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Dirac operators, Boundary Value Problems, and the b-Calculus

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ABSTRACT. It is well-known that the index of a Dirac operator with augmented Atiyah-Patodi-Singer (= APS) boundary conditions on a compact manifold with boundary can be identified with the L^2 index of a corresponding operator on a manifold with cylindrical ends. The augmented APS condition is a specific example of an "ideal boundary condition," which is a boundary condition that differs from the APS condition by a projection on the kernel of the boundary Dirac operator. Following Melrose and Piazza [**42**] we show that the index and eta invariants of a Dirac operator on a compact manifold with boundary with any ideal boundary condition can be identified with parallel invariants of a perturbation of the corresponding Dirac operator on the manifold with cylindrical ends with L^2 domain by a b-smoothing operator constructed from the ideal boundary condition. In this sense, the "b-category" of objects is able to give a complete description of index and eta invariants for all ideal boundary conditions, and not just the augmented APS condition.

1. Introduction

Two of the most basic and most studied invariants describing to different degrees the *spectral asymmetry* of Dirac operators are the index and eta invariants. The index describes the asymmetry of the kernel and the eta invariant describes the asymmetry of the entire spectrum of Dirac operators. The purpose of this article is to relate these invariants on manifolds with boundary to corresponding invariants on manifolds with cylindrical ends.

1.1. Ideal boundary conditions and the index theorem. The seminal papers [**4, 5**] by Atiyah, Patodi, and Singer created immense investigations into index theory on manifolds with boundary and singularities; to name only a few extensions of their work, see for instance, Cheeger [**14**], Atiyah, Donnelly, and Singer [**3**], Müller [**43**], Stern [**57**], Brüning [**11**], Bismut and Cheeger [**7, 8**], Grubb and Seeley [**22**] and many others; for survey articles treating various aspects of index theory, see Müller [**46**], Piazza [**49**], Seeley [**55**], or Loya [**32**]. In this paper

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we shall focus our attention on Melrose's [**39, 40**] b-calculus reinterpretation of the Atiyah, Patodi, and Singer (henceforth APS) index theorem, especially in regards to the so-called "ideal boundary conditions", which we now review.

We first set the stage by stating our assumptions and then we recall the augmented APS boundary condition. Let $D : C^\infty(M,E) \longrightarrow C^\infty(M,F)$ be an admissible (also called compatible) Dirac operator associated to a $\mathbb{Z}_2$-graded Hermitian Clifford module $E \oplus F \to M$ over a compact, *even*-dimensional, Riemannian manifold with boundary. We assume that all the geometric structures are of product type on a collar $[0,1)_x \times Y$ of the boundary Y of M, where x is a boundary defining function which is everywhere positive on the interior of M. Therefore, on this collar we assume that $E \cong E|_{x=0}$, $F \cong F|_{x=0}$, the metric g on M takes the form

$$g = dx^2 + h$$

with h a metric on Y, and finally,

$$D = \Gamma(\partial_x + D_Y), \tag{1.1}$$

where $\Gamma : E|_{x=0} \longrightarrow F|_{x=0}$ is a unitary isomorphism (Clifford multiplication by dx) and where D_Y is a Dirac operator on Y. Since D_Y is a self-adjoint elliptic first order differential operator on a compact manifold without boundary, it has discrete spectrum consisting of real numbers extending above and below zero to $\pm\infty$. Let Π_+ denote the orthogonal projection of $L^2(Y,E_Y)$, where $E_Y := E|_{x=0}$, onto the eigenspaces of D_Y corresponding to the positive eigenvalues.

Let $V = \ker D_Y$ and let Π_0 denote the orthogonal projection of $L^2(Y,E_Y)$ onto V. Then there is a distinguished subspace Λ_C of V (see Melrose [**40**], Müller [**45, 44**]) defined by

$$\Lambda_C := \left\{ \Pi_0 u|_{x=0} \, ; \, u \in H^1(M,E), \; Du = 0, \; \Pi_+ u|_{x=0} = 0 \right\}. \tag{1.2}$$

If Π_C is the orthogonal projection onto Λ_C, then $C := 2\Pi_C - \mathrm{Id}$, acting on V, is a unitary map on V with eigenvalues ± 1 and with $+1$ eigenspace exactly Λ_C. The unitary map C is called the *scattering matrix* and Λ_C is called the *scattering Lagrangian.*[1] This subspace is also known as the subspace corresponding to the "limiting values of extended L^2 solutions of $Du = 0$" for the following reason. Let $\widehat{M}$ be the manifold formed by taking the infinite cylinder $(-\infty,0]_x \times Y$ and gluing it onto the end of the collar $[0,1)_x \times Y$ of M:

$$\widehat{M} = (-\infty,0]_x \times Y \sqcup_{\partial M} M.$$

Since all the geometric structures and the Dirac operator are of product type on the collar of M, they all have natural extensions to the manifold $\widehat{M}$. We denote these extended structures on $\widehat{M}$ using the same notations used for the original objects on M; however, since the extended Dirac operator on $\widehat{M}$ acts on a different domain than the Dirac operator on M, namely sections on $\widehat{M}$ rather than on M, we denote the extension of the Dirac operator by $\widehat{D}$. Then [**40**], [**45**]

$$\Lambda_C = \left\{ \lim_{x \to -\infty} u(x,y) \, ; \, u \in C^\infty(\widehat{M},E) \text{ is bounded and } \widehat{D}u = 0 \right\}. \tag{1.3}$$

The manifold $\widehat{M}$ will play very important roles as our story unfolds.

[1] This is really "half" of the true Lagrangian, which is $\Lambda_C \oplus \Gamma\Lambda_C \subset E \oplus F$ for the symplectic structure given by the L^2 inner product and the Clifford action $\Gamma \oplus (-\Gamma^*) : E \oplus F \longrightarrow F \oplus E$.

We now define the augmented APS boundary condition. The Dirac operator D with domain

$$\operatorname{Dom}(D_{\mathrm{APS}}) := \{u \in H^1(M,E)\ ;\ (\Pi_+ + \Pi_C)u|_{x=0} = 0\} \tag{1.4}$$

defines the Dirac operator D_{APS} with the *augmented APS boundary condition*. The APS index theorem can then be written as

$$\operatorname{ind} D_{\mathrm{APS}} = \int_M \mathrm{AS} - \frac{1}{2}\eta(D_Y),$$

where AS is the Atiyah-Singer index density manufactured from the Clifford modules and connection and where $\eta(D_Y)$ is the eta invariant of D_Y, to be discussed more fully in Section 1.2.

Although C is in some sense canonical, it is possible, of course, to choose other projections onto subspaces in V in (1.4) to define Dirac operators with other domains. Let T be a self-adjoint isomorphism on V with $T^2 = \mathrm{Id}$. Then T has ± 1 eigenvalues. We define D_T as the Dirac operator D with domain

$$\operatorname{Dom}(D_T) := \{u \in H^1(M,E)\ ;\ \Pi_+^T u|_{x=0} = 0\}, \tag{1.5}$$

where $\Pi_+^T := \Pi_+ + \Pi_T^\perp$ with $\Pi_T^\perp$ the orthogonal projection onto the -1 eigenspace of T. Such a boundary condition is called an *ideal boundary condition*. Thus, D_{APS} is really just D_{-C} with C the scattering matrix.

Before stating the first result in this paper we make a couple remarks about some natural objects on $\widehat{M}$. First, we remark that there is a natural scale of L^2 based Sobolev spaces on $\widehat{M}$, where square integrability means with respect to the (extended) measure dg on $\widehat{M}$. In particular, the natural domain of $\widehat{D}$ is $H^1(\widehat{M},E)$, which consists of those sections u on $\widehat{M}$ such that $\widehat{D}u$ is L^2. Second, we remark that various classes of pseudodifferential operators on $\widehat{M}$ that preserve these Sobolev spaces have been developed by many different authors, to name a few, Egorov and Schulze [**17**], Melrose [**40**], Melrose and Mendoza [**41**], Plamenevskij [**50**], Rempel and Schulze [**52**], and Schulze [**54**]. These operators are more or less the same (cf. Lauter and Seiler [**26**]), but the operators we choose for the purposes of this paper are the (small calculus of) b-pseudodifferential operators of Melrose, to be explained in Section 2.2. Of these operators, the b-smoothing operators (b-pseudodifferential operators of order $-\infty$) will serve as a natural class of "perturbations" in the sequel.

With this background, we are ready to state the first theorem in this paper.

THEOREM 1.1. *Let D and $\widehat{D}$ be as above and let T be a self-adjoint isomorphism on V with $T^2 = \mathrm{Id}$. Then there exists a b-smoothing operator $\widehat{T}$ such that the L^2 based operator $\widehat{D} - \widehat{T}$ on the complete manifold $\widehat{M}$ and the operator D_T on the compact manifold M have the same index theoretic properties:*

(a) $\ker(\widehat{D} - \widehat{T}) \cong \ker D_T$ *and* $\ker(\widehat{D} - \widehat{T})^* \cong \ker(D_T)^*$.

(b) *The operators*

$$\widehat{D} - \widehat{T} : H^1(\widehat{M},E) \longrightarrow L^2(\widehat{M},F),$$
$$D_T : \operatorname{Dom}(D_T) \longrightarrow L^2(M,F)$$

are Fredholm with (by (a)) equal indices.

(c) *The following index formula holds:*

$$\operatorname{ind}(\widehat{D} - \widehat{T}) = \operatorname{ind} D_T = \int_M \mathrm{AS} - \frac{1}{2}[\eta(D_Y) - \operatorname{sign} T].$$

The equality in (a) was proved in joint work with Melrose [**33**]. In [**42**], Melrose and Piazza prove a families index theorem, which in the simplest case when the base manifold is a point, consists of statements (b) and (c) but with T a very general finite rank operator (connected with the notion of a *spectral section*). Therefore, the equality of indices and the index formula in Properties (b) and (c) are special cases of Melrose and Piazza's theorem. In Theorem 3.4, we shall extend the index formula (c) for $\operatorname{ind}(\widehat{D} - \widehat{T})$ to a slightly more general class of perturbations.

1.2. Ideal boundary conditions and the eta invariant. The eta invariant for self-adjoint Dirac operators on *odd*-dimensional manifolds shares many properties with the index (cf. Singer [**56**]) and has also become an area of much interest since the publication of the seminal work of Atiyah, Patodi, and Singer [**5**]. Recently, many authors have focused on understanding the decomposition of the eta invariant under gluing of manifolds. Such "gluing problems" have been investigated by, for instance, Dai and Freed [**15**], Müller [**45**], Wojciechowski [**59, 60, 61**], Bunke [**13**], Lesch and Wojciechowski [**27**], Mazzeo and Melrose [**36**], Hassell, Mazzeo, and Melrose [**23**], Brüning and Lesch [**12**], Kirk and Lesch [**25**], and Loya and Park [**34**]. For surveys on such "cut and paste" formulas of the eta invariant see Mazzeo and Piazza [**37**] or Bleecker and Booß-Bavnbek [**9**]. One aspect of this gluing problem involves the eta invariant on a manifold with boundary and its dependence on boundary conditions. Lagrangian subspaces and ideal boundary conditions come into the picture in order to get a self-adjoint Dirac operator.

Let D be a Dirac operator as considered in Section 1.1, however we now assume that $E = F$ and M is odd-dimensional. Of course, we still assume product structures near the boundary as in (1.1), but now Γ is a unitary isomorphism on E_Y only, since $E = F$. Moreover, Clifford and self-adjointness considerations impose the following relations:

$$\Gamma^2 = -\mathrm{Id}, \quad \Gamma^* = -\Gamma, \quad \Gamma\, D_Y = -D_Y \Gamma. \tag{1.6}$$

As before, we set $V := \ker D_Y$. Then the last equality in (1.6) implies that Γ acts on V. The set of unitary isomorphisms T on V such that $\Gamma\, T = -T\, \Gamma$ and $T^2 = \mathrm{Id}$ is denoted by $\mathcal{L}(V)$. Such isomorphisms can be constructed as follows. First, observe that $V = V^+ \oplus V^-$, where $V^\pm$ are the $\pm i$ eigenspaces of Γ. Note that $\dim V^+ = \dim V^-$ by the cobordism invariance of the index, see Theorem 21.5 of Booß-Bavnbek and Wojciechowski [**10**]. Then given any unitary isomorphism $T^+ : V^+ \longrightarrow V^-$ and setting $T^- := (T^+)^{-1}$, the map $T := T^+ + T^-$ is in $\mathcal{L}(V)$. Conversely, every element of $\mathcal{L}(V)$ arises in this way. Given $T \in \mathcal{L}(V)$, we denote the $+1$ eigenspace of T by Λ_T. Note that $\Gamma \Lambda_T$ is the -1 eigenspace of T and that $V = \Lambda_T \oplus \Gamma \Lambda_T$ is an orthogonal decomposition. One can check that $\Omega(v, w) := \mathrm{Re}(\Gamma v, w)_Y$, where $v, w \in V$ and where $(\ ,\ )_Y$ is the L^2 inner product on Y, is a symplectic form on V, and that the subspaces of V that are Lagrangian with respect to Ω are exactly those of the form Λ_T for some $T \in \mathcal{L}(V)$. Moreover, the scattering matrix C is an element of $\mathcal{L}(V)$ with associated Lagrangian Λ_C [**45**].

We now recall the definition of the eta invariant in the context of ideal boundary conditions as presented in Appendix 1 of Douglas and Wojciechowski [**16**]. Given $T \in \mathcal{L}(V)$, recall that D_T is the Dirac operator D with domain given in (1.5). Because $T \in \mathcal{L}(V)$ and therefore Λ_T is Lagrangian, it turns out that the operator D_T is self-adjoint and has real discrete spectrum. If $\{\lambda_j\}$ are the eigenvalues of

D_T, then the eta function of D_T,

$$\eta(z, D_T) := \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}, \tag{1.7}$$

extends from $\operatorname{Re} z >> 0$ to be a meromorphic function of $z \in \mathbb{C}$ that is regular at $z = 0$. The eta invariant, $\eta(D_T)$, is defined as the number $\eta(0, D_T)$. Thus, formally speaking, "$\eta(D_T) = \sum_{\lambda_j \neq 0} \operatorname{sign} \lambda_j$" and hence, $\eta(D_T)$ is a measure of the spectral asymmetry of D_T.

As the scattering Lagrangian is canonically associated to the Dirac operator, one can argue that the eta invariant with the augmented APS condition, $\eta(D_{-C})$, provides an "origin" to which to compare eta invariants defined using other Lagrangian subspaces. To support this statement, in [**45**] Müller proves that $\eta(D_{-C})$ is equal to the b-eta invariant ${}^b\eta(\widehat{D})$ of the Dirac operator $\widehat{D}$ on the corresponding cylindrical end manifold $\widehat{M}$. The b-eta invariant of $\widehat{D}$ is not defined via a formula of the sort (1.7) because $\widehat{D}$ has continuous and not discrete spectrum (as $\widehat{M}$ is not compact), but nevertheless is the natural generalization of the eta invariant to $\widehat{M}$, see Section 4. Before presenting our second theorem, we need the following function introduced by Lesch and Wojciechowski in [**27**]: For $T, S \in \mathcal{L}(V)$, we define

$$m(\Lambda_T, \Lambda_S) := -\frac{1}{i\pi} \sum_{\substack{e^{i\theta} \in \operatorname{spec}(-T^- S^+) \\ \theta \in (-\pi, \pi)}} i\theta. \tag{1.8}$$

Here, T^- and S^+ are the restrictions of T and S to the $-i$ and $+i$ eigenspaces of Γ respectively. The second result of this note is the following.

THEOREM 1.2. *Let D and $\widehat{D}$ be as above and let $T \in \mathcal{L}(V)$. Then there exists a b-smoothing operator $\widehat{T}$ such that D_T and the perturbed Dirac operator $\widehat{D} - \widehat{T}$ have the same eta invariant theoretic properties:*

(a) $\ker(\widehat{D} - \widehat{T}) \cong \ker(D_T)$.
(b) ${}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_T)$.
(c) *The following surgery formula holds:*

$$ {}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_T) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C),$$

where C is the scattering matrix.

The equality

$$ {}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C)$$

was proved in joint work with Melrose [**33**].

As a corollary of Theorem 1.2, we obtain a formula for the dependence of the eta invariant on different Lagrangian subspaces. To this end, let τ denote the triple Maslov index, defined on a triple $(\Lambda_A, \Lambda_B, \Lambda_C)$ of Lagrangian subspaces of V by

$$\tau(\Lambda_A, \Lambda_B, \Lambda_C) := m(\Lambda_A, \Lambda_B) + m(\Lambda_B, \Lambda_C) + m(\Lambda_C, \Lambda_A).$$

Although the function m is real-valued, the triple index is integer-valued, see Bunke [**13**] and Lion and Vergne [**29**]. Theorem 1.2, the definition of τ, and the fact that m is antisymmetric, imply the following corollary.

COROLLARY 1.3. *For any $T, S \in \mathcal{L}(V)$, we have*

$$\eta(D_T) - \eta(D_S) = m(\Lambda_T, \Lambda_S) + \tau(\Lambda_T, \Lambda_C, \Lambda_S),$$

where C is the scattering matrix.

1.3. Final remarks and outline of paper. The proofs of Theorem 1.1 and 1.2 use the so-called *heat kernel method* which relies on taking traces of suitable heat operators, cf. Mckean and Singer [**38**], Patodi [**48**], Atiyah, Bott, and Patodi [**2**], and Gilkey [**19**]. However, since $\widehat{M}$ is not compact, it turns out that the relevant heat operators are not trace class on $\widehat{M}$. To overcome this obstacle, we utilize a regularized trace called the b-trace[2] developed by Melrose [**40**]. In many respects, the b-trace is the "hero" of this paper: We use it to make non-trace class heat operators "(b-) trace class;" give a *direct* derivation of the APS index formula with the eta invariant emerging as a *direct* computation from the b-trace's key feature, the trace-defect formula; we use it to define the b-eta invariant (without requiring the Dirac operators to have discrete spectrum), and finally, we use it and the trace-defect formula to prove the variation formula for the eta invariant via a *direct* computation.

The outline of this paper is as follows. We begin this paper in Section 2 by proving a simplified version of Theorem 1.1 in the case that the boundary Dirac operator is invertible. We also delve into a detailed study of b-pseudodifferential operators and we introduce the b-trace and derive its key feature: the trace-defect formula. In Section 3, we define the b-smoothing perturbation $\widehat{T}$ in Theorem 1.1 and we prove this theorem via the general index theorem 3.4. The proof of Theorem 1.2 is based on Vishik's technique of rotating boundary conditions, see Section 1 of [**58**], originally developed to prove gluing formulas for torsion invariants on manifolds with boundary. In Section 4, we review this technique as refined by Brüning and Lesch [**12**] for the study of eta invariants to prove Theorem 1.2.

