THE ζ-DETERMINANT OF GENERALIZED APS BOUNDARY PROBLEMS OVER THE CYLINDER

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Abstract. In this note, we explicitly compute the ζ-determinant of a Dirac Laplacian with APS boundary conditions over a finite cylinder. Using this exact result, we illustrate the gluing and comparison formulas for the ζ-determinants of Dirac Laplacians proved in [12] and [14].

1. Introduction

The ζ-function technique of regularizing determinants entered the mathematical world in Ray and Singer's celebrated article [16] on the analytic torsion, and in the physics world commencing with the groundbreaking works of Dowker and Critchley [6] and Hawking [9] (for a recent review, see [10]). The power of this technique can be appreciated by the now well-known fact that *any* quantum field theory can be renormalized to the theory of one loops via ζ-regularization. Because of their facility in mathematics and physics, there has been immense research in computing ζ-determinants under a variety of conditions, cf. Elizalde *et al.* [8] for such techniques. Of particular importance is the Dirac Laplacian with non-local Atiyah-Patodi-Singer (APS) boundary conditions, which arises in a variety of situations; for instance, one-loop quantum cosmology [3, 4, 5], spectral branes [18], and the study of Dirac fields in the background of a magnetic flux [2].

However, the value of the ζ-determinant for a Dirac Laplacian with APS boundary conditions over a finite cylinder has remained an open question, partly because it is not possible to compute the eigenvalues of the Dirac operator "explicitly" under these conditions. The main purpose of this note is to answer this question and compute this ζ-determinant. Because in general it is not possible to compute the eigenvalues of the Dirac operator explicitly, we have to proceed using a totally different method from the conventional ones used to compute ζ-determinants. The method we use is the method of *adiabatic decomposition*, pioneered in the work of Douglas and Wojciechowski [7] for the eta invariant, and by the second author and Wojciechowski [15] for the ζ-determinant. The second purpose of this paper is to elucidate the effectiveness of the adiabatic method in a concrete situation (see Section 4).

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Finally, we investigate the *gluing problem* for the ζ-determinant (see Section 5), which can be stated as follows: Given a partitioned compact manifold $M = M_- \cup M_+$ into manifolds with boundaries, describe the ζ-determinant of a Dirac Laplacian on M in terms of the ζ-determinants on $M_\pm$ with suitable boundary conditions. This gluing problem has remained an open problem partly because of the highly nonlocal nature of the ζ-determinant and its variation and partly because of the technical aspects inherent with the nonlocal pseudodifferential boundary conditions required for Dirac type operators. In [12] we solve this problem and the third purpose of this paper is to illustrate our gluing formula in the concrete situation of a partitioned finite cylinder. We also illustrate the so-called *comparison, or relative invariant, formula* proved in [14].

We now describe our set up. Let $\mathcal{D}_R : C^\infty(N_R, S) \to C^\infty(N_R, S)$ be a Dirac type operator where $N_R = [-R, R] \times Y$ is a finite cylinder with $R > 0$, Y a closed compact Riemannian manifold (of arbitrary dimension), and S a Clifford bundle over N_R. We assume that $\mathcal{D}_R$ is of product form

$$\mathcal{D}_R = G(\partial_u + D_Y) \tag{1.1}$$

where G is a bundle automorphism of $S_0 := S|_Y$ and D_Y is a Dirac operator acting on $C^\infty(Y, S_0)$ such that $G^2 = -Id$ and $GD_Y = -D_Y G$. Since the finite cylinder N_R has boundaries, we have to impose boundary conditions. An important boundary condition for applications is the non-local generalized APS spectral condition, which is defined as follows. We assume that $\dim \ker(G + i) \cap \ker(D_Y) = \dim \ker(G - i) \cap \ker(D_Y)$. Then we can fix two involutions σ_1, σ_2 over $\ker(D_Y)$ such that $\sigma_1 G = -G\sigma_1$ and $\sigma_2 G = -G\sigma_2$, and impose the boundary conditions given by the following *generalized APS spectral projections*,

$$\begin{aligned} \Pi_{\sigma_1} &= \Pi_> + \frac{1+\sigma_1}{2}\Pi_0 \quad \text{at } \{-R\} \times Y, \\ \Pi_{\sigma_2} &= \Pi_< + \frac{1+\sigma_2}{2}\Pi_0 \quad \text{at } \{R\} \times Y \end{aligned} \tag{1.2}$$

where $\Pi_>, \Pi_<, \Pi_0$ denote the orthogonal projections onto the positive, negative, and zero eigenspaces of D_Y. We denote by $\mathcal{D}_{R,P}$ the resulting operator with these boundary conditions, that is,

$$\mathcal{D}_{R,P} := \mathcal{D}_R : \mathrm{dom}(\mathcal{D}_{R,P}) \to L^2(N_R, S)$$

where

$$\mathrm{dom}(\mathcal{D}_{R,P}) := \Big\{ \phi \in H^1(N_R, S) \mid \Pi_{\sigma_1}\phi|_{u=-R} = 0 \ , \ \Pi_{\sigma_2}\phi|_{u=R} = 0 \Big\}.$$

Then the spectrum of the Dirac Laplacian $\mathcal{D}^2_{R,P}$ consists of discrete real eigenvalues $\{\lambda_k\}$. The ζ-function of $\mathcal{D}^2_{R,P}$ is defined by

$$\zeta_{\mathcal{D}^2_{R,P}}(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$

which is a priori defined for $\Re(s) \gg 0$ and has a meromorphic extension to $\mathbb{C}$ with 0 as a regular point. Then the ζ-determinant of $\mathcal{D}_{R,P}^2$ is defined by

$$\det{}_\zeta \mathcal{D}_{R,P}^2 := \exp\big(- \zeta'_{\mathcal{D}_{R,P}^2}(0)\big).$$

As we already mentioned, since we imposed APS spectral boundary conditions, it is not possible to compute the eigenvalues $\{\lambda_k\}$ explicitly, so there is no direct way to compute the ζ-determinant $\det_\zeta \mathcal{D}_{R,P}^2$ from the eigenvalues. However, using adiabatic and gluing techniques proved in [15], [11], [13], we compute $\det_\zeta \mathcal{D}_{R,P}^2$, which we now explain. We denote by $(\sigma_1\sigma_2)_-$ the restriction of $\sigma_1\sigma_2$ to $\ker(G+i)\cap\ker(D_Y)$. For a linear operator L over a finite-dimensional vector space, $\det^*(L)$ denotes the determinant of the invertible operator $(L|_{\ker(L)^\perp})$. The following theorem is the main result of this note.

