# *Index theory on singular manifolds II*

# *Introduction to R.B. Melrose's $b$-calculus*

Paul Loya

# *Main idea*

**Definition**

**cal·cu·lus**

n. pl. **cal·cul·li** or **cal·cu·lu·ces**

*A method of analysis or calculation using a special symbolic notation.*

- Today we'll study the $b$-**calculus**: A method of (pseudodifferential) analysis on manifolds with boundary using special "singular" $b$-notation.

# Outline of talk: Four main points

I. The $b$-geometry

# *Outline of talk: Four main points*

I. The $b$-geometry

II. $b$-pseudodifferential operators

# *Outline of talk: Four main points*

I. The $b$-geometry

II. $b$-pseudodifferential operators

III. $b$-trace and normal operators

# Outline of talk: Four main points

I. The $b$-geometry

II. $b$-pseudodifferential operators

III. $b$-trace and normal operators

IV. $b$-proof of APS

## Preview of Part I

- Through a simple change of variables, we take the APS theorem for

*noncompact* manifold with cylindrical ends

and turn it into a result for

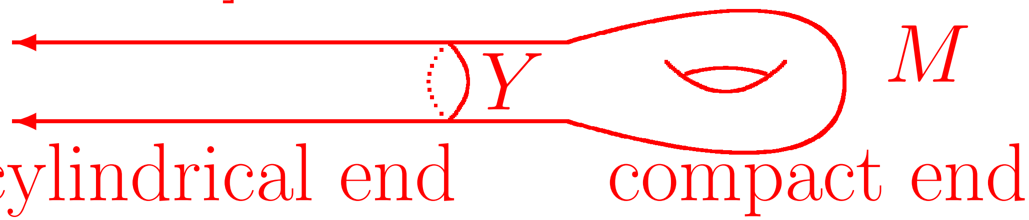
*compact* manifolds with boundary.

- We will be thrown into the new and exciting "$b$-world."

**We start with manifold with cylindrical ends** ...

# I. The $b$-geometry

$$M \cong (-\infty, 0]_s \times Y$$




- Top/Geo Data: Let $E, F$ be Hermitian vector bundles over an even-dimensional, compact, oriented, Riemannian manifold $M$ with cylindrical end.

On the cylindrical end, assume

$$g = ds^2 + h$$

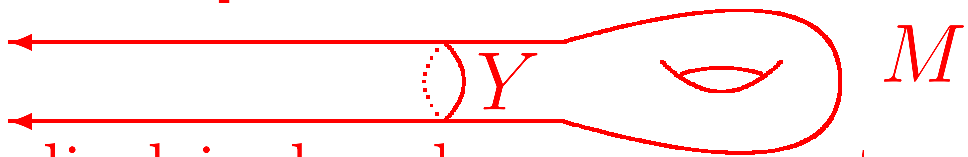
$$E \cong E|_{s=0} \ , \ F \cong F|_{s=0}$$

$$dg = ds\, dh.$$

**Note:** For notational simplicity we drop most “hats” $\widehat{\ }$ from last lecture.

## *I. The $b$-geometry*

$$M \cong (-\infty, 0]_s \times Y$$




cylindrical end compact end

- Functional Ana. Data: Let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a Dirac operator. ($D$ elliptic and $\sigma(D^* D)(\xi) = |\xi|^2$.)

On the cylindrical end, assume

$$\boxed{D = \Gamma(\partial_s + D_Y),}$$

where

$$\Gamma : E|_{s=0} \to F|_{s=0} \quad , \quad \Gamma^* \Gamma = \mathrm{Id},$$

and $D_Y : C^\infty(Y, E|_{s=0}) \to C^\infty(Y, E|_{s=0})$

is a self-adjoint Dirac operator on $Y$.

Let
$\widehat{C}^\infty(M,E) =$ smooth sections that $\to 0$ exp. as $s \to -\infty$.

**Atiyah-Patodi-Singer index theorem (1975):**
$D : \widehat{C}^\infty(M,E) \to \widehat{C}^\infty(M,F)$ is Fredholm, and

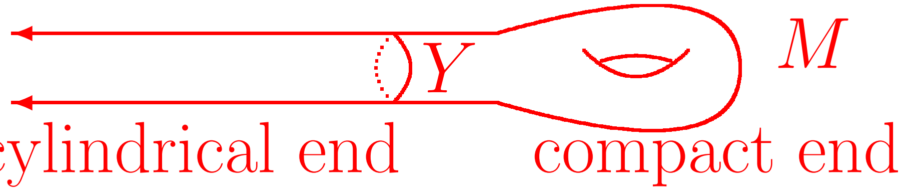
$$\boxed{\mathrm{ind} D = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big),}$$

where $K_{AS}$ is the Atiyah-Singer polynomial and $\eta(D_Y)$ is the eta invariant of $D_Y$:

$$\begin{aligned}\eta(D_Y) &= \text{“}\# \text{ of pos. e.v.} \;-\; \# \text{ of neg. e.v.”} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}\left(D_Y e^{-tD_Y^2}\right) dt.\end{aligned}$$