In conclusion, I wish to thank Richard Melrose for sharing his insights into many of these problems, Paolo Piazza for looking over parts of the manuscript, and Krzysztof Wojciechowski for his support and encouragement in proving Theorem 1.2. I also thank the referee for thoughtful comments, finding many mistakes, and for very helpful suggestions all of which led to many improvements. Finally, I thank the organizers of the conference, Bernhelm Booß-Bavnbek, Gerd Grubb, and Krzysztof Wojciechowski, for allowing me to participate.

2. Transformation to b-objects I

In this section we prove a simplified version of Theorem 1.1 in the case that the boundary Dirac operator is invertible, which is a special case of Atiyah, Patodi, and Singer's classic result [**5**]. We begin by reviewing the necessary ingredients and then we move into a detailed study of b-pseudodifferential operators. Next, we introduce the hero of this paper, the b-trace and its main feature: the trace-defect formula. Finally, we prove the index formula for the case that the boundary Dirac operator is invertible. Here we see our first instance of the trace-defect formula in action in the direct manner in which the eta invariant appears in APS formula.

[2]The b-trace is almost the same as the *relative trace* found in Müller [**47**].

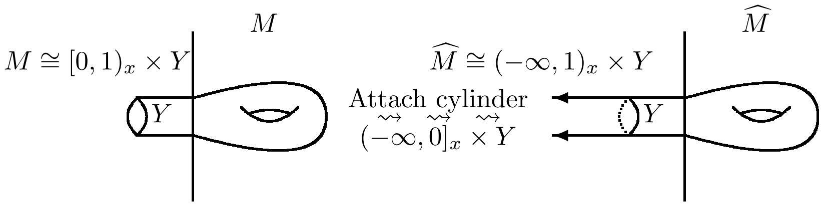

FIGURE 1. Attaching the infinite cylinder $(-\infty, 0]_x \times Y$ to M produces the manifold with cylindrical end $\widehat{M}$.

2.1. APS and cylindrical ends. We shall work under the same assumptions and notations introduced in Section 1.1. Thus, let $D : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ be an admissible Dirac operator associated to a $\mathbb{Z}_2$-graded Hermitian Clifford module $E \oplus F \to M$ over a compact, even-dimensional, Riemannian manifold with boundary and we assume product structures near the boundary such as in (1.1). We also assume, for simplicity at least for this section, that D_Y is invertible (which is equivalent to saying that it has no kernel). Let Π_+ denote the orthogonal projection onto the eigenspaces of D_Y corresponding to the positive eigenvalues. Then the operator D acting on the domain

$$\mathrm{Dom}(D_{\Pi_+}) := \{u \in H^1(M, E)\ ;\ \Pi_+ u|_{x=0} = 0\}$$

defines the operator D_{Π_+} with APS boundary conditions.

As noted in [**5**], we can view the APS boundary condition as an L^2 condition on an enlarged noncompact manifold with cylindrical end $\widehat{M}$ which we now review. Let $\widehat{M}$ be the manifold formed by taking the infinite cylinder $(-\infty, 0]_x \times Y$ and gluing it onto the end of the collar $[0, 1)_x \times Y$ of M as shown in Figure 1:

$$\widehat{M} = (-\infty, 0]_x \times Y \sqcup_{\partial M} M.$$

Then all the geometric structures have natural extensions to the manifold $\widehat{M}$ and are denoted by the same notations used for the original objects on M except for the Dirac operator which we denote by $\widehat{D}$. Recall that the natural domain of $\widehat{D}$ is $H^1(\widehat{M}, E)$, which consists of those sections u on $\widehat{M}$ such that $\widehat{D}u$ is square integrable with respect to the measure dg on $\widehat{M}$.

We now show how the L^2 kernel of $\widehat{D}$ (the noncompact problem) relates to the kernel of D_{Π_+} (the compact problem). With this in mind, let $u \in C^\infty(\widehat{M}, E)$ with $\widehat{D}u = 0$. Let $\{\lambda_j\}$ be the set of eigenvalues of D_Y, which are real and nonzero, with corresponding eigenvectors $\{\varphi_j\}$ so that $D_Y \varphi_j = \lambda_j\, \varphi_j$. On the collar $(-\infty, 0]_x \times Y$ we can expand u in terms of $\{\varphi_j\}$ as $u = \sum_j u_j(x)\, \varphi_j(y)$, and therefore on the collar,

$$0 = \widehat{D}u = \Gamma(\partial_x + D_Y)\Big(\sum_j u_j(x)\, \varphi_j(y)\Big) = \Gamma \sum_j \big(u_j'(x) + \lambda_j u_j(x)\big)\, \varphi_j(y),$$

which implies that for each j, $u_j'(x) + \lambda_j\, u_j(x) = 0$, or $u_j(x) = c_j e^{-\lambda_j x}$ for some constant c_j. Since

$$\lim_{x \to -\infty} e^{-\lambda_j x} = \begin{cases} \infty & \lambda_j > 0, \\ 0 & \lambda_j < 0, \end{cases}$$

it follows that

$$u \in L^2(\widehat{M}, E) \quad \Longleftrightarrow \quad c_j = 0 \ \text{ for } \lambda_j > 0 \quad \Longleftrightarrow \quad \Pi_+ u|_{x=0} = 0.$$

Thus,

$$\ker \widehat{D} \cong \ker D_{\Pi_+}, \tag{2.1}$$

where the left-hand side is the kernel of $\widehat{D}$ on its natural domain. Similarly, one can show that the kernels of the adjoints are isomorphic: $\ker \widehat{D}^* \cong \ker D^*_{\Pi_+}$.

Before stating the APS index theorem for the operators $\widehat{D}$ and D_{Π_+}, we go into more depth on the eta invariant than was described in Section 1.2. The original way to define the eta invariant was through the eta function, $\eta(z)$, which is defined as the meromorphic function

$$\eta(z) = \sum_j \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}. \tag{2.2}$$

In the general case when D_Y has a kernel, we only sum over $\lambda_j \neq 0$. Weyl asymptotics show that $\eta(z)$ is holomorphic for $\operatorname{Re} z > \dim Y$, but one of the main accomplishments of [**5**] was the proof that $\eta(z)$ in fact defines a meromorphic function on $\mathbb{C}$ that is regular at $z = 0$. The *eta invariant* of D_Y is the value of the eta function at zero, $\eta(D_Y) = \eta(0)$, which represents a formal signature of the operator D_Y:

$$\text{“ } \eta(D_Y) = \sum_j \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}\Big|_{z=0} = \sum_j \operatorname{sign} \lambda_j = \#\{\lambda_j > 0\} - \#\{\lambda_j < 0\}. \text{ ”}$$

The reason for the quotation marks is that there are an infinite number of positive and negative eigenvalues, so this equation really reads $\infty - \infty$! Nonetheless, $\eta(D_Y)$ can be interpreted as a measurement of the signature or spectral asymmetry of D_Y. We can also express the eta function in terms of the heat operator via

$$\eta(z) = \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \operatorname{Tr}(D_Y\, e^{-tD_Y^2})\, dt, \tag{2.3}$$

where $\Gamma(z)$ is the Gamma function. To see this, we notice that

$$\operatorname{Tr}(D_Y\, e^{-tD_Y^2}) = \sum_j \lambda_j\, e^{-t\lambda_j^2}$$

and that

$$\frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \lambda_j\, e^{-t\lambda_j^2}\, dt = \frac{\lambda_j}{|\lambda_j|^{z+1}} \cdot \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} e^{-t}\, dt = \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z},$$

where we made the change of variables $t \mapsto t/|\lambda_j|^2$. In particular, according to [**6**], we can set $z = 0$ in (2.3) to obtain the following important formula:

$$\eta(D_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(D_Y\, e^{-tD_Y^2})\, dt. \tag{2.4}$$

The advantage of this formula is that it naturally falls out of the heat kernel proof of the APS index formula as we shall see in Section 2.5.

We are now ready to state the (simplified) Atiyah-Patodi-Singer index theorem, which is a special case of Theorem 1.1 when D_Y is invertible, and hence $V = \ker D_Y$ is just the zero vector space. The following statement is not how their theorem originally appeared, but all its content were part of the original paper [**5**].

THEOREM 2.1. *Let D be a Dirac operator on an even-dimensional, compact, oriented, Riemannian manifold with boundary with product type structures specified as in Section 1.1 and such that the boundary operator D_Y is invertible. Then*

(a) $\ker \widehat{D} \cong \ker D_{\Pi_+}$ *and* $\ker \widehat{D}^* \cong \ker D^*_{\Pi_+}$.
(b) *The operators*

$$\widehat{D} : H^1(\widehat{M}, E) \longrightarrow L^2(\widehat{M}, F),$$
$$D_{\Pi_+} : \mathrm{Dom}(D_{\Pi_+}) \longrightarrow L^2(M, F)$$

are Fredholm with (by (a)) equal indices.
(c) *The following index formula holds:*

$$\operatorname{ind} \widehat{D} = \operatorname{ind} D_{\Pi_+} = \int \mathrm{AS} - \frac{1}{2}\eta(D_Y).$$

We already proved Part (a) around (2.1). For Part (b), the Fredholm property of $\widehat{D}$ follows from Lemma 2.3 below; for the Fredholm properties of D_{Π_+}, see [**5**] or Booß-Bavnbek and Wojciechowski [**10**]. We shall prove Part (c) in Section 2.5. To do so, we use the machinery of Melrose's b-calculus [**40**] which we describe next.

2.2. b-pseudodifferential operators. We only describe the so-called "small over-blown" calculus of b-pseudodifferential operators. There are also "big" calculi which are useful for establishing Fredholm properties of b-pseudodifferential operators or for obtaining precise information concerning Green operators, but for the goals of this paper, these larger calculi are unnecessary.

So what exactly is a b-pseudodifferential operator? Perhaps the best way to think of a b-pseudodifferential operator is as a usual pseudodifferential operator on $\widehat{M}$ that is exponentially uniform as $x \to -\infty$ on the cylindrical end. As with all algebras of operators defined on noncompact manifolds, cf. Lockhart and McOwen [**30**], Rabinovič [**51**], and Schrohe [**53**], we need to have control in asymptotic behavior at infinity in order to get a well-behaved class of operators. This is no different in our situation, where we require "exponential asymptotics". Therefore, before describing the small calculus, we need to fix the notion of exponential decay. Later, we shall see that these conditions can be interpreted quite naturally as C^∞ conditions on a related *compact* manifold.

We say that a smooth function $u(x, y)$ on the infinite cylinder $(-\infty, 0]_x \times Y$ can be *expanded exponentially* on the cylindrical end if

$$u(x, y) \sim \sum_{k=0}^{\infty} e^{kx}\, u_k(y) = u_0(y) + e^x\, u_1(y) + e^{2x}\, u_2(y) + \cdots , \tag{2.5}$$

where $u_k \in C^\infty(Y)$ for each k. This asymptotic sum means that for any N,

$$u(x, y) - \sum_{k=0}^{N-1} e^{kx}\, u_k(y) = e^{Nx}\, r_N(x, y), \tag{2.6}$$

where all derivatives of the remainder $r_N(x, y)$ in x and y are bounded. In particular, $u(x, y) = u_0(y)$ modulo an exponentially decaying term. We say that u *vanishes to infinite exponential order* on the cylindrical end if u can be expanded exponentially as $x \to -\infty$ with all the $u_k(y)$'s zero; equivalently, u with all its derivatives vanish faster on the cylindrical end than any exponential power.

Let $\mathcal{S}(\widehat{M})$ denote the set of all smooth functions on $\widehat{M}$ that vanish to infinite exponential order on the cylindrical end. A *b-pseudodifferential operator A of order* $m \in \mathbb{R}$ is a continuous linear map on $\mathcal{S}(\widehat{M})$ that has the following properties: Given any compactly supported $\psi \in C^\infty_c(\widehat{M})$ equal to 1 on M, and smooth $\varphi \in C^\infty(\widehat{M})$

supported on the cylindrical end $(-\infty,0]_x \times Y$ with $\varphi(x,y)=1$ for x sufficiently negative with support disjoint to ψ, we have

(I) $(1-\varphi)A(1-\varphi)$ is a usual pseudodifferential operator of order m on $\widehat{M}$.
(II) $\varphi A\psi$ is a smoothing operator with an integral kernel $K_{\varphi A\psi}(x,y,q)$, where $(x,y)\in(-\infty,0]\times Y$ and $q\in\widehat{M}$, that vanishes to infinite exponential order as $x\to-\infty$ and is compactly supported in $q\in\widehat{M}$.
(III) $\psi A\varphi$ is a smoothing operator with an integral kernel $K_{\psi A\varphi}(p,x,y)$, where $p\in\widehat{M}$ and $(x,y)\in(-\infty,0]\times Y$, that is compactly supported in $p\in\widehat{M}$ and vanishes to infinite exponential order as $x\to-\infty$.
(IV) $\tilde{A}=(1-\psi)A(1-\psi)$ is a pseudodifferential operator on $(-\infty,0]\times Y$ satisfying two "lacunary" type conditions (cf. Hörmander [**24**]) that we now describe.

First, given a coordinate neighborhood $\mathbb{R}^{n-1}_y$ on the cross section Y and a compactly supported smooth function ω on $\mathbb{R}^{n-1}$, for any $u\in\mathcal{S}(\widehat{M})$ supported on $\mathcal{U}=(-\infty,0]\times\mathbb{R}^{n-1}$, we can write

$$\big(\omega\tilde{A}\,\omega\big)u=\int e^{i(x,y)\cdot(\tau,\eta)}\,a(x,y,\tau,\eta)\,\widehat{u}(\tau,\eta)\,đ\tau đ\eta, \tag{2.7}$$

where $đ(\;)$ means to divide by as many 2π's as there are variables in $(\;)$, $\widehat{u}(\tau,\eta)$ is the Fourier transform of $u(x,y)$:

$$\widehat{u}(\tau,\eta)=\int e^{-i(x,y)\cdot(\tau,\eta)}\,u(x,y)\,dxdy,$$

and where $a(x,y,\tau,\eta)$ is a symbol in (τ,η) of order m. However, $a(x,y,\tau,\eta)$ is no standard symbol, we require it to be entire in τ such that given any constant R, for τ in the strip $|\operatorname{Im}\tau|\le R$, the following properties hold:

(1) $a(x,y,\tau,\eta)$ is a symbol of order m:

$$\Big|\big(\partial_x\partial_y\big)^{\alpha}\big(\partial_\tau\partial_\eta\big)^{\beta}a(x,y,\tau,\eta)\Big|\le C(1+|\tau|+|\eta|)^{m-|\beta|} \tag{2.8}$$

where this estimate holds for all $(x,y)\in\mathcal{U}$, $|\operatorname{Im}\tau|\le R$, and $\eta\in\mathbb{R}^{n-1}$.

(2) $a(x,y,\tau,\eta)$ can be expanded exponentially on the cylindrical end where the coefficients of e^{kx} and remainder in (2.6) are also symbols of order m satisfying estimates of the sort (2.8).

Second, given any compactly supported smooth functions ω and $\widetilde{\omega}$ on Y with supports in disjoint coordinate patches with coordinates y and $\tilde{y}$, respectively, for any $u\in\mathcal{S}(\widehat{M})$ supported on $(-\infty,0]\times Y$, we can write

$$\big(\omega\tilde{A}\,\widetilde{\omega}\big)u=\int e^{ix\,\tau}\,a(x,y,\tilde{y},\tau)\,\widehat{u}(\tau,\tilde{y})\,đ\tau\,dh(\tilde{y}), \tag{2.9}$$

where $\widehat{u}(\tau,\tilde{y})$ is the Fourier transform of $u(x,\tilde{y})$ in only the x-variable and where $a(x,y,\tilde{y},\tau)$ is a symbol in τ of order $-\infty$ satisfying properties (1) and (2) above (of course, for arbitrary m, and now there are derivatives in the extra variable $\tilde{y}$).

The set of all such operators is denoted[3] by $\Psi_b^m(\widehat{M})$. A *b-smoothing operator* is a b-pseudodifferential operator of order $-\infty$. Just like the usual class of pseudodifferential operators on compact manifolds without boundary, the space of all b-pseudodifferential operators forms a $*$-filtered algebra of operators with the filtration determined by the *principal symbol*. One can also define a corresponding space of *classical* operators. See [**40**] for more information. Also, there exist natural Sobolev spaces $H^s(\widehat{M})$ on $\widehat{M}$ with $H^0(\widehat{M}) = L^2(\widehat{M})$ and if $A \in \Psi_b^m(\widehat{M})$, then A defines a continuous linear map,

$$A : H^s(\widehat{M}) \longrightarrow H^{s-m}(\widehat{M}). \tag{2.10}$$

We now describe the normal operator, which captures the dominant behavior of $A \in \Psi_b^m(\widehat{M})$ as $x \to -\infty$. Let $a_0(y,\tau,\eta)$ be the first term in the expansion (2.5) for $b(x,y,\tau,\eta)$ on the cylindrical end. Then $b(x,y,\tau,\eta) = a_0(y,\tau,\eta)$ modulo an exponentially decaying term as $x \to -\infty$. The *normal or indicial operator* of A is the entire family $N(A)(\tau) \in \Psi^m(Y)$ defined locally by

$$N(A)(\tau)\psi = \int e^{iy\cdot\eta}\, a_0(y,\tau,\eta)\, \hat{\psi}(\eta)\, đ\eta; \tag{2.11}$$

there is a similar formula in disjoint coordinate patches using (2.9). The normal operator preserves composition:

$$N(AB)(\tau) = N(A)(\tau) \circ N(B)(\tau)$$

and in a certain respect, adjoints:

$$N(A^*)(\tau) = N(A)(\overline{\tau})^*,$$

where $\overline{\tau}$ is the complex conjugate of τ. Finally, we remark that the normal operator governs the Fredholm properties of b-pseudodifferential operators, see Mazzeo [**35**], Melrose [**40**], or Loya and Melrose [**33**]:

Theorem 2.2. *Let $A \in \Psi_b^m(\widehat{M})$ with $m \in \mathbb{R}$. Then*

$$A : H^m(\widehat{M}) \longrightarrow L^2(\widehat{M})$$

is Fredholm if and only if A is elliptic and for all $\tau \in \mathbb{R}$,

$$N(A)(\tau) : H^m(Y) \longrightarrow L^2(Y)$$

is invertible.

