X)$ is not spectrally invariant in the sense that this set is not closed under inversion, when inverses exist. Inverses are in the calculus with bounds, which we now describe. Let $\theta > 0$. We define $\Psi_b^{-\infty,\theta}(X)$ as those operators R such that with respect to coordinates of the sort described in (2.3), we can write

$$R\,u = \int_{\mathcal{U}'} k\Big(x, \frac{x}{x'}, y, y'\Big)\, u(x', y')\, \mathfrak{m}(x', y'),$$

where $k(x, z, y, y')$ has the following 'boundedness' properties: It is smooth for x, y, y' in the interior of X and in $z > 0$, and there is an $\varepsilon > 0$ such that given any constant coefficient b-differential operator P in the variables x, z, y, y', the function $(Pk)(x, z, y, y')$ is bounded for all x, z, y, y', continuous at each $x_i = 0$, vanishes to order $z_i^{\theta+\varepsilon}$ and $z_i^{-\theta-\varepsilon}$ at each $z_i = 0$ and as $z_i \to \infty$, respectively, and vanishes to order $y_i^{\theta+\varepsilon}$ at any $y_i = 0$ and $(y_i')^{\theta+\varepsilon}$ at any $y_i' = 0$ if any of these sets represent boundary hypersurfaces of X. This definition of the calculus with bounds is not as fine as that found in [22, 20, 21], but we shall not need the extra structures found in these papers. For any $m \in \mathbb{R}$, we define

$$\Psi_b^{m,\theta}(X) = \Psi_b^m(X) + \Psi_b^{-\infty,\theta}(X).$$

These spaces form the *calculus with bounds* and they too form an algebra in the sense that $\Psi_b^{m,\theta}(X) \circ \Psi_b^{m',\theta'}(X) \subset \Psi_b^{m+m',\theta''}(X)$, where $\theta'' = \min\{\theta, \theta'\}$.

2.2. **Parameter-dependent operators and the normal operator.** Let $S \subset \mathbb{C}^k$ be a *horizontal strip*, a subset $S \subset \mathbb{C}^k$ of the form $S = \{\tau \in \mathbb{C}^k\,;\, a < \operatorname{Im}\tau < b\}$ for some $a, b \in [-\infty, \infty]^k$. Here, for any $v, w \in [-\infty, \infty]^k$, we define $v < w$ if and only if $v_i < w_i$ for each $i = 1, \ldots, k$. For instance, $\mathbb{C}^k$ itself is a horizontal strip.

If $m \in \mathbb{R}$, then we define the space $\Psi_{b,S}^m(X)$ as those operators $A(\tau) \in \Psi_b^m(X)$ depending holomorphically in $\tau \in S$ such that in local coordinates of the sort described in (2.3) and (2.4) (but now with the terms depending on the extra parameter τ), the operator $R(\tau)$ and symbol $a(x, y, \tau, \xi)$ have the following properties: Let $a < a' < b' < b$. Then $R(\tau)$ vanishes in the topology of $\Psi_b^{-\infty}(X)$ to infinite order with all derivatives in τ as $|\operatorname{Re}\tau| \to \infty$ with $a' \le \operatorname{Im}\tau \le b'$, and where $a(x, y, \tau, \xi)$ is, with all its derivatives in x and y and for $|\operatorname{Im}\xi'|$ bounded by any fixed number, a symbol of order m in (τ, ξ) as $|\operatorname{Re}\tau| \to \infty$ with $a' \le \operatorname{Im}\tau \le b'$ and $|\operatorname{Re}\xi'|, |\xi''| \to \infty$. Given $\theta > 0$, it is also possible to define a calculus with bounds space $\Psi_{b,S}^{-\infty,\theta}(X)$, see [22]. Finally, we define

$$\Psi_{b,S}^{m,\theta}(X) = \Psi_{b,\mathbb{C}^k}^m(X) + \Psi_{b,S}^{-\infty,\theta}(X).$$

Using the techniques of the b-calculus, see the appendix of [29], one can establish the following lemma.

Lemma 2.1. *The space $\Psi_{b,S}^{\bullet,\bullet}(X)$ is closed under composition; and under inversion in the following sense. Let $A(\tau) \in \Psi_{b,S}^{m,\theta}(X)$ and suppose that $A(\tau)$ is invertible for τ on some horizonal plane $\{\operatorname{Im}\tau = c\}$ in the strip S. Then there is a horizontal strip $S' \subset S$ containing the plane and a $\theta' > 0$ such that $A(\tau)^{-1}$ is defined for $\tau \in S'$ and $A(\tau)^{-1} \in \Psi_{b,S'}^{-m,\theta'}(X)$.*

Let $M \in M_k(X)$ be a component of $H_{i_1} \cap \cdots \cap H_{i_k}$, where $i_1 < \cdots < i_k$, so that $\rho_{i_1}, \ldots, \rho_{i_k}$ are defining functions for M. Then near M (cf. (2.1)) we have a decomposition

$$X \cong [0, \epsilon)_x^k \times M, \quad x = (\rho_{i_1}, \ldots, \rho_{i_k}). \tag{2.8}$$

Given $A \in \Psi_b^m(X)$, the normal operator of A at M is defined as follows. Given a function $v \in C^\infty(M)$, let $u \in C^\infty(X)$ be any smooth function such that $u|_M = v$. The properties of the small calculus imply that given any fixed $\tau \in \mathbb{C}^k$, the function $x^{-i\tau} A(x^{i\tau} u)$ defines a smooth function on X. Restricting this function to M defines the *normal operator* of A:

$$N_M(A)(\tau)v := \Big(x^{-i\tau} A(x^{i\tau}u)\Big)\Big|_M.$$

This operator is defined independent of the choice of extension u. In local coordinates $N_M(A)(\tau)$ is simple to describe. For instance, consider a coordinate patch $\mathcal{U} = [0,\epsilon)^\ell_{\tilde{x}} \times \mathbb{R}^{n-\ell}_y$ on X where x equals the first k coordinate functions of $\tilde{x}$ and where the Schwartz kernel of A is given on a compact neighborhood in $\mathcal{U} \times \mathcal{U}$ by (cf. (2.6))

$$K_A = \int \left(\frac{\tilde{x}}{\tilde{x}'}\right)^{i\xi'} e^{i(y-y')\cdot\xi''} a(\tilde{x}, y, \xi)\, đ\xi \cdot \frac{d\tilde{x}'}{\tilde{x}'} dy'$$

Writing $\tilde{x} = (x, \tilde{y})$, we have $\mathcal{U} = [0,\epsilon)^k_x \times \widetilde{\mathcal{U}}$ where $\widetilde{\mathcal{U}} = [0,\epsilon)^{\ell-k}_{\tilde{y}} \times \mathbb{R}^{n-\ell}_y$ is a coordinate patch on M. Let $\xi = (\tau_1, \ldots, \tau_k, \eta_1, \ldots, \eta_{n-k})$. Then,

$$K_A = \int \left(\frac{x}{x'}\right)^{i\tau} \left(\frac{\tilde{y}}{\tilde{y}'}\right)^{i\eta'} e^{i(y-y')\cdot\eta''} a(x, \tilde{y}, y, \tau, \eta)\, đ\tau\, đ\eta \cdot \frac{dx'}{x'} \frac{d\tilde{y}'}{\tilde{y}'} dy', \tag{2.9}$$

where $\eta' = (\eta_1, \ldots, \eta_{\ell-k})$ and $\eta'' = (\eta_{\ell-k+1}, \ldots, \eta_{n-k})$. Then by definition of the normal operator, it seems reasonable, and in fact can be proved, that the Schwartz kernel of $N_M(A)(\tau)$ in a compact neighborhood in $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}$ is given by

$$K_{N_M(A)(\tau)} = \int \left(\frac{\tilde{y}}{\tilde{y}'}\right)^{i\eta'} e^{i(y-y')\cdot\eta''} a(0, \tilde{y}, y, \tau, \eta)\, đ\eta \cdot \frac{d\tilde{y}'}{\tilde{y}'} dy',$$

Properties of the normal operator can be found in [22], [17], or [29]. From this explicit description of the Schwartz kernel of $N_M(A)(\tau)$ (at least near the diagonal of $M \times M$ and with a similar analysis in coordinate patches away from the diagonal) and the definition of our parameter-dependent operators above, it follows that $N_M(A)(\tau) \in \Psi^m_{b,\mathbb{C}^k}(M)$. Moreover, it readily follows that

$$N_M(A^*)(\tau) = N_M(A)(\overline{\tau})^*,\ N_M(AB)(\tau) = N_M(A)(\tau) N_M(B)(\tau) \tag{2.10}$$

for any $B \in \Psi_b^{m'}(X)$ and all $\tau \in \mathbb{C}^k$. For instance, to prove the composition property, let u be an extension of v as before, and note that

$$\begin{aligned} N_M(AB)(\tau)v = \Big(x^{-i\tau} ABx^{i\tau} u\Big)\Big|_M &= \Big(x^{-i\tau} A x^{i\tau} \Big(x^{-i\tau} B x^{i\tau} u\Big)\Big)\Big|_M \\ &= N_M(A)(\tau)\Big(\Big(x^{-i\tau} B x^{i\tau} u\Big)\Big|_M\Big) \\ &= N_M(A)(\tau)(N_M(B)(\tau)v). \end{aligned}$$

The following convenient formula, which follows directly from the definition of normal operator, is used throughout this paper. If α is a multi-index, then

$$N_M(\rho^{-\alpha} A \rho^\alpha)(\tau) = \varrho^{-\beta} N_M(A)(\tau - i\gamma)\varrho^\beta \tag{2.11}$$

where $\gamma = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, $\beta = \{\alpha_j\,;\, j \neq i_1, \ldots, i_k\}$, and

$$\varrho = \prod_{j \neq i_1, \ldots, i_k} \rho_j|_M, \qquad \varrho^\beta = \prod_{j \neq i_1, \ldots, i_k} \rho_j^{\alpha_j}|_M. \tag{2.12}$$

The normal operator also extends to the calculus with bounds. In the following lemma we summarize some of the main properties of the normal operator.

Lemma 2.2. *If $M \in M_k(X)$, then for any ϑ,*

$$\partial_\tau^\vartheta N_M \colon \Psi_b^{m,\theta}(X) \longrightarrow \bigcup_{\varepsilon>0} \Psi_{b,S_\varepsilon}^{m-|\vartheta|,\theta}(M),$$

where S_ε is the strip: $-\theta - \varepsilon < \operatorname{Im} \tau_j < \theta + \varepsilon$. Moreover, with $\vartheta = 0$, N_M is surjective and satisfies the properties (2.10).

2.3. **Fredholm properties and the b-trace.** In this section we list some Fredholm properties of b-pseudodifferential operators. Proofs can be found in the appendix of [22]. The following theorem characterizes Fredholm operators on nonweighted Sobolev spaces.

Theorem 2.3. *Let $A \in \Psi_b^m(X)$, $m \in \mathbb{R}^+$. Then the following are equivalent:*
(1) $A\colon H_b^m(X) \longrightarrow L_b^2(X)$ is a Fredholm.
(2) A is elliptic and for each $H \in M_1(X)$, $N_H(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}$, in which case $N_H(A)(\tau)^{-1} \in \Psi_{b,S}^{-m,\varepsilon}(H)$ for some $\varepsilon > 0$ and some horizontal strip S containing $\mathbb{R}$ (cf. Lemma 2.1).
(3) A is elliptic and for each $M \in M_k(X)$, $k \geq 1$, $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k$, in which case $N_M(A)(\tau)^{-1} \in \Psi_{b,S}^{-m,\varepsilon}(M)$ for some $\varepsilon > 0$ and some horizontal strip S containing $\mathbb{R}^k$ (cf. Lemma 2.1).