Theorem 1.1. *The following equality holds:*

$$\det{}_\zeta \mathcal{D}_{R,P}^2 = (2R)^{2h}\, e^{2CR}\, 2^{\zeta_{D_Y^2}(0)+h_Y}\, \det{}^*\Big(\frac{2Id - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4}\Big)$$

where h is the number of $(+1)$-eigenvalues of $(\sigma_1\sigma_2)_-$, $h_Y = \dim\ker(D_Y)$ and $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ with $\zeta_{D_Y^2}(s)$ the ζ-function of D_Y^2.

This exact value is used to determine certain constants appearing in the gluing formulas of the ζ-determinants of Dirac Laplacians in [12], [13]. Finally, the authors thank the referees for helpful comments.

2. Asymptotics of $\det_\zeta \mathcal{D}_{R,P}^2$ as $R \to \infty$

In this section, we derive the asymptotics of $\det_\zeta \mathcal{D}_{R,P}^2$ as $R \to \infty$. This is one of the main ingredients in the proof of our main theorem.

We decompose $L^2(N_R, S)$ as follows:

$$L^2(N_R,S) = L^2([-R,R];\ker(D_Y)) \oplus L^2([-R,R];\ker(D_Y)^\perp) \tag{2.1}$$

where $\ker(D_Y)^\perp$ is the orthogonal complement of $\ker(D_Y)$ in $L^2(Y, S_0)$. We denote by $\mathcal{D}_{R,P}(0)$ the restriction of $\mathcal{D}_{R,P}$ to the first component of the decomposition (2.1). Since $D_Y = 0$ on $\ker(D_Y)$, the operator $\mathcal{D}_{R,P}(0)$ is $G\partial_u$ with the boundary conditions at $\{\pm R\}\times Y$ determined by σ_1, σ_2.

For $\mathcal{D}_{R,P}(0)$, we can compute all the eigenvalues of $\mathcal{D}_{R,P}(0)$ explicitly using elementary ordinary differential equations and we obtain

Lemma 2.1. *The spectrum of $\mathcal{D}_{R,P}(0)$ is given by*

$$\Big\{ \Big(k\pi - \frac{\alpha_j}{2}\Big)(2R)^{-1} \;\Big|\; k \in \mathbb{Z}\ ,\ \alpha_j \in (-\pi, \pi]\ ,\ e^{i\alpha_j} \in Spec(\sigma_1\sigma_2)_- \Big\}.$$

Therefore, we can also compute $\det_\zeta \mathcal{D}_{R,P}(0)^2$ explicitly as we now show.

Proposition 2.2. *We have the following equality:*

$$\det{}_\zeta \mathcal{D}_{R,P}(0)^2 = (2R)^{2h}\, 2^{h_Y} \det{}^* \Big(\frac{2Id - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4} \Big)$$

with h the number of $(+1)$-eigenvalues of $(\sigma_1\sigma_2)_-$ and $h_Y = \dim\ker(D_Y)$.

Proof. By Lemma 2.1, the ζ-function of $\mathcal{D}_{R,P}(0)^2$ is given by

$$\zeta_{\mathcal{D}_{R,P}(0)^2}(s) = 2h \cdot (2R)^{2s}\pi^{-2s}\zeta(2s) + F(s),$$

where $\zeta(s)$ is the Riemann zeta function and the second term is given by

$$F(s) = (2R)^{2s}\pi^{-2s} \sum_{j=1}^{h_Y/2-h} \sum_{k\in\mathbb{Z}} \Big(k - \frac{\alpha_j}{2\pi}\Big)^{-2s}$$

with $\alpha_j \neq 0$ in the sum. For the first term, using that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, we obtain

$$-\frac{d}{ds}\Big|_{s=0} 2h \cdot (2R)^{2s}\pi^{-2s}\zeta(2s) = \log(4R)^{2h}. \tag{2.2}$$

To compute $-F'(0)$, we use the Hurwitz zeta function defined by

$$\zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s} \quad \text{for } 0 < a < 1,$$

which has the properties $\zeta(0, a) = \frac{1}{2} - a$ and $\zeta'(0, a) = \log(\Gamma(a)) - \frac{1}{2}\log(2\pi)$. Then $F(s)$ can be written in terms of the Hurwitz function as

$$F(s) = (2R)^{2s}\pi^{-2s} \sum_{j=1}^{h_Y/2-h} \Big(\zeta(2s, \frac{\alpha_j}{2\pi}) + \zeta(2s, 1 - \frac{\alpha_j}{2\pi})\Big),$$

where we assumed that $\alpha_j > 0$ since $\sum_{k\in\mathbb{Z}}(k - a)^{-2s} = \sum_{k\in\mathbb{Z}}(k + a)^{-2s}$. Using the properties of the Hurwitz zeta function, we have

$$-F'(0) = -2 \sum_{j=1}^{h_Y/2-h} \Big(\zeta'(0, \frac{\alpha_j}{2\pi}) + \zeta'(0, 1 - \frac{\alpha_j}{2\pi})\Big)$$

$$= -2 \sum_{j=1}^{h_Y/2-h} \Big(\log\Big(\Gamma(\frac{\alpha_j}{2\pi})\Gamma(1 - \frac{\alpha_j}{2\pi})\Big) - \log(2\pi)\Big) = \sum_{j=1}^{h_Y/2-h} \log\big(4\sin^2(\alpha_j/2)\big)$$

where we used $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$. Combining this derivative with the derivative (2.2) and the fact that

$$\sin^2(\alpha_j/2) = \Big(\frac{e^{i\alpha_j/2} - e^{-i\alpha_j/2}}{2i}\Big)^2 = \frac{2 - e^{i\alpha_j} - e^{-i\alpha_j}}{4},$$

completes the proof. □

Since we can split the contributions of $\det_\zeta \mathcal{D}_{R,P}(0)^2$ over each subspace in the decomposition (2.1) and we already obtained the exact value of $\det_\zeta \mathcal{D}_{R,P}(0)^2$, it remains to compute the ζ-determinant of the restriction of $\mathcal{D}_{R,P}^2$ to the second component in the decomposition (2.1). Therefore, from now on, we can assume:

(2.3) *The tangential operator D_Y is invertible.*

We previously remarked that it is not possible to get the exact form of all the eigenvalues of $\mathcal{D}_{R,P}$, so we can not compute $\det_\zeta \mathcal{D}_{R,P}^2$ in a direct way. For this reason, we first consider the asymptotics of $\det_\zeta \mathcal{D}_{R,P}^2$ as $R \to \infty$.