Of course, everything that we have said so far for operators on functions works equally well for operators mapping between sections of vector bundles. We now consider the Dirac operator, our main example of a b-pseudodifferential operator. In this case, it is easy to see that $\widehat{D}$ is a first order b-pseudodifferential operator. Indeed, since $\widehat{D} = \Gamma(\partial_x + D_Y)$ on the cylinder, given any u supported on a coordinate patch with local coordinates (x,y), writing u as the inverse Fourier transform

[3] This space usually has a subscript "ob", $\Psi_{ob}^m(\widehat{M})$, for "over-blown." When the cross section Y is disconnected, this is a slightly larger class of operators than described by Melrose in [**40**]; when Y is connected, this over-blown space is exactly the same as in loc. cit. Finally, we remark that "over-blown" has to do with the blown-up space X_b^2 to be described in Section 2.3.

of its Fourier transform, we have

$$\widehat{D}u(x,y) = \Gamma(\partial_x + D_Y)\int e^{i(x,y)\cdot(\tau,\eta)}\,\widehat{u}(\tau,\eta)\,đ\tau đ\eta$$
$$= \int e^{i(x,y)\cdot(\tau,\eta)}\,\Gamma(i\tau + b(y,\eta))\,\widehat{u}(\tau,\eta)\,đ\tau đ\eta,$$

where $b(y,\eta) = D_Y\big(e^{iy\cdot\eta}\big)$ is the symbol of D_Y in local coordinates. The function $\Gamma(i\tau + b(y,\eta))$ is certainly a symbol of order 1 satisfying all of (1) and (2) around (2.8). Thus, $\widehat{D} \in \Psi^1_b(\widehat{M},E,F)$. Moreover, by definition of the normal operator (2.11),

$$N(\widehat{D})(\tau)\psi = \int e^{iy\cdot\eta}\,\Gamma(i\tau + b(y,\eta))\,\widehat{\psi}(\eta)\,đ\eta = \Gamma(i\tau + D_Y)\psi.$$

Thus,

$$N(\widehat{D})(\tau) = \Gamma(i\tau + D_Y), \tag{2.12}$$

a formula that we shall use later in our proof of the APS theorem. Let $E_Y = E|_{x=0}$ and $F_Y = F|_{x=0}$. Since D_Y is self-adjoint,

$$i\tau + D_Y : H^m(Y,E_Y) \longrightarrow L^2(Y,E_Y)$$

is automatically invertible for $\tau \neq 0$ and is invertible at $\tau = 0$ if and only if D_Y is invertible. It follows that

$$N(\widehat{D})(\tau) = \Gamma(i\tau + D_Y) : H^m(Y,E_Y) \longrightarrow L^2(Y,F_Y)$$

is invertible for all $\tau \in \mathbb{R}$ if and only if D_Y is invertible. Although we have been working under the even-dimensional assumption, this whole argument works for any dimensional manifold. Thus, Theorem 2.2 immediately gives the following theorem, which proves the first part in Part (b) of Theorem 2.1.

Lemma 2.3. *Let D be a Dirac operator on a compact Riemannian manifold with boundary with product type structures near the boundary. Then $\widehat{D}$ is Fredholm on its natural (that is, Sobolev) domain if and only if D_Y is invertible.*

Of course, in this section we have been assuming that D_Y is invertible, but the proof of this lemma does not require this assumption.

2.3. The compact picture. Now what is all this stuff about exponential asymptotics? Isn't there a more natural way to describe these operators? There is such a way, which we now explain. First we need to make a simple campactification of our noncompact manifold $\widehat{M}$. On the cylindrical end $(-\infty,0]_x \times Y$ of $\widehat{M}$ we make the change of variables $r = e^x$. Notice that as $x \to -\infty$, $r \to 0$. Thus, under this change of variables, $\widehat{M}$ transforms into the interior of the compact manifold with boundary X, where X has the same compact end as $\widehat{M}$ but with the cylindrical end $(-\infty,0]_x \times Y$ replaced with the compact end $[0,1]_r \times Y$, see Figure 2.

Observe that under the change of variables $r = e^x$, the asymptotic expansion (2.5) transforms to

$$u(x,y) \sim \sum_{k=0}^{\infty} e^{kx}\,u_k(y) \;\rightsquigarrow\; \tilde{u}(r,y) \sim \sum_{k=0}^{\infty} r^k\,u_k(y), \qquad \tilde{u}(r,y) = u(\log r, y); \tag{2.13}$$

in other words, $\tilde{u}(r,y)$ is smooth in the usual sense at $r = 0$! (We do have to think a little about the remainders in (2.6), but this is not difficult.) In particular, u

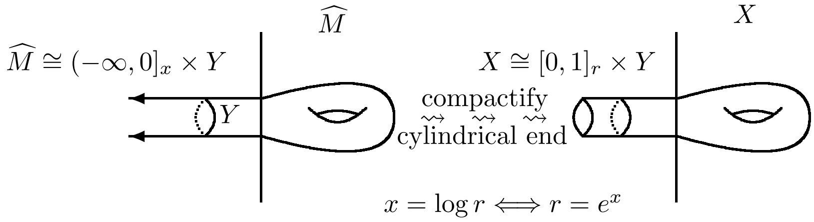

FIGURE 2. The compact manifold with boundary X is the compactification of the manifold with cylindrical end $\widehat{M}$.

vanishes to infinite exponential order just means that $\tilde{u}$ vanishes to infinite order in the usual sense at $r = 0$! Thus, the "exponential asymptotics" as $x \to -\infty$ corresponds to the usual smoothness regularity at $r = 0$.

We now see how the geometry on $\widehat{M}$ transform to X. Since $r = e^x$, we have $dx = dr/r$ and $\partial_x = r\partial_r$, therefore under this change of variables,

$$g = dx^2 + h \;\rightsquigarrow\; g = \Big(\frac{dr}{r}\Big)^2 + h,$$
$$dg = dx\, dh \;\rightsquigarrow\; dg = \frac{dr}{r}\, dh,$$

where the objects on the right are referred to as an *(exact) b-metric* and *b-measure or b-density*, respectively. Also, the Dirac operator transforms to

$$\widehat{D} = \Gamma(\partial_x + D_Y) \;\rightsquigarrow\; \widehat{D} = \Gamma(r\partial_r + D_Y),$$

where the object on the right is referred to as a (first order) *b-differential or totally characteristic operator*. Thus, the geometric objects on the manifold with cylindrical end $\widehat{M}$ transform into corresponding singular geometric "b-objects" on the compact manifold with boundary X.

Let us see how b-pseudodifferential operators transform to X. It is customary to concentrate on the Schwartz kernels. Let $A \in \Psi^m_b(\widehat{M})$. Then, from (2.7), we see that the Schwartz kernel of A on the product of the cylindrical ends is given by

$$K_A(x, y, x', y') = \int e^{i(x-x', y-y')\cdot(\tau,\eta)}\, a(x, y, \tau, \eta)\, đ\tau đ\eta, \tag{2.14}$$

where $a(x, y, \tau, \eta)$ is a symbol of order m satisfying the "lacunary" type conditions we discussed and where (x, y) and (x', y') denote the same coordinates on the left and right factors of $\widehat{M}^2$. Setting $z = (x-x', y-y')$, which define normal coordinates to the diagonal $\{z = 0\} = \{x = x',\, y = y'\}$, we can write

$$K_A = \int e^{iz\cdot(\tau,\eta)}\, a(x, y, \tau, \eta)\, đ\tau đ\eta.$$

Because z defines normal coordinates to the diagonal, using the terminology of distributions, we say that K_A is *conormal* to the diagonal. Now making the substitutions $r = e^x$ and $r' = e^{x'}$, we can transform K_A to X^2, obtaining

$$K_A = \int e^{iw\cdot(\tau,\eta)}\, \tilde{a}(r, y, \tau, \eta)\, đ\tau đ\eta, \qquad w = (s, y - y'), \quad s = \log(r/r'), \tag{2.15}$$

where $\tilde{a}(r, y, \tau, \eta) = a(\log r, y, \tau, \eta)$ is smooth at $r = 0$ (see (2.13)). Note that we cannot interpreted K_A as conormal to the diagonal $\{r = r',\, y = y'\}$ since

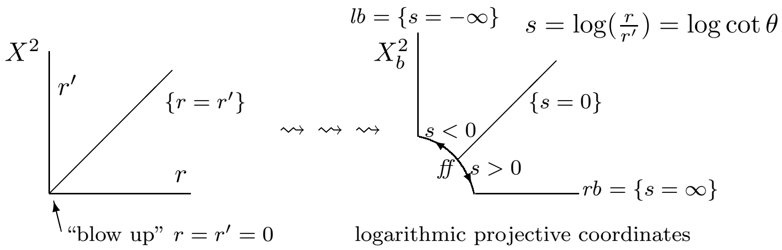

FIGURE 3. The manifold X_b^2 is obtained from X^2 by blowing-up (introducing polar coordinates at) $r = r' = 0$.

$w = (s, y - y')$ fails to define normal coordinates to the diagonal at $\{r = r' = 0\}$ where $s = \log(r/r')$ has a "nasty" singularity. However, Melrose [**40**] showed that after blowing up the "nasty" set $\{r = r' = 0\}$, the kernel (2.15) can be interpreted as a conormal distribution. To explain this, we first write X^2 near $r = r' = 0$:

$$X^2 \cong [0,1]_r \times [0,1]_{r'} \times Y^2.$$

We define the *b-stretched product* X_b^2 by blowing up the origin $\{r = r' = 0\}$ in the $[0,1]_r \times [0,1]_{r'}$ factor of X^2, which geometrically is just replacing the origin with a quarter circle, called the *front face* and denoted by *ff*; see Figure 3. Analytically speaking, X_b^2 is obtained from X^2 by simply introducing polar coordinates at the origin in the $[0,1]_r \times [0,1]_{r'}$ factor of X^2:

$$r = \rho\cos\theta, \qquad r' = \rho\sin\theta,$$

where $\rho \geq 0$ and $0 \leq \theta \leq \pi/2$. The *right boundary* and *left boundary*, denoted *rb* and *lb*, are where $\theta = 0$ and $\theta = \pi/2$, respectively. In summary, $X_b^2 \equiv X^2$ away from $r = r' = 0$ and near $r = r' = 0$, by definition, we have

$$X_b^2 \cong [0,\varepsilon)_\rho \times [0,\pi/2]_\theta \times Y^2, \qquad \varepsilon > 0.$$

The face $\{\rho = 0\} \times [0,\pi/2]_\theta \times Y^2$ (the quarter circle in Figure 3) is the front face of X_b^2. There are other coordinates we can use instead of polar coordinates. For instance, instead of using $\theta = \operatorname{arccot}(r/r')$, we can use $s = \log(r/r')$, because

$$s = \log\left(\frac{r}{r'}\right) = \log\cot\theta \quad\Longleftrightarrow\quad \theta = \operatorname{arccot}\left(\frac{r}{r'}\right) = \operatorname{arccot}(e^s)$$

implies that we can use s instead of θ as the angular variable, at least away from $\theta = 0$. Note that $s = 0$ corresponds to $\theta = \pi/4$, $s = \infty$ to *rb*, and $s = -\infty$ to *lb*. Also, one can check that away from *rb* and *lb*, (r, s) can be used as coordinates instead of (ρ, θ), and in the coordinates (r, s), the coordinate s represents an angular variable and r the boundary defining function to the front face. Moreover, in view of Figure 3, it follows that s can be considered a normal coordinate to the set $\{r = r'\}$, considered as a subset of X_b^2. This implies that $w = (s, y - y')$ defines normal coordinates to the diagonal considered as a subset of X_b^2, which we denote by Δ_b. Therefore, in view of the formula (2.15), we see that the Schwartz kernel K_A can be considered conormal to the diagonal Δ_b in X_b^2! Furthermore, directly from the expression (2.15) one can check that K_A is smooth at $r = 0$ and the "lacunary" condition implies that K_A vanishes to infinite order at *lb* and *rb*. Summarizing our discussion (modulo some details that need to be checked), we conclude:

Geometric definition: $\Psi_b^m(\widehat{M})$ consists of operators whose Schwartz kernels are distributions on X_b^2 conormal to Δ_b, smooth at *ff*, and vanishing to infinite order at *lb* and *rb*.

In this paper we shall work with $\widehat{M}$ leaving the interested reader to check out Melrose [**40**] or Mazzeo [**35**] for this conormal distribution viewpoint.

2.4. The *b*-trace. We now introduce the star of the show: The *b*-trace. By (2.10) any $A \in \Psi_b^{-\infty}(\widehat{M})$ defines a continuous linear map

$$A : L^2(\widehat{M}) \longrightarrow L^2(\widehat{M}),$$

but we note that this map is in general not trace class (or even compact). The *b*-trace is designed to make *b*-smoothing operators (*b*-pseudodifferential operators of order $-\infty$) trace-class even though they are really not trace-class! Indeed, we would like to define the trace of A via a Lidskiĭ type formula, cf. [**28**]:

$$\text{“ } \operatorname{Tr}(A) = \int_{\widehat{M}} K_A|_\Delta, \text{ ”}$$

where K_A denotes the Schwartz kernel of A and Δ is the diagonal in $\widehat{M} \times \widehat{M}$ and we make the identification $\widehat{M} \equiv \Delta$. The reason for the quotation marks is that this integral does not exist because the integral over the cylindrical end actually diverges (in general)! To see this, note that by the definition of *b*-pseudodifferential operators, in coordinates (x, y) on the cylindrical end we can write

$$K_A|_\Delta = a_0(y) + a_1(x, y), \tag{2.16}$$

where both a_0 and a_1 are smooth and $a_1(x, y)$ decays like e^x as $x \to -\infty$. Therefore, since $\int_{-\infty}^0 dx$ diverges, the integral

$$\begin{aligned} \int_{\widehat{M}} K_A|_\Delta &= \int_{(-\infty,0]_x \times Y} K_A|_\Delta \, dx\, dh + \int_M K_A|_\Delta \\ &= \int_{(-\infty,0]_x \times Y} a_0(y)\, dx\, dh + \int_{(-\infty,0]_x \times Y} a_1(x,y)\, dx\, dh + \int_M K_A|_\Delta \end{aligned} \tag{2.17}$$

in general diverges since the first integral on the right does not exist in general, except of course when $a_0(y) = 0$. This discussion shows that the function $a_0(y)$ is the problem to the non-trace class nature of A. As we all know, one way to solve a problem is to simply get rid of it, and this is exactly what we shall do in our situation: We throw out $a_0(y)$ in (2.17) to get a convergent integral, which we call the *b-trace* of A:

$${}^b\operatorname{Tr} A := \int_{(-\infty,0]_x \times Y} a_1(x,y)\, dx\, dh + \int_M K_A|_\Delta.$$

In particular, if $a_0(y) = 0$, which happens if A vanishes exponentially at the end of the cylinder, then $K_A|_\Delta = a_1(x, y)$ on the cylinder, so in this case ${}^b\operatorname{Tr} A$ equals

$$\int_{(-\infty,0]_x \times Y} K_A|_\Delta \, dx\, dh + \int_M K_A|_\Delta = \int_{\widehat{M}} K_A|_\Delta,$$

the trace of A in the usual sense. In the following lemma we show how the *b*-trace is related to elementary complex analysis.

Lemma 2.4. *Let $A \in \Psi_b^{-\infty}(\widehat{M})$. Then for all complex numbers z with $\operatorname{Re} z > 0$, the operator $e^{zx}A$ is trace class and the integral*

$$F(z) = \int_{\widehat{M}} e^{zx} K_A|_\Delta$$

exists. Moreover, $F(z)$ extends from $\operatorname{Re} z > 0$ to be a meromorphic function on the half-plane $\operatorname{Re} z > -1$ with only a simple pole at $z = 0$.[4] *Furthermore, the regular value of $F(z)$ at $z = 0$ is just the b-trace of A,*

$$ {}^b\mathrm{Tr}\, A = \operatorname{Reg}_{z=0} F(z), \tag{2.18}$$

and the residue of $F(z)$ at $z = 0$ is given in terms of the normal operator of A via

$$\operatorname{Res}_{z=0} F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(N(A)(\tau))\, d\tau. \tag{2.19}$$

Proof. Let $\operatorname{Re} z > 0$. To see that $e^{zx}A$ is trace class, we write

$$e^{zx}A = e^{zx/2} \cdot \left(e^{zx/2} A e^{-zx/2}\right) \cdot e^{zx/2}. \tag{2.20}$$

Now the Schwartz kernel of A in local coordinates on the infinite cylinder is of the form, see (2.14)

$$K_A(x, y, x', y') = \int e^{i(x-x', y-y')\cdot(\tau,\eta)}\, a(x, y, \tau, \eta)\, đ\tau đ\eta, \tag{2.21}$$

where the primes denote the same coordinates (x, y) but on the right factor of $\widehat{M} \times \widehat{M}$, and where $a(x, y, \tau, \eta)$ is a symbol of order $-\infty$ satisfying properties (1) and (2) around (2.8). In these coordinates, it follows that the Schwartz kernel of $A_z := e^{zx/2} A e^{-zx/2}$ is given by

$$\begin{aligned} K_{A_z} = e^{zx/2} K_A(x, y, x', y') e^{-zx'/2} &= e^{(x-x')z/2} K_A \\ &= \int e^{i(x-x')(\tau - iz/2)} e^{i(y-y')\cdot\eta}\, a(x, y, \tau, \eta)\, đ\tau đ\eta \\ &= \int e^{i(x-x')\tau} e^{i(y-y')\cdot\eta}\, a(x, y, \tau + iz/2, \eta)\, đ\tau đ\eta, \end{aligned} \tag{2.22}$$

where we used the fact that the symbol is entire in τ. It follows that A_z is also a b-smoothing operator. In view of (2.20), the Schwartz kernel of $e^{zx}A$ is given by

$$K_{e^{zx}A}(x, y, x', y') = e^{zx/2} \cdot K_{A_z} \cdot e^{zx'/2},$$

which vanishes exponentially like $e^{zx/2}$ on the cylinders in both factors of $\widehat{M} \times \widehat{M}$. Therefore, $e^{zx}A$ is trace class for $\operatorname{Re} z > 0$.

We now verify the properties of $F(z)$. Assuming that $\operatorname{Re} z > 0$, the function e^{zx} is integrable on $(-\infty, 0]_x$ and for such z,

$$\int_{-\infty}^0 e^{zx}\, dx = \frac{e^{zx}}{z}\bigg|_{x=-\infty}^{x=0} = \frac{1}{z}.$$

[4] It turns out that $F(z)$ extends to be meromorphic on $\mathbb{C}$ with only simple poles at the points $0, -1, -2, \ldots$, but we don't need this fact.