We note again that by *convention*, the words ‘Fredholm’ or ‘invertible’ mean with respect to the natural, that is *unweighted*, Sobolev space domains unless explicitly stated otherwise. The next theorem describes the generalized inverse of Fredholm operators.

Theorem 2.4. *If $A \in \Psi_b^m(X)$, $m \in \mathbb{R}^+$, be Fredholm. Then its generalized inverse, G, is in the full calculus: $G \in \Psi_b^{-m,\varepsilon}(X)$ for some $\varepsilon > 0$. Here, the generalized inverse is defined by the equations*

$$AG = \mathrm{Id} - \Pi_1, \quad GA = \mathrm{Id} - \Pi_0,$$

where Π_0 and Π_1 are the orthogonal projections onto the null space and off the range in $H_b^m(X)$ and $L_b^2(X)$, respectively. Moreover, $\ker A \subset \rho^\varepsilon H_b^\infty(X)$, *and* $\operatorname{coker} A \cong \ker A^* \subset \rho^\varepsilon H_b^\infty(X)$.

Finally, we describe the b-integral and b-trace. It turns out that elements of $\Psi_b^{-\infty}(X)$ are not trace class. Indeed, consider the Schwartz kernel of an element $R \in \Psi_b^{-\infty}(X)$ given in the coordinates (2.3),

$$K_R = k\Big(x, \frac{x}{x'}, y, y'\Big)\, \mathfrak{m}(x', y').$$

Then restricting to the diagonal, we see that

$$K_R\Big|_{\mathrm{Diag}} = k\Big(x, \mathbf{1}, y, y\Big)\, \mathfrak{m}(x, y), \quad \mathbf{1} = (1, \dots, 1) \ \ (k \text{ ones}).$$

Because $k(x, z, y, y')$ is only assumed to be smooth at $x = 0$ and $\mathfrak{m}$ has factors of x_i^{-1}, the integral $\int_X K_R\big|_{\mathrm{Diag}}$, where we identity the diagonal with X, in general does not exist. However, such operators do have a b-trace as we now define. Let

$$\mathbb{C}_+^N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N \,;\, \operatorname{Re} z_i > 0 \text{ for all } i\}.$$

Given any function $u \in C^\infty(X)$, it is straightforward to prove that the integral

$$\int_X \rho^z u \,\mathfrak{m}, \qquad \rho^z = \rho^{z_1} \cdots \rho^{z_N},$$

is well-defined for $z \in \mathbb{C}^N_+$ and extends to be a meromorphic function of z for z in all of $\mathbb{C}^N$. We define the b-integral of u over X to be the regular value of this family at $z = 0$,

$$ {}^b\!\!\int_X u \,\mathfrak{m} := \operatorname{Reg}_{z=0} \int \rho^z u \,\mathfrak{m}.$$

Given $R \in \Psi_b^{-\infty}(X)$, we define the *$b$-trace* of R by

$${}^b\mathrm{Tr}\, R := {}^b\!\!\int_X K_R\Big|_{\mathrm{Diag}}.$$

The b-trace is also defined on elements of $\Psi_b^m(X)$ with $m < -n = -\dim X$. The following theorem is proved in [22], and it gives a formula for the defect in the b-trace of a commutator.

Theorem 2.5. *If $A \in \Psi_b^m(X)$ and $B \in \Psi_b^{m'}(X)$ with $m + m' < -n$, then*

$$ {}^b\mathrm{Tr}[A, B] = -\sum_{M \in M'(X)} \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\,\mathbf{D}^k_\tau N_M(A)(\tau)\, N_M(B)(\tau)\,)\, đ\tau, \tag{2.13}$$

where $đ\tau = d\tau/(2\pi)^k$ and $\mathbf{D}^k_\tau = D_{\tau_1} \cdots D_{\tau_k}$ with $D_{\tau_j} = \frac{1}{i}\partial_{\tau_j}$, and the sum is over every $k \geq 1$ and $M \in M_k(X)$.

3. An index formula for b-pseudodifferential operators

In Section 3.1 we prove sufficiency in Theorem 1.1. Then in Section 3.2 we state the index theorem giving a precise description of each term in the formula. Finally, in Section 3.3 we give various applications of the index formula.

3.1. The Fredholm condition on weighted Sobolev spaces.

In Proposition 3.3 below, we give a sufficient condition on which an operator is Fredholm on weighted Sobolev spaces. In Section 5, we show that this condition is also necessary. Proposition 3.3 was given in [22], but as the lemmas used in its proof are needed later we provide a complete proof here. We begin with the following lemma.

Lemma 3.1. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$, be elliptic. Let $M \in M_k(X)$ with $k \geq 1$ and let ϱ be the function in (2.12). Then $N_M(A)(\tau) \in \Psi_b^m(M, E, F)$ is elliptic for any $\tau \in \mathbb{C}^k$ and given any $c > 0$,*

$$N_M(A)(\tau)\colon \varrho^\beta H_b^m(M, E) \longrightarrow \varrho^\beta L_b^2(M, F)$$

is invertible for all $\tau \in \mathbb{C}^k$ for $|\operatorname{Re}\tau|$ sufficiently large with $|\operatorname{Im}\tau|, |\beta| \leq c$.

Proof. Being elliptic, A has a small parametrix $B \in \Psi_b^{-m}(X, F, E)$ such that $\mathrm{Id} - AB = R_1 \in \Psi_b^{-\infty}(X, F)$ and $\mathrm{Id} - BA = R_2 \in \Psi_b^{-\infty}(X, E)$. Hence,

$$N_M(A)(\tau) N_M(B)(\tau) = \mathrm{Id} - N_M(R_1)(\tau)$$
$$N_M(B)(\tau) N_M(A)(\tau) = \mathrm{Id} - N_M(R_2)(\tau).$$

Since $N_M(R_1)(\tau)$ (and $N_M(R_2)(\tau)$) is an operator of order $-\infty$, it follows that $N_M(A)(\tau)$ is elliptic for all $\tau \in \mathbb{C}^k$. Conjugating the above equations with appropriate factors of $\varrho^{\pm\beta}$, we see that

$$\begin{aligned} \Big\{\varrho^{-\beta} N_M(A)(\tau)\varrho^{\beta}\Big\}\Big\{\varrho^{-\beta} N_M(B)(\tau)\varrho^{\beta}\Big\} &= \mathrm{Id} - \varrho^{-\beta} N_M(R_1)(\tau)\varrho^{\beta} \\ \Big\{\varrho^{-\beta} N_M(B)(\tau)\varrho^{\beta}\Big\}\Big\{\varrho^{-\beta} N_M(A)(\tau)\varrho^{\beta}\Big\} &= \mathrm{Id} - \varrho^{-\beta} N_M(R_2)(\tau)\varrho^{\beta}. \end{aligned} \tag{3.1}$$

Now we know that as $|\operatorname{Re}\tau| \to \infty$ with $|\operatorname{Im}\tau|, |\beta| \le c$, $\varrho^{-\beta} N_M(R_i)(\tau)\varrho^{\beta} \to 0$ uniformly in the topology of $\Psi_b^{-\infty}$, and hence, in the topology of L^2_b. Thus, we can invert each operator on the right in (3.1) on L^2_b for $|\operatorname{Re}\tau|$ sufficiently large with $|\operatorname{Im}\tau| \le c$. Our lemma follows. ☐

Lemma 3.2. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$, be elliptic and suppose that for each $M \in M_2(X)$, $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then there is an $\varepsilon > 0$ having the following properties:*

(1) Let $M \in M_k(X)$ with $k \ge 2$. Then for all $\tau \in \mathbb{C}^k$ and β with $|\operatorname{Im}\tau|, |\beta| < \varepsilon$,

$$\varrho^{-\beta} N_M(A)(\tau)\varrho^{\beta} \colon H_b^m(M, E) \longrightarrow L_b^2(M, F) \tag{3.2}$$

is invertible, where ϱ is the function in (2.12).

(2) Let $M \in M_1(X)$. Then for $|\beta| < \varepsilon$, the operator (3.2) *has a meromorphic inverse on the strip $|\operatorname{Im}\tau| < \varepsilon$ with finitely many poles on the real line defined independent of β with finite rank residues of the form $\varrho^{-\beta} K \varrho^{\beta}$ where K is the corresponding residue of $N_M(A)(\tau)^{-1}$.*

Proof. First of all, given $M \in M_k(X)$ with $k > 2$, we show that $N_M(A)(\tau)$ is invertible all real τ. To see this, choose $M' \in M_2(X)$ such that $M \subset M'$ and for any $v \in \mathbb{R}^2$, define $A_v := N_{M'}(A)(v)$. Then by assumption, A_v is invertible and hence Fredholm, so by Theorem 2.3, all its normal operators are invertible for all real parameters. In particular, its normal operator at $M \subset M'$, $N_M(A_v)(u) = N_M(N_{M'}(A)(v))(u)$, is invertible for all real u. Since $N_M(N_{M'}(A)) = N_M(A)$, it follows that $N_M(A)(\tau)$ is invertible all real τ.

Applying Lemma 3.1 with $c = 1$ to each face of X, we see that there is an $r > 0$ such that for each $k \ge 1$, the map (3.2) is invertible if $|\operatorname{Im}\tau|, |\beta| \le 1$ and $|\operatorname{Re}\tau| > r$. Let $M \in M_k(X)$ with $k \ge 2$. Then we know that $N_M(A)(\tau)$ is invertible for all real τ. Since the invertible operators form an open set and $N_M(A)(\tau)$ depends continuously on τ, it follows that there is a $0 < \delta_M \le 1$ such that the map (3.2) is invertible if $|\operatorname{Re}\tau| \le r$ and $|\operatorname{Im}\tau|, |\beta| \le \delta_M$. Now let $M = H \in M_1(X)$. We claim that $N_H(A)(\tau)$ is Fredholm for all $\tau \in \mathbb{R}$. To see this, let M' be a hypersurface of H. Then $M' \in M_2(X)$ and so by assumption, $N_{M'}(N_H(A)) = N_{M'}(A)$ is invertible for all real normal parameters. Thus by Theorem 2.3, $N_H(A)(\tau)$ is Fredholm for all real τ; hence, by analytic Fredholm theory (cf. Theorem 2.4), for some $0 < \delta_H \le 1$, with $|\beta| < \delta_H$, $\varrho^{-\beta} N_H(A)(\tau)\varrho^{\beta}$ is a meromorphic function on a rectangle $|\operatorname{Im}\tau| \le \delta_H$, $|\operatorname{Re}\tau| \le r$ with finitely many poles on the real line defined independent of β with finite rank residues of the form $\varrho^{-\beta} K \varrho^{\beta}$ where K is the corresponding residue of $N_H(A)(\tau)^{-1}$. Let ε be the minimum of all the δ_M's chosen for each $M \in M_k(X)$, $k \ge 1$. This ε has all the properties requested and completes our proof. ☐

Proposition 3.3. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$, be elliptic and suppose that for each $M \in M_2(X)$, $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. For $\varepsilon > 0$ given in*

Lemma 3.2,

$$A\colon \rho^\alpha H_b^m(X,E) \longrightarrow \rho^\alpha L_b^2(X,F)$$

is Fredholm for all multi-indices α with $0<|\alpha|<\varepsilon$.

