While not all of the assumptions of the model will be satisfied in every case, nevertheless, this analysis might be used to justify limiting the size of committees. Now, if we could only come up with an analysis to justify limiting the number of committees we are assigned to!

Acknowledgment. The authors would like to express their appreciation to the reviewers for many helpful suggestions.

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Dirichlet and Fresnel Integrals via Iterated Integration

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Many articles [**2, 3, 4, 6**] have been devoted to establishing the values of some important improper integrals:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}.$$

The first integral is called the Dirichlet integral and the other two are called Fresnel integrals. One way to establish these formulas is to consider the iterated integrals of the functions $f(x, y) = e^{-xy}\sin x$ and $g(x, y) = y^{-1/2}e^{-xy+ix}$, respectively, over $[0, \infty) \times [0, \infty)$. For instance, if only we could justify switching the order of integration, we would evaluate the Dirichlet integral like this:

$$\int_0^\infty \left(\int_0^\infty e^{-xy}\sin x\,dy\right)dx = \int_0^\infty \left(\int_0^\infty e^{-xy}\sin x\,dx\right)dy. \tag{1}$$

Since $\int_0^\infty e^{-xy}\sin x\,dy = \sin x/x$, the left-hand integral is $\int_0^\infty (\sin x)/x\,dx$. In view of the integration formula

$$\int e^{-xy}\sin x\,dx = -\frac{e^{-xy}}{1+y^2}\,(y\sin x + \cos x) + C, \tag{2}$$

which is proved using integration by parts, the right-hand integral in (1) is

$$\int_0^\infty \frac{1}{1+y^2}\,dy = \lim_{t\to\infty} \tan^{-1}(y)\Big|_{y=0}^{y=t} = \frac{\pi}{2}.$$

Hence, we have computed the value of the Dirichlet integral: $\int_0^\infty (\sin x)/x\,dx = \pi/2$. Unfortunately, justification for these steps is not at all obvious! The reason is that the standard hypotheses justifying iterating improper integrals, namely Fubini's theorem, which requires absolute integrability, do not apply to the above mentioned f and g. After all, $f(x, 0)$ is $\sin(x)$, which is certainly not integrable over the whole line, and $g(x, 0)$ is not even defined.

However, in this paper we have just the right theorem to justify the desired steps. This theorem applies to the functions f and g and is useful and appropriate in an undergraduate analysis course for two reasons: (1) The hypotheses are straightforward to verify and they apply to many important examples (see Examples 1–4); (2) The proof is very short (given certain well-known results).

THEOREM. *Let $F(x, y)$ be a continuous function on $(a, \infty) \times (\alpha, \infty)$, where a and α are real numbers, and suppose that the improper integrals*

$$G(x) = \int_{\alpha^+}^{\infty} F(x, y)\, dy \quad \text{and} \quad H(y) = \int_{a^+}^{\infty} F(x, y)\, dx \tag{3}$$

exist and converge uniformly for x and y restricted to compact subintervals of (a, ∞) and (α, ∞), respectively. In addition, suppose that for all $b, c > a$,

$$\left| \int_b^c F(x, y)\, dx \right| \le M(y), \tag{4}$$

where $\int_{\alpha^+}^{\infty} M(y)\, dy$ exists. Then the improper integrals

$$\int_{a^+}^{\infty} G(x)\, dx \quad \text{and} \quad \int_{\alpha^+}^{\infty} H(y)\, dy \tag{5}$$

exist and are equal.

The integrals in this theorem are Riemann integrals and they are improper at a, α, and ∞; hence the plus signs on a and α. Since the integrals in (3) are uniformly convergent, $G(x)$ and $H(y)$ are continuous on their respective domains [**1**, Th. 33.6], which guarantees they are Riemann integrable over compact subintervals of their respective domains. If all integrals are understood as Kurzweil-Henstock integrals or (improper) Lebesgue integrals, or if more knowledge concerning Riemann integrals is assumed, then the hypotheses can be weakened considerably. We invite those readers familiar with more advanced theories to formulate such generalizations.

Before proving our theorem, we need the following standard results (the reader not interested in the proof can skip to Example 1 below):

LEMMA 1.

(a) *If $f(x, y)$ is continuous on a finite rectangle $[a, b] \times [\alpha, \beta]$, then*

$$\int_{\alpha}^{\beta} \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_{\alpha}^{\beta} f(x, y)\, dy \right) dx,$$

and the inner integrals are continuous functions of y and x, respectively.