Hence, writing $K_A|_\Delta$ as in (2.16), we see that

$$
\begin{aligned}
F(z) &= \int_{\widehat{M}} e^{zx} K_A|_\Delta \\
&= \int_{(-\infty,0]_x \times Y} e^{zx}(a_0(y) + a_1(x,y))\, dx\, dh + \int_M e^{zx} K_A|_\Delta \\
&= \frac{1}{z}\int_Y a_0(y)\, dh + \int_{(-\infty,0]_x\times Y} e^{zx} a_1(x,y)\, dx\, dh + \int_M e^{zx} K_A|_\Delta,
\end{aligned}
$$

where both a_0 and a_1 are smooth and $a_1(x,y)$ vanishes like e^x as $x \to -\infty$. It follows that $F(z)$ extends to be a meromorphic function on the strip $\operatorname{Re} z > -1$, with regular value at $z = 0$ equal to ${}^b\mathrm{Tr}\, A$, and with only a simple pole at $z = 0$ with residue given by

$$\int_Y a_0(y)\, dh.$$

To show that this integral is related to the normal operator of A, observe that setting $(x, y) = (x', y')$ in (2.21), we obtain

$$K_A(x, y, x, y) = \int a(x, y, \tau, \eta)\, đ\tau đ\eta,$$

so if $a_0(y, \tau, \eta)$ is the limiting term in the expansion (2.5) for $a(x, y, \tau, \eta)$ as $x \to -\infty$, then

$$\int_Y a_0(y)\, dh = \int_Y \int a_0(y, \tau, \eta)\, đ\tau đ\eta\, dh(y). \tag{2.23}$$

On the other hand, by definition of $N(A)(\tau)$ (see (2.11)), we have

$$K_{N(A)(\tau)}(y, y') = \int e^{i(y-y')\cdot\eta}\, a_0(y, \tau, \eta)\, đ\eta,$$

which implies that

$$\mathrm{Tr}\, N(A)(\tau) = \int_Y K_{N(A)(\tau)}(y, y)\, dh(y) = \int_Y \int a_0(y, \tau, \eta)\, đ\eta\, dh(y). \tag{2.24}$$

Equating (2.23) and (2.24) proves (2.19) and completes the proof. □

It is well-known that the trace functional on genuine (not b-) smoothing operators on a compact manifold is the unique functional, up to multiplicative constant, that vanishes on commutators. The formula (2.25) below is sometimes called the *trace-defect formula* because it gives a formula for the nonvanishing of the b-trace on commutators, and hence measures the "non-trace like nature" of the b-trace.

Theorem 2.5. *If $A \in \Psi_b^m(\widehat{M})$ and $B \in \Psi_b^{m'}(\widehat{M})$ with $m + m' = -\infty$, then*

$$
\begin{aligned}
{}^b\mathrm{Tr}[A, B] &= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\, \partial_\tau N(A)(\tau) \circ N(B)(\tau)\,\right) d\tau \\
&= -\frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\, N(A)(\tau) \circ \partial_\tau N(B)(\tau)\,\right) d\tau.
\end{aligned} \tag{2.25}
$$

Proof. Integration by parts shows that the two integrals on the right are equal. Throughout this proof we assume that A is of order $-\infty$ and we shall prove the first equality. To prove this theorem, we use Lemma 2.4, which states that

$$ {}^b\mathrm{Tr}[A, B] = \mathrm{Reg}_{z=0}\, \mathrm{Tr}\big(e^{zx}[A, B]\big).$$

To evaluate the right-hand side, we first rewrite $e^{zx}[A,B]$ as

$$e^{zx}[A,B] = [e^{zx}, A]B + [A, e^{zx}B].$$

Since the trace vanishes on commutators when one operator is a bounded operator and the other is trace class, we have $\mathrm{Tr}([A, e^{zx}B]) = 0$ for $\mathrm{Re}\, z > 0$, and hence its analytic continuation at $z = 0$ is zero also. Therefore, since

$$[e^{zx}, A]B = e^{zx}[AB - e^{-zx}Ae^{zx}B] = e^{zx}\{AB - A(z)B\},$$

where $A(z) = e^{-zx}Ae^{zx}$, we have

$$ {}^b\mathrm{Tr}[A,B] = \mathrm{Reg}_{z=0}\ \mathrm{Tr}\left(e^{zx}\{AB - A(z)B\}\right). \tag{2.26}$$

Writing the Schwartz kernel of A as in (2.21) and using the same computation found around (2.22), we can write the Schwartz kernel of $A(z)$ as

$$K_{A(z)} = \int e^{i(x-x')\tau} e^{i(y-y')\cdot\eta}\, a(x, y, \tau - iz, \eta)\, đ\tau đ\eta.$$

It follows that $A(z) = e^{-zx}Ae^{zx}$ is also a b-pseudodifferential operator of order $-\infty$ that is holomorphic in z such that $A(0) = A$. Moreover, if $A'(z)$ denotes the derivative of $A(z)$ with respect to z, then

$$K_{A'(z)} = -i \int e^{i(x-x')\tau} e^{i(y-y')\cdot\eta}\, \partial_\tau a(x, y, \tau - iz, \eta)\, đ\tau đ\eta,$$

therefore, by definition of the normal operator,

$$K_{N(A'(0))(\tau)} = -i \int e^{i(y-y')\cdot\eta}\, \partial_\tau a(0, y, \tau, \eta)\, đ\eta = -iK_{\partial_\tau N(A)(\tau)}. \tag{2.27}$$

Since $A(0) = A$, expanding $A(z)$ in Taylor series at $z = 0$, we can write

$$AB - A(z)B = -zA'(0)B - z^2 C(z),$$

where $C(z)$ is a b-smoothing operator that is holomorphic in z. In view of (2.26), we have

$$ {}^b\mathrm{Tr}[A,B] = -\mathrm{Reg}_{z=0}\left\{z\,\mathrm{Tr}(e^{zx}A'(0)B) + z^2\,\mathrm{Tr}(e^{zx}C(z))\right\}.$$

By Lemma 2.4, the traces on the right have at most simple poles at $z = 0$, so in particular, the second term $z^2\,\mathrm{Tr}(e^{zx}C(z))$ vanishes at $z = 0$, while for the first term, by Lemma 2.4, we obtain

$$\begin{aligned} {}^b\mathrm{Tr}[A,B] &= -\mathrm{Reg}_{z=0}\, z\,\mathrm{Tr}(e^{zx}A'(0)B) = -\mathrm{Res}_{z=0}\ \mathrm{Tr}(e^{zx}A'(0)B) \\ &= -\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\, N(A'(0)B)(\tau)\, d\tau = -\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(N(A'(0))(\tau)\, N(B)(\tau))\, d\tau. \end{aligned}$$

By (2.27), we see that $N(A'(0))(\tau) = -i\partial_\tau N(A)(\tau)$, which finishes our proof. □

2.5. The b-proof of the index theorem. We now give Melrose's proof of the APS index formula in Theorem 2.1. For this we need the heat operators

$$e^{-t\widehat{D}^*\widehat{D}} \quad \text{and} \quad e^{-t\widehat{D}\widehat{D}^*},$$

both of which exist via the usual arguments and moreover, are b-smoothing operators for $t > 0$ (see Melrose [**40**] and Loya [**31**]). The key idea behind the heat kernel proof of the index formula is to consider the difference of the heat traces:

$$\text{“ } h(t) = \mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - \mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}). \text{ ”}$$

The reason for the quotation marks is that, as we observed earlier, b-smoothing operators are in general not trace class; this is true in the present situation too for the heat operators. However, the following b-object is well-defined:

$$h(t) = {}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}),$$

and we shall use this object instead of the ill-defined one above. We now indicate how $h(t)$ has the following amazing properties (however, we really only focus on Property (3)):

$$\begin{aligned}
&(1) \quad \lim_{t\to\infty} h(t) = \operatorname{ind} \widehat{D}\\
&(2) \quad \lim_{t\to 0} h(t) = \int_M \mathrm{AS}\\
&(3) \quad h'(t) = -\frac{1}{2\sqrt{\pi}}\, t^{-1/2}\mathrm{Tr}(D_Y\, e^{-tD_Y^2}).
\end{aligned}$$

These three facts together with the basic fundamental theorem of calculus easily prove the APS formula:

$$\begin{aligned}
\operatorname{ind}\widehat{D} = h(\infty) &= h(0) + \int_0^\infty h'(t)\, dt\\
&= \int_M \mathrm{AS} + \int_0^\infty -\frac{1}{2\sqrt{\pi}}\, t^{-1/2}\mathrm{Tr}(D_Y\, e^{-tD_Y^2})\, dt\\
&= \int_M \mathrm{AS} - \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-1/2}\mathrm{Tr}(D_Y\, e^{-tD_Y^2})\, dt\\
&= \int_M \mathrm{AS} - \frac{1}{2}\eta(D_Y).
\end{aligned}$$

It turns out that the proofs of Property (1) and Property (2) follow basically the same lines of reasoning as for the corresponding statements in the compact manifold without boundary case. For this reason, we shall not discuss them in detail and leave the reader to check Chapters 8 and 9 in [**40**] for the arguments. However, we remark that Property (2) follows from the the so-called *local index theorem*, which states that for $p \in \widehat{M}$,

$$\lim_{t\to 0}\Big\{ \mathrm{tr}\, e^{-t\widehat{D}^*\widehat{D}}(p,p) - \mathrm{tr}\, e^{-t\widehat{D}\widehat{D}^*}(p,p)\Big\} = \mathrm{AS}(p)$$

uniformly in t, where the lower case tr denotes the fiber-wise trace, and where the right-hand side really represents the coefficient of the volume form component of the differential form $\mathrm{AS}(p)$, cf. McKean and Singer [**38**], Gilkey [**19**], Patodi [**48**], Alvarez-Gaumé [**1**], and Getzler [**18**]. Note that AS vanishes on the cylindrical end by our product type hypothesis.

We now prove (3), which is where we see our hero, the b-trace, in action in the direct manner by which (3) is derived. First, recalling the elementary identity[5]

[5]This identity follows from uniqueness of solutions to the heat equation, cf. [**40**, p. 271].

$\widehat{D}^*\widehat{D}e^{-t\widehat{D}^*\widehat{D}} = \widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}\widehat{D}$, we take the derivative of $h(t)$:

$$
\begin{aligned}
h'(t) &= \frac{d}{dt}\Big({}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*})\Big) \\
&= {}^b\mathrm{Tr}(\, -\widehat{D}^*\widehat{D}e^{-t\widehat{D}^*\widehat{D}}\,) + {}^b\mathrm{Tr}(\,\widehat{D}\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}\,) \\
&= {}^b\mathrm{Tr}(\, -\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}\widehat{D}\,) + {}^b\mathrm{Tr}(\,\widehat{D}\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}\,) \\
&= {}^b\mathrm{Tr}(\,[\widehat{D},\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}]\,),
\end{aligned}
\tag{2.28}
$$

where $[\widehat{D},\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}]$ is the commutator of $\widehat{D}$ and $\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}$. Second, by the trace-defect formula, we find

$$
\begin{aligned}
h'(t) &= {}^b\mathrm{Tr}(\,[\widehat{D},\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}]\,) \\
&= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\partial_\tau N(\widehat{D})(\tau)\, N(\widehat{D}^*)(\tau)\, N(e^{-t\widehat{D}\widehat{D}^*})(\tau)\right) d\tau.
\end{aligned}
$$

Third, we complete the verification of (3) by *directly* computing the right-hand side of $h'(t)$ to equal $-\frac{1}{2\sqrt{\pi}}\, t^{-1/2}\mathrm{Tr}(D_Y\, e^{-tD_Y^2})$. To see this, observe that for $\tau \in \mathbb{R}$, by the formula (2.12) for the normal operator of $\widehat{D}$, we have

$$
N(\widehat{D})(\tau) = \Gamma(i\tau + D_Y) \quad \text{and} \quad N(\widehat{D}^*)(\tau) = -(-i\tau + D_Y)\Gamma^*.
$$

It follows that

$$
N(\widehat{D}\widehat{D}^*)(\tau) = N(\widehat{D})(\tau)N(\widehat{D}^*)(\tau) = \Gamma(\tau^2 + D_Y^2)\Gamma^*.
$$

Now it is easily proved from the continuity properties of the normal operator that

$$
N(e^{-t\widehat{D}\widehat{D}^*})(\tau) = e^{-tN(\widehat{D}\widehat{D}^*)(\tau)} = \Gamma e^{-t\tau^2}e^{-tD_Y^2}\Gamma^*.
$$

Multiplying this with $\partial_\tau N(\widehat{D})(\tau) = i\Gamma$ and $N(\widehat{D}^*)(\tau)$, we obtain

$$
\partial_\tau N(\widehat{D})(\tau)\, N(\widehat{D}^*)(\tau)\, N(e^{-t\widehat{D}\widehat{D}^*})(\tau) = i\Gamma(-i\tau + D_Y)e^{-t\tau^2}e^{-tD_Y^2}\Gamma^*.
$$

Finally, using the facts that $\int_{\mathbb{R}} \tau e^{-t\tau^2}\, d\tau = 0$, $\int_{\mathbb{R}} e^{-t\tau^2}\, d\tau = t^{-1/2}\sqrt{\pi}$, and Γ is unitary, we get our desired result:

$$
\begin{aligned}
h'(t) = {}^b\mathrm{Tr}(\,[\widehat{D},\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}]\,) &= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\partial_\tau N(\widehat{D})(\tau)\, N(\widehat{D}^*)(\tau)\, N(e^{-t\widehat{D}\widehat{D}^*})(\tau)\right) d\tau \\
&= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(\, i\Gamma(-i\tau + D_Y)e^{-t\tau^2}e^{-tD_Y^2}\Gamma^*\,)\, d\tau \\
&= \frac{i^2}{2\pi} t^{-1/2}\sqrt{\pi}\ \mathrm{Tr}(\, D_Y e^{-tD_Y^2}\,). \\
&= -\frac{1}{2\sqrt{\pi}} t^{-1/2}\ \mathrm{Tr}(\, D_Y e^{-tD_Y^2}\,).
\end{aligned}
$$

3. Transformation to b-objects II

In this section we prove Theorem 1.1. Needless to say, we now drop all assumptions about the invertibility of D_Y. We start this section by defining the b-smoothing perturbation $\widehat{T}$ in Theorem 1.1 and then we relate the kernels of $\widehat{D} - \widehat{T}$ and D_T. One of the main goals in this section is to prove the general index theorem 3.4 from which the index formula in Theorem 1.1 follows. The direct (but somewhat complicated) computation of the eta invariant portion of the general index formula

(3.8) boils down once again to the trace-defect formula. We end this section by proving this general index formula.

3.1. Definition of $\widehat{T}$. We begin by defining the b-smoothing perturbation $\widehat{T}$ in Theorem 1.1. Let D and $\widehat{D}$ be as in Section 1.1 and let T be a self-adjoint isomorphism on $V = \ker D_Y$ with $T^2 = \mathrm{Id}$.

The b-smoothing operator $\widehat{T}$ is really very simple to define. First, we define an auxiliary b-smoothing operator on the half-line $(-\infty, 0]$. Let $\chi \in C^\infty(\mathbb{R})$, where $\chi(x) = 1$ for $x \leq -2$ and $\chi(x) = 0$ for $x \geq -1$. Let $\varphi \geq 0$ be a smooth compactly supported even function on $\mathbb{R}$ with $\varphi(0) > 0$. Then $\widehat{\varphi}(\tau)$ is an even entire function. Define $Q \in \Psi_b^{-\infty}((-\infty, 0])$ by

$$Qu = \chi(x) \int_{\mathbb{R}} e^{ix\tau}\, \widehat{\varphi}(\tau)\, \widehat{\chi u}(\tau)\, đ\tau, \tag{3.1}$$

where $\widehat{\chi u}$ is the Fourier transform of χu:

$$\widehat{\chi u}(\tau) = \int_{\mathbb{R}} e^{-ix\tau}\, \chi(x)\, u(x)\, dx.$$

The Schwartz kernel of Q is

$$K_Q = \chi(x) \cdot \int_{\mathbb{R}} e^{i(x-x')\tau}\, \widehat{\varphi}(\tau)\, đ\tau \cdot \chi(x').$$

Since φ is compactly supported, $\widehat{\varphi}(\tau)$ vanishes to infinite order as $|\tau| \to \infty$ for $|\operatorname{Im}\tau|$ within any fixed bound and therefore, $Q \in \Psi_b^{-\infty}((-\infty, 0])$ by definition of this space. Moreover, since φ is even, $\widehat{\varphi}(\tau)$ is also even, so $K_Q(x, x') = \overline{K_Q(x', x)}$, which implies that Q is self-adjoint. Finally, by definition of the normal operator, we have

$$N(Q)(\tau) = \widehat{\varphi}(\tau).$$

Second, we note that T is a smoothing operator. To see this, observe that we can identify $T : V \longrightarrow V$ with $T \circ \pi : L^2(Y, E_Y) \longrightarrow V$, with π the orthogonal projection of $L^2(Y, E_Y)$ onto V. Since $V \subset C^\infty(Y, E_Y)$ is finite dimensional it follows that T is a (finite rank) smoothing operator.

Third, we define the b-smoothing operator $\widehat{T} \in \Psi_b^{-\infty}(\widehat{M}, E, F)$ supported on the cylindrical end $(-\infty, 0]_x \times Y$ by

$$\widehat{T} = \Gamma\, Q^2 T \in \Psi_b^{-\infty}(\widehat{M}, E, F), \tag{3.2}$$

where Q is the self-adjoint operator given in (3.1). Note that T acts on the cross section Y while Q acts on the cylinder part $(-\infty, 0]_x$, and Γ simply maps E to F. The normal operator of $\widehat{T}$ is given by

$$N(\widehat{T})(\tau) = N(\Gamma\, Q^2 T)(\tau) = \Gamma\, N(Q)(\tau)^2 T = \Gamma\, \widehat{\varphi}(\tau)^2 T. \tag{3.3}$$

Now to the proof of Theorem 1.1. We shall prove Property (a) of Theorem 1.1 in Section 3.2 and then Properties (b) and (c) in Section 3.3.

3.2. Relation of kernels and indices of $\widehat{D} - \widehat{T}$ and D_T. Throughout this section we denote the $+1$ eigenspace of T by Λ_T. To prove that $\ker(\widehat{D} - \widehat{T}) \cong \ker D_T$ and $\ker(\widehat{D} - \widehat{T})^* \cong \ker D_T^*$, we start off with the following lemma.

LEMMA 3.1. *If T is a self-adjoint linear map on V and W is a subspace of V, then given any $v_0 \in W$, the boundary value problem*

$$v \in H^1((-\infty, 0], V), \quad \Big(\partial_x - Q^2 T\Big) v = 0, \quad v|_{x=1} = v_0,$$

has a non-trivial solution if and only if $v_0 \in \Lambda_T \cap W$, in which case, the solution is unique and also takes values in $\Lambda_T \cap W$.

PROOF. We can decompose Λ_T as $\Lambda_T = U_0 \oplus U_1$, where $U_0 = \Lambda_T \cap W$ and U_1 is the orthogonal complement in Λ_T of U_0. Thus, we can decompose the vector space V and matrix T as

$$V = U_0 \oplus U_1 \oplus \Lambda_T^{\perp}, \quad T = \mathrm{Id} \oplus \mathrm{Id} \oplus -\mathrm{Id}.$$

Since we can decompose any element of $H^1((-\infty, 0], V)$ into functions taking values in U_0, U_1, and $\Lambda_T^{\perp}$, our lemma is proved once we show that there are exactly $\dim(U_0)$ non-trivial solutions to the boundary value problem

$$v \in H^1((-\infty, 0], V), \quad \Big(\partial_x - Q^2 T\Big) v = 0, \quad v|_{x=1} \in W \tag{3.4}$$

if v takes values in U_0, and has no solutions otherwise. First suppose that v takes values in $\Lambda_T^{\perp}$. Since $T = -1$ on $\Lambda_T^{\perp}$, we have

$$\Big(\partial_x + Q^2\Big) v(x) = 0. \tag{3.5}$$

By choosing a basis for W, we may assume that v is a scalar function. Also, since ∂_x and Q are real, we may assume that v is a real-valued function. Since v is an L^2 solution of (3.5), elementary use of the Fourier transform can be used to show that $v(x) \to 0$ as $x \to -\infty$. This implies that $\int_{-\infty}^0 v'\, v dx = \frac{1}{2} v(0)^2$. Thus, multiplying (3.5) by $v\, dx$, integrating from $-\infty$ to 0, and using that Q is self-adjoint, we obtain

$$\frac{1}{2} v(0)^2 + \int |Qv|^2 dx = 0. \tag{3.6}$$

Thus, $v(0) = 0$ and $Qv = 0$. Setting $Qv = 0$ in (3.5), we see that v must be constant. As $v(0) = 0$, v must be the constant 0.