For functions $f(R) > 0, g(R) > 0$ defined over $(0,\infty)$, $f(R) \sim g(R)$ means

$$\lim_{R\to\infty} |\log f(R) - \log g(R)| = 0 \quad \Longleftrightarrow \quad \lim_{R\to\infty} \frac{f(R)}{g(R)} = 1.$$

The following proposition is the main result of this section.

Proposition 2.3. *When D_Y is invertible, we have*

$$\det_\zeta \mathcal{D}_{R,P}^2 \ \sim \ 2^{\zeta_{D_Y^2}(0)} e^{2CR}$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ with $\zeta_{D_Y^2}(s)$ the ζ-function of D_Y^2.

Proof. With $\mathcal{D}_R = G(\partial_u + D_Y)$ over $[-R,R]\times Y$, let $\mathcal{D}_{R,-}$ denote the restriction of $\mathcal{D}_R$ to $[-R,0]\times Y$ with boundary condition $\Pi_<$ at $\{0\}\times Y$, and $\mathcal{D}_{R,+}$ denote the restriction of $\mathcal{D}_R$ to $[0,R]\times Y$ with boundary condition $\Pi_>$ at $\{0\}\times Y$. We take the square of these operators and impose Dirichlet boundary conditions over $\{\pm R\}\times Y$ and denote by $\mathcal{D}_{R,d}^2$, $\mathcal{D}_{R,d,-}^2$, and $\mathcal{D}_{R,d,+}^2$ the resulting operators. Then by Proposition 4.1 to be proved later,

$$\frac{\det_\zeta \mathcal{D}_{R,d}^2}{\det_\zeta \mathcal{D}_{R,d,-}^2 \cdot \det_\zeta \mathcal{D}_{R,d,+}^2} \sim 2^{-\zeta_{D_Y^2}(0)}. \tag{2.4}$$

By Proposition 7.1 in [11], we know that

$$\det_\zeta \mathcal{D}_{R,d}^2 \ = \ \Big(\det_\zeta \sqrt{D_Y^2}\Big)^{-1} e^{2RC} \prod_{k=1}^\infty (1 - e^{-4R\mu_k})^2$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ and $\{\mu_k\}$ are the positive eigenvalues of D_Y. It follows that

$$\det{}_\zeta \mathcal{D}^2_{R,d} \ \sim \ \left(\det{}_\zeta \sqrt{D_Y^2}\right)^{-1} e^{2RC}.$$

Combining this with (2.4), we conclude that

$$\det{}_\zeta \mathcal{D}^2_{R,d,-} \cdot \det{}_\zeta \mathcal{D}^2_{R,d,+} \sim 2^{\zeta_{D_Y^2}(0)} \left(\det{}_\zeta \sqrt{D_Y^2}\right)^{-1} e^{2RC}. \tag{2.5}$$

Now let $\mathcal{D}^2_{R,-,d}$ and $\mathcal{D}^2_{R,+,d}$ denote the restrictions of $\mathcal{D}^2_{R,P}$ to $[-R,0]\times Y$ and $[0,R]\times Y$, respectively, with the Dirichlet condition at $\{0\}\times Y$. Then according to the main result in [11], which also holds for this case, we have

$$\frac{\det{}_\zeta \mathcal{D}^2_{R,P}}{\det{}_\zeta \mathcal{D}^2_{R,-,d} \cdot \det{}_\zeta \mathcal{D}^2_{R,+,d}} = 2^{-\zeta_{D_Y^2}(0)} \det{}_\zeta \mathcal{R}_R, \tag{2.6}$$

where $\mathcal{R}_R$ is the sum of the Dirichlet to Neumann operators for the restriction of $\mathcal{D}^2_{R,P}$ to $[-R,0]\times Y$ and $[0,R]\times Y$. By a direct computation, we find that

$$\begin{aligned} \det{}_\zeta \mathcal{R}_R &= 2^{\zeta_{D_Y^2}(0)} \left(\det{}_\zeta \sqrt{D_Y^2}\right) \prod_{k=1}^{\infty} (1-e^{-2\mu_k R})^{-2} \\ &\sim 2^{\zeta_{D_Y^2}(0)} \left(\det{}_\zeta \sqrt{D_Y^2}\right) \end{aligned} \tag{2.7}$$

where $\{\mu_k\}$ are the positive eigenvalues of D_Y. Finally, noting that we have $\det_\zeta \mathcal{D}^2_{R,d,-} = \det_\zeta \mathcal{D}^2_{R,+,d}$ and $\det_\zeta \mathcal{D}^2_{R,d,+} = \det_\zeta \mathcal{D}^2_{R,-,d}$, in view of (2.5), (2.6), and (2.7), we obtain

$$\begin{aligned} \det{}_\zeta \mathcal{D}^2_{R,P} &\sim \left(\det{}_\zeta \sqrt{D_Y^2}\right)^{-1} e^{2RC} \det{}_\zeta \mathcal{R}_R \\ &\sim \left(\left(\det{}_\zeta \sqrt{D_Y^2}\right)^{-1} e^{2RC}\right)\left(2^{\zeta_{D_Y^2}(0)}\left(\det{}_\zeta \sqrt{D_Y^2}\right)\right) = 2^{\zeta_{D_Y^2}(0)} e^{2RC}. \end{aligned}$$