Proof. Let $\varepsilon>0$ be the number given in Lemma 3.2. We shall prove that for any $H \in M_1(X)$, $N_H(\rho^{-\alpha}A\rho^\alpha)(\tau)$ is invertible for all real τ from $H_b^m(H,E)$ onto $L_b^2(H,F)$ for all multi-indices α such that $0<|\alpha|<\varepsilon$. If $H=H_j$, then with the notation of (2.11), we have

$$N_H(\rho^{-\alpha}A\rho^\alpha)(\tau) = \varrho^{-\beta} N_H(A)(\tau - i\alpha_j)\varrho^\beta.$$

By Lemma 3.2, this operator is invertible for all real τ and $0<|\alpha|<\varepsilon$, which implies that $\rho^{-\alpha}A\rho^\alpha\colon H_b^m(X,E)\longrightarrow L_b^2(X,F)$ is Fredholm and proves the proposition. □

3.2. **Statement of the index theorem.** We repeat the index formula presented in the introduction, but now spelling out all the terms. See Section 4 for the proof.

Theorem 3.4. *Let $A\in\Psi_b^m(X,E,F)$, $m\in\mathbb{R}^+$, be elliptic. Suppose that for each $M\in M_2(X)$, $N_M(A)(\tau)$ is invertible for all $\tau\in\mathbb{R}^2$. Then for $\varepsilon>0$ given in Lemma 3.2, for all multi-indices α with $0<|\alpha|<\varepsilon$,*

$$A\colon \rho^\alpha H_b^m(X,E) \longrightarrow \rho^\alpha L_b^2(X,F)$$

is Fredholm. Moreover,

$$\operatorname{ind}_\alpha A = {}^b\!\!\int_X \omega_A - \frac{1}{2}\sum_{H\in M_1(X)} \Big\{ {}^b\tilde{\eta}_H + \operatorname{sgn}(\alpha_H)\cdot \sum_{z\in \operatorname{spec}_H(A)} \operatorname{rk}_H(z) + \beta_H \Big\} - \frac{1}{2}\sum_{\substack{M\in M_k(X)\\ k\geq 2}} {}^b\eta_M.$$

We now explain the meaning of each term. As we already know, ω_A is the 'analytic' Atiyah-Singer density of A manufactured from the local symbols of A. The other terms are defined as follows. To describe the last sum, let $M\in M_k(X)$ where $k\geq 2$ and define

$$(3.3)\qquad {}^b\eta_M(t) = 2\int_{\mathbb{R}^k} {}^b\mathrm{Tr}\big(\mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\big)\, đ\tau,$$

where $đ\tau = d\tau/(2\pi)^k$ and $\mathbf{D}_\tau^k = D_{\tau_1}\cdots D_{\tau_k}$ with $D_{\tau_j} = \frac{1}{i}\partial_{\tau_j}$. In Lemma 4.2 we show that ${}^b\eta_M(t)$ has an asymptotic expansion as $t\to 0$ in powers and log-powers of $t^{1/2m}$. We define

$$(3.4)\qquad {}^b\eta_M := \text{constant term in the expansion of } {}^b\eta_M(t) \text{ as } t\to 0.$$

We now describe the middle term. Let $H\in M_1(X)$. Then by Lemma 3.2, the inverse $N_H(A)(\tau)^{-1}$ is meromorphic on the strip $|\operatorname{Im}\tau|<\varepsilon$ with finitely many poles on the real line with finite rank residues. We define $\operatorname{spec}_H(A)\subset\mathbb{R}$ as the set of poles of $N_H(A)(\tau)^{-1}$ on the real line. In particular, we can write

$$(3.5)\qquad D_\tau N_H(A)(\tau)\, N_H(A)(\tau)^{-1} = S(\tau) + G(\tau),$$

where $G(\tau)$ is holomorphic on the strip and $S(\tau)$ is of the form

$$S(\tau) = \sum_z S_z(\tau), \qquad S_z(\tau) = \sum_{j=0}^{\nu-1} \frac{K_{z,j}}{(\tau-z)^{j+1}},$$

where the first sum is a finite sum over z in $\operatorname{spec}_H(A)$, and in the second sum, ν is an integer depending on z and the $K_{z,j}$'s are finite rank operators. The rank of the pole at z is the dimension of the singular range of $D_\tau N_H(A)(\tau)\, N_H(A)(\tau)^{-1}$,

$$\operatorname{rk}_H(z) := \sum_{j=0}^{\nu-1} \dim \operatorname{Im}\Big(\sum_{k=j}^{\nu-1} K_{z,k} \Big). \tag{3.6}$$

The eta term ${}^b\tilde{\eta}_H$ is defined like the terms ${}^b\eta_M$ but replacing the meromorphic operator $D_\tau N_H(A)(\tau)\, N_H(A)(\tau)^{-1}$ by its holomorphic part $G(\tau)$. Thus, setting

$${}^b\tilde{\eta}_H(t) = 2 \int_{\mathbb{R}} {}^b\mathrm{Tr}(\, G(\tau)\, N_H(e^{-tAA^*})(\tau)\,)\, đ\tau, \tag{3.7}$$

in Lemma 4.4 we show that ${}^b\tilde{\eta}_H(t)$ has an asymptotic expansion as $t \to 0$ in powers and log-powers of $t^{1/2m}$ (up to a term vanishing at $t = 0$); then we define

$${}^b\tilde{\eta}_H := \text{constant term in the expansion of } {}^b\tilde{\eta}_H(t) \text{ as } t \to 0. \tag{3.8}$$

Finally, we describe the third hypersurface term β_H, which depends on the principal symbol of A and the singular term $S(\tau)$. First define the inclusion

$$\iota_H \colon {}^bN^*H \hookrightarrow {}^bT^*X, \qquad \iota_H\Big(\frac{d\rho_H}{\rho_H}\Big) = \frac{d\rho_H}{\rho_H},$$

where ${}^bN^*H$ is the span of $d\rho_H/\rho_H$ on the hypersurface H. Let a_m denote the principal symbol of A and set

$$b_H := \frac{1}{\pi} {}^b\!\!\int_{\mathbb{R}} e^{-\iota_H^*(a_m a_m^*)(\tau)}\, \frac{d\tau}{\tau} \in C^\infty(H, \hom(F)), \tag{3.9}$$

where the b-integral means that the integral is regularized at $\tau = 0$ as follows:

$${}^b\!\!\int_{\mathbb{R}} e^{-\iota_H^*(a_m a_m^*)(\tau)}\, \frac{d\tau}{\tau} = \int_{-r}^{r} \frac{e^{-\iota_H^*(a_m a_m^*)(\tau)} - 1}{\tau}\, d\tau + \Big(\int_{-\infty}^{-r} + \int_r^\infty \Big) e^{-\iota_H^*(a_m a_m^*)(\tau)}\, \frac{d\tau}{\tau}.$$

One can check that the right-hand side is defined independent of r. The function b_H is described locally as follows. If $[0,\epsilon)_x \times \mathbb{R}^{n,k-1}_y$ are local coordinates on X and $H = \{x = 0\}$ in this coordinate patch with x representing the boundary defining function for H, then

$$b_H = \frac{1}{\pi} {}^b\!\!\int_{\mathbb{R}} e^{-a_m(0,y,\tau,0)\, a_m(0,y,\tau,0)^*}\, \frac{d\tau}{\tau},$$

where $a_m(x, y, \tau, \eta)$ is the principal symbol of A in the coordinates. From this formula it is clear that b_H is a smooth $\hom(F)$-valued function on H. We define

$$\beta_H := \sum_{z \in \operatorname{spec}_H(A)} \mathrm{Tr}(b_H K_{z,0}), \tag{3.10}$$

where $K_{z,0}$ is the finite rank residue of $D_\tau N_H(A)(\tau)\, N_H(A)(\tau)^{-1}$ at $\tau = z$.

3.3. **Various applications.** Before presenting the proof of Theorem 3.4 in Section 4, we give some applications. The following theorem is an immediate consequence of Theorem 3.4. The formula (3.11) below is called the *relative index formula*.

Theorem 3.5. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$, be elliptic. Suppose that for each $M \in M_2(X)$, $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then for $\varepsilon > 0$ given in Lemma 3.2, for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$A\colon \rho^\alpha H_b^m(X, E) \longrightarrow \rho^\alpha L_b^2(X, F)$$

is Fredholm. Moreover, given any two such multi-indices α, α', we have

$$\operatorname{ind}_\alpha A - \operatorname{ind}_{\alpha'} A = - \sum_{\substack{\operatorname{sgn}\, \alpha_H \neq \operatorname{sgn}\, \alpha'_H, \\ z \in \operatorname{spec}_H(A),\, H \in M_1(X)}} \operatorname{sgn}(\alpha_H - \alpha'_H) \cdot \operatorname{rk}_H(z). \tag{3.11}$$

The β_H terms vanish for many operators. For instance, consider the following.

Lemma 3.6. *If $\iota_H^*(a_m a_m^*)(\tau)$ is an even function in τ, then $b_H = 0$. In particular, if m is a positive integer and $\iota_H^* a_m(\tau)$ is a polynomial in τ of order m, then $b_H = 0$.*

Proof. If $\iota_H^*(a_m a_m^*)(\tau)$ is even, then $e^{-\iota_H^*(a_m a_m^*)(\tau)}/\tau$ is odd; hence, the integral defining b_H vanishes. □

In particular, the β_H terms vanish if A is of the form $P + R$ where P is a b-differential operator and R is a b-pseudodifferential operator of order $-\infty$. Here, R can be thought of as a 'perturbation' of P. In this case, we also have $\omega_A = \omega_P$ since the index density only depends on finitely many homogeneous terms in the local symbols of the operator. Therefore, Theorem 3.4 implies the following.

Theorem 3.7. *Let $P \in \operatorname{Diff}_b^m(X, E, F)$ be an elliptic b-differential operator of positive order and let $R \in \Psi_b^{-\infty}(X, E, F)$ be such that for each $M \in M_2(X)$, $N_M(P + R)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$. Then for some $\varepsilon > 0$, for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$P + R\colon \rho^\alpha H_b^m(X, E) \longrightarrow \rho^\alpha L_b^2(X, F)$$

is Fredholm. Moreover,

$$\operatorname{ind}_\alpha(P + R) = {}^b\!\!\int_X \omega_P - \frac{1}{2} \sum_{H \in M_1(X)} \Big\{ {}^b\tilde{\eta}_H + \operatorname{sgn}(\alpha_H) \cdot \sum_{z \in \operatorname{spec}_H(P+R)} \operatorname{rk}_H(z) \Big\} - \frac{1}{2} \sum_{\substack{M \in M_k(X) \\ k \geq 2}} {}^b\eta_M,$$

where ω_P is local index density of P, and where ${}^b\tilde{\eta}_H$, $\operatorname{rk}_H(z)$, and ${}^b\eta_M$ are defined with respect to the normal operators of $A = P + R$.

In joint work with Melrose [22], we prove a similar index formula for P a Dirac operator and R specific b-pseudodifferential operators of order $-\infty$; in this case all the terms in the index formula involve only the Dirac operator and certain Lagrangian subspaces of the null spaces of the corner Dirac operators. For now we shall consider Theorem 3.7 in case P is a Dirac operator and $R = 0$. For more on the subject of perturbed Dirac operators, we refer the reader to [22].