Now suppose that v takes values in U_1. Since $U_1 \cap W = 0$ and since $v(0) \in W$, we have $v(0) = 0$. By choosing a basis for U_1, we may assume that v is a scalar function, and assuming as before that v is real, a similar argument used to prove (3.6) shows that

$$\frac{1}{2} v(0)^2 - \int |Qv|^2 dx = 0.$$

Since $v(0) = 0$, $Qv = 0$. Arguing as in the previous case shows that $v = 0$.

Thus, we are left with the case that v takes values in U_0. As before, we may assume that v is a scalar function by choosing a basis for U_0, in which case v is in the kernel of the one-dimensional operator

$$A = \partial_x - Q^2 \quad \text{on } (-\infty, 0].$$

So, once we show that $\dim \ker A = 1$ on $H^1((-\infty, 0])$, our proof is complete. Moreover, since $Q = 0$ near $x = 0$, a function in $\ker A$ must be constant near $x = 0$, so for purposes of investigating $\ker A$, we can put Neumann boundary conditions at

$x = 0$. Recall that the normal operator of A is invertible for all real parameters. Thus, Theorem 2.2 implies that[6]

$$A : H^1((-\infty, 0]) \longrightarrow L^2((-\infty, 0])$$

is Fredholm. Now observe that

$$A^* = -\partial_x - Q^2,$$

as an operator, together with the Dirichlet boundary condition at $x = 0$. The same argument used to prove that there are no solutions to (3.5) proves that $\ker A^* = \{0\}$. Since A is Fredholm, it follows that $\operatorname{ind} A = \dim\ker A$. Thus, it remains to show that $\operatorname{ind} A = 1$. To see this, consider equation (3.1) for Q:

$$Qu = \chi(x) \int_{\mathbb{R}} e^{ix\tau}\, \widehat{\varphi}(\tau)\, \widehat{\chi u}(\tau)\, đ\tau$$

and consider the following deformation of Q:

$$Q_t u = \chi(x) \int_{\mathbb{R}} e^{ix\tau}\, \widehat{\varphi}(t\tau)^2\, \widehat{\chi u}(\tau)\, đ\tau, \qquad t \in [0, 1].$$

Then, since $N(\partial_x - Q_t)(\tau) = -i\tau - \widehat{\varphi}(t\tau)^2$ is invertible for all $0 \le t \le 1$ and $\tau \in \mathbb{R}$, it follows that $\partial_x - Q_t$ is a continuous family of Fredholm operators for each $0 \le t \le 1$. Equating the indices at $t = 1$ and $t = 0$, we obtain $\operatorname{ind} A = \operatorname{ind}(\partial_x - \chi^2)$. Again using the fact that the index is stable under compact perturbations, we can replace χ^2 with H, where $H(x) = 1$ for $x \le -1$, and $H(x) = 0$ for $x > -1$ and conclude that $\operatorname{ind} A = \operatorname{ind} \tilde{A}$, where $\tilde{A} = \partial_x - H$. Since $\tilde{A}^* = -\partial_x - H$, the same argument used to prove that there are no solutions to (3.5) proves that $\ker \tilde{A}^* = \{0\}$. Suppose that $\tilde{A} f = 0$. Then,

$$\partial_x f - H(x) f = 0.$$

Solving this equation, we find that for some $c \in \mathbb{C}$, $f = c\, e^x$ for $x \le -1$ and $f = c\, e^{-1}$ for $x > -1$. Thus, $\dim\ker \tilde{A} = 1$. Hence, $\operatorname{ind} A = 1$. □

We now come to the main theorem in this section.

THEOREM 3.2. *The kernels D_T and $\widehat{D} - \widehat{T}$ are canonically isomorphic. In fact, each is canonically isomorphic to $(\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$. In particular,*

$$\dim\ker D_T = \dim\ker(\widehat{D} - \widehat{T}) = \dim(\Lambda_T \cap \Lambda_C) + \dim\ker D_{-C}.$$

PROOF. We first prove that $\ker D_T$ is isomorphic to $(\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$, then we prove the same for $\widehat{D} - \widehat{T}$.

Proof for D_T: If $u \in \ker D_T$, then $Du = 0$ and $\Pi_+^T u|_{x=0}$, so that $\Pi_+ u|_{x=0} = 0$ and $\Pi_T^{\perp} u|_{x=0} = 0$. In particular, $\Pi_0 u|_{x=0} \in \Lambda_T$. On the other hand, since $Du = 0$ and $\Pi_+ u|_{x=0} = 0$, by definition of Λ_C, $\Pi_0 u|_{x=0} \in \Lambda_C$. Thus, $\Pi_0 u|_{x=0} \in \Lambda_T \cap \Lambda_C$.

The previous paragraph implies that $\ker D_T$ can be identified with the space of pairs (v, w) in

$$(\Lambda_T \cap \Lambda_C) \oplus \{w \in H^1(M, E)\,;\, Dw = 0,\ \Pi_+ w|_{x=0} = 0\}$$

such that $v = \Pi_C w|_{x=0}$. Denote this space by U. To see that $U \cong (\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$, consider the map $A \ni (v, w) \overset{\pi_1}{\longmapsto} v \in \Lambda_T \cap \Lambda_C$. Since $v = \Pi_C w|_{x=0}$, and since $\Pi_C = \Pi_{-C}^{\perp}$, the kernel of this map is exactly $\ker D_{-C}$. This map is also

[6] Of course, here we have an extra boundary at $x = 0$ but this is the "compact end" where Q vanishes and the usual elliptic theory can be implemented.

surjective, for if $v \in \Lambda_T \cap \Lambda_C$, then by definition of Λ_C, there is a $w \in H^1(M, E)$ with $Dw = 0$ and $\Pi_+ w = 0$ such that $v = \Pi_C w|_{x=0}$. Thus, the following sequence is exact:

$$0 \longrightarrow \ker D_{-C} \longrightarrow U \xrightarrow{\pi_1} \Lambda_T \cap \Lambda_C \longrightarrow 0.$$

This proves our theorem for D_T.

Proof for $\widehat{D} - \widehat{T}$: Suppose that $u \in \ker \widehat{D}$. Let $\{\varphi_j\} \subset C^\infty(Y, E_Y)$ be the eigenvectors of D_Y with corresponding real eigenvalues $\{\lambda_j\}$. Then on the product decomposition, $\widehat{M} \cong (-\infty, 0]_x \times Y$, we can write $u = \sum_j f_j(x)\varphi_j(y)$ for some $f_j \in L^2((-\infty, 0])$. Since $\widehat{D} = \Gamma[\partial_x + D_Y]$ on the collar and since $\widehat{D}u = 0$, one concludes that $f_j(x) = 0$ if $\lambda_j \geq 0$, and $f_j(x) = c_j e^{-\lambda_j x}$ if $\lambda_j < 0$, where c_j is a constant. Thus, $u = \sum_{\lambda_j < 0} c_j e^{-\lambda_j x}\varphi_j(y)$ on the collar. Since T acts only on V, and since $\widehat{T}$ is supported on the collar, it follows that $\widehat{T}u = 0$. Thus, $(\widehat{D} - \widehat{T})u = 0$ and $u \in \ker(\widehat{D} - \widehat{T})$.

Suppose that $u \in \ker(\widehat{D} - \widehat{T}) \setminus \ker \widehat{D}$. Then, as $\widehat{D} - \widehat{T} = \Gamma[\partial_x + D_Y] - \widehat{T}$ and $\widehat{T}$ acts only on the kernel of D_Y, as in the previous paragraph one can show that on the collar, $u = v(x, y) + \sum_{\lambda_j < 0} c_j e^{-\lambda_j x}\varphi_j(y)$, where $v(x, y) \neq 0$ takes values in V and $[\Gamma(\partial_x) - \widehat{T}]v = 0$. Since $\widehat{T}$ is supported on $(-\infty, -1]$, $v(x, y)$ must be constant off the support of $\widehat{T}$. Now define $\widehat{v} = u$ off of the collar and $\widehat{v} = v(0, y) + \sum_{\lambda_j < 0} e^{-\lambda_j x} c_j \varphi_j(y)$ on the collar. Then $\widehat{v}$ and $\widehat{D}\widehat{v} = 0$. Thus, by definition of the scattering Lagrangian, $v(0, y) \in \Lambda_C$. Thus, v is a non-trivial solution to the boundary value problem

$$[\Gamma(\partial_x) - \widehat{T}]v = 0, \quad v|_{x=0} \in \Lambda_C.$$

By Lemma 3.1, there are exactly $\dim(\Lambda_T \cap \Lambda_C)$ independent solutions to this boundary value problem, occurring only when $v \in \Lambda_T \cap \Lambda_C$. It follows that $\ker(\widehat{D} - \widehat{T}) \setminus \ker \widehat{D} \equiv \Lambda_T \cap \Lambda_C$. Finally, by definition of the scattering Lagrangian (see (1.3)) it follows that $\ker \widehat{D} \equiv \ker D_{-C}$. Thus, $\ker(\widehat{D} - \widehat{T}) \equiv (\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$ and our proof is now complete. ∎

Remark 3.3. A similar proof can be used to show that the kernels of the adjoints $(D_T)^*$ and $(\widehat{D} - \widehat{T})^*$ are canonically isomorphic. The exact same proof gives the same result in case M is odd-dimensional.

3.3. Proof of Theorem 1.1 and a general index theorem. We are now ready to prove Theorem 1.1. We begin by recalling Theorem 1.1, which states that there exists a b-smoothing operator $\widehat{T}$ such that D_T and the perturbed Dirac operator $\widehat{D} - \widehat{T}$ have the same index theoretic properties:.

(a) $\ker(\widehat{D} - \widehat{T}) \cong \ker D_T$ and $\ker(\widehat{D} - \widehat{T})^* \cong \ker(D_T)^*$.

(b) The operators

$$\widehat{D} - \widehat{T} : H^1(\widehat{M}, E) \longrightarrow L^2(\widehat{M}, F),$$
$$D_T : \mathrm{Dom}(D_T) \longrightarrow L^2(M, F)$$

are Fredholm with (by (a)) equal indices.

(c) The following index formula holds:

$$\operatorname{ind}(\widehat{D} - \widehat{T}) = \operatorname{ind} D_T = \int_M \mathrm{AS} - \frac{1}{2}[\eta(D_Y) - \operatorname{sign} T]. \tag{3.7}$$

Theorem 3.2 (see also Remark 3.3) proves Part (a), so we just need to prove Parts (b) and (c). The statement about D_T in Property (b) follows from work in, for instance, Atiyah, Patodi, and Singer [**5**] or Booß-Bavnbek and Wojciechowski [**10**], so we shall omit the proof of this fact. The Fredholm property of $\widehat{D} - \widehat{T}$ in Property (b) and the formula (3.7) follow from the next theorem, whose proof is found in Sections 3.4 and 3.5.

Theorem 3.4. *Let* $R \in \Psi_b^{-\infty}(\widehat{M}, E, F)$ *and suppose that* $N(R)(\tau) = \Gamma \widetilde{R}(\tau)$, *where* $\widetilde{R}(\tau) \in \Psi^{-\infty}(Y, E_Y)$ *is self-adjoint for* $\tau \in \mathbb{R}$, *and if* $R_Y = \widetilde{R}(0)$, *then* $D_Y + R_Y$ *is invertible. Then*

$$\widehat{D} + R : H^1(\widehat{M}, E) \longrightarrow L^2(\widehat{M}, F)$$

is Fredholm and its index is given by

$$\operatorname{ind}(\widehat{D} + R) = \int_M \mathrm{AS} - \frac{1}{2}\eta(D_Y + R_Y), \tag{3.8}$$

where AS *is the Atiyah-Singer density and* $\eta(D_Y + R_Y)$ *is the eta invariant of* $D_Y + R_Y$, *defined through any of the definitions* (2.2), (2.3), (2.4), *for instance,*[7]

$$\eta(D_Y + R_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(\,(D_Y + R_Y)e^{-t(D_Y + R_Y)^2}\,)\, dt. \tag{3.9}$$

Remark 3.5. The index formula (3.8) is similar to Melrose and Piazza's celebrated families index theorem [**42**, Th. 1] in the simplest case when the base manifold is a point. In this case (cf. Lemmas 8 and 9 in [**42**]), they consider a b-smoothing operator of the form like our $\widehat{T}$ in (3.2), but where T in (3.2) is a very general finite rank operator adapted to a finite rank perturbation of the APS projection Π_+ connected with the notion of a *spectral section*.

To see that $\widehat{D} + R$ is Fredholm, we observe that

$$N(\widehat{D} + R)(\tau) = \Gamma(i\tau + D_Y) + \Gamma \widetilde{R}(\tau) = \Gamma(i\tau + D_Y + \widetilde{R}(\tau)).$$

For $\tau \in \mathbb{R}$, the operator $D_Y + \widetilde{R}(\tau)$ is self-adjoint, so $N(\widehat{D} + R)(\tau)$ is automatically invertible for $\tau \in \mathbb{R}$ not zero and is invertible at $\tau = 0$ if and only if

$$D_Y + R_Y : H^m(Y, E_Y) \longrightarrow L^2(Y, E_Y)$$

is invertible. But this is invertible by assumption, therefore $\widehat{D} + R$ is Fredholm. The proof of the index formula (3.8) is given in Section 3.5 after we prove some preliminary lemmas in Section 3.4.

Let us apply Theorem 3.4 to prove (b) and (c) above. Recall from (3.3) that

$$N(\widehat{T})(\tau) = \Gamma\, \widehat{\varphi}(\tau)^2 T.$$

Then $\widehat{\varphi}(\tau)^2 T \in \Psi^{-\infty}(Y, E_Y)$ is self-adjoint for $\tau \in \mathbb{R}$ and since $\widehat{\varphi}(0) = \int \varphi(x)\, dx > 0$ (recall that $\varphi \geq 0$ with $\varphi(0) > 0$) and T is an isomorphism on $V = \ker D_Y$,

$$D_Y - \widehat{\varphi}(0)^2 T : H^m(Y, E_Y) \longrightarrow L^2(Y, E_Y)$$

is invertible. Thus, according to Theorem 3.4, $\widehat{D} - \widehat{T}$ is Fredholm and

$$\operatorname{ind}(\widehat{D} - \widehat{T}) = \int_M \mathrm{AS} - \frac{1}{2}\eta(D_Y - \widehat{\varphi}(0)^2 T).$$

[7]It may not be entirely obvious that the integral (3.9) is well-defined, but we shall see this integral is well-defined in Proposition 3.9.

If $\{\lambda_j\}$ denotes the eigenvalues of D_Y and $\{\mu_j\}$ those of the finite dimensional matrix T, then the eta function of $D_Y - \widehat{\varphi}(0)^2 T$ is

$$\sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z} - \sum_j \frac{\operatorname{sign} \mu_j}{|\mu_j|^z},$$

which implies that

$$\eta(D_Y - \widehat{\varphi}(0)^2 T) = \eta(D_Y) - \operatorname{sign} T.$$

This completes the proof of Theorem 1.1.

Remark 3.6. The proof that $\widehat{D} - \widehat{T}$ is Fredholm also works when M is odd-dimensional. However, in this case $\widehat{D} - \widehat{T}$ is self-adjoint so has trivial index.

3.4. Preliminary lemmas for the general index formula. The following lemmas will be used in the next section to prove the general index formula (3.8).

Lemma 3.7. *Given any $S \in \Psi^{-\infty}(Y, E_Y)$, we have*

$$e^{-t(D_Y^2 + S)} = e^{-tD_Y^2} + t\, T(t),$$

where $T(t) \in C^\infty([0,\infty); \Psi^{-\infty}(Y, E_Y))$. Moreover, if S depends continuously on parameters, then so does $T(t)$.

Proof. With $F(t) = e^{-t(D_Y^2+S)} - e^{-tD_Y^2}$, we obtain

$$(\partial_t + (D_Y^2 + S))F(t) = -Se^{-tD_Y^2}.$$

As $F(0) = 0$, by Duhamel's Principle, $F(t) = -\int_0^t e^{-(t-s)(D_Y^2+S)} S e^{-sD_Y^2}\, ds$. Since $S \in \Psi^{-\infty}(Y, E_Y)$, we have $Se^{-sD_Y^2} \in C^\infty([0,\infty)_s; \Psi^{-\infty}(Y, E_Y))$ by the properties of the heat operator $e^{-sD_Y^2}$. It follows that $F(t) = t\,T(t)$, where $T(t) \in C^\infty([0,\infty); \Psi^{-\infty}(Y, E_Y))$. By our proof, it follows that if S depends continuously on parameters, then so does $T(t)$. □

Remark 3.8. We have stated this theorem for operators on Y because we will use this lemma immediately in Lemma 3.10, but this argument works equally well on $\widehat{M}$: Given any $S \in \Psi_b^{-\infty}(\widehat{M}, E)$, we have

$$e^{-t(\widehat{D}^* \widehat{D} + S)} = e^{-t\widehat{D}^* \widehat{D}} + t\, T(t),$$

where $T(t) \in C^\infty([0,\infty); \Psi_b^{-\infty}(\widehat{M}, E))$. Moreover, if S depends continuously on parameters, then so does $T(t)$. A similar statement holds for $e^{-t(\widehat{D}\widehat{D}^* + S)}$ when $S \in \Psi_b^{-\infty}(\widehat{M}, F)$.

Proposition 3.9. *If $S \in \Psi^{-\infty}(Y, E_Y)$ is self-adjoint, then the eta integral*

$$\eta(D_Y + S) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(\,(D_Y + S)e^{-t(D_Y+S)^2}\,)\, dt \tag{3.10}$$

is absolutely convergent.

Proof. Since $D_Y + S$ is elliptic and Y is compact without boundary, $D_Y + S$ is Fredholm, so the usual arguments [**40**, Ch. 9] show that $\operatorname{Tr}(\,(D_Y + S)e^{-t(D_Y+S)^2}\,)$ decays exponentially as $t \to \infty$.