This completes our proof. □

3. Proof of Theorem 1.1

Let us consider the Dirac type operator $G(\partial_u + D_Y)$ on the infinite cylinder $M = \big((-\infty,0]\cup[0,\infty)\big)\times Y$ with boundary conditions $\Pi_<$ and $\Pi_>$ at the left and right, respectively, of the two copies of $\{0\}\times Y$ and we denote by $\hat{\mathcal{D}}_P$ the resulting operator. We decompose M into $M_{2R} = \big([-2R,0]\cup[0,2R]\big)\times Y$ and $M_{2R,\infty} = \big((-\infty,-2R]\cup[2R,\infty)\big)\times Y$ and obtain Dirac operators over these by restricting $\hat{\mathcal{D}}_P$. On M_{2R}, we then impose the boundary conditions given by $\Pi_>$ at the boundary $\{-2R\}\times Y$ and $\Pi_<$ at the boundary $\{2R\}\times Y$, and on $M_{2R,\infty}$, we put $\Pi_<$ at the boundary $\{-2R\}\times Y$ and $\Pi_>$ at the boundary

$\{2R\} \times Y$. Then the resulting operator over M_{2R} is equivalent to two copies of $\mathcal{D}_{R,P}$. We denote the resulting operator over $M_{2R,\infty}$ by $\hat{\mathcal{D}}_{2R,P}$.

As remarked in the proof of Lemma 8.3 of [13], it follows that

$$\det{}_\zeta(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2) \left(\det{}_\zeta \mathcal{D}_{R,P}^2\right)^{-2} \quad \text{is independent of } R, \tag{3.1}$$

where $\det_\zeta(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2)$ denotes the relative ζ-determinant of $(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2)$ defined by

$$\det{}_\zeta(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2) := \exp\left(-\zeta'(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2, 0)\right)$$

with

$$\zeta(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2, s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}\left(e^{-t\hat{\mathcal{D}}_P^2} - e^{-t\hat{\mathcal{D}}_{2R,P}^2}\right) dt.$$

In the following lemma we compute this relative ζ-determinant explicitly.

Lemma 3.1. *When D_Y is invertible, the following equality holds:*

$$\det{}_\zeta(\hat{\mathcal{D}}_P^2, \hat{\mathcal{D}}_{2R,P}^2) = e^{4CR}$$

where $C = -(2\sqrt{\pi})^{-1}(\Gamma(s)^{-1}\Gamma(s-1/2)\zeta_{D_Y^2}(s-1/2))'(0)$ with $\zeta_{D_Y^2}(s)$ the ζ-function of D_Y^2.

Proof. Let $\{(\mu_k, \varphi_k)\}$ be the spectral resolution of D_Y. Then as shown in [1], for $\big((u,y),(u',y')\big) \in ([0,\infty) \times Y)^2$, we have

$$\begin{aligned} e^{-t\hat{\mathcal{D}}_P^2} &= \sum_{\mu_k>0} \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[e^{-(u-u')^2/4t} - e^{-(u+u')^2/4t}\Big] \varphi_k(y) \otimes \varphi_k(y') \\ &+ \sum_{\mu_k>0} \bigg\{ \frac{e^{-t\mu_k^2}}{\sqrt{4\pi t}} \Big[e^{-(u-u')^2/4t} + e^{-(u+u')^2/4t}\Big] \\ &- \mu_k\, e^{\mu_k(u+u')} \operatorname{erfc}\Big(\frac{u+u'}{2\sqrt{t}} + \mu_k\sqrt{t}\Big)\bigg\} G\varphi_k(y) \otimes G\varphi_k(y'); \end{aligned} \tag{3.2}$$

with a similar formula for $\big((u,y),(u',y')\big) \in ((-\infty,0] \times Y)^2$. Since the heat kernel of $\hat{\mathcal{D}}_{2R,P}^2$ is obtained from $e^{-t\hat{\mathcal{D}}_P^2}$ by shifts of $\pm 2R$, it follows that

$$\operatorname{Tr}\left(e^{-t\hat{\mathcal{D}}_P^2} - e^{-t\hat{\mathcal{D}}_{2R,P}^2}\right) = 4R \cdot \frac{1}{\sqrt{4\pi t}} \operatorname{Tr}_Y\left(e^{-tD_Y^2}\right).$$

From this, the claim follows by the standard computation. $\square$

Now taking the logarithm of (3.1) and using Lemma 3.1 and Proposition 2.3, we see that

$$2CR - \log\det{}_\zeta \mathcal{D}_{R,P}^2 = -\zeta_{D_Y^2}(0)\log 2 + \mathcal{E}(R) \quad \text{is independent of } R, \tag{3.3}$$

where $\mathcal{E}(R) \to 0$ as $R \to \infty$. Since $\mathcal{E}(R)$ vanishes as $R \to \infty$, and the expression (3.3) is constant in R, it follows that $\mathcal{E}(R)$ is in fact identically zero. Then setting $\mathcal{E}(R) = 0$ in (3.3) and then solving for $\log \det_\zeta \mathcal{D}^2_{R,P}$ completes the proof of Theorem 1.1.

4. Adiabatic decomposition of ζ-determinant

The aim of this section is to prove the following proposition, which was used in the proof of Proposition 2.3.

Proposition 4.1. *When D_Y is invertible, we have*

$$\frac{\det_\zeta \mathcal{D}^2_{R,d}}{\det_\zeta \mathcal{D}^2_{R,d,-} \cdot \det_\zeta \mathcal{D}^2_{R,d,+}} \sim 2^{-\zeta_{D_Y^2}(0)}.$$

For simplicity we use the notation $\mathcal{D}^2_{R,\sqcup}$ for the operator

$$\mathcal{D}^2_{R,d,-} \oplus \mathcal{D}^2_{R,d,+} : \mathrm{dom}(\mathcal{D}^2_{R,d,-}) \oplus \mathrm{dom}(\mathcal{D}^2_{R,d,+}) \\ \to L^2([-R,0] \times Y, S) \oplus L^2([0,R] \times Y, S).$$

Then the log of the left-hand side of Proposition 4.1 can be written as

$$\begin{aligned} \text{(4.1)} \quad & \log \det_\zeta \mathcal{D}^2_{R,d} - \log \det_\zeta \mathcal{D}^2_{R,d,-} - \log \det_\zeta \mathcal{D}^2_{R,d,+} \\ & = - \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}\left(e^{-t\mathcal{D}^2_{R,d}} - e^{-t\mathcal{D}^2_{R,\sqcup}}\right) dt. \end{aligned}$$

The fundamental idea to prove Proposition 4.1 is to construct a parametrix for $e^{-t\mathcal{D}^2_{R,d}} - e^{-t\mathcal{D}^2_{R,\sqcup}}$ up to an error term that vanishes as $R \to \infty$. Because the arguments below are similar to those in [15], we shall omit some details which the reader can find in [15].