Assume now that X is even-dimensional. A metric on the b-tangent bundle ${}^bTX$ is said to be *exact* if it takes the form

$$g = \sum_{i=1}^{N} \Big(\frac{d\rho_i}{\rho_i}\Big)^2 + g',$$

where $g' \in C^\infty(X, T^*X \otimes T^*X)$. Let $\eth^+ \colon C^\infty(X, E^+) \longrightarrow C^\infty(X, E^-)$ be a Dirac operator associated to g, where E^+ and E^- are the chiral parts of a Clifford bundle over X. Let $M \in M_2(X)$ be defined by ρ_{i_1} and ρ_{i_2} with $i_1 < i_2$, and let $X \cong [0,\epsilon)^2_x \times M$ near M where $x_j = \rho_{i_j}$ and where $E^\pm$ are isomorphic to their restrictions on M. Then modulo a b-differential operator that vanishes on M, we can write

$$\eth^+ = \sigma_1\, x_1 D_{x_1} + \sigma_2\, x_2 D_{x_2} + B_M, \qquad D_{x_j} = \frac{1}{i}\partial_{x_j},$$

where σ_j is Clifford multiplication by dx_j/x_j and B_M is the restriction of $\eth^+$ to sections on M. By definition of the normal operator, we have

$$N_M(\eth^+)(\tau) = \Big(x^{-i\tau}\eth^+ x^{i\tau}\Big)\Big|_M = \sigma_1\tau + \sigma_2\tau_2 + B_M.$$

We call $\eth_M = i\sigma_2 B_M$ the *induced Dirac operator on* M. Using properties of Dirac operators, it follows that $N_M(\eth^+)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ if and only if B_M is invertible, if and only if $\eth_M$ is invertible. Consider now a hypersurface $H = H_j \in M_1(X)$. Let $X \cong [0,\epsilon)_x \times H$ be a collar neighborhood of H in X where $x = \rho_j$ and let B_H be the restriction of $\eth^+$ to sections on H. Then we can write

$$N_H(\eth^+)(\tau) = \sigma\tau + B_H = \Gamma\big[i\tau + \eth_H\big], \tag{3.12}$$

where σ is Clifford multiplication by dx/x, $\Gamma = \sigma/i$, and $\eth_H = i\sigma B_H$ is, by definition, the *induced Dirac operator on* H.

The following theorem contains the direct analog of the formula (1.2).

Theorem 3.8. *If $M \in M_2(X)$, then $N_M(\eth^+)(\tau)$ is invertible for all $\tau \in \mathbb{R}^2$ if and only if $\eth_M$ is invertible. Suppose that $\eth_M$ is invertible for each $M \in M_2(X)$. Then for some $\varepsilon > 0$, for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$\eth^+ \colon \rho^\alpha H^1_b(X, E^+) \longrightarrow \rho^\alpha L^2_b(X, E^-)$$

is Fredholm, and if we denote its index by $\operatorname{ind}_\alpha \eth^+$, then

$$\operatorname{ind}_\alpha \eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2}\sum_{H \in M_1(X)} \Big\{ {}^b\eta(\eth_H) + \operatorname{sgn}(\alpha_H)\cdot \dim\ker \eth_H \Big\},$$

where AS *is the Atiyah-Singer density of E, and where*

$${}^b\eta(\eth_H) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\ {}^b\mathrm{Tr}(\,\eth_H e^{-t\eth_H^2}\,)\, dt.$$

Proof. We already discussed the equivalence of the statements that for each $M \in M_2(X)$, $N_M(\eth^+)(\tau)$ is invertible for all real τ and the invertibility of $\eth_M$. Therefore, with the notation of Theorem 3.4, we have

$$\operatorname{ind}_\alpha \eth^+ = {}^b\!\!\int_X \omega_{\eth^+} - \frac{1}{2}\sum_{H\in M_1(X)}\Big\{ {}^b\tilde{\eta}_H + \operatorname{sgn}(\alpha_H)\cdot \sum_{z\in \operatorname{spec}_H(A)} \operatorname{rk}_H(z) + \beta_H \Big\} \\ - \frac{1}{2}\sum_{\substack{M\in M_k(X)\\ k\geq 2}} {}^b\eta_M .$$

We analyze each term on the right as follows. To start with, since for any $M \in M_k(X)$, $N_M(\eth^+)(\tau)$ is a degree one polynomial in τ, we have $\mathbf{D}^k_\tau N_M(\eth^+)(\tau) = 0$ if $k \geq 2$, and so ${}^b\eta_M = 0$ for $k \geq 2$. The analysis in [24] extends from manifolds with boundary to manifolds with corners showing that $\omega_{\eth^+} = \mathrm{AS}$. From Lemma 3.6 we know that the β_H terms vanish. Now by (3.12) and the definition of boundary spectrum and rank, we have $\operatorname{spec}_H(\eth^+) = \{0\}$ and $\operatorname{rk}_H(0) = \dim\ker \eth_H$. Finally, it remains to prove that ${}^b\tilde{\eta}_H = {}^b\eta(\eth_H)$. To see this, let Π_0 be the orthogonal projection onto $\ker \eth_H$ and observe that

$$\int_t^\infty \Pi_0^\perp e^{-s(\tau^2+\eth_H^2)}\, ds = \Pi_0^\perp (\tau^2+\eth_H^2)^{-1}\, e^{-t(\tau^2+\eth_H^2)}.$$

As $D_\tau N_H(\eth^+)(\tau)\, N_H(\eth^+)(\tau)^{-1} = \Gamma(i\tau + \eth_H)^{-1}\Gamma^{-1}$, using the notation of (3.5), we have $G(\tau) = \Pi_0^\perp \Gamma(i\tau+\eth_H)^{-1}\Gamma^{-1}$, and therefore

$$\begin{aligned} {}^b\tilde{\eta}_H(t) &= 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(\, \Pi_0^\perp (i\tau+\eth_H)^{-1}\, e^{-t(\tau^2+\eth_H^2)}\,)\, đ\tau \\ &= 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(\, (-i\tau+\eth_H)\Pi_0^\perp (\tau^2+\eth_H^2)^{-1}\, e^{-t(\tau^2+\eth_H^2)}\,)\, đ\tau \\ &= 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(\, \eth_H\Pi_0^\perp (\tau^2+\eth_H^2)^{-1}\, e^{-t(\tau^2+\eth_H^2)}\,)\, đ\tau \\ &= 2\int_{\mathbb{R}}\int_t^\infty {}^b\mathrm{Tr}(\, \eth_H\Pi_0^\perp\, e^{-s(\tau^2+\eth_H^2)}\,)\, ds\, đ\tau \\ &= \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-s\eth_H^2}\,)\, ds. \end{aligned}$$

The 'odd-dimensional local index theorem' [24, Th. 8.36] extends to manifolds with corners and implies that the last integral displayed is continuous at $t = 0$. This shows that ${}^b\tilde{\eta}_H = {}^b\eta(\eth_H)$ and our proof is complete. □

4. Proof of the index formula

We now prove Theorem 3.4. Let $A \in \Psi^m_b(X, E, F)$, $m \in \mathbb{R}^+$, be elliptic with $N_M(A)(\tau)$ invertible at all corners M of X of codimension two and for all $\tau \in \mathbb{R}^2$. Choose $\varepsilon > 0$ as in Lemma 3.2. Throughout this section, we shall henceforth *fix* a multi-index α with $0 < |\alpha| < \varepsilon$. Then setting $A_\alpha = \rho^{-\alpha} A \rho^\alpha$,

$$A_\alpha \colon H^m_b(X, E) \longrightarrow L^2_b(X, F)$$

is Fredholm and $\operatorname{ind}_\alpha A = \operatorname{ind} A_\alpha$. The merit of working with A_α is that A_α has a constant domain while the domains of the weighted Sobolev spaces change with α. We shall break up the proof of the index formula into three parts: First, in Section 4.1 we use the heat kernel method to find a formula for $\operatorname{ind}_\alpha A$ in terms of the local index density of A_α and its eta invariants. Second, we analyze these terms as the weight parameter tends to zero, which we accomplish in Sections 4.2 and 4.3.

4.1. **The heat kernel proof of the index theorem.** Since the index is stable under continuous variations, it follows that

$$\operatorname{ind}_\alpha A = \operatorname{ind} A_\alpha = \operatorname{ind} A_a,$$

where $A_a = \rho^{-a} A \rho^a$ and a is a multi-index with $0 < |a| < \varepsilon$ and $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$; that is, $\operatorname{sgn}(a_H) = \operatorname{sgn}(\alpha_H)$ for each H.

Step 1: We compute $\operatorname{ind} A_a = \operatorname{ind}_\alpha A$ in the usual way by considering the "McKean-Singer" function:

$$h_a(t) = {}^b\mathrm{Tr}(e^{-tA_a^* A_a}) - {}^b\mathrm{Tr}(e^{-tA_a A_a^*}). \tag{4.1}$$

According to Lemma 7.3 in [20], we have

$$\lim_{t\to\infty} h_a(t) = \operatorname{ind} A_a = \operatorname{ind}_\alpha A.$$

Thus by the Fundamental Theorem of Calculus, for any $t > 0$,

$$\operatorname{ind}_\alpha A - h_a(t) = \int_t^\infty h_a'(s)\, ds.$$

Since $A_a^* A_a e^{-sA_a^* A_a} = A_a^* e^{-sA_a A_a^*} A_a$, we have

$$\begin{aligned} h_a'(s) &= {}^b\mathrm{Tr}(\, -A_a^* A_a e^{-sA_a^* A_a} + A_a A_a^* e^{-sA_a A_a^*}\,) \\ &= {}^b\mathrm{Tr}(\, -A_a^* e^{-sA_a A_a^*} A_a + A_a A_a^* e^{-sA_a A_a^*}\,) \\ &= {}^b\mathrm{Tr}[A_a, A_a^* e^{-sA_a A_a^*}]. \end{aligned}$$

To evaluate this expression, we use the trace-defect formula (2.13):

$$h_a'(s) = -\sum_{M\in M'(X)} \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a^*)(\tau)\, N_M(e^{-sA_a A_a^*})(\tau))\, đ\tau,$$

where $\mathbf{D}_\tau^k = D_{\tau_1}\cdots D_{\tau_k}$ with $D_{\tau_j} = i^{-1}\partial_{\tau_j}$. Properties of the normal operator imply that $N_M(e^{-sA_a A_a^*})(\tau) = e^{-sN_M(A_a A_a^*)(\tau)}$ (cf. Lemma 4.4 in [19] for manifolds with corners or equation (7.91) in [24] for the manifold with boundary case), and that $e^{-sN_M(A_a A_a^*)(\tau)} \to 0$ exponentially as $s \to \infty$ since $N_M(A_a A_a^*)(\tau)$ is invertible for all real τ. Thus,

$$\int_t^\infty N_M(e^{-sA_a A_a^*})(\tau) ds = N_M(A_a A_a^*)(\tau)^{-1} N_M(e^{-tA_a A_a^*})(\tau),$$

and so

$$\begin{aligned} \int_t^\infty \int_{\mathbb{R}^k} & {}^b\mathrm{Tr}(\, \mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a^*)(\tau)\, N_M(e^{-sA_a A_a^*})(\tau)\,)\, đ\tau\, ds \\ &= \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, \mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1}\, N_M(e^{-tA_a A_a^*})(\tau)\,)\, đ\tau. \end{aligned}$$

