Arguing as before, we find that (4) holds, and from it we again infer that

$$f(x)f(-y) = f(y)f(-x)$$

for all real $x$ and $y$.

Now setting $y = t$ and $x = -t$ yields $f(-t)^2 = 1$, so $f(-t) = \pm 1$. The choice $f(-t) = 1$ leads to the conclusion that $f$ is even and $g$ is constant, which is not the case. Thus $f$ is odd and (17) and (4) become the "−" half of (3) and (2), respectively. This furnishes a solution to Klee's problem. Note that we have used the conditions $f(t) = 1$, $g(t) = 0$ a couple of times.

As remarked at the end of the solution of E1079, the usual formula for $\cos(x \pm y)$ and $\sin(x \pm y)$ follow purely algebraically from the formula for $\cos(x - y)$.

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# Green's Theorem and the Fundamental Theorem of Algebra

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One proof of the fundamental theorem of algebra uses Liouville's theorem, which follows from Cauchy's theorem, which in turn can be derived from Green's theorem; see, for instance, the beautiful book [**1**]. The purpose of this note is to show that Green's theorem is sufficient. The proof does not use any topology or analytic function theory.

Let $p(w)$ be a polynomial in $w = x + iy$ of degree $n \geq 1$ with complex coefficients. We show that $p(w)$ must have a zero. As usual we proceed by contradiction, assuming that $p(w)$ is not zero for any $w$. Then $f(x, y) := \ln |p(x + iy)|$ is a smooth function of $(x, y)$ in $\mathbb{R}^2$. For $r > 0$, let $D_r$ be the disk of radius $r$ centered at the origin, and let $C_r$ be the boundary of $D_r$ oriented counterclockwise. We will work out both sides of Green's formula

$$\int_{C_r} P\,dx + Q\,dy = \iint_{D_r} (\partial_x Q - \partial_y P)\,dx\,dy, \tag{1}$$

with $P = -\partial_y f$ and $Q = \partial_x f$. We first work out the right-hand side.

In fact, we show that the right-hand side is zero. Note that $\partial_x Q - \partial_y P = \partial_x^2 f + \partial_y^2 f$. Since $f(x, y) = (1/2) \ln |p(x + iy)|^2$, we have

$$\partial_x f = \frac{1}{2} \frac{\partial_x |p(x + iy)|^2}{|p(x + iy)|^2}, \quad \partial_y f = \frac{1}{2} \frac{\partial_y |p(x + iy)|^2}{|p(x + iy)|^2}. \tag{2}$$

Thus,

$$\partial_x^2 f = \frac{1}{2} \frac{\partial_x^2 |p|^2}{|p|^2} - \frac{1}{2} \frac{(\partial_x |p|^2)^2}{|p|^4}, \quad \partial_y^2 f = \frac{1}{2} \frac{\partial_y^2 |p|^2}{|p|^2} - \frac{1}{2} \frac{(\partial_y |p|^2)^2}{|p|^4}. \tag{3}$$

We claim that $\partial_y p = i \partial_x p$ and $\partial_y \overline{p} = -i \partial_x \overline{p}$. Indeed, by linearity of derivatives, it suffices to prove these identities for $q = (x + iy)^k$. In this case, $q = \sum_{l=0}^{k} \binom{k}{l} x^{k-l}\, i^l\, y^l$ by the binomial theorem. Hence, $\partial_y q = \sum_{l=1}^{k} \binom{k}{l} l\, x^{k-l}\, i^l\, y^{l-1}$. Setting $l = j + 1$ and noting that

$$\binom{k}{j+1}(j + 1) = \binom{k}{j}(k - j)$$

gives

$$\partial_y q = \sum_{j=0}^{k-1} \binom{k}{j} (k - j)\, x^{k-j-1}\, i^{j+1}\, y^j,$$

which is just $i \partial_x q$. The second of these identities is proved similarly or can be proved by taking the complex conjugate of the first identity. Now employing these identities and Leibniz's rule on $|p|^2 = p\,\overline{p}$, the reader can verify with a straightforward computation using the equations for $\partial_x^2 f$ and $\partial_y^2 f$ found in (3) that $\partial_y^2 f = -\partial_x^2 f$. Thus, $\partial_x Q - \partial_y P = 0$ and so the right-hand side of (1) is zero.

We now estimate the left-hand side of (1). Without loss of generality, we assume that the leading coefficient of $p(w)$ is one. Then $p(x + iy)$ is of the form $(x + iy)^n$ plus a polynomial in $x + iy$ of degree at most $n - 1$. Hence, we can write

$$|p(x + iy)|^2 = p(x + iy)\,\overline{p(x + iy)} = (x^2 + y^2)^n + \widetilde{p}(x, y), \tag{4}$$

where $\widetilde{p}(x, y)$ is a polynomial in the variables $x$ and $y$ of degree at most $2n - 1$. Taking the partials of $|p(x + iy)|^2 = (x^2 + y^2)^n + \widetilde{p}(x, y)$ with respect to $x$ and $y$

and plugging the results into (2), we see that

$$P = -\partial_y f = -\frac{ny}{x^2+y^2} + \widetilde{P}, \qquad Q = \partial_x f = \frac{nx}{x^2+y^2} + \widetilde{Q},$$

where

$$\widetilde{P} = -\frac{(x^2+y^2)\,\partial_y \widetilde{p}(x,y) - 2ny\,\widetilde{p}(x,y)}{2|p(x+iy)|^2\,(x^2+y^2)},$$
$$\widetilde{Q} = \frac{(x^2+y^2)\,\partial_x \widetilde{p}(x,y) - 2nx\,\widetilde{p}(x,y)}{2|p(x+iy)|^2\,(x^2+y^2)}.$$

Using the curve $c_r(t) := (r\cos t, r\sin t)$, which traces out $C_r$ for $0 \le t \le 2\pi$, a direct computation gives

$$\int_{C_r} \frac{-ny}{x^2+y^2}\,dx + \frac{nx}{x^2+y^2}\,dy = 2\pi n.$$

Thus,

$$\int_{C_r} P\,dx + Q\,dy = 2\pi n + g(r), \tag{5}$$

where

$$g(r) = \int_{C_r} \widetilde{P}\,dx + \widetilde{Q}\,dy.$$

We analyze $g(r)$ as follows. Since $\widetilde{p}(x,y)$ is a polynomial in $x$ and $y$ of degree at most $2n-1$, $\partial_x \widetilde{p}(x,y)$ and $\partial_y \widetilde{p}(x,y)$ are polynomials in $x$ and $y$ of degree at most $2n-2$. Hence, the numerators of $\widetilde{P}$ and $\widetilde{Q}$ are polynomials in $x$ and $y$ of degree at most $2n$. As a result, these numerators are each bounded, in absolute value, by a constant times $(x^2+y^2)^n$. Since $\widetilde{P}$ and $\widetilde{Q}$ contain $|p(x+iy)|^2\,(x^2+y^2)$ in their denominators, in view of (4) it follows that $|\widetilde{P}|$ and $|\widetilde{Q}|$ are each bounded by a constant times $(x^2+y^2)^{-1}$. These estimates on $|\widetilde{P}|$ and $|\widetilde{Q}|$ imply that $|g(r)|$ is bounded by a constant times $r^{-1}$. Since the right-hand side of (1) was shown to be zero, letting $r \to \infty$ in (5) gives the contradiction $0 = 2\pi n$. Thus, our original assumption that $p(w)$ has no zero must be false.

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