We introduce a smooth even function $\rho(a,b) : \mathbb{R} \to [0,1]$ that is equal to 0 for $-a \le u \le a$ and equal to 1 for $b \le |u|$. We now define

$$\phi_1 = 1 - \rho((5/7)R, (6/7)R)\ , \quad \psi_1 = 1 - \psi_2,$$
$$\phi_2 = \rho((1/7)R, (2/7)R)\ , \qquad \psi_2 = \rho((3/7)R, (4/7)R).$$

We now define parametrices of the heat kernels $\mathcal{E}_R(t;x,x')$ of $\mathcal{D}^2_{R,d}$ where $(x,x') \in N_R^2$ and $\mathcal{E}_{R,\sqcup}(t;x,x')$ of $\mathcal{D}^2_{R,\sqcup}$ where $(x,x') \in M_R^2$. To do so, we consider the heat kernel of $-\partial_u^2 + D_Y^2$ over $\mathbb{R} \times Y$, which we denote by

$$\text{(4.2)} \qquad \mathcal{E}(t;x,x') := \frac{1}{\sqrt{4\pi t}} e^{-(u-u')^2/4t}\, e^{-tD_Y^2}(y,y')$$

where $(x,x') \in (\mathbb{R}\times Y)^2$ with $x=(u,y)$, $x'=(u',y')$. For $e^{-t\hat{\mathcal{D}}_P^2}$ defined in the previous section, we put $\mathcal{E}_P(t;x,x') := e^{-t\hat{\mathcal{D}}_P^2}(x,x')$ where $(x,x')\in M^2$. Now we define the parametrices by

$$\begin{aligned} Q_R(t;x,x') &= \phi_1(x)\,\mathcal{E}(t;x,x')\,\psi_1(x') + \phi_2(x)\,\mathcal{E}_R(t;x,x')\,\psi_2(x'),\\ Q_{R,\sqcup}(t;x,x') &= \phi_1(x)\,\mathcal{E}_P(t;x,x')\,\psi_1(x') + \phi_2(x)\,\mathcal{E}_{R,\sqcup}(t;x,x')\,\psi_2(x'), \end{aligned}$$

where $\phi_i(x)=\phi_i(u)$ with $x=(u,y)$ and $\psi_i(x')$ is defined similarly. By Duhamel's principle, we can estimate the difference of the real heat kernels and these parametrices. We refer the proof of the following lemma to [15, Lem. 1.5].

Lemma 4.2. *For any $t>0$, there are positive constants c_1,c_2,c_3 such that*

$$\begin{aligned} \big|\big|\ \mathcal{E}_R(t;x,x') - Q_R(t;x,x')\ \big|\big| &\le c_1 e^{c_2 t - c_3(R^2/t)},\\ \big|\big|\ \mathcal{E}_{R,\sqcup}(t;x,x') - Q_{R,\sqcup}(t;x,x')\ \big|\big| &\le c_1 e^{c_2 t - c_3(R^2/t)} \end{aligned}$$

where $(x,x')\in N_R^2$, M_R^2, respectively, and $||\cdot||$ denotes the norm for an element in $End(S_{x'},S_x)$.

We are now ready to prove Proposition 4.1. First, we note that since D_Y is invertible by assumption, as $R\to\infty$ all the eigenvalues of $\mathcal{D}_{R,d}^2$ and $\mathcal{D}_{R,\sqcup}^2$ are bounded below by a positive constant c. Hence we have

$$\begin{aligned} \big|\operatorname{Tr}\big(e^{-t\mathcal{D}_{R,d}^2} - e^{-t\mathcal{D}_{R,\sqcup}^2}\big)\big| \le\ & e^{-c(t-1)}\big|\operatorname{Tr}\big(e^{-\mathcal{D}_{R,d}^2} - e^{-\mathcal{D}_{R,\sqcup}^2}\big)\big|\\ \le\ & c'\mathrm{vol}(N_R)e^{-c(t-1)} \le c'' R e^{-ct} \end{aligned}$$

for positive constants c', c''. Henceforth we fix $0<\varepsilon<1$. Then from these inequalities, it is straightforward to show that

$$\frac{1}{\Gamma(s)}\int_{R^\varepsilon}^\infty t^{s-1}\operatorname{Tr}\big(e^{-t\mathcal{D}_{R,d}^2} - e^{-t\mathcal{D}_{R,\sqcup}^2}\big)\,dt \to 0 \quad \text{as } R\to\infty.$$

Here, the convergence means that this holomorphic function and its derivative converge to the zero function uniformly over some compact neighborhood of $s=0$. Thus, for the purpose of evaluating the asymptotics of (4.1), we can ignore this large time integral and focus on the small time integral

$$\frac{1}{\Gamma(s)}\int_0^{R^\varepsilon} t^{s-1}\operatorname{Tr}\big(e^{-t\mathcal{D}_{R,d}^2} - e^{-t\mathcal{D}_{R,\sqcup}^2}\big)\,dt. \tag{4.3}$$

Applying Lemma 4.2, this integral is equal to

$$\frac{1}{\Gamma(s)}\int_0^{R^\varepsilon} t^{s-1}\operatorname{Tr}\big(Q_R - Q_{R,\sqcup}\big)\,dt \tag{4.4}$$

modulo a term vanishing as $R\to\infty$, where again, vanishing means that the concerned error function and its derivative converge to the zero function uniformly over some compact neighborhood of $s=0$. From the explicit formulas (4.2) and (3.2), and recalling that (3.2) only represents $e^{-t\hat{\mathcal{D}}_P^2}$ for