Hence, for any $t > 0$,

$$\operatorname{ind}_\alpha A = h_a(t) - \frac{1}{2} \sum_{M\in M'(X)} {}^b\eta_M(a,t), \tag{4.2}$$

where for $M \in M_k(X)$,

$$ {}^b\eta_M(a,t) = 2 \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\mathbf{D}^k_\tau N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1}\, N_M(e^{-tA_aA_a^*})(\tau))\, đ\tau. \tag{4.3} $$

The analysis leading up to [21, Cor. 6.4] or [20, Cor. 5.54] imply that $h_a(t)$ has an asymptotic expansion as $t \to 0$ in powers and log-powers of $t^{1/2m}$ and the constant term is given by ${}^b\!\int \omega_{A_a}$ where ω_{A_a} denotes the 'analytic' index density of A_a manufactured from the local symbols of A_a. Also, by Proposition 4.1 to be proved shortly, see equation (4.8), ${}^b\eta_M(a,t)$ has an asymptotic expansion as $t \to 0$ in powers and log-powers of $t^{1/2m}$; we denote the constant term by ${}^b\eta_{M,0}(a)$. Thus, taking the constant term in the expansion as $t \to 0$ on the right-hand side of (4.2), we obtain

$$ \mathrm{ind}_\alpha A = {}^b\!\!\int_X \omega_{A_a} - \frac{1}{2} \sum_{M \in M(X)} {}^b\eta_{M,0}(a). $$

<u>*Step 2:*</u> We now take $a \to 0$:

$$ \mathrm{ind}_\alpha A = \lim_{a\to 0} \left\{ {}^b\!\!\int_X \omega_{A_a} - \frac{1}{2} \sum_{M \in M'(X)} {}^b\eta_{M,0}(a) \right\}, $$

where the limit as the multi-index $a \to 0$ is taken so that $0 < |a| < \varepsilon$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$. Since ω_{A_a} is constructed from the local symbols of $A_a = \rho^{-a} A \rho^a$, we have $\lim_{a\to 0} \omega_{A_a} = \omega_A$, which implies that

$$ \mathrm{ind}_\alpha A = {}^b\!\!\int_X \omega_A - \frac{1}{2} \lim_{a\to 0} \left\{ \sum_{M \in M'(X)} {}^b\eta_{M,0}(a) \right\}. \tag{4.4} $$

<u>*Step 3:*</u> Finally, we analyze the limits of the eta invariants, which is done in Section 4.3. In particular, in Lemma 4.2 we prove that if $M \in M_k(X)$ with $k \geq 2$, then

$$ \lim_{a\to 0} {}^b\eta_{M,0}(a) = {}^b\eta_M, $$

where ${}^b\eta_M$ is defined in (3.4), and if $M = H \in M_1(X)$, then in Lemma 4.4 we prove that

$$ \lim_{a\to 0} {}^b\eta_{H,0}(a) = {}^b\tilde{\eta}_H + \mathrm{sgn}(\alpha_H) \cdot \sum_{z \in \mathrm{spec}_H(A)} \mathrm{rk}_H(z) + \beta_H, $$

where ${}^b\tilde{\eta}_H$ is defined in (3.8), $\mathrm{rk}_H(z)$ in (3.6), and β_H in (3.9). These two limits together with the formula (4.4) establish our Index Theorem 3.4. The proofs of the limits of ${}^b\eta_{M,0}(a)$ and ${}^b\eta_{H,0}(a)$ as $a \to 0$ turn out to be quite technical, and can be skipped over without losing continuity in the paper.

4.2. **Parameter-dependent trace expansions.** In this section we prove a general result that will be used in the next section to analyze the smoothness properties of the functions ${}^b\eta_{M,0}(a)$ in the multi-index a.

Proposition 4.1. *Let $M \in M_k(X)$ with $k \geq 1$ and let $C_s(\tau) \in \Psi^{m',\delta}_{b,S}(M,F)$ depend smoothly on a parameter s, where $m' \in \mathbb{R}$, $\delta > 0$, and S is a strip containing $\mathbb{R}^k$.*

Let $B_s \in \Psi_b^m(X,F)$, $m \in \mathbb{R}^+$, be elliptic, depend smoothly on the same parameter s, and have a positive definite principal symbol. Then as $t \to 0$ we have

$$\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, C_s(\tau)\, N_M(e^{-tB_s})(\tau)\,)\, đ\tau \sim \sum_{j=-n}^{\infty} t^{(j+m')/2m}\, \xi_j(s) + \sum_{j=0}^{\infty} (\eta_j(s) + \zeta_j(s) \log t)\, t^j,$$

where $\xi_j(s)$, $\eta_j(s)$, and $\zeta_j(s)$ depend smoothly on s, and where the log coefficient $\zeta_j(s)$ is nonzero only if $2mj - m' + n \in \mathbb{N}_0$.

Proof. First of all, by [21] the heat operator e^{-tB_s} exists. It suffices to prove this proposition for s restricted to any compact neighborhood in its parameter space. In this case we can choose $C_s \in \Psi_b^{m',\delta'}(X,F)$, for some $\delta' > 0$, depending smoothly in s such that $N_M(C_s)(\tau) = C_s(\tau)$ for real τ (cf. Lemma 2.2). Since the normal operator preserves composition,

$$\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, C_s(\tau)\, N_M(e^{-tB_s})(\tau)\,)\, đ\tau = \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\, N_M(\, C_s\, e^{-tB_s}\,)(\tau)\,)\, đ\tau.$$

If M is a component of $H_{i_1} \cap \cdots \cap H_{i_k}$, then it follows from the definition of the b-trace (see [22] or [19, Lem. 6.5]) that

$$\int_{\mathbb{R}^k} {}^b\mathrm{Tr}(N_M(\, C_s\, e^{-tB_s}\,)(\tau))\, đ\tau = \mathrm{Reg}_{z=0} z_{i_1} \cdots z_{i_k} \int_X (\, \rho^z\, C_s\, e^{-tB_s}\,)|_{\mathrm{Diag}}. \tag{4.5}$$

Since $C_s \in \Psi_b^{m',\delta'}(X,F)$, by Corollary 6.4 in [21],

$$(\, C_s\, e^{-tB_s}\,)|_{\mathrm{Diag}} \sim \sum_{j=-n}^{\infty} t^{(j+m')/2m}\, \xi_j(s,x) + \sum_{j=0}^{\infty} (\eta_j(s,x) + \zeta_j(s,x) \log t)\, t^j, \tag{4.6}$$

for some smooth functions $\eta_j(s,x)$, $\eta_j'(s,x)$, $\eta_j''(s,x)$ of $x \in X$, where the log coefficient $\zeta_j(s,x)$ is nonzero only if $2mj - m' + n \in \mathbb{N}_0$. Moreover, since the operators C_s and B_s are smooth in s, the analysis in [21] shows that the coefficients in the asymptotic expansion are also smooth functions in the parameter s. Substituting the expansion (4.6) into the equation (4.5) now proves the result. □

4.3. **Analysis of the eta invariants.** We now compute the limits

$$\lim_{a \to 0} {}^b\eta_{M,0}(a), \quad \text{where } M \in M_k(X) \text{ with } k \geq 1,$$

of the eta invariants appearing in (4.4). Recall that $\varepsilon > 0$ is chosen as in Lemma 3.2 and a multi-index α is fixed with $0 < |\alpha| < \varepsilon$. If $0 < |a| < \varepsilon$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$, then by definition,

$${}^b\eta_M(a,t) = 2 \int_{\mathbb{R}^k} {}^b\mathrm{Tr}(\mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1}\, N_M(e^{-tA_a A_a^*})(\tau))\, đ\tau, \tag{4.7}$$

where $M \in M_k(X)$ with $k \geq 1$. By Theorem 2.3 and the composition properties in Lemma 2.1, $C_a(\tau) = \mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1}$ is an element of $\Psi_{b,S}^{-k,\delta}(M,F)$ for some $\delta > 0$ and some strip S containing $\mathbb{R}^k$. Therefore, applying Proposition

4.1 with $B_a = A_a A_a^*$, whose principal symbol is $a_m a_m^*$ where a_m is the principal symbol of A, we see that for fixed a, we have an expansion as $t \to 0$,

$$
{}^b\eta_M(a,t) \sim \sum_{-n \le j < k} t^{(j-k)/2m}\, {}^b\xi_{M,j}(a) + \sum_{j=0}^{\infty} ({}^b\eta_{M,j}(a) + {}^b\zeta_{M,j}(a) \log t)\, t^j. \tag{4.8}
$$

In the following lemma we prove that ${}^b\eta_M(a,t)$ *extends* from the a priori assumptions that $|a| > 0$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$, to define a smooth function of the multi-index a where $|a| < \varepsilon$ without any conditions on $\mathrm{sgn}(a)$. If $k \ge 2$, then we prove that ${}^b\eta_{M,0}(a)|_{a=0} = {}^b\eta_M$, where ${}^b\eta_M$ is defined in (3.4). If $k = 1$, then ${}^b\eta_{M,0}(a)|_{a=0}$ is much harder to analyze and we leave this case to Lemma 4.4.

Lemma 4.2. *Let ${}^b\eta_M(a,t)$ be the function in* (4.7) *where $M \in M_k(X)$ with $k \ge 1$, $0 < |a| < \varepsilon$, and* $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$. *Then ${}^b\eta_M(a,t)$ and the coefficients ${}^b\xi_{M,j}(a)$, ${}^b\eta_{M,j}(a)$, and ${}^b\zeta_{M,j}(a)$, extend from the a priori conditions that $|a| > 0$ and* $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$ *to define smooth functions of the multi-index a where $|a| < \varepsilon$. In particular, ${}^b\eta_{M,0}(a)$ defines a smooth function of a with $|a| < \varepsilon$. Moreover, if $k \ge 2$, then*

$$
{}^b\eta_{M,0}(a)|_{a=0} = {}^b\eta_M, \quad k \ge 2,
$$

where ${}^b\eta_M$ is defined in (3.4).

Proof. Using the notation of (2.11), if $M \in M_k(X)$ is defined by $\rho_{i_1}, \dots, \rho_{i_k}$, then we can write $N_M(\rho^{-a} A \rho^a)(\tau) = \varrho^{-\beta} N_M(A)(\tau - i\gamma)\varrho^{\beta}$ where β and γ depend on the multi-index a by $\gamma = \{a_{i_1}, \dots, a_{i_k}\}$ and $\beta = \{a_j \,;\, j \ne i_1, \dots, i_k\}$. Assume that $k \ge 2$. Then by Lemma 3.2, $\varrho^{-\beta} N_M(A)(\tau - i\xi)\varrho^{\beta}$ is invertible for all $\tau, \xi \in \mathbb{R}^k$ with $|\xi| < \varepsilon$. Thus, for a restricted to any compact subset of $|a| < \varepsilon$, even without the a priori assumptions that $|a| > 0$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$, the operator

$$
C_a(\tau) = \mathbf{D}_\tau^k N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1},
$$

is, according to Theorem 2.3 and the composition properties in Lemma 2.1, an element of $\Psi_{b,S}^{-k,\delta}(M,F)$ for some $\delta > 0$ and some strip S containing $\mathbb{R}^k$. Applying Proposition 4.1 with this $C_a(\tau)$ and $B_a = A_a A_a^*$, we see that ${}^b\eta_M(a,t)$ has the expansion (4.8) with coefficients smooth in a.