We now prove that $t^{-1/2} \operatorname{Tr}((D_Y + S)e^{-t(D_Y+S)^2})$ is also absolutely integrable near $t = 0$. If $\widetilde{S} = D_Y S + S D_Y + S^2 \in \Psi^{-\infty}(Y, E_Y)$, then $e^{-t(D_Y+S)^2} =$

$e^{-t(D_Y^2+\widetilde{S})}$. Thus, by Lemma 3.7, $e^{-t(D_Y+S)^2} = e^{-tD_Y^2} + t\,T(t)$, where $T(t) \in C^\infty([0,\infty);\Psi^{-\infty}(Y,E_Y))$. Therefore,

$$(D_Y+S)e^{-t(D_Y+S)^2} = D_Y e^{-tD_Y^2} + K(t),$$

where $K(t) = Se^{-tD_Y^2}+t(D_Y+S)T(t)$. Note that $K(t) \in C^\infty([0,\infty);\Psi^{-\infty}(Y,E_Y))$, which implies that $\operatorname{Tr}(K(t)) \in C^\infty([0,\infty)_t)$, and therefore $t^{-1/2}\operatorname{Tr}(K(t))$ is absolutely integrable near $t = 0$. By the 'local index theorem for odd-dimensional manifolds', see Bismut and Freed [**6**] or Melrose [**40**, Th. 8.36], $\operatorname{Tr}(D_Y e^{-tD_Y^2}) \in t^{1/2}C^\infty([0,\infty)_t)$. Thus, $t^{-1/2}\operatorname{Tr}\left((D_Y+S)e^{-t(D_Y+S)^2}\right)$ is integrable near $t=0$. □

We need two more lemmas before presenting the proof of the general index formula (3.8). These proofs can be skipped without losing continuity.

LEMMA 3.10. *Let $A(r,\tau) = D_Y+T(r,\tau)$, where $T(r,\tau)$ is continuous in $(r,\tau) \in [0,1]\times\mathbb{R}$ and bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$. Assume that $T(r,\tau)$ is self-adjoint and $A(r,\tau)$ is invertible for all $(r,\tau) \in [0,1]\times\mathbb{R}$. Let*

(1) $B(r,\tau) = A(r,\tau)$, or let

(2) $B(r,\tau)$ be continuous in $(r,\tau) \in [0,1]\times\mathbb{R}$ and bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$.

Then for all $(r,\tau) \in [0,1]\times\mathbb{R}$, the integral

$$\eta(r,t) = \int_{\mathbb{R}} \operatorname{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau$$

exists as an absolutely convergent integral and $\eta(r,t)$ decays exponentially as $t\to\infty$ and is $\mathcal{O}(t^{-1/2})$ as $t\to 0$, both uniformly in $r\in[0,1]$.

PROOF. We begin by splitting up the integral into the parts where the integration variable is bounded and unbounded:

$$\eta(r,t) = \eta_1(r,t)+\eta_2(r,t),$$

where, provided the following integrals exist,

$$\eta_1(r,t) = \int_{|\tau|\geq 1} \operatorname{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau,$$

$$\eta_2(r,t) = \int_{|\tau|\leq 1} \operatorname{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau.$$

We shall analyze each of these integrals separately. Consider first the analysis of η_1. For this, we need some bounds on the heat operator $e^{-t\tau^2}e^{-tA(r,\tau)^2}$. Since $A(r,\tau) = D_Y+T(r,\tau)$ is self-adjoint and invertible, the operator $(D_Y+T(r,\tau))^2$ is positive, so we can write

$$e^{-tA(r,\tau)^2} = \frac{i}{2\pi}\int_\Upsilon e^{-t\lambda}\Big((D_Y+T(r,\tau))^2-\lambda\Big)^{-1}d\lambda \tag{3.11}$$

where Υ is any counter-clockwise contour in the complex plane around the positive real axis. Since $T(r,\tau)$ is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and is bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$, the explicit resolvent construction in, for example Grubb [**20**], Grubb and Seeley [**21**], or Loya [**31**], shows that we can write

$$((D_Y+T(r,\tau))^2-\lambda)^{-1} = Q(\lambda)+R(r,\tau,\lambda),$$

where $Q(\lambda)$ is a pseudodifferential operator of order -2 living in an appropriate parameter-dependent pseudodifferential calculi, and where $R(r,\tau,\lambda)$ is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and is bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$ and decays in λ to order -1 uniformly as $|\lambda|\to\infty$ in sectors bounded away from the positive real axis. Using the contour integral (3.11) and repeated integration by parts, one can show that for any $\varepsilon>0$, the heat operator $e^{-tA(r,\tau)^2}$ is of the form $e^{\varepsilon t}\times$ a function that is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and is bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$. With this fact established, we can now analyze $\eta_1(r)$. Observe that

$$\int_{|\tau|\geq 1} e^{-t\tau^2}d\tau=\frac{2}{\sqrt{t}}\int_{\sqrt{t}}^{\infty}e^{-\tau^2}d\tau\leq\frac{2}{\sqrt{t}}\int_{\sqrt{t}}^{\infty}\tau e^{-\tau^2}d\tau=\frac{2}{\sqrt{t}}e^{-t}.$$

It follows that for any $0<\varepsilon<1$, for some constant C we have

$$\left|\int_{|\tau|\geq 1}\mathrm{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau\right|\leq\frac{C}{\sqrt{t}}e^{(\varepsilon-1)t}.$$

The function on the right decays exponentially as $t\to\infty$ and is $\mathcal{O}(t^{-1/2})$ as $t\to 0$, both uniformly in $r\in[0,1]$.

We now analyze η_2:

$$\eta_2(r,t)=\int_{|\tau|\leq 1}\mathrm{Tr}(B(r,\tau)e^{-t\tau^2}e^{-tA(r,\tau)^2})\,d\tau dt. \tag{3.12}$$

First of all, we note that since $A(r,\tau)$ is by assumption continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and invertible, it follows that $e^{-tA(r,\tau)^2}$ vanishes exponentially as $t\to\infty$ uniformly in $(r,\tau)\in[0,1]\times\mathbb{R}$. In particular, the integral (3.12) defining $\eta_2(r,t)$ is absolutely convergent for t integrated over $[1,\infty)$. Thus, it remains to analyze the integral (3.12) for t over the bounded interval $[0,1]$. If $B(r,\tau)$ is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ and is bounded as a function with values in $\Psi^{-\infty}(Y,E_Y)$, then the integrand of $\eta_2(r,t)$ involves the trace of an operator of order $-\infty$; this trace is certainly a continuous function of $(r,\tau)\in[0,1]\times[-1,1]$ and $t\in[0,1]$. Suppose now that $B(r,\tau)=A(r,\tau)=D_Y+T(r,\tau)$. Since $A(r,\tau)^2=D_Y^2+R(r,\tau)$ where $R(r,\tau)$ is continuous in $(r,\tau)\in[0,1]\times\mathbb{R}$ with values in $\Psi^{-\infty}(Y,E_Y)$, according to our Lemma 3.7, for some $S(r,\tau,t)$ that is continuous in $(r,\tau,t)\in[0,1]\times\times\mathbb{R}\times[0,\infty)$ with values in $\Psi^{-\infty}(Y,E_Y)$, we can write

$$e^{-tA(r,\tau)^2}=e^{-tD_Y^2}+t\,S(r,\tau,t).$$

Hence, we can write

$$\mathrm{Tr}(A(r,\tau)e^{-tA(r,\tau)^2})=\mathrm{Tr}(D_Ye^{-tD_Y^2})+t\,\tilde{S}(r,t,\tau),$$

where $\tilde{S}(r,\tau,t)$ is continuous in $(r,t,\tau)\in[0,1]\times\mathbb{R}\times[0,\infty)$. Since $\mathrm{Tr}(D_Ye^{-tD_Y^2})=\mathcal{O}(\sqrt{t})$ near $t=0$ (the 'local index theorem for odd-dimensional manifolds') it follows that the integral (3.12) decays exponentially as $t\to\infty$ and is $\mathcal{O}(t^{-1/2})$ as $t\to 0$, both uniformly in $r\in[0,1]$. Our proof is now complete. □

We need one more lemma.

Lemma 3.11. *For $r\in[0,1]$ and $t>0$, define*

$$\zeta(r,t)=\int_{\mathbb{R}}\mathrm{Tr}((D_Y+\widetilde{R}(r\tau))e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(r\tau))^2})\,d\tau \\ -\int_{\mathbb{R}}\mathrm{Tr}(\tau\partial_\tau\widetilde{R}(r\tau)\,e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(r\tau))^2})\,d\tau,$$

where $\widetilde{R}(\tau)$ is given in Theorem 3.4. Then

$$\frac{d}{dr}\zeta(r,t) = \frac{d}{dt}\left\{2tr\int_{\mathbb{R}} \mathrm{Tr}(\tau\partial_\tau \widetilde{R}(r\tau)\, e^{-t\tau^2 - t(D_Y+\widetilde{R}(r\tau))^2}\,)\, d\tau\right\}. \tag{3.13}$$

Proof. This proof is similar to that found in Melrose and Piazza [**42**, Prop. 13]. First of all, by Lemma 3.10, both integrals defining $\zeta(r,t)$ is absolutely convergent. To simplify the above formulas, define $B = B(r,t,\tau) = t^{1/2}(D_Y + \widetilde{R}(r\tau))$ and $L = 2t^{1/2}\partial_t - t^{-1/2}\tau\partial_\tau$. Then it is a straightforward to show that $LB = D_Y + \widetilde{R}(r\tau) - (\tau\partial_\tau R)(r\tau)$. Hence, we can write

$$\zeta(r,t) = \int_{\mathbb{R}} \mathrm{Tr}(LB\, e^{-t\tau^2 - B^2}\,)\, d\tau.$$

We now prove (3.13). To simplify notation, we shall denote derivatives with respect to r by dots. Observe that

$$\dot{\zeta}(r,t) = \int_{\mathbb{R}} \mathrm{Tr}(L\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau + \int_{\mathbb{R}} \mathrm{Tr}\left(LB\,\frac{d}{dr}e^{-t\tau^2-B^2}\right)\, d\tau. \tag{3.14}$$

By Duhamel's principle, the second term on the right is given by

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{Tr}\left(LB\,\frac{d}{dr}e^{-t\tau^2-B^2}\right)\, d\tau &= \\ &-\int_{\mathbb{R}}\int_0^1 e^{-t\tau^2}\,\mathrm{Tr}(LB\, e^{-uB^2}(\dot{B}\cdot B + B\cdot\dot{B})e^{-(1-u)B^2})\, dud\tau.\end{aligned} \tag{3.15}$$

Plugging in $L = 2t^{1/2}\partial_t - t^{-1/2}\tau\partial_\tau$ and then integrating by parts in the τ variable shows that the first term on the right-hand side of (3.14) is

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{Tr}(L\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau = \\ &\quad\frac{d}{dt}\left\{\int_{\mathbb{R}} \mathrm{Tr}(2t^{1/2}\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau\right\} - \int_{\mathbb{R}} \mathrm{Tr}(\dot{B}\, Le^{-t\tau^2-B^2}\,)\, d\tau.\end{aligned}$$

One can check that $Le^{-t\tau^2-B^2} = e^{-t\tau^2}Le^{-B^2}$, so Duhamel's principal applied to the second term on the right of this equation gives

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{Tr}(L\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau &= \frac{d}{dt}\left\{\int_{\mathbb{R}} \mathrm{Tr}(2t^{1/2}\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau\right\} \\ &+ \int_{\mathbb{R}}\int_0^1 e^{-t\tau^2}\,\mathrm{Tr}(\dot{B}\, e^{-uB^2}(LB\cdot B + B\cdot LB)e^{-(1-u)B^2})\, dud\tau.\end{aligned} \tag{3.16}$$

Combining (3.16) and (3.15) and using the fact that the trace vanishes on commutators, we arrive at

$$\dot{\zeta}(t,r) = \frac{d}{dt}\left\{\int_{\mathbb{R}} \mathrm{Tr}(2t^{1/2}\dot{B}\, e^{-t\tau^2-B^2}\,)\, d\tau\right\}.$$

Finally, using the definition of B, we get (3.13). ∎

3.5. Proof of the general index theorem 3.4. The proof of the general index formula (3.8) in Theorem 3.4 proceeds as in Section 2.5. Define $A = \widehat{D} + R$ and consider the difference of the heat b-traces

$$h(t) = {}^b\mathrm{Tr}(e^{-tA^*A}) - {}^b\mathrm{Tr}(e^{-tAA^*}).$$

We shall prove that $h(t)$ has following amazing properties:

$$(1) \quad \lim_{t\to\infty} h(t) = \mathrm{ind}(\widehat{D} + R)$$
$$(2) \quad \lim_{t\to 0} h(t) = \int_M \mathrm{AS}$$
$$(3) \quad \int_0^\infty h'(t)\, dt = -\frac{1}{2}\eta(D_Y + R_Y).$$

We verify these properties one by one. Now just as in the invertible boundary operator case (Section 2.5), Property (1) follows from the "standard" theory [**33**, Appendix]. Consider Property (2). To prove this, we first observe that

$$A^*A = (\widehat{D} + R)^*(\widehat{D} + R) = \widehat{D}^*\widehat{D} + S,$$

where $S = R^*\widehat{D} + \widehat{D}^*R + R^*R \in \Psi_b^{-\infty}(\widehat{M}, E)$. Second, we apply Remark 3.8 to conclude that

$$e^{-tA^*A} = e^{-t\widehat{D}^*\widehat{D}} + t\, T_1(t),$$

where $T_1(t) \in C^\infty([0,\infty); \Psi_b^{-\infty}(\widehat{M}, E))$. Applying the same arguments to AA^*, we obtain a similar conclusion:

$$e^{-tAA^*} = e^{-t\widehat{D}\widehat{D}^*} + t\, T_2(t),$$

where $T_2(t) \in C^\infty([0,\infty); \Psi_b^{-\infty}(\widehat{M}, F))$. It follows that

$$\begin{aligned} h(t) &= {}^b\mathrm{Tr}(e^{-tA^*A}) - {}^b\mathrm{Tr}(e^{-tAA^*}) \\ &= {}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) + t\, {}^b\mathrm{Tr}(T_1(t)) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}) - t\, {}^b\mathrm{Tr}(T_2(t)) \\ &= {}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}) + \mathcal{O}(t). \end{aligned}$$

Therefore, by the local index theorem as discussed in Section 2.5, we obtain

$$\lim_{t\to 0} h(t) = \lim_{t\to 0} \Big({}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}) \Big) = \int_M \mathrm{AS}$$

as required, and (2) is proved.

It remains to prove Property (3) where we again see the b-trace in action. Following the exact same argument as we did in (2.28), we obtain

$$h'(t) = {}^b\mathrm{Tr}(\,[A, A^*e^{-tAA^*}]\,),$$

which, by the trace-defect formula, is given by

$$h'(t) = \frac{i}{2\pi}\int_{\mathbb{R}} \eta(t,\tau)\, d\tau = -\frac{1}{2}\cdot\frac{1}{i\pi}\int_{\mathbb{R}} \eta(t,\tau)\, d\tau,$$

where

$$\eta(t,\tau) = \mathrm{Tr}\left(\partial_\tau N(A)(\tau)\, N(A^*)(\tau)\, N(e^{-tAA^*})(\tau)\right).$$

Thus,

$$
\begin{aligned}
\operatorname{ind}(\widehat{D}+R) &= h(\infty) = h(0) + \int_0^\infty h'(t)\,dt \\
&= \int_M \mathrm{AS} - \frac{1}{2}\cdot \eta_A \quad \text{with } \eta_A = \frac{1}{i\pi}\int_0^\infty \int_{\mathbb{R}} \eta(t,\tau)\,d\tau\,dt. \qquad (3.17)
\end{aligned}
$$

Therefore, it remains to *directly* work out the integral η_A and prove it is equal to $\eta(D_Y + R_Y)$. This proves (3) and completes the proof of Theorem 3.4.

Lemma 3.12. *We have*

$$
\frac{1}{i\pi}\int_0^\infty \int_{\mathbb{R}} \eta(t,\tau)\,d\tau\,dt = \eta(D_Y + R_Y). \qquad (3.18)
$$

Proof. In order to lessen confusion when we get to equation (3.19) below, we first claim that we may assume $\widetilde{R}(\tau)$ is even. To see this, we write $\widetilde{R}(\tau) = S(\tau) + T(\tau)$ where $S(\tau)$ is even in τ and $T(\tau)$ is odd in τ and both are self-adjoint for $\tau \in \mathbb{R}$. We can choose a smooth family of b-smoothing operators R_t with $t \in [0,1]$ such that $R_1 = R$ and $N(R_t)(\tau) = \Gamma \widetilde{R}_t(\tau)$, and satisfies

$$
\widetilde{R}_t(\tau) = S(\tau) + t\,T(\tau).
$$

Since $T(\tau)$ is odd in τ, $T(0) = 0$, so $\widetilde{R}_t(0) = \widetilde{R}(0) = R_Y$. By the Fredholm part of Theorem 3.4 (which we already proved) it follows that

$$
\widehat{D} + R_t : H^1(\widehat{M}, E) \longrightarrow L^2(\widehat{M}, F)
$$

is a continuous family of Fredholm operators. Since the index is invariant under continuous Fredholm perturbations, we have

$$
\operatorname{ind}(\widehat{D} + R_t) = \operatorname{ind}(\widehat{D} + R)
$$

for all $t \in [0,1]$ and therefore $\operatorname{ind}(\widehat{D} + R_0) = \operatorname{ind}(\widehat{D} + R)$. Applying the same argument as we did to prove (3.17) but now to $\widehat{D}+R_0$ and using that $\operatorname{ind}(\widehat{D}+R_0) = \operatorname{ind}(\widehat{D}+R)$, we see that $\eta_A = \eta_{\widehat{D}+R_0}$. Thus, by proving that $\eta_{\widehat{D}+R_0} = \eta(D_Y + R_Y)$, we may assume that $\widetilde{R}(\tau)$ is even in τ.