$u, u' \geq 0$ and there is a similar formula for $u, u' \leq 0$, it follows that (4.4) is equal to

$$\frac{1}{\Gamma(s)} \int_0^{R^\varepsilon} t^{s-1} \int_0^\infty 2 \sum_{\mu_k > 0} \psi_1(u) \mu_k \, e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) \, du \, dt$$

modulo a term vanishing as $R \to \infty$. To evaluate the right-hand side, we integrate by parts to get

$$\begin{aligned} &\int_0^\infty 2 \sum_{\mu_k > 0} \psi_1(u) \mu_k \, e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) \, du \\ =& \frac{1}{\sqrt{\pi t}} \, \mathrm{Tr}(e^{-tD_Y^2}) \int_0^\infty \psi_1(u) e^{-u^2/t} \, du - \sum_{\mu_k > 0} \mathrm{erfc}\Big(\mu_k \sqrt{t} \Big) \\ &- \int_0^\infty \sum_{\mu_k > 0} \psi_1'(u) e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) \, du. \end{aligned}$$

Now by Proposition 2.1 of [15],

$$-\frac{1}{\Gamma(s)} \int_0^{R^\varepsilon} t^{s-1} \int_0^\infty \sum_{\mu_k > 0} \psi_1'(u) e^{2\mu_k u} \mathrm{erfc}\Big(\frac{u}{\sqrt{t}} + \mu_k \sqrt{t} \Big) \, du \, dt$$

vanishes as $R \to \infty$. Therefore, the nontrivial contribution to the asymptotics of (4.3) is given by

$$\frac{1}{\Gamma(s)} \int_0^{R^\varepsilon} t^{s-1} \left(\frac{\mathrm{Tr}(e^{-tD_Y^2})}{\sqrt{\pi t}} \int_0^\infty \psi_1(u) e^{-u^2/t} \, du - \sum_{\mu_k > 0} \mathrm{erfc}\Big(\mu_k \sqrt{t} \Big) \right) \, dt.$$

Up to a term vanishing as $R \to \infty$, we can remove $\psi_1(u)$ and then adding the large time integral $\int_{R^\varepsilon}^\infty$, which gives rise to another term vanishing as $R \to \infty$, we can see that the final contribution to (4.3) is given by the integral

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{\mathrm{Tr}(e^{-tD_Y^2})}{\sqrt{\pi t}} \int_0^\infty e^{-u^2/t} \, du - \sum_{\mu_k > 0} \mathrm{erfc}\Big(\mu_k \sqrt{t} \Big) \right) \, dt.$$

Using an integration by parts argument (or a table of Mellin transforms), we can evaluate this integral as $\frac{1}{2}\Big(1 - \frac{\Gamma(s+1/2)}{\Gamma(s+1)\sqrt{\pi}}\Big)\zeta_{D_Y^2}(s)$. Finally we obtain

$$-\frac{d}{ds}\bigg|_{s=0} \left(\frac{1}{2}\Big(1 - \frac{\Gamma(s+1/2)}{\Gamma(s+1)\sqrt{\pi}}\Big)\zeta_{D_Y^2}(s) \right) = -\zeta_{D_Y^2}(0) \log 2,$$

which completes our proof.

5. Gluing and comparison formulæ of the ζ-determinant

In this section, for the case of the finite cylinder we illustrate the gluing and comparison formulas of the ζ-determinant proved in [12] and [14].

Let $\mathcal{D}$ be a Dirac type operator acting on $C^\infty(M,S)$ where M is a closed compact Riemannian manifold of arbitrary dimension and S is a Clifford bundle over M. Suppose that $M = M_- \cup M_+$ is partitioned into a union of manifolds with a common boundary $Y = \partial M_- = \partial M_+$. We assume that all geometric structures are of product type over a tubular neighborhood N of Y where $\mathcal{D}$ takes the product form (1.1). By restriction of $\mathcal{D}$, we obtain Dirac type operators $\mathcal{D}_\pm$ over $M_\pm$. We impose the boundary conditions given by the orthogonalized Calderón projectors $\mathcal{C}_\pm$ for $\mathcal{D}_\pm$ and we denote by $\mathcal{D}_{\mathcal{C}_\pm}$ the resulting operators,

$$\mathcal{D}_{\mathcal{C}_\pm} = \mathcal{D}_\pm \quad \text{with} \quad \operatorname{dom}(\mathcal{D}_{\mathcal{C}_\pm}) := \{\, \phi \in H^1(M_\pm, S) \mid \mathcal{C}_\pm(\phi|_Y) = 0 \,\}.$$

Here, we recall that the Calderón projectors $\mathcal{C}_\pm$ are the projectors defined intrinsically as the unique orthogonal projectors onto the infinite-dimensional *Cauchy data spaces* of $\mathcal{D}_\pm$:

$$\{\, \phi|_Y \mid \phi \in C^\infty(M_\pm, S) \ , \ \mathcal{D}_\pm \phi = 0 \,\} \subset C^\infty(Y, S_0),$$

where $S_0 := S|_Y$. The *gluing problem* for the ζ-determinant is to describe the "defect"

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_+} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = \, ?$$

in terms of recognizable data. To describe the solution in [12], we need to introduce some notations. The Calderón projectors $\mathcal{C}_\pm$ have the matrix forms

$$\mathcal{C}_\pm = \frac{1}{2} \begin{pmatrix} Id & \kappa_\pm^{-1} \\ \kappa_\pm & Id \end{pmatrix} \tag{5.1}$$

with respect to the decomposition $C^\infty(Y,S_0) = C^\infty(Y,S^+) \oplus C^\infty(Y,S^-)$, where $S^\pm \subset S_0$ are the subbundles defined as the $(\pm i)$-eigenspaces of G. Here, the maps $\kappa_\pm : C^\infty(Y,S^+) \to C^\infty(Y,S^-)$ are isometries, so that $U := -\kappa_- \kappa_+^{-1}$ is a unitary operator over $C^\infty(Y,S^-)$. Furthermore, U is of Fredholm determinant class. We denote by $\widehat{U}$ the restriction of U to the orthogonal complement of its (-1)-eigenspace. We also put

$$\mathcal{L} := \sum_{k=1}^{h_M} \gamma_0 U_k \otimes \gamma_0 U_k$$

where $h_M = \dim \ker(\mathcal{D})$, γ_0 is the restriction map from M to Y, and $\{U_k\}$ is an orthonormal basis of $\ker(\mathcal{D})$. Then $\mathcal{L}$ is a positive operator on the finite-dimensional vector space $\gamma_0(\ker(\mathcal{D}))$. We now have all the ingredients to state the following *gluing formula* [12]:

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_+}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L})^{-2} \det_F \Big(\frac{2Id + \widehat{U} + \widehat{U}^{-1}}{4} \Big) \tag{5.2}$$

where $h_Y = \dim \ker(D_Y)$ and $\det_F$ denotes the Fredholm determinant. There is a similar formula for manifolds with cylindrical ends [13].