(b) *If $\{f_n\}$ is a sequence of continuous functions on a finite interval $[a, b]$ that converges uniformly on $[a, b]$ to a limit function f, then f is continuous and*

$$\int_a^b f\, dx = \lim_{n \to \infty} \int_a^b f_n\, dx.$$

(c) DOMINATED CONVERGENCE THEOREM. *Suppose that* $f(x) = \lim f_n(x)$ *for all* $x > a$ *where* f *and* f_n, $n \in \mathbb{N}$, *are continuous on* (a, ∞). *Suppose that* $|f_n(x)| \leq M(x)$ *for* $x > a$ *and* $n \in \mathbb{N}$ *where* $\int_{a+}^{\infty} M(x)\,dx$ *exists. Then* f *has an integral over* $[a, \infty)$ *and*

$$\int_{a^+}^{\infty} f\,dx = \lim_{n\to\infty} \int_{a^+}^{\infty} f_n\,dx.$$

Statements (a) and (b) are found in most elementary analysis books, see, for instance, Bartle [**1**]. The Dominated Convergence Theorem can be found there as Theorem 33.10 and it follows directly from the usual one on compact intervals, a simple proof of which is given by Lewin [**5**]. (Technically, Bartle's Theorem 33.10 is stated for integrals that are improper only at infinity, but an analogous proof works for integrals improper at both limits of integration.)

Now to the proof of the theorem: Let $a < a_n < b_n$ be sequences with $a_n \to a$ and $b_n \to \infty$, and let $\alpha < \alpha_n < \beta_n$ be sequences with $\alpha_n \to \alpha$ and $\beta_n \to \infty$. Since F is continuous on the rectangle $[a_m, b_m] \times [\alpha_n, \beta_n]$, by (a) of Lemma 1,

$$\int_{\alpha_n}^{\beta_n} \left(\int_{a_m}^{b_m} F(x, y)\,dx \right) dy = \int_{a_m}^{b_m} \left(\int_{\alpha_n}^{\beta_n} F(x, y)\,dy \right) dx,$$

and the inner integrals are continuous functions of y and x, respectively. As $n \to \infty$, the inner integral on the right converges uniformly to $G(x)$, so by (b) of Lemma 1, the limit as $n \to \infty$ of the right-hand integral exists and equals $\int_{a_m}^{b_m} G(x)dx$. Thus, the improper integral of the inner integral on the left exists, and

$$\int_{\alpha^+}^{\infty} \left(\int_{a_m}^{b_m} F(x, y)\,dx \right) dy = \int_{a_m}^{b_m} G(x)\,dx. \tag{6}$$

As $m \to \infty$, the continuous function on (α, ∞) given by the inner integral on the left in (6) converges to the continuous function $H(y)$ and by (4), the inner integral is dominated by a function that has an integral over $[\alpha, \infty)$. Thus, (c) of Lemma 1 implies that as $m \to \infty$ the limit of the left-hand side in (6) exists and equals $\int_{\alpha+}^{\infty} H(y)dy$. It follows that the improper integral $\int_{a+}^{\infty} G(x)dx$ exists and equals $\int_{\alpha+}^{\infty} H(y)\,dy$. This completes the proof.

We now demonstrate how easy it is to use this theorem. Henceforth we drop the plus signs on the lower limits of integration to simplify notation.

EXAMPLE 1. For our first example, consider once again $f(x, y) = e^{-xy} \sin x$ on $[0, \infty) \times [0, \infty)$. One can check that f is not absolutely integrable over $[0, \infty) \times [0, \infty)$, so the usual Fubini's theorem does not imply the existence or equality of the iterated integrals of f over this quadrant. However, we can apply our theorem, as we now show. First, because of the exponentially decaying factor, it follows that $\int_0^{\infty} e^{-xy} \sin x\,dy$ and $\int_0^{\infty} e^{-xy} \sin x\,dx$ are uniformly convergent for x and y restricted to compact subintervals of $(0, \infty)$. Second, using the formula (2), one can verify that for all $b, c > 0$,

$$\left| \int_b^c f(x, y)\,dx \right| \leq \frac{K}{1 + y^2}, \quad \text{for some } K > 0,$$

which has an integral on $[0, \infty)$. Thus, the conditions of the theorem are satisfied, and so the formula (1) at the beginning of this paper is indeed true! We can now proceed exactly as we did before to derive the value of the Dirichlet integral:

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

EXAMPLE 2. Now let $g(x, y) = y^{-1/2}e^{-xy+ix}$ on $[0, \infty) \times [0, \infty)$. As with the previous example, the usual Fubini's theorem does not apply to this function, but we shall see that our theorem does apply. Because of the exponentially decaying factor, it follows that $\int_0^\infty y^{-1/2}e^{-xy+ix}\,dy$ and $\int_0^\infty y^{-1/2}e^{-xy+ix}\,dx$ are uniformly convergent for x and y restricted to compact subintervals of $(0, \infty)$. Moreover, one can easily check that for any $b, c > 0$,

$$\left|\int_b^c y^{-1/2}e^{-xy+ix}\,dx\right| = \frac{y^{-1/2}}{|-y+i|}|e^{-cy+ic} - e^{-by+ib}|$$
$$\leq \frac{1}{\sqrt{y}\sqrt{1+y^2}}(e^{-cy} + e^{-by})$$
$$\leq \frac{2}{\sqrt{y}\sqrt{1+y^2}},$$

which has an integral over $[0, \infty)$. Thus, the conditions of the theorem are met, and so

$$\int_0^\infty \left(\int_0^\infty y^{-1/2}e^{-xy+ix}\,dy\right)dx = \int_0^\infty \left(\int_0^\infty y^{-1/2}e^{-xy+ix}\,dx\right)dy.$$