Now back to our proof. First, we work out

$$
\eta(t,\tau) = \operatorname{Tr}\big(\partial_\tau N(A)(\tau)\, N(A^*)(\tau)\, N(e^{-tAA^*})(\tau)\big).
$$

Observe that for $\tau \in \mathbb{R}$, we have $N(A)(\tau) = \Gamma(i\tau + D_Y + \widetilde{R}(\tau))$ and $N(A^*)(\tau) = (-i\tau + D_Y + \widetilde{R}(\tau))\Gamma^*$. Hence, $\partial_\tau N(A)(\tau) = \Gamma(i + \partial_\tau \widetilde{R}(\tau))$ and $N(AA^*)(\tau) = \Gamma(\tau^2 + (D_Y + \widetilde{R}(\tau))^2)\Gamma^*$. Thus, after simplification, we can write

$$
\begin{aligned}
\eta(t,\tau) &= \operatorname{Tr}((i + \partial_\tau \widetilde{R}(\tau))(-i\tau + D_Y + \widetilde{R}(\tau))e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(\tau))^2}) \\
&= \operatorname{Tr}((D_Y + \widetilde{R}(\tau))e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(\tau))^2}) \\
&\qquad - \operatorname{Tr}(\tau\partial_\tau \widetilde{R}(\tau)e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(\tau))^2}) + \xi(t,\tau),
\end{aligned}
$$

where $\xi(t,\tau)$ is odd in τ. Since the integral of an odd function over $\mathbb{R}$ is zero, we have

$$\eta_A = \frac{1}{\pi}\int_0^\infty\!\!\int_{\mathbb{R}} \operatorname{Tr}((D_Y + \widetilde{R}(\tau))e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(\tau))^2})\,d\tau dt \\ - \frac{1}{\pi}\int_0^\infty\!\!\int_{\mathbb{R}} \operatorname{Tr}(\tau\partial_\tau\widetilde{R}(\tau)e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(\tau))^2})\,d\tau dt. \tag{3.19}$$

By the way, according to Lemma 3.10, each of the integrands on the right are absolutely integrable over $(t,\tau)\in[0,\infty)\times\mathbb{R}$. Second, we now use a homotopy argument to complete our proof that η_A is the eta invariant of D_Y+R_Y. To this end, for $r\in[0,1]$, let $\zeta(r,t)$ be the function in Lemma 3.11 and define

$$\zeta(r) = \frac{1}{\pi}\int_0^\infty \zeta(r,t)\,dt = \frac{1}{\pi}\int_0^\infty\!\!\int_{\mathbb{R}} \operatorname{Tr}((D_Y + \widetilde{R}(r\tau))e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(r\tau))^2})\,d\tau dt \\ - \frac{1}{\pi}\int_0^\infty\!\!\int_{\mathbb{R}} \operatorname{Tr}(\tau\partial_\tau\widetilde{R}(r\tau)e^{-t\tau^2}e^{-t(D_Y+\widetilde{R}(r\tau))^2})\,d\tau dt.$$

Then $\eta(1)=\eta_A$ and by Lemma 3.10, $\zeta(r)$ is a continuous function of $r\in[0,1]$. Since the second term of $\zeta(r)$ vanishes at $r=0$ and since $\int_{\mathbb{R}} e^{-t\tau^2}d\tau = \sqrt{\pi/t}$, we have $\eta(0)=\eta(D_Y+R)$. Third, we shall prove that $\zeta(r)$ is in fact constant, which implies that

$$\eta_A = \zeta(1) = \zeta(0) = \eta(D_Y+R_Y).$$

To prove that $\zeta(r)$ is constant, we begin with the derivative formula (3.13) in Lemma 3.11 which shows that for any $a,\varepsilon>0$,

$$\frac{d}{dr}\int_\varepsilon^a \zeta(r,t)dt = \int_\varepsilon^a \frac{d}{dr}\zeta(r,t)dt = 2ar\int_{\mathbb{R}} \operatorname{Tr}(\tau\partial_\tau\widetilde{R}(r\tau)\,e^{-a\tau^2-a(D_Y+\widetilde{R}(r\tau))^2})\,d\tau \\ - 2\varepsilon r\int_{\mathbb{R}} \operatorname{Tr}(\tau\partial_\tau\widetilde{R}(r\tau)\,e^{-\varepsilon\tau^2-\varepsilon(D_Y+\widetilde{R}(r\tau))^2})\,d\tau.$$

Now Lemma 3.10 shows that the two terms in this equation vanish as $a\to\infty$ and as $\varepsilon\to 0$, respectively, uniformly in $r\in[0,1]$. Interchanging derivatives and limits it follows that $\zeta(r)$ is differentiable in r, and

$$\frac{d}{dr}\zeta(r) = \frac{1}{\pi}\frac{d}{dr}\int_0^\infty \zeta(r,t)\,dt = \frac{1}{\pi}\lim_{a\to\infty,\ \varepsilon\to 0}\frac{d}{dr}\int_\varepsilon^a \zeta(r,t)\,dt = 0,$$

so $\zeta(r)$ is constant, and our proof is complete. □

4. The eta invariant

In this section we prove Theorem 1.2. We start this section by defining b-eta invariants for Dirac operators perturbed by b-smoothing operators. We then review Vishik's technique of rotating boundary conditions as a means to prove Theorem 1.2. We end this section with one more presentation of the trace-defect formula in action by deriving the variation formula for eta invariants.

4.1. b-eta invariants. Throughout this section we will use the same assumptions and notations from Section 1.2 of the introduction. Let D and $\widehat{D}$ be Dirac operators on sections of a (single) vector bundle $E=F$ over an odd-dimensional Riemannian manifold with boundary as discussed in Section 1.2 and let $T\in\mathcal{L}(V)$.

Define a b-smoothing operator $\widehat{T} \in \Psi_b^{-\infty}(\widehat{M}, E, E) = \Psi_b^{-\infty}(\widehat{M}, E)$ just like in Section 3.1. We shall prove that this operator satisfies the conditions of Theorem 1.2; namely that D_T and the perturbed Dirac operator $\widehat{D} - \widehat{T}$ have the same eta invariant theoretic properties:

(a) $\ker(\widehat{D} - \widehat{T}) \cong \ker(D_T)$.
(b) ${}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_T)$.
(c) The following surgery formula holds:

$$ {}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_T) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C), $$

where C is the scattering matrix.

By Theorem 3.2 in Section 3.2, see Remark 3.3, we conclude that Property (a) holds. Before considering Parts (b) and (c), we first need to define the b-eta invariant of the operator $\widehat{D} - \widehat{T}$, cf. Melrose [**40**, Sec. 9.7]. It turns out that the operator $\widehat{D} - \widehat{T}$, although Fredholm (see Remark 3.6), has continuous spectrum and not discrete spectrum, due to the fact that $\widehat{M}$ has infinite volume. Thus, its eta invariant cannot be defined in terms of a corresponding eta function in the same way as in the case of a manifold with boundary with boundary condition as in (1.7) or a manifold without boundary as in (2.2). However, we can still try to define the eta invariant via the heat trace integral (2.4):

$$ \text{“ } \eta(\widehat{D} - \widehat{T}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(\,(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}\,)\, dt. \text{ ”} $$

For $t > 0$, the operator $(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}$ is a b-smoothing operator which is not trace class, cf. our discussion in Section 2.4; thus, the quotation marks. However, being a b-smoothing operator, the b-trace of $(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}$ does exist. Replacing the trace Tr with the b-trace ${}^b\mathrm{Tr}$ in the above formula defines what we call the b-eta invariant of $\widehat{D} - \widehat{T}$:

PROPOSITION 4.1. *The integral*

$$ {}^b\eta(\widehat{D} - \widehat{T}) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\ {}^b\mathrm{Tr}(\,(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}\,)\, dt \tag{4.1} $$

is absolutely convergent. This integral defines the b-eta invariant of $\widehat{D} - \widehat{T}$.

PROOF. By Remark 3.6, $\widehat{D} - \widehat{T}$ is Fredholm, so the standard arguments [**40**, Ch. 9] show that ${}^b\mathrm{Tr}(\,(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}\,)$ decays exponentially as $t \to \infty$.

The proof that ${}^b\mathrm{Tr}(\,(\widehat{D} - \widehat{T}) e^{-t(\widehat{D} - \widehat{T})^2}\,)$ is also absolutely integrable near $t = 0$ follows the same argument used in Proposition 3.9. □

Now back to Parts (b) and (c). We already mentioned in the introduction that the formula

$$ {}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C) $$

is proved in Loya and Melrose [**33**]. Thus, to complete the proof of our theorem, we just need to prove that

$$ \eta(D_T) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C). $$

In order to prove this, we will employ Vishik's technique of rotating boundary conditions [**58**] as presented by Brüning and Lesch [**12**].

4.2. Review of rotating boundary conditions. Fix $T \in \mathcal{L}(V)$, and fix $0 < a < 1$. We will denote the left half of $x = a$, $\{p \in M\,;\, x(p) \leq a\}$, by M_1, and the right half of $x = a$, $\{p \in M\,;\, x(p) \geq a\}$, by M_2. (Here, we assume that $x \geq 1$ off the collar.) Thus, $M_1 \cong [0, a]_x \times Y$, and near ∂M_2, we have $M_2 \cong [a, 1)_x \times Y$. The main idea in proving Theorem 1.2 is to separate the eta invariant $\eta(D_T)$ into an eta invariant on M_1 and an eta invariant on M_2. This separation is achieved by "twisting off" the end M_1 from the manifold M, and thus separating M into the two parts M_1 and M_2. Analytically, this twisting is carried out by defining a family of rotating boundary conditions at $x = a$ as follows.

Given $\theta \in [0, \pi/4]$, we denote by $D(\theta)$, the Dirac operator D with domain $\mathrm{Dom}(D(\theta))$ consisting of pairs of functions $(u_1, u_2) \in H^1(M_1, E_Y) \oplus H^1(M_2, E)$, where $E_Y := E|_{x=0}$, such that at $x = 0$, we have

$$\Pi_+^T u_1|_{x=0} = 0, \tag{4.2}$$

and at $x = a$, we have

$$\sin\theta\, \Pi_+^{-C} u_1|_{x=a} = \cos\theta\, \Pi_+^{-C} u_2|_{x=a}, \tag{4.3}$$

$$\cos\theta\, \Pi_-^{C} u_1|_{x=a} = \sin\theta\, \Pi_-^{C} u_2|_{x=a}. \tag{4.4}$$

Here, C is the scattering matrix, and for any $S \in \mathcal{L}(V)$, $\Pi_\pm^S := \Pi_\pm + \Pi_S^\perp$, where $\Pi_\pm$ are the orthogonal projections of $L^2(Y, E_Y)$ onto the positive and negative eigenspaces of D_Y respectively, and where $\Pi_S^\perp$ the orthogonal projection onto $\Gamma\Lambda_S$. Then,

$$D(\theta) : \mathrm{Dom}(D(\theta)) \longrightarrow L^2(M_1, E_Y) \oplus L^2(M_2, E),$$

by restricting D to each of the factors $H^1(M_1, E_Y)$ and $H^1(M_2, E)$.

To understand the boundary conditions above, let us consider them at the two end points, $\theta = \pi/4$ and $\theta = 0$. Note that at $\theta = \pi/4$, (4.3) and (4.4) imply that $u_1|_{x=a} = u_2|_{x=a}$. Thus, (u_1, u_2) can be considered as a function on M that is continuous across $x = a$. For this reason, the boundary condition at $x = a$ for $\theta = \pi/4$ represents the *continuous transmission condition*. In particular, it follows that $\mathrm{Dom}(D(\pi/4))$ can be identified with $\mathrm{Dom}(D_T)$, and hence,

$$\eta(D(\pi/4)) = \eta(D_T).$$

Consider now $D(0)$. Indeed, from (4.2), (4.3) and (4.4), we can write

$$\mathrm{Dom}(D(0)) = \{(u_1, u_2) \in H^1(M_1, E_Y) \oplus H^1(M_2, E)\ ;$$
$$\Pi_+^T u_1|_{x=0} = 0,\ \Pi_-^C u_1|_{x=a} = 0,\ \Pi_+^{-C} u_2|_{x=a} = 0\}.$$

Thus, $\mathrm{Dom}(D(0))$ separates into two parts:

$$\mathrm{Dom}(D(0)) = \mathrm{Dom}(D(T, C)) \oplus \mathrm{Dom}(\widetilde{D}_{-C}), \tag{4.5}$$

where $D(T, C)$ is the Dirac operator $D = \Gamma(\partial_x + D_Y)$ restricted to M_1 with domain

$$\mathrm{Dom}(D(T, C)) = \{u \in H^1(M_1, E_Y)\ ;\ \Pi_+^T u|_{x=0} = 0,\ \Pi_-^C u|_{x=a} = 0\},$$

and where $\widetilde{D}_{-C}$ is the Dirac operator D restricted to M_2 with domain consisting of those $u \in H^1(M_2, E)$ such that $\Pi_+^{-C} u|_{x=a} = 0$. Hence, the boundary conditions (4.3) and (4.4) "twist" from the continuous transmission conditions to separation boundary conditions as the parameter θ decreases from $\pi/4$ to 0. From (4.5), it follows that $\eta(D(0))$ also splits into two pieces:

$$\eta(D(0)) = \eta(D(T, C)) + \eta(\widetilde{D}_{-C}).$$

By Proposition 2.16 of Müller [**45**], we have $\eta(\widetilde{D}_{-C}) = \eta(D_{-C})$, where D_{-C} is the Dirac operator on M with domain consisting of those $u \in H^1(M, E)$ such that $\Pi_+^{-C} u|_{x=0} = 0$. Also, by Theorem 2.1 of Lesch and Wojciechowski [**27**], we have $\eta(D(T, C)) = m(\Lambda_T, \Lambda_C)$. We will see in the next section that the eta invariant of $D(\theta)$, for each $\theta \in [0, \pi/4]$, can be defined, and that $\eta(D(\theta))$ is constant in θ. Thus, once we show that $\eta(D(\theta))$ is constant, Theorem 1.2 is proved:

$$\eta(D_{-C}) + m(\Lambda_T, \Lambda_C) = \eta(D(0)) = \eta(D(\pi/4)) = \eta(D_T).$$

We show that $\eta(D(\theta))$ is constant in the next section.

4.3. Proof of Eta Invariant Theorem 1.2. The proof that

$$\eta(D_T) = \eta(D_{-C}) + m(\Lambda_T, \Lambda_C).$$

follows the basic outline found in Brüning and Lesch [**12**, Th. 3.9], which concerns the eta invariant of the following problem. We will only present this result as it pertains to our problem; the actual theorem is more general. Let D' be a Dirac operator associated to a Hermitian Clifford module E' over a *closed* compact manifold M' such that Y is a hypersurface in M'. Suppose that $Y = \{x' = a\}$ for some function x' on M', and that M_2 is on the right-hand side of Y in M' in the sense that the set $\{x' \geq a\} \subset M'$ can be identified with M_2, and such that over this set, $E' \equiv E$, and $D' \equiv D$. Also suppose that Y has a collar $(0, 1)_{x'} \times Y$ in M' over which $D' = \Gamma(\partial_{x'} + D_Y)$. Let $M' = \{x' \leq a\} \subset M'$ be the left half of M'. We define $D'(\theta)$ to be the Dirac operator D' with domain

$$\begin{aligned}\mathrm{Dom}(D'(\theta)) := \{(u_1, u_2) \in H^1(M', E') \oplus H^1(M_2, E)\,;& \\ u_1,\, u_2 \text{ satisfy (4.3) and (4.4) at } x' = a\}&.\end{aligned}$$

In [**12**], the eta function (1.7) for $D'(\theta)$ is shown to exist as a meromorphic function on $\mathbb{C}$. (Actually, in [**12**], the boundary conditions on $V = \ker D_Y$ were held constant, unlike in (4.3) and (4.4) where the boundary conditions on V are also rotating; however, the arguments in [**12**] also go through for the situation described here.) For $\theta \in (0, \pi/4)$, the eta function will in general have a pole at $z = 0$. However, the eta invariant of $D'(\theta)$, for each $\theta \in [0, \pi/4]$, can still be defined as the regular value of the eta function at $z = 0$. Then Theorem 3.9 of [**12**] states that if $\dim \ker D'(\theta)$ is constant, then $\eta(D'(\theta))$ is also constant.

The only difference between our problem considered in Section 4.2 and the problem considered in [**12**] is that the compact manifold with boundary M' is replaced by the compact cylinder M_1 with a boundary condition at the left end of the cylinder given by Π_+^T. For this reason, proving that the eta invariant of the operator $D(\theta)$ considered in Section 4.2 can be defined for each $\theta \in [0, \pi/4]$, and that $\eta(D(\theta))$ is constant if $\dim \ker D(\theta)$ is constant, can proceed exactly as in [**12**]. The details will be omitted in order to avoid reproducing the arguments of [**12**]. Thus, to prove Theorem 1.2 we are left to prove that $\dim \ker D(\theta)$ is constant. This is proved in the following lemma, which completes the proof of Theorem 1.2.

THEOREM 4.2. *For each $\theta \in [0, \pi/4]$, we have*

$$\dim \ker D(\theta) = \dim(\Lambda_T \cap \Lambda_C) + \dim \ker D_{-C}.$$

In fact, $\ker D(\theta)$ is canonically isomorphic to $(\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$.

PROOF. Fix $\theta \in [0, \pi/4]$ and let $u \in \ker D(\theta)$. Then we can write $u = (u_1, u_2) \in H^1(M_1, E_Y) \oplus H^1(M_2, E)$, where $Du_1 = 0$, $Du_2 = 0$, and

$$\Pi_+^T u_1|_{x=0} = 0, \tag{4.6}$$

$$\sin\theta\, \Pi_+^{-C} u_1|_{x=a} = \cos\theta\, \Pi_+^{-C} u_2|_{x=a}, \tag{4.7}$$

$$\cos\theta\, \Pi_-^C u_1|_{x=a} = \sin\theta\, \Pi_-^C u_2|_{x=a}. \tag{4.8}$$

To prove this lemma, we need to analyze exactly what these boundary conditions imply. Thus, recall that $D = \Gamma(\partial_x + D_Y)$ on the collar $[0,1)_x \times Y$ in M. Since $Du_1 = 0$, it follows that $u_1 = v_1 + w_1$, where $v_1 \in \ker D_Y$ is constant in x, and where $w_1(x)$ takes values in the nonzero eigenvectors of D_Y and satisfies $(\partial_x + D_Y)w_1(x) = 0$. Also, since $Du_2 = 0$, on the collar $[a,1)_x \times Y$ in M_2 we have $u_2 = v_2 + w_2$, where $v_2 \in \ker D_Y$ is constant in x, and where $w_2(x)$ takes values in the nonzero eigenvectors of D_Y and satisfies $(\partial_x + D_Y)w_2(x) = 0$. To show that $\ker D(\theta) \equiv (\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$, we first show that w_1 is determined by w_2.

To see this, let Λ be the eigenspace of D_Y corresponding to a nonzero eigenvalue λ. Then the equation $(\partial_x + D_Y)w_1(x) = 0$ implies that when projected onto Λ, we have $w_1(x) = e^{-x\,\lambda} w_{1\lambda}$ where $w_{1\lambda} \in \Lambda$ is constant in x. Similarly, when projected onto Λ, we have $w_2(x) = e^{-x\,\lambda} w_{2\lambda}$ where $w_{2\lambda} \in \Lambda$ is constant in x. By condition (4.6), we must have $w_{1\lambda} = 0$ for $\lambda > 0$. Thus, by condition (4.7), we also have $w_{2\lambda} = 0$ for $\lambda > 0$. Now condition (4.8) implies that for $\lambda < 0$, we have $\cos\theta\, w_{1\lambda} = \sin\theta\, w_{2\lambda}$, or $w_{1\lambda} = \tan\theta\, w_{2\lambda}$. Hence, $w_1 = \tan\theta\, w_2$, and w_1 is determined uniquely by w_2. Moreover, we also proved that $\Pi_+ w_1 = \Pi_+ w_2 = 0$.