Using Theorem 1.1, let us verify the gluing formula (5.2) for the Dirac type operator $\mathcal{D}_{R,P}$ of the form (1.1) on $N_R = [-R, R] \times Y$ with boundary conditions (1.2), where we partition N_R into

$$N_R = N_{R,-} \cup N_{R,+}, \qquad N_{R,-} = [-R, 0] \times Y, \quad N_{R,+} = [0, R] \times Y.$$

We denote by $\mathcal{D}_{R,-}$ and $\mathcal{D}_{R,+}$ the restrictions of $\mathcal{D}_{R,P}$ to $N_{R,-}$ and $N_{R,+}$, respectively, with the boundary conditions at $\{0\} \times Y$ given by their corresponding Calderón projectors $\mathcal{C}_{R,-}$ and $\mathcal{C}_{R,+}$, respectively. It is easy to check that $\mathcal{C}_{R,-} = \Pi_< + \frac{Id - \sigma_1}{2} \Pi_0$ and $\mathcal{C}_{R,+} = \Pi_> + \frac{Id - \sigma_2}{2} \Pi_0$. Now it is straightforward to confirm that

$$\frac{\det_\zeta \mathcal{D}^2_{R,P}}{\det_\zeta \mathcal{D}^2_{R,-} \cdot \det_\zeta \mathcal{D}^2_{R,+}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (2R)^{2h} \det{}^* \Big(\frac{2Id - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4} \Big)$$

where we used Theorem 1.1 to compute the left-hand side. Comparing this and (5.2), we see that the following equalities should hold:

$$\begin{aligned} (\det \mathcal{L})^{-2} &= (2R)^{2h}, \\ \det_F \Big(\frac{2Id + \widehat{U} + \widehat{U}^{-1}}{4} \Big) &= \det{}^* \Big(\frac{2Id - (\sigma_1\sigma_2)_- - (\sigma_1\sigma_2)_-^{-1}}{4} \Big), \end{aligned} \tag{5.3}$$

where U and $\mathcal{L}$ are the operators defined before, but now for our finite cylinder operator $\mathcal{D}_{R,P}$. To verify the first equality in (5.3), we note by definition of $\mathcal{D}_{R,P}$,

$$\ker(\mathcal{D}_{R,P}) = \{\varphi \in \ker(D_Y) \mid \sigma_1 \varphi = -\varphi \ \text{ and } \ \sigma_2 \varphi = -\varphi\}. \tag{5.4}$$

It follows that projecting onto S^- gives an isomorphism of $\ker(\mathcal{D}_{R,P})$ to the $(+1)$-eigenspace of $(\sigma_1\sigma_2)_-$, thus $\dim \ker(\mathcal{D}_{R,P}) = h$. Moreover, if $\{\varphi_k\}$ is an orthonormal basis for the right-hand side of (5.4), then the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \sum_{k=1}^{h} \frac{1}{\sqrt{2R}} \varphi_k \otimes \frac{1}{\sqrt{2R}} \varphi_k.$$

This implies the first equality in (5.3). To verify the second equality in (5.3), note that by the definition of U and the formulas for $\mathcal{C}_{R,\pm}$, we have

$$U = Id \quad \text{over } P^-(\ker(D_Y))^\perp, \quad U = -(\sigma_1\sigma_2)_- \quad \text{over } P^-(\ker(D_Y))$$

where $P^- = \frac{Id+iG}{2}$ is the projection onto S^-. This implies the second equality in (5.3). In conclusion, we can see that the gluing formula (5.2) is compatible with Theorem 1.1 for the case of $\mathcal{D}^2_{R,P}$ over N_R.

We now explain the *comparison formula* proved in [14]. To this end, we consider the *smooth, self-adjoint Grassmannian* $Gr^*_\infty(\mathcal{D}_\pm)$, which consists of orthogonal projections $\mathcal{P}_\pm$ such that $G\mathcal{P}_\pm = (Id - \mathcal{P}_\pm)G$ and $\mathcal{P}_\pm - \mathcal{C}_\pm$ are smoothing operators. For $\mathcal{P}_1 \in Gr^*_\infty(\mathcal{D}_-)$, let $\kappa_1 : C^\infty(Y, S^+) \to C^\infty(Y, S^-)$ be the map that determines $\mathcal{P}_1$ as $\kappa_\pm$ does $\mathcal{C}_\pm$ in (5.1). Let $\mathcal{D}_{\mathcal{P}_1}$ denote the operator $\mathcal{D}_-$ on M_- with the boundary condition given by $\mathcal{P}_1$. Let P_1 be the orthogonal projection of $C^\infty(Y, S_0)$ onto the finite-dimensional vector space $\ker(\mathcal{D}_{\mathcal{P}_1})|_Y$. Then we introduce a linear map

$$\mathcal{L}_1 = -P_1\, G\, \mathcal{R}_-^{-1}\, G\, P_1 \quad \text{over} \quad \ker(\mathcal{D}_{\mathcal{P}_1})|_Y$$

where $\mathcal{R}_-$ is the sum of the Dirichlet to Neumann maps on the double of M_- defined as follows. If we denote the double of M_- by $\widetilde{M} = M_- \cup (-M_-)$ and the double of $\mathcal{D}_-$ by $\widetilde{\mathcal{D}}$, then for any $\varphi \in C^\infty(Y, S_0)$, there are unique $\phi_1 \in C^\infty(M_-, S)$ and $\phi_2 \in C^\infty(-M_-, S)$ that are continuous at Y with value φ such that $\widetilde{\mathcal{D}}^2\phi_i = 0$, $i = 1, 2$, off of Y. Then

$$\mathcal{R}_-\,\varphi := \partial_u\phi_1\Big|_Y - \partial_u\phi_2\Big|_Y. \tag{5.5}$$

In [14], we prove that $\mathcal{L}_1$ is a positive operator so that $\det\mathcal{L}_1$ is a positive real number. Now the main result of [14] states that

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1}}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = (\det\mathcal{L}_1)^2 \cdot \det{}_F\Big(\frac{2Id + \widehat{U}_1 + \widehat{U}_1^{-1}}{4}\Big) \tag{5.6}$$

where $\widehat{U}_1$ is the restriction of $U_1 := \kappa_-\kappa_1^{-1}$ to the orthogonal complement of its (-1)-eigenspace. The formula (5.6) generalizes Scott's formula [17] to the case when $\mathcal{D}_{\mathcal{P}_1}$ is not invertible.