Since $\int_0^\infty y^{-1/2}e^{-xy+ix}\,dy = e^{ix}\int_0^\infty y^{-1/2}e^{-xy}\,dy = \sqrt{\pi}\,x^{-1/2}e^{ix}$, where we made the change of variables $y \mapsto x^{-1}y^2$ and used the Gaussian integral $\int_0^\infty e^{-y^2}\,dy = \sqrt{\pi}/2$, the left-hand integral is

$$\int_0^\infty \sqrt{\pi}\,x^{-1/2}e^{ix}\,dx = \sqrt{\pi}\int_0^\infty x^{-1/2}\cos x\,dx + i\sqrt{\pi}\int_0^\infty x^{-1/2}\sin x\,dx;$$

on the other hand, changing variables $y \mapsto y^2$, the right-hand integral is

$$\int_0^\infty y^{-1/2}\frac{-1}{-y+i}\,dy = \int_0^\infty \frac{2}{y^2-i}\,dy = \int_0^\infty \frac{2y^2}{1+y^4}\,dy + i\int_0^\infty \frac{2}{1+y^4}\,dy.$$

Each integral on the right has the value $\pi/\sqrt{2}$, which can be found using the method of partial fractions as in [**4**]. Thus, we have computed the Fresnel integrals:

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}.$$

EXAMPLE 3. Let $0 < a < 1$. Then, arguing as in Example 2, we can apply our theorem to the function $y^{-a}\,e^{-xy+ix}$ on $[0, \infty) \times [0, \infty)$. Working out the iterated integrals in the same spirit as we did in Example 2 and using some elementary properties of the Gamma and Beta functions, we arrive at the following "generalized" Fresnel integrals:

$$\int_0^\infty x^{a-1}\,\cos x\,dx = \Gamma(a)\cos\left(\frac{a\,\pi}{2}\right) \quad \text{and} \quad \int_0^\infty x^{a-1}\,\sin x\,dx = \Gamma(a)\sin\left(\frac{a\,\pi}{2}\right),$$

where $\Gamma(a)$ is the Gamma function evaluated at a.

EXAMPLE 4. We remark that our theorem immediately implies the celebrated "Weierstrass M-Test" for iterated integrals [**1**, Th. 33.13]: Let $F(x, y)$ be a continuous function on $(a, \infty) \times (\alpha, \infty)$ and suppose that

$$\left|F(x, y)\right| \leq L(x) \cdot M(y),$$

where $L(x)$ and $M(y)$ have improper integrals over $[a, \infty)$ and $[\alpha, \infty)$, respectively. Then the iterated integrals in (5) exist and are equal.

Finally, we end with a brief outline of how the Dirichlet and Fresnel integrals can be derived from the standard Fubini's theorem. As we already mentioned, both $f(x, y) = e^{-xy} \sin x$ and $g(x, y) = y^{-1/2} e^{-xy+ix}$ are not absolutely integrable on $[0, \infty) \times [0, \infty)$, so, without using the theorem, some ingenious trick is usually required to justify the iteration of integrals. For instance, Bartle [**1**] integrates the function $f(x, y)$ over $[s, \infty) \times [t, \infty)$ with $s, t > 0$, where Fubini's theorem is valid, and after integration is performed, one takes the limits as $s, t \to 0$ to establish Dirichlet's integral. Leonard [**4**] applies Fubini's theorem to $e^{-tx} g(x, y)$ with $t > 0$, which is absolutely integrable on $[0, \infty) \times [0, \infty)$, and, after integrating, takes the limit as $t \to 0$ to establish the Fresnel integrals.

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A Publishing Paradox

Alert reader Jack C. Abad and his brother Victor Abad send the following:

> A recent Birkhäuser-Verlag book list included *Unpublished Philosophical Essays/Kurt Gödel*, edited by Francisco A. Rodriquez-Consuegra, 1995. If the title is accurate, it might make appropriate barbershop reading in that town where the barber shaves everyone who doesn't shave himself.

A quick search of Amazon.com turns up a wealth of similar material—unpublished recordings by Elizabeth Schwartzkopf and Marian Anderson, unpublished letters from General Robert E. Lee to Jefferson Davis, and unpublished opinions of the Warren Court. Author Michael McMullen is more scrupulously logical: The title of his book is *The Blessing of God: Previously Unpublished Sermons of Jonathan Edwards*.