Assume now that $k = 1$ and let us again start with the assumption that $0 < |a| < \varepsilon$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$. In this case, the above argument doesn't apply to prove the smoothness of ${}^b\eta_M(a,t)$ or the trace coefficients, since $N_M(A)(\tau)$ is not invertible on $\mathbb{R}$. However, we can argue as follows. Using our usual notation for β and γ, observe that $N_M(\rho^{-a} A \rho^a)(\tau) = \varrho^{-\beta} N_M(A)(\tau - i\gamma)\varrho^{\beta}$ and

$$
\varrho^{\beta} N_M(e^{-tA_a A_a^*})(\tau)\varrho^{-\beta} = N_M(e^{-tB_a})(\tau),
$$

where $B_a = \rho^{\beta} A_a A_a^* \rho^{-\beta}$. Therefore,

$$
\begin{aligned}
{}^b\eta_M(a,t) &= 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(D_\tau N_M(A_a)(\tau)\, N_M(A_a)(\tau)^{-1}\, N_M(e^{-tA_a A_a^*})(\tau))\, đ\tau \\
&= 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(D_\tau N_M(A)(\tau - i\gamma)\, N_M(A)(\tau - i\gamma)^{-1}\, N_M(e^{-tB_a})(\tau))\, đ\tau. \qquad (4.9)
\end{aligned}
$$

Recall that $\gamma = a_M$ has the same sign as α_M. Assume that $\mathrm{sgn}(\alpha_M) > 0$ so that $\gamma > 0$; the opposite sign is handled similarly. Fix any γ_0 with $0 < \gamma_0 < \varepsilon$. We shall prove that ${}^b\eta_M(a,t)$ and the trace coefficients with $0 < |a| < \gamma_0$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$ extend to define smooth functions of the multi-index a where $-(\varepsilon - \gamma_0) < |a| < \gamma_0$; this will complete our proof. So, assume for the moment that $0 < |a| < \gamma_0$ and

$\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$. In particular, since $0 < a_M = \gamma < \gamma_0 < \varepsilon$, we can shift the contour $\operatorname{Im}\tau = 0$ (the real line) in the integral for ${}^b\eta_M(a,t)$ down to $\operatorname{Im}\tau = -(\varepsilon - \gamma_0)$:

$$ {}^b\eta_M(a,t) = 2\int_{\operatorname{Im}\tau=-(\varepsilon-\gamma_0)} {}^b\mathrm{Tr}(D_\tau N_M(A)(\tau - i\gamma)N_M(A)(\tau - i\gamma)^{-1}N_M(e^{-tB_a})(\tau))\, đ\tau. $$

This shift is valid because $N_M(A)(\tau - i\gamma)^{-1}$ is holomorphic for $-\varepsilon < \operatorname{Im}\tau - \gamma < 0$, which holds in particular for any τ with $-(\varepsilon - \gamma_0) = -\varepsilon + \gamma_0 \le \operatorname{Im}\tau \le 0$ since $0 < \gamma < \gamma_0$. We can write this integral as

$$ {}^b\eta_M(a,t) = 2\int_{\mathbb{R}} {}^b\mathrm{Tr}\big(D_\tau N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))\cdot \\ N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))^{-1}\, N_M(e^{-tB_a})(\tau - i(\varepsilon - \gamma_0))\big)\, đ\tau. $$

If $\widetilde{B}_a = \rho_H^{-(\varepsilon-\gamma_0)} B_a \rho_H^{\varepsilon-\gamma_0}$, where ρ_H is the defining function for H, then we can further write

$$ {}^b\eta_M(a,t) = 2\int_{\mathbb{R}} {}^b\mathrm{Tr}\big(D_\tau N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))\cdot \\ N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))^{-1}\, N_M(e^{-t\widetilde{B}_a})(\tau)\big)\, đ\tau. $$

If $-(\varepsilon - \gamma_0) < \xi < \gamma_0$, then $0 < \xi + \varepsilon - \gamma_0 < \varepsilon$, so $N_M(A)(\tau - i(\xi + \varepsilon - \gamma_0))^{-1}$ is invertible for all $\tau \in \mathbb{R}$. Thus, for γ restricted to any compact subset of the interval $(-(\varepsilon - \gamma_0), \gamma_0)$, even without the a priori assumption that $\gamma > 0$, the operator

$$ C_a(\tau) = D_\tau N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))\, N_M(A)(\tau - i(\gamma + \varepsilon - \gamma_0))^{-1}, $$

is, according to Theorem 2.3 and the composition properties in Lemma 2.1, an element of $\Psi^{-1,\delta}_{b,S}(M,F)$ for some $\delta > 0$ and some strip S containing $\mathbb{R}$. Applying Proposition 4.1, we see that ${}^b\eta_M(a,t)$ has the expansion (4.8) where the coefficients have the required smoothness. □

Let $H \in M_1(X)$. We now analyze the constant term ${}^b\eta_{H,0}(a)$ in (4.8) at $a = 0$ (here we change M in (4.8) to H). Recall that we are working under the assumption that $0 < |a| < \varepsilon$ and $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$. The key to analyzing ${}^b\eta_{H,0}(a)$ is formula (4.9) in the previous lemma (after changing M to H):

$$ {}^b\eta_H(a,t) = 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(D_\tau N_H(A)(\tau - i\gamma)\, N_H(A)(\tau - i\gamma)^{-1}N_H(e^{-tB_a})(\tau))\, đ\tau, $$

where $B_a = \rho^\beta A_a A_a^* \rho^{-\beta}$ with $\gamma = a_H$ and $\beta = \{\alpha_{H'}\,;\, H' \neq H\}$. By Lemma 4.2, ${}^b\eta_H(a,t)$ and the coefficients in its trace expansion as $t \to 0$ extend from the a priori assumptions that $|a| > 0$ and $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$, to be smooth functions of the multi-index a where $|a| < \varepsilon$ without any conditions on $\operatorname{sgn}(a)$. Now according to (3.5), we can write

$$ D_\tau N_H(A)(\tau)\, N_H(A)(\tau)^{-1} = S(\tau) + G(\tau), $$

where $G(\tau)$ is holomorphic on the strip $|\operatorname{Im}\tau| < \varepsilon$ and $S(\tau)$ is of the form

$$ S(\tau) = \sum_z S_z(\tau), \qquad S_z(\tau) = \sum_{j=0}^{\nu-1} \frac{K_{z,j}}{(\tau - z)^{j+1}}, $$

where the first sum is over z in $\mathrm{spec}_H(A)$, and in the second sum, ν is an integer depending on z and the $K_{z,j}$'s are finite rank operators. Thus,

$$
{}^b\eta_H(a,t) = 2\int_{\mathbb{R}} \mathrm{Tr}(\, S(\tau - i\gamma)\, N_H(e^{-tB_a})(\tau)\,)\, đ\tau \\ + 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(\, G(\tau - i\gamma)\, N_H(e^{-tB_a})(\tau)\,)\, đ\tau, \tag{4.10}
$$

where we used the fact that $S(\tau)$ consists of finite rank operators, so the b-trace in the first term is just the regular trace. In the following (unfortunately long and technical, but) crucial lemma we analyze the first term on the right.

Lemma 4.3. *We can write*

$$
2\int_{\mathbb{R}} \mathrm{Tr}(\, S(\tau - i\gamma)\, N_H(e^{-tB_a})(\tau)\,)\, đ\tau \\ = \beta_H + \mathrm{sgn}(\alpha_H)\cdot \sum_{z\in \mathrm{spec}_H(A)} \mathrm{rk}_H(z) + f(a,t), \tag{4.11}
$$

where $\mathrm{rk}_H(z)$ *is defined in* (3.6), β_H *in* (3.9), *and where* $f(a,t)$ *extends from the a priori conditions that* $|a| > 0$ *and* $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$ *to define a smooth function of the multi-index* a *where* $|a| < \varepsilon$ *and* $t \in (0,\infty)$. *Moreover,* $f(a,t)$ *is continuous for* $|a| < \varepsilon$ *and* $t \in [0,\infty)$ *with* $f(a,0) = 0$ *for all* a *with* $|a| < \varepsilon$.

Proof. Assume that $\mathrm{sgn}(\alpha_H) > 0$ so that $\gamma > 0$; we shall comment on the case with the opposite sign at the end of the proof. Let $0 < \gamma < \gamma_0 < \varepsilon$. Then using the same trick of shifting the contour as explained in the previous lemma, starting from equation (4.9) and below, we can write the integral (4.10) as

$$
2\int_{\mathbb{R}} \mathrm{Tr}(\, S(\tau - i(\gamma + \varepsilon - \gamma_0))N_H(e^{-t\widetilde{B}_a})(\tau)\,)\, đ\tau, \tag{4.12}
$$

where $\widetilde{B}_a = \rho_H^{-(\varepsilon-\gamma_0)} B_a \rho_H^{\varepsilon-\gamma_0}$. This shows that the integral (4.11) extends to be a smooth function of the multi-index a where $-(\varepsilon - \gamma_0) < |a| < \gamma_0$ for $t > 0$. The arbitrariness of γ_0 implies that the integral (4.11) extends to be a smooth function of the multi-index a where $|a| < \varepsilon$ for $t > 0$. We also need to prove that this integral extends to define a continuous function of a with $|a| < \varepsilon$ and of t down to $t = 0$, with (4.11) holding.

Substituting in the formula for $S(\tau)$ in terms of the $K_{z,j}$'s, and setting $\widetilde{z} = z + i\{\gamma + \varepsilon - \gamma_0\}$, the integral (4.12) becomes

$$
\sum_{z,j} 2\int_{\mathbb{R}} \mathrm{Tr}\left(\frac{K_{z,j}\, N_H(e^{-t\widetilde{B}_a})(\tau)}{(\tau - \widetilde{z})^{j+1}} \right) đ\tau,
$$

where the sum of over all $z \in \mathrm{spec}_H(A)$ and where, for fixed z, the index j goes from 0 to the order of the pole at z minus one. Because τ^{-2} integrable near $|\tau| = \infty$ it follows that the sum over $j \geq 1$ is continuous at $t = 0$ with

$$
\sum_{z,j\geq 1} 2\int_{\mathbb{R}} \mathrm{Tr}\left(\frac{K_{z,j}\, N_H(e^{-t\widetilde{B}_a})(\tau)}{(\tau - \widetilde{z})^{j+1}} \right) đ\tau \Bigg|_{t=0} \\ = \sum_{z,j\geq 1} 2\int_{\mathbb{R}} \mathrm{Tr}\left(\frac{K_{z,j}}{(\tau - \widetilde{z})^{j+1}} \right) đ\tau = 0,
$$

where we used the fact that $e^{-t\widetilde{B}_a} = \mathrm{Id}$ at $t = 0$ and $\operatorname{Tr} K_{z,j} = 0$ for $j \geq 1$ [10, eqn. (2.6)]. Therefore, it remains to prove that

$$\sum_z 2 \int_{\mathbb{R}} \operatorname{Tr}\left(\frac{K_{z,0}\, N_H(e^{-t\widetilde{B}_a})(\tau)}{\tau - \widetilde{z}} \right) đ\tau$$

is continuous at $t = 0$ with value equal to the right-hand side in (4.11) at $t = 0$. This fact is not at all obvious because formally setting $t = 0$ into the integral results in an integral that diverges logarithmically at $|\tau| = \infty$! However, the integral actually is continuous at $t = 0$ and to prove this we proceed as follows. First, we make the change of variables, $\tau \longmapsto t^{-1/2m}\tau$, which results in the integral

$$\sum_z 2 \int_{\mathbb{R}} \operatorname{Tr}\left(\frac{K_{z,0}\, f(a,t,\tau)}{\tau - t^{1/2m}\widetilde{z}} \right) đ\tau, \quad f(a,t,\tau) = N_H(e^{-t\widetilde{B}_a})(t^{-1/2m}\tau). \tag{4.13}$$