Let us verify the comparison formula in (5.6) for $\mathcal{D}_{R,-}$ on $N_{R,-}$ using Theorem 1.1. To this end, we define $\mathcal{D}_{R,1}$ by replacing the boundary condition $\mathcal{C}_{R,-} = \Pi_< + \frac{Id-\sigma_1}{2}\Pi_0$ with $\Pi_< + \frac{Id+\widetilde{\sigma}_1}{2}\Pi_0$ at $\{0\}\times Y$ where $\widetilde{\sigma}_1$ is an involution over $\ker(D_Y)$ anticommuting with G. Then

$$\frac{\det_\zeta \mathcal{D}^2_{R,1}}{\det_\zeta \mathcal{D}^2_{R,-}} = R^{2h_1}\det{}^*\Big(\frac{2Id - (\sigma_1\widetilde{\sigma}_1)_- - (\sigma_1\widetilde{\sigma}_1)_-^{-1}}{4}\Big) \tag{5.7}$$

with h_1 is the number of $(+1)$-eigenvalues of $(\sigma_1\widetilde{\sigma}_1)_-$ and where we used Theorem 1.1 to compute the left-hand side. Hence, comparing the formulas (5.6) and (5.7), we can see that the following equalities should hold:

$$\begin{aligned} &(\det\mathcal{L}_1)^2 = R^{2h_1}, \\ &\det{}_F\Big(\frac{2Id + \widehat{U}_1 + \widehat{U}_1^{-1}}{4}\Big) = \det{}^*\Big(\frac{2Id - (\sigma_1\widetilde{\sigma}_1)_- - (\sigma_1\widetilde{\sigma}_1)_-^{-1}}{4}\Big), \end{aligned} \tag{5.8}$$

where U_1 and $\mathcal{L}_1$ are the operators explained above, but now for our operators $\mathcal{D}_{R,1}$, $\mathcal{D}_{R,-}$. The second equality in (5.8) holds by the same reason as we gave for the operator $\widehat{U}$ before. For the first equality in (5.8), we note that $\ker(\mathcal{D}_{R,1})$ is given by a similar formula to (5.4) but with σ_2 replaced with $\widetilde{\sigma}_1$. This implies that $\dim\ker(\mathcal{D}_{R,1}) = h_1$. To find the operator $\mathcal{L}_1$, we recall that $\mathcal{L}_1 = -P_1\, G\, \mathcal{R}_-^{-1}\, G\, P_1$ and now P_1 denotes the projection onto $\ker(\mathcal{D}_{R,1})|_{\{0\}\times Y}$. Since G exchanges $\operatorname{Im}(P_1)$ and $G\big(\operatorname{Im}(P_1)\big)$, we need to know how $\mathcal{R}_-$ acts over $G\big(\operatorname{Im}(P_1)\big)$. To do so, we note that the double of $N_{R,-}$ is just N_R and the double of $\mathcal{D}_{R,-}$ is just $\mathcal{D}_R$ together with the boundary conditions $\Pi_> + \frac{Id+\sigma_1}{2}\Pi_0$ at $\{-R\}\times Y$ and $\Pi_< + \frac{Id-\sigma_1}{2}\Pi_0$ at $\{R\}\times Y$. We denote this operator by $\widetilde{\mathcal{D}}_{R,-}$. Then, given $\varphi \in G\big(\operatorname{Im}(P_1)\big)$, one can easily check that $\phi_1 \in C^\infty(N_{R,-}, S)$ and $\phi_2 \in C^\infty(N_{R,+}, S)$ defined by

$$\phi_1(u,y) = \varphi + (u/R)\varphi \quad , \quad \phi_2(u,y) = \varphi$$

satisfy $\widetilde{\mathcal{D}}^2_{R,-}\phi_i = 0$, $i = 1,2$, off of $\{0\}\times Y$. Thus, we have

$$\mathcal{L}_1 = -P_1\, G\, \mathcal{R}_-^{-1}\, G\, P_1 = RP_1.$$

One can also derive this formula from Proposition 7.3 in [11]. This shows that the first equality in (5.8) holds, and verifies the compatibility of the comparison formula (5.6) with Theorem 1.1.

We remark that an equality similar to (5.6) holds for the corresponding objects over M_+ with the proper changes taking care of the orientation. Let $\mathcal{P}_2 \in Gr^*_\infty(\mathcal{D}_+)$ and let κ_2, U_2, and $\mathcal{L}_2$ be the corresponding objects for the pair $(\mathcal{D}_+, \mathcal{P}_2)$ defined as we did for $(\mathcal{D}_-, \mathcal{P}_1)$ before. Then combining (5.2) with (5.6) and the comparison formula for $(\mathcal{D}_+, \mathcal{P}_2)$, one can check that

$$\begin{aligned}\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1}\cdot \det_\zeta \mathcal{D}^2_{\mathcal{P}_2}} &= 2^{-\zeta_{D_Y^2}(0)-h_Y}(\det\mathcal{L})^{-2}\det{}_F\Big(\frac{2Id + \widehat{U} + \widehat{U}^{-1}}{4}\Big)\\ &\quad \cdot \prod_{i=1}^{2}(\det\mathcal{L}_i)^{-2}\cdot \det{}_F\Big(\frac{2Id + \widehat{U}_i + \widehat{U}_i^{-1}}{4}\Big)^{-1}.\end{aligned}$$

For more details on this *general gluing formula*, see [12]. As with our previous examples, one can also verify that this general gluing formula is compatible with Theorem 1.1.

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