We now show that $v_2 \in \Lambda_C$. Indeed, define $\tilde{u}_2 \in H^1(M, E)$ by $\tilde{u}_2 := v_2 + w_2$ for $x \in [0, a)$, and $\tilde{u}_2 := u_2$ for $x \geq a$. Then it follows that $D\tilde{u}_2 = 0$ and $\Pi_+ \tilde{u}_2|_{x=0} = 0$. Thus, by definition of Λ_C, $v_2 = \Pi_0 \tilde{u}_2|_{x=0} \in \Lambda_C$.

We next show that $v_1 \in \Lambda_T \cap \Lambda_C$ and $\sin\theta\, v_1 = \cos\theta\, v_2$. First, by (4.6), we must have $v_1 \in \Lambda_T$. Hence, as $\Pi_C^\perp v_2 = 0$ since $v_2 \in \Lambda_C$, by condition (4.8) we obtain $\Pi_C^\perp v_1 = 0$, or $v_1 \in \Lambda_C$. Thus, $v_1 \in \Lambda_T \cap \Lambda_C$. Finally, as $\Pi_{-C}^\perp = \Pi_C$, we have $\Pi_{-C}^\perp v_1 = v_1$ and $\Pi_{-C}^\perp v_2 = v_2$, and hence by (4.7), $\sin\theta\, v_1 = \cos\theta\, v_2$.

The previous three paragraphs imply that $\ker D(\theta)$ can be identified with the space of pairs (v, w) in

$$(\Lambda_T \cap \Lambda_C) \oplus \{w \in H^1(M, E)\,;\, Dw = 0,\ \Pi_+ w|_{x=0} = 0\}$$

such that $\sin\theta\, v = \cos\theta\, \Pi_C w|_{x=0}$. Denote this space by $U(\theta)$. Observe that since $\Pi_C = \Pi_{-C}^\perp$, if $\theta = 0$, then $U(0) = (\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$. To see that $U(\theta) \cong (\Lambda_T \cap \Lambda_C) \oplus \ker D_{-C}$ for $\theta \in (0, \pi/4]$, for such θ, consider the map $U(\theta) \ni (v, w) \xmapsto{\pi_1} v \in \Lambda_T \cap \Lambda_C$. Since $\sin\theta\, v = \cos\theta\, \Pi_C w|_{x=0}$, and since $\Pi_C = \Pi_{-C}^\perp$, the kernel of this map is exactly $\ker D_{-C}$. This map is also surjective, for if $v \in \Lambda_T \cap \Lambda_C$, then by definition of Λ_C, there is a $w \in H^1(M, E)$ with $Dw = 0$ and $\Pi_+ w = 0$ such that $\sin\theta\, v = \cos\theta\, \Pi_C w|_{x=0}$. Thus, the following sequence is exact:

$$0 \longrightarrow \ker D_{-C} \longrightarrow U(\theta) \xrightarrow{\pi_1} \Lambda_T \cap \Lambda_C \longrightarrow 0.$$

Our lemma is now proved. $\square$

4.4. Variation of the eta-invariant. In this last section we use the b-trace and the identity ${}^b\eta(\widehat{D} - \widehat{T}) = \eta(D_T)$ to derive the variation formula for the eta invariant $\eta(D_T)$.

THEOREM 4.3. *Let $T(r) \in \mathcal{L}(V)$, $r \in [0,1]$, be a smooth family such that* $\dim\ker(\widehat{D} - \widehat{T(r)})$ *is constant, where $\widehat{T(r)}$ is as in Section 3.1. Then ${}^b\eta(\widehat{D} - \widehat{T(r)})$ is smooth for $r \in [0,1]$, and*

$$\frac{d}{dr}\,{}^b\eta(\widehat{D} - \widehat{T(r)}) = \frac{1}{i\pi}\operatorname{tr}(\dot{T}^+(r)T^+(r)^{-1}), \tag{4.9}$$

where $\dot{T}^+(r) = \frac{d}{dr}T^+(r)$.

Using the equality ${}^b\eta(\widehat{D} - \widehat{T(r)}) = \eta(D_{T(r)})$, we immediately get the following variation result first proved by Lesch and Wojciechowski [**27**].

COROLLARY 4.4. *Let $T(r) \in \mathcal{L}(V)$, $r \in [0,1]$, be any smooth family such that* $\dim\ker D_{T(r)}$ *is constant. Then $\eta(D_{T(r)})$ is smooth for $r \in [0,1]$, and*

$$\frac{d}{dr}\,\eta(D_{T(r)}) = \frac{1}{i\pi}\operatorname{tr}(\dot{T}^+(r)T^+(r)^{-1}), \tag{4.10}$$

where $\dot{T}^+(r) = \frac{d}{dr}T^+(r)$.

We can also use the surgery formula $\eta(D_{T(r)}) = \eta(D_{-C}) + m(\Lambda_{T(r)}, \Lambda_C)$ and the variation of the matrix quantity $m(\Lambda_{T(r)}, \Lambda_C)$ to derive the same result, but the proof of Theorem 4.3 might be of interest since the proof is independent of this formula and it gives one last performance of our hero, the b-trace.[8]

The proof of this theorem is a consequence of the next two lemmas. In the following two lemmas, we denote $\widehat{D} - \widehat{T(r)}$ by $A = A(r)$.

LEMMA 4.5. *With $\dot{A} = \frac{d}{dr}A$, we have*

$$\frac{d}{dr}\Big\{t^{-1/2}\ {}^b\mathrm{Tr}(Ae^{-tA^2})\Big\} = \frac{d}{dt}\Big\{2t^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-tA^2})\Big\} + \alpha_r(t), \tag{4.11}$$

where $\alpha_r(t)$ is given in terms of the b-trace of commutators via

$$\begin{aligned}\alpha_r(t) = t^{-1/2}&\int_0^t {}^b\mathrm{Tr}[Ae^{-(t-s)A^2}, \dot{A}Ae^{-sA^2}]ds \\ &- t^{-1/2}\int_0^t {}^b\mathrm{Tr}[A^2e^{-(t-s)A^2}, \dot{A}e^{-sA^2}]ds.\end{aligned} \tag{4.12}$$

PROOF. We follow the proof of Melrose [**40**, Prop. 8.39]. First, note that

$$\frac{d}{dr}\Big\{t^{-1/2}\ {}^b\mathrm{Tr}(Ae^{-tA^2})\Big\} = t^{-1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-tA^2}) + t^{-1/2}\ {}^b\mathrm{Tr}\left(A\frac{d}{dr}e^{-tA^2}\right). \tag{4.13}$$

Second, noting that

$$(\partial_t + A^2)\left(\frac{d}{dr}e^{-tA^2}\right) = (\dot{A}A + A\dot{A})e^{-tA^2},$$

we use Duhamel's principal to obtain

$$\frac{d}{dr}e^{-tA^2} = -\int_0^t e^{-(t-s)A^2}(\dot{A}A + A\dot{A})e^{-sA^2}ds. \tag{4.14}$$

Plugging (4.14) into the second term on the right of (4.13), and then using the trace identities

$${}^b\mathrm{Tr}(Ae^{-(t-s)A^2}\dot{A}Ae^{-sA^2}) = {}^b\mathrm{Tr}(\dot{A}A^2e^{-tA^2}) + {}^b\mathrm{Tr}[Ae^{-(t-s)A^2}, \dot{A}Ae^{-sA^2}];$$

[8]This technique might also be useful to those who find finite dimensional matrices difficult :)

$$ {}^b\mathrm{Tr}(Ae^{-(t-s)A^2}A\dot{A}e^{-sA^2}) = {}^b\mathrm{Tr}(\dot{A}A^2e^{-tA^2}) + {}^b\mathrm{Tr}[A^2e^{-(t-s)A^2}, \dot{A}e^{-sA^2}], $$

it follows that

$$ \frac{d}{dr}\Big\{t^{-1/2}\ {}^b\mathrm{Tr}(Ae^{-tA^2})\Big\} = \frac{d}{dt}\Big\{2t^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-tA^2})\Big\} $$
$$ t^{-1/2}\int_0^t {}^b\mathrm{Tr}[Ae^{-(t-s)A^2}, \dot{A}Ae^{-sA^2}]ds - t^{-1/2}\int_0^t {}^b\mathrm{Tr}[A^2e^{-(t-s)A^2}, \dot{A}e^{-sA^2}]ds. $$

This proves our lemma. □

The following lemma proves Theorem 4.3.

Lemma 4.6. *If* $r_0,\, r_1 \in [0,1]$, *then*

$$ {}^b\eta(\widehat{D} - \widehat{T(r_1)}) - {}^b\eta(\widehat{D} - \widehat{T(r_0)}) = \frac{1}{i\pi}\int_{r_0}^{r_1} \mathrm{tr}(\dot{T}^+(r)T^+(r)^{-1})\, dr. $$

Proof. Using the definition of the b-eta invariant, we can write

$$ {}^b\eta(A(r_1)) - {}^b\eta(A(r_0)) = \lim_{\substack{a\to\infty \\ \varepsilon\downarrow 0}} \frac{1}{\sqrt{\pi}}\int_\varepsilon^a\int_{r_0}^{r_1} \frac{d}{dr}t^{-1/2}\ {}^b\mathrm{Tr}(Ae^{-tA^2})\, dr\, dt. $$

Now by (4.11),

$$ \int_\varepsilon^a \frac{d}{dr}\Big\{t^{-1/2}\ {}^b\mathrm{Tr}(Ae^{-tA^2})\Big\}dt = 2a^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-aA^2}) $$
$$ - 2\varepsilon^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-\varepsilon A^2}) \quad + \int_\varepsilon^a \alpha_r(t)\, dt, $$

where $\alpha_r(t)$ is given in (4.12). Since $A(r)$ is Fredholm and its kernel has constant dimension by assumption, the proof used in Proposition 8.39 of [**40**] shows that $2a^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-aA^2}) \to 0$ as $a \to \infty$ and since $\dot{A} = \frac{d}{dr}\widehat{T(r)} \in \Psi_b^{-\infty}(\widehat{M}, E)$, it follows that $2\varepsilon^{1/2}\ {}^b\mathrm{Tr}(\dot{A}e^{-\varepsilon A^2}) \to 0$ as $\varepsilon \downarrow 0$. Now the fact that

$$ \int_0^\infty \alpha_r(t)\, dt = \frac{1}{i\pi}\mathrm{tr}(\dot{T}^+(r)T^+(r)^{-1}) $$

is a *direct* computation using the explicit formula (4.12) for $\alpha_r(t)$ in terms of b-traces of commutators and the trace-defect formula in Theorem 2.5. The details of this computation can be found in Loya and Melrose [**33**] or left for the reader. □

References

1. L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, Comm. Math. Phys. **90** (1983), no. 2, 161–173.
2. M. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.
3. M. F. Atiyah, H. Donnelly, and I. M. Singer, *Eta invariants, signature defects of cusps, and values of L-functions*, Ann. of Math. (2) **118** (1983), no. 1, 131–177.
4. M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry*, Bull. London Math. Soc. **5** (1973), 229–234.
5. ———, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
6. J.-M. Bismut and D.S. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Comm. Math. Phys. **107** (1986), no. 1, 103–163.
7. J.-M. Bismut and J. Cheeger, *Families index for manifolds with boundary, superconnections, and cones. I. Families of manifolds with boundary and Dirac operators*, J. Funct. Anal. **89** (1990), no. 2, 313–363.

8. ______, *Families index for manifolds with boundary, superconnections and cones. II. The Chern character*, J. Funct. Anal. **90** (1990), no. 2, 306–354.
9. D. Bleecker and B. Booß-Bavnbek, *Spectral invariants of operators of Dirac type on partitioned manifolds*, Aspects of Boundary Problems in Analysis and Geometry, Birkhäuser, Boston, 2004, pp. 1–130.
10. B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser Boston Inc., Boston, MA, 1993.
11. J. Brüning, *L^2-index theorems on certain complete manifolds*, J. Differential Geom. **32** (1990), no. 2, 491–532.
12. J. Brüning and M. Lesch, *On the η-invariant of certain nonlocal boundary value problems*, Duke Math. J. **96** (1999), no. 2, 425–468.
13. U. Bunke, *On the gluing problem for the η-invariant*, J. Differential Geom. **41** (1995), no. 2, 397–448.
14. J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657 (1984).
15. X. Dai and D. S. Freed, *η-invariants and determinant lines*, J. Math. Phys. **35** (1994), no. 10, 5155–5194, Topology and physics.
16. R. G. Douglas and K. P. Wojciechowski, *Adiabatic limits of the η-invariants. The odd-dimensional Atiyah-Patodi-Singer problem*, Comm. Math. Phys. **142** (1991), no. 1, 139–168.
17. Y. V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Birkhäuser Verlag, Basel, 1997.
18. E. Getzler, *A short proof of the local Atiyah-Singer index theorem*, Topology **25** (1986), no. 1, 111–117.
19. P. B. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math. **10** (1973), 344–382.
20. G. Grubb, *Functional calculus of pseudodifferential boundary problems*, second ed., Progress in Math., Birkhäuser, Boston, 1996.
21. G. Grubb and R.T. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patoti-Singer operators*, Invent. Math. **121** (1995), 481–529.
22. ______, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. Anal. **6** (1996), no. 1, 31–77.
23. A. Hassell, R. Mazzeo, and R. B. Melrose, *Analytic surgery and the accumulation of eigenvalues*, Comm. Anal. Geom. **3** (1995), no. 1-2, 115–222.
24. L. Hörmander, *The analysis of linear partial differential operators. III*, second ed., Springer-Verlag, Berlin, 1985, Pseudo-Differential Operators.
25. P. Kirk and M. Lesch, *The η-invariant, Maslov index, and spectral flow for Dirac–type operators on manifolds with boundary*, preprint, math.DG/0012123, 2001.
26. R. Lauter and J. Seiler, *Pseudodifferential analysis on manifolds with boundary—a comparison of b-calculus and cone algebra*, Approaches to singular analysis (Berlin, 1999), Birkhäuser, Basel, 2001, pp. 131–166.
27. M. Lesch and K. P. Wojciechowski, *On the η-invariant of generalized Atiyah-Patodi-Singer boundary value problems*, Illinois J. Math. **40** (1996), no. 1, 30–46.
28. V. B. Lidskiĭ, *Non-selfadjoint operators with a trace*, Dokl. Akad. Nauk SSSR **125** (1959), 485–487.
29. G. Lion and M. Vergne, *The Weil representation, Maslov index and theta series*, Birkhäuser Boston, Mass., 1980.
30. R. B. Lockhart and R. C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 409–447.
31. P. Loya, *Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators*, Comm. Partial Differential Equations **26** (2001), no. 7 & 8, 1253–1321.
32. ______, *Index theory of Dirac operators on manifolds with corners up to codimension two*, Aspects of Boundary Problems in Analysis and Geometry, Birkhäuser, Boston, 2004, pp. 131–169.
33. P. Loya and R. Melrose, *Fredholm perturbations of Dirac operators on manifolds with corners*, preprint, 2003.
34. P. Loya and J. Park, *On the gluing problem for the spectral invariants of Dirac operators*, preprint, 2004.

35. R. Mazzeo, *Elliptic theory of differential edge operators. I*, Comm. Partial Differential Equations **16** (1991), no. 10, 1615–1664.
36. R. Mazzeo and R. B. Melrose, *Analytic surgery and the eta invariant*, Geom. Funct. Anal. **5** (1995), no. 1, 14–75.
37. R. Mazzeo and P. Piazza, *Dirac operators, heat kernels and microlocal analysis. II. Analytic surgery*, Rend. Mat. Appl. (7) **18** (1998), no. 2, 221–288.
38. H.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geometry **1** (1967), 43–69.
39. R. B. Melrose, *Transformation of boundary problems*, Acta Math. **147** (1981), no. 3-4, 149–236.
40. ______, *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, 1993.
41. R.B. Melrose and G.A. Mendoza, *Elliptic pseudodifferential operators of totally characteristic type*, MSRI preprint, 1983.
42. R. B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, J. Differential Geom. **46** (1997), no. 1, 99–180.
43. W. Müller, *Signature defects of cusps of Hilbert modular varieties and values of L-series at $s = 1$*, J. Differential Geom. **20** (1984), no. 1, 55–119.
44. ______, *L^2-index and resonances*, Geometry and analysis on manifolds (Katata / Kyoto, 1987), Lecture Notes in Math., vol. 1339, Springer, Berlin, 1988, pp. 203–211.
45. ______, *Eta invariants and manifolds with boundary*, J. Differential Geom. **40** (1994), no. 2, 311–377.
46. ______, *The eta invariant (some recent developments)*, Astérisque (1995), no. 227, Exp. No. 787, 5, 335–364, Séminaire Bourbaki, Vol. 1993/94.
47. ______, *Relative zeta functions, relative determinants and scattering theory*, Comm. Math. Phys. **192** (1998), no. 2, 309–347.
48. V. K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Differential Geometry **5** (1971), 233–249.
49. P. Piazza, *Dirac operators, heat kernels and microlocal analysis. I. Family index theory*, Proceedings of the Workshop on Differential Geometry and Topology (Palermo, 1996), no. 49, 1997, pp. 187–201.
50. B.A. Plamenevskij, *Algebras of pseudodifferential operators*, Kluwer Academic Publishers, Dordrecht, 1989, Published originally in Nauka, Moscow, 1986.
51. V. S. Rabinovič, *Pseudodifferential operators on a certain class of noncompact manifolds*, Mat. Sb. (N.S.) **89(131)** (1972), 46–60, 165.
52. S. Rempel and B.-W. Schulze, *Complete mellin and green symbolic calculus in spaces with conormal asymptotics*, Ann. Global Anal. Geom. **4** (1986), 137–224.
53. E. Schrohe, *Complex powers on noncompact manifolds and manifolds with singularities*, Math. Ann. **281** (1988), no. 3, 393–409.
54. B.-W. Schulze, *Boundary value problems and singular pseudodifferential operators*, J. Wiley, Chichester, 1998.
55. R.T. Seeley, *Asymptotic expansions for Atiyah-Patodi-Singer problems*, Harmonic analysis and partial differential equations (Chicago, IL, 1996), Univ. Chicago Press, Chicago, IL, 1999, pp. 275–283.
56. I.M. Singer, *The eta invariant and the index*, Mathematical aspects of string theory, World Scientific, Singapore, 1988, pp. 239–258.
57. M. Stern, *L^2-index theorems on locally symmetric spaces*, Invent. Math. **96** (1989), no. 2, 231–282.
58. S. M. Vishik, *Generalized Ray-Singer conjecture. I. A manifold with a smooth boundary*, Comm. Math. Phys. **167** (1995), no. 1, 1–102.
59. K. P. Wojciechowski, *The additivity of the η-invariant: the case of an invertible tangential operator*, Houston J. Math. **20** (1994), no. 4, 603–621.
60. ______, *The additivity of the η-invariant. The case of a singular tangential operator*, Comm. Math. Phys. **169** (1995), no. 2, 315–327.
61. ______, *The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian*, Comm. Math. Phys. **201** (1999), no. 2, 423–444.

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