We claim that for $\tau \neq 0$ (see around (3.9) for the notation):

$$f(a,t,\tau) = e^{-\iota_H^*(a_m a_m^*)(\tau)} + g(a,t,\tau), \quad \tau \neq 0, \tag{4.14}$$

where $g(a,t,\tau)$ is smooth in a, continuous in $t \in [0,1]$ and vanishing at $t = 0$, vanishes like $|\tau|^{-1}$ as $|\tau| \to \infty$, and finally, is bounded uniformly for a within any bounded set and for $t \in [0,1]$ and $\tau \in \mathbb{R} \setminus 0$. To prove this, note that according to the construction of the heat kernel found in [21], we know that the heat kernel $e^{-t\widetilde{B}_a}$ localizes near the diagonal in X^2 as $t \to 0$ up to a term supported off the diagonal of order $-\infty$ that vanishes at $t = 0$. It is also clear from the construction that the heat kernel depends smoothly on the parameter a. Since the normal operator of an operator of order $-\infty$ vanishes to infinite order as $|\tau| \to \infty$, to prove the properties of $f(a,t,\tau)$ it suffices to consider the Schwartz kernel of $e^{-t\widetilde{B}_a}$ near the diagonal on a coordinate patch. Let $\mathcal{U} = [0,\epsilon)_s \times \mathcal{V}_y$ be a coordinate patch on X where $s = \rho_H$. Since the principal symbol of $\widetilde{B}_a$ equals $a_m\, a_m^*$ where a_m is the principal symbol of A, by the structure of the heat kernel found in [21], we can write (cf. (2.7) and (2.9)),

$$e^{-t\widetilde{B}_a}|_{\mathcal{U}^2} = \int e^{i(\log s - \log s', w)\cdot(\tau,\eta)}\, h(a,t,s,y,\tau,\eta)\, đ\tau đ\eta,$$

where w is the normal to the diagonal in $\mathcal{V}^2$ in logarithmic coordinates, and where $h(a,t,s,y,\tau,\eta)$ is a symbol of order zero in (τ,η) uniformly in s, y and $t \geq 0$ and of order $-\infty$ for t bounded below by any fixed, but arbitrary, positive constant. Moreover,

$$h(a,t,s,y,\tau,\eta) = e^{-t\, a_m(s,y,\tau,\eta)\, a_m(s,y,\tau,\eta)^*} + h_1(a,t,s,y,\tau,\eta),$$

where $h_1(t,s,y,\tau,\eta)$ vanishes at $t = 0$ and is a symbol of order -1 in (τ,η) uniformly in s, y. Thus, taking normal operators, we obtain

$$f(a,t,\tau)|_{\mathcal{V}^2} = \int e^{iw\cdot\eta}\, h(a,t,0,y,\tau,\eta)\, đ\eta.$$

Replacing τ with $t^{-1/2m}\tau$ and using the homogeneity of the principal symbol a_m, we get

$$f(a,t,\tau)|_{\mathcal{V}^2} = \int e^{iw\cdot\eta}\, e^{-a_m(0,y,\tau,t^{1/2m}\eta)\, a_m(0,y,\tau,t^{1/2m}\eta)^*}\, đ\eta$$
$$+ \int e^{iw\cdot\eta}\, h_1(a,t,0,y,t^{-1/2m}\tau,\eta)\, đ\eta.$$

Using this equation, together with the properties of h_1, and the fact that as $t \to 0$,

$$\int e^{iw\cdot\eta}\, e^{-a_m(0,y,\tau,t^{1/2m}\eta)\, a_m(0,y,\tau,t^{1/2m}\eta)^*}\, đ\eta \to e^{-a_m(0,y,\tau,0)\, a_m(0,y,\tau,0)^*}$$

(since $\int e^{iw\cdot\eta}\, đ\eta$ is the Schwartz kernel of the identity operator in local coordinates) we obtain the decomposition (4.14). Now we write (4.13) as

$$(4.15)\quad \sum_z 2\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right) \mathrm{Tr}\left(\frac{K_{z,0}\, f(a,t,\tau)}{\tau - t^{1/2m}\widetilde{z}}\right) đ\tau$$
$$+ \sum_z 2\int_{-1}^{1} \mathrm{Tr}\left(\frac{K_{z,0}\, \{f(a,t,\tau)-1\}}{\tau - t^{1/2m}\widetilde{z}}\right) đ\tau + \sum_z 2\int_{-1}^{1} \mathrm{Tr}\left(\frac{K_{z,0}}{\tau - t^{1/2m}\widetilde{z}}\right) đ\tau.$$

By elementary complex analysis, we see that

$$(4.16)\quad 2\int_{-1}^{1} \mathrm{Tr}\left(\frac{K_{z,0}}{\tau - t^{1/2m}\widetilde{z}}\right) đ\tau = \frac{\mathrm{Tr}(K_{z,0})}{\pi}\Big\{\log(1 - t^{1/2m}\widetilde{z})$$
$$- \log(-1 - t^{1/2m}\widetilde{z})\Big\} \xrightarrow{t\to 0} \frac{\mathrm{Tr}(K_{z,0})}{\pi}\{0 - (-\pi)\} = \mathrm{Tr}(K_{z,0}),$$

where at this point we used the fact that $\widetilde{z} = z + i\{\gamma + \varepsilon - \gamma_0\}$ has positive imaginary part under the assumption that we started with $\mathrm{sgn}(\alpha_H) > 0$. Thus, taking $t \to 0$ in (4.15) and using (4.14), the sum (4.15) approaches the following limit:

$$2\sum_z \mathrm{Tr}\left(K_{z,0}\cdot\left\{\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right) e^{-\iota_H^*(a_m a_m^*)(\tau)}\, \frac{đ\tau}{\tau}\right.\right.$$
$$\left.\left. + \int_{-1}^{1} \frac{e^{-\iota_H^*(a_m a_m^*)(\tau)} - 1}{\tau}\, đ\tau\right\}\right) + \sum_z \mathrm{Tr}(K_{z,0}) = \beta_H + \sum_{z\in \mathrm{spec}_H(A)} \mathrm{rk}_H(z),$$

by definition of β_H and the fact that $\mathrm{Tr}(K_{z,0}) = \mathrm{rk}_H(z)$ [10, eqn. (2.5)]. If $\mathrm{sgn}(\alpha_H) < 0$, then the plus sign on the right becomes a minus sign in view of our discussion following (4.16). Our lemma is now proved. □

Lemma 4.4. *Let $H \in M_1(X)$ and let ${}^b\widetilde{\eta}_H(t)$ be defined as in* (3.7). *Then as $t \to 0$,*

$${}^b\widetilde{\eta}_H(t) \sim \sum_{-n\le j<1} t^{(j-1)/2m}\, {}^b\xi_{H,j} + {}^b\widetilde{\eta}_H + {}^b\zeta_H \log t + o(t),$$

where (see (3.8)*) ${}^b\widetilde{\eta}_H$ is by definition the constant coefficient in this expansion. Let ${}^b\eta_{H,0}(a)$ be the constant coefficient in the expansion* (4.8) *for $M = H$ with $0 < |a| < \varepsilon$ and $\mathrm{sgn}(a) = \mathrm{sgn}(\alpha)$. Then,*

$${}^b\eta_{H,0}(a)|_{a=0} = {}^b\widetilde{\eta}_H + \mathrm{sgn}(\alpha_H)\cdot \sum_{z\in\mathrm{spec}_H(A)} \mathrm{rk}_H(z) + \beta_H.$$

Proof. Since $G(\tau)$ is holomorphic on the strip $|\operatorname{Im}\tau| < \varepsilon$, taking $a \to 0$ in (4.10) and using our previous lemma, we obtain

$$(4.17)\quad {}^b\eta_H(0,t) = \beta_H + \mathrm{sgn}(\alpha_H)\cdot\sum_{z\in\mathrm{spec}_H(A)} \mathrm{rk}_H(z) + f(t)$$
$$+ 2\int_{\mathbb{R}} {}^b\mathrm{Tr}(\, G(\tau)\, N_M(e^{-tAA^*})(\tau)\,)\, đ\tau,$$

where the limit is taken so that $0 < |a| < \varepsilon$ and $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$, and where $f(t)$ is continuous in $t \in [0, \infty)$ vanishing at $t = 0$. By Lemma 4.2, we know that ${}^b\eta_H(0,t)$ has a full asymptotic expansion as $t \to 0$ of the form

$$ {}^b\eta_H(0,t) \sim \sum_{-n \le j < 1} t^{(j-1)/2m}\, {}^b\xi_{H,j}(0) + \sum_{j=0}^{\infty} ({}^b\eta_{H,j}(0) + {}^b\zeta_{H,j}(0) \log t)\, t^j. $$

Substituting in this expansion for the left-hand side of (4.17) and solving for ${}^b\tilde{\eta}_H(t)$ proves our lemma. □

5. Characterization of Fredholm b-pseudodifferential operators

We begin with the following technical, but fundamental lemma, whose proof can be skipped over without losing continuity in the paper.

Lemma 5.1. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$. Suppose that there is an $\varepsilon > 0$ such that for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$ A\colon \rho^\alpha H_b^m(X,E) \longrightarrow \rho^\alpha L_b^2(X,F) \tag{5.1} $$

is Fredholm, and for some fixed $k \ge 2$, $N_M(A)(\tau)$ is Fredholm for all $M \in M_k(X)$ and $\tau \in \mathbb{R}^k$. Then $N_M(A)(\tau)$ is in fact invertible for all $M \in M_k(X)$ and $\tau \in \mathbb{R}^k$.

Proof. Throughout this proof, we shall use formula (2.11): If $M \in M_k(X)$ is defined by $\rho_{i_1}, \dots, \rho_{i_k}$ where $i_1 < \cdots < i_k$, in other words M is one of the connected components of $H_{i_1} \cap \cdots \cap H_{i_k}$, then

$$ N_M(\rho^{-\alpha} A \rho^\alpha)(\tau) = \varrho^{-\beta} N_M(A)(\tau - i\gamma) \varrho^\beta, \tag{5.2} $$

where $\gamma = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ and $\beta = \{\alpha_j\,;\, j \ne i_1, \dots, i_k\}$. The proof of our lemma is based on the following inductive statement: We show that for $\ell = 1, 2, \dots, k$ and for $M \in M_k(X)$, using the notation (5.2),

$$ \begin{aligned} &\varrho^{-\beta} N_M(A)(\tau - i\xi)\varrho^\beta \text{ is invertible for all } \tau \in \mathbb{R}^k,\ 0 < |\beta| < \varepsilon, \text{ and} \\ &\qquad 0 \le |\xi| < \varepsilon \text{ where at most } \ell \text{ of the } \xi_j\text{'s can be set to zero at once.} \end{aligned} \tag{5.3} $$

To start, we must prove that (5.3) holds where at most one of the ξ_j's can be zero. Consider first the case when only ξ_1 can be set to zero and when $H_1 \cap \cdots \cap H_k$ is nonempty; we shall prove (5.3) with $\ell = 1$ and where ξ_1 can be set to zero for any component of this intersection. A similar argument proves this statement for components of $H_{i_1} \cap \cdots \cap H_{i_k}$ where $i_1 < \cdots < i_k$ and where at most one ξ_j can be set to zero. Let $M' \in M_{k-1}(X)$ be defined by $\rho_2, \dots, \rho_k$ such that the hypersurface H_1 intersects M'. Then M' is a manifold with corners whose boundary hypersurfaces are elements of $M_k(X)$, at least one of which is in $H_1 \cap \cdots \cap H_k$. Let $r = \rho_2 \cdot \rho_3 \cdots \rho_N$ and for any multi-index $\alpha' = \{\alpha_j\,;\, j \ne 1\}$, we define

$$ A_{\alpha',v} := N_{M'}(r^{-\alpha'} A r^{\alpha'})(v), \quad v \in \mathbb{R}^{k-1}. $$

For any $v \in \mathbb{R}^{k-1}$ and α with $0 < |\alpha| < \varepsilon$ where $\alpha = \{\alpha_1\} \cup \alpha'$, we claim that

$$ \rho_1^{-\alpha_1} A_{\alpha',v} \rho_1^{\alpha_1}\colon H_b^m(M',E) \longrightarrow L_b^2(M',F) $$

is Fredholm. Indeed, a hypersurface $M \in M_1(M')$ is just an element $M \in M_k(X)$ with $M \subset M'$. It follows that $N_M(\rho_1^{-\alpha_1} A_{\alpha',v} \rho_1^{\alpha_1})(u) = N_M(\rho^{-\alpha} A \rho^\alpha)(u,v)$, which is invertible for all $(u,v) \in \mathbb{R}^k$, since by assumption the map (5.1) is Fredholm. This proves our claim. Now by Lemma 3.1, for all α bounded by a fixed constant,

$\rho_1^{\mp\alpha_1} A_{\alpha',v} \rho_1^{\pm\alpha_1}$ is invertible for all $v \in \mathbb{R}^{k-1}$ with $|v|$ sufficiently large. Since the index is invariant under continuous deformations, it follows that $\mathrm{ind}_{\pm\alpha_1} A_{\alpha',v} = 0$ for all $v \in \mathbb{R}^{k-1}$ and α_1, α' with $0 < |\alpha_1|, |\alpha'| < \varepsilon$. In particular, by the relative index formula (3.11), we have

$$0 = 0 - 0 = \mathrm{ind}_{\alpha_1} A_{\alpha',v} - \mathrm{ind}_{-\alpha_1} A_{\alpha',v} = -\operatorname{sgn}(\alpha_1) \sum \mathrm{rk}_M(z), \tag{5.4}$$

where the sum is over those $M \in M_1(M')$ such that considered as elements of $M_k(X)$ are defined by $\rho_1, \rho_2, \ldots, \rho_k$, and where $\mathrm{rk}_M(z)$ denotes the rank of a pole at $u = z \in \mathbb{R}$ of the inverse $u \longmapsto N_M(A_{\alpha',v})(u)^{-1} = N_M(r^{-\alpha'} A r^{\alpha'})(u,v)^{-1}$. Of course, as the left hand side of (5.4) is zero, it follows that $N_M(r^{-\alpha'} A r^{\alpha'})(u,v)^{-1}$ has no poles for any $u \in \mathbb{R}$; that is, $N_M(r^{-\alpha'} A r^{\alpha'})(u,v)$ is invertible for $u \in \mathbb{R}$. In view of (5.2), we conclude that $\varrho^{-\beta} N_M(A)(\tau - i\gamma)\varrho^{\beta}$ is invertible for all $\tau \in \mathbb{R}^k$ with $0 < |\gamma|, |\beta| < \varepsilon$ and even with $\gamma_1 = 0$. This proves (5.3) for $\ell = 1$.

Assume that (5.3) holds for $\ell - 1$; we prove that it holds for ℓ. Again suppose that $H_1 \cap \cdots \cap H_k$ is nonempty. Fix any ℓ of these indices, say $j_1 < \cdots < j_\ell$; we shall prove that (5.3) holds for ℓ such that only $\xi_{j_\nu} = 0$ where $\nu = 1, \ldots, \ell$ and M is a component of $H_1 \cap \cdots \cap H_k$. A similar argument proves this statement for components of $H_{i_1} \cap \cdots \cap H_{i_k}$ where $i_1 < \cdots < i_k$. Let $M' \in M_{k-1}(X)$ be a component of $\bigcap_{j \neq j_1} H_j$ such that H_{j_1} intersects M'. Then M' is a manifold with corners whose boundary hypersurfaces are elements of $M_k(X)$, at least one of which is in $H_1 \cap \cdots \cap H_k$. Let r be the product of all the boundary defining functions on X except $\rho_{j_1}, \ldots, \rho_{j_\ell}$. For a multi-index $\alpha' = \{\alpha_j \,;\, j \neq j_1, \ldots, j_\ell\}$, we define $A_{\alpha',v} := N_{M'}(r^{-\alpha'} A r^{\alpha'})(v)$ where $v \in \mathbb{R}^{k-1}$. For any $v \in \mathbb{R}^{k-1}$ and α with $0 < |\alpha| < \varepsilon$ where $\alpha = \{\alpha_{j_1}\} \cup \alpha'$ (with $\alpha_{j_2}, \ldots, \alpha_{j_\ell}$ set equal to zero), we claim that the operator

$$\rho_{j_1}^{-\alpha_{j_1}} A_{\alpha',v} \rho_{j_1}^{\alpha_{j_1}} \colon H_b^m(M', E) \longrightarrow L_b^2(M', F)$$

is Fredholm. Indeed, in view of (5.2), the normal operator of $\rho_{j_1}^{-\alpha_{j_1}} A_{\alpha',v} \rho_{j_1}^{\alpha_{j_1}} = \rho_{j_1}^{-\alpha_{j_1}} N_{M'}(r^{-\alpha'} A r^{\alpha'})(v) \rho_{j_1}^{\alpha_{j_1}}$ at any hypersurface of M' (which are codimension k faces of X) will be of the form (5.3) with at most $\ell - 1$ of the ξ_j's zero, namely those corresponding to $j_2, j_3, \ldots, j_\ell$. By our induction hypothesis, these normal operators are invertible and so proves our claim. Then by similar arguments found in the previous paragraph for the $\ell = 1$ case, it follows that for $v \in \mathbb{R}^{k-1}$ and $0 < |\alpha| < \varepsilon$, $\rho_{j_1}^{-\alpha_{j_1}} A_{\alpha',v} \rho_{j_1}^{\alpha_{j_1}}$ has index zero and using the relative index theorem exactly as we did in (5.4), one proves that for each component $M \in M_k(X)$ of $H_1 \cap \cdots \cap H_k$, the inverse $N_M(r^{-\alpha'} A r^{\alpha'})(\tau)^{-1}$ has no poles for $\tau_{j_1} \in \mathbb{R}$. Using the notation of (5.2) for β and γ, we conclude that $\varrho^{-\beta} N_M(A)(\tau - i\gamma)\varrho^{\beta}$ is invertible for all $\tau \in \mathbb{R}^k$ with $0 < |\gamma|, |\beta| < \varepsilon$ and even with $\gamma_{j_1} = 0$. Hence, $\varrho^{-\beta} N_M(A)(\tau - i\xi)\varrho^{\beta}$ is invertible for all $\tau \in \mathbb{R}^k$, $0 < |\beta| < \varepsilon$, and $0 \leq |\xi| < \varepsilon$ where only $\xi_{j_1}, \ldots, \xi_{j_\ell}$ can be set to zero. This proves (5.3) for each $\ell = 1, \ldots, k$.

We now finish our proof. Setting $\ell = k$ in (5.3), we conclude that for each $M \in M_k(X)$, $\varrho^{-\beta} N_M(A)(\tau)\varrho^{\beta}$ is invertible for all $\tau \in \mathbb{R}^k$ and $0 < |\beta| < \varepsilon$. We need to prove that $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k$. To see this, we know by Lemma 3.2 that $N_M(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}^k$ sufficiently large. By assumption, $N_M(A)(\tau)$ is Fredholm, so by the invariance of the index, $N_M(A)(\tau)$ has index zero for all $\tau \in \mathbb{R}^k$. Thus, $N_M(A)(\tau)$ is invertible if and only if it has no null space. So, suppose that $N_M(A)(\tau_0)u = 0$ for some u and $\tau_0 \in \mathbb{R}^k$. Then by Theorem 2.4, we know that $u \in \varrho^\delta H_b^\infty(M)$ for some $\delta > 0$, and by taking δ smaller if necessary we may assume that $\delta < \varepsilon$. In particular, we can write $u = \varrho^\delta v$ for some $v \in H_b^\infty(M)$. This implies that $\varrho^{-\delta} N_M(A)(\tau_0)\varrho^\delta v = 0$, which implies that $v = 0$ since $\varrho^{-\delta} N_M(A)(\tau_0)\varrho^\delta$ is known to be invertible. Thus, $u = 0$ and so $N_M(A)(\tau_0)$ is invertible. Our proof is now complete. □

Theorem 5.2. *Let $A \in \Psi_b^m(X, E, F)$, $m \in \mathbb{R}^+$. Then there is an $\varepsilon > 0$ such that for all multi-indices α with $0 < |\alpha| < \varepsilon$,*

$$A\colon \rho^\alpha H_b^m(X, E) \longrightarrow \rho^\alpha L_b^2(X, F)$$

is Fredholm, if and only if, A is elliptic and $N_M(A)(\tau)\colon H_b^m(M, E) \longrightarrow L_b^2(M, F)$ is invertible for each $M \in M_2(X)$ and $\tau \in \mathbb{R}^2$.

Proof. By Proposition 3.3, we need only prove necessity. By Theorem 2.3, $\rho^{-\alpha} A \rho^\alpha$ is elliptic. Since the principal symbol of $\rho^{-\alpha} A \rho^\alpha$ equals the symbol of A, it follows that A is elliptic. In particular, by Lemma 3.1, $N_M(A)(\tau)$ is elliptic for all $M \in M'(X)$ and all normal parameters τ.

Let $n' = \operatorname{codim}(X)$ with $n' \leq 2$ and let $M \in M_{n'}(X)$. Since $N_M(A)(\tau)$ is elliptic and M has no boundary, $N_M(A)(\tau)$ is a family of Fredholm operators on M. Therefore by our lemma, $N_M(A)(\tau)$ is invertible for all real τ. Let $M \in M_{n'-1}(X)$ and assume now that $n' < 2$. Then $N_M(A)(\tau)$ is elliptic for all real τ and by the fact proved in the previous paragraph, all normal operators of $N_M(A)(\tau)$ are invertible for all real normal parameters. It follows that $N_M(A)(\tau)$ is Fredholm for all real τ, and hence by our lemma, invertible. Using our lemma, we can continue by induction on $k = n'-2, n'-3, \ldots, 2$ proving that for each $M \in M_k(X)$, $N_M(A)(\tau)$ is invertible for all real normal parameters. This completes the proof of our theorem. □

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