## I. The $b$-geometry

$$M \cong (-\infty, 0]_s \times Y$$

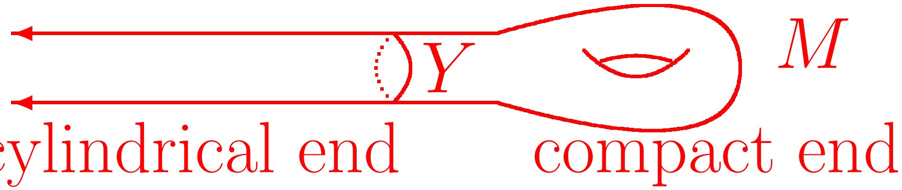



- Common to use Sobolev spaces. Let $H^k(M) =$ usual space on compact part of $M$ and on cylinder part,

$$u \in H^k(M) \iff \partial_s^j \partial_y^\alpha u \in L^2(M) \text{ for } j + |\alpha| \le k.$$

## I. The $b$-geometry

$$M \cong (-\infty, 0]_s \times Y$$




- Common to use Sobolev spaces. Let $H^k(M) =$ usual space on compact part of $M$ and on cylinder part,

$$u \in H^k(M) \iff \partial_s^j \partial_y^\alpha u \in L^2(M) \text{ for } j + |\alpha| \le k.$$

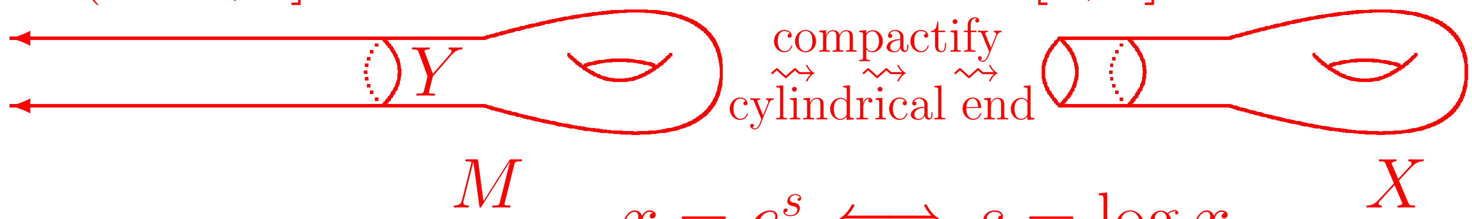
- The APS theorem for $\widehat{C}^\infty$ is in fact equivalent to
**APS index theorem:** For $\varepsilon > 0$ suff. small,
$D : e^{\varepsilon s} H^1(M, E) \to e^{\varepsilon s} L^2(M, F)$ is Fredholm, and

$$\boxed{\operatorname{ind} D = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

- We now change variables to get into the "$b$-world."

- Transformations to "$b$-objects":

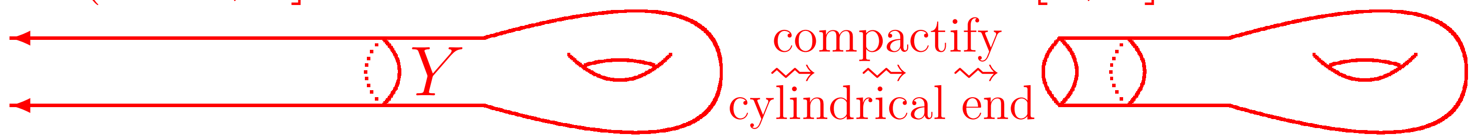
$M \cong (-\infty, 0]_s \times Y$ $\qquad$ $X \cong [0,1]_x \times Y$




$$x = e^s \iff s = \log x$$

- Transformations to "$b$-objects":

$M \cong (-\infty, 0]_s \times Y$ $\qquad\qquad$ $X \cong [0,1]_x \times Y$




$M$ $\qquad\qquad\qquad\qquad$ $X$

$$x = e^s \iff s = \log x$$

$$g = ds^2 + h \;\rightsquigarrow\; g = \left(\frac{dx}{x}\right)^2 + h \quad (b\text{-metric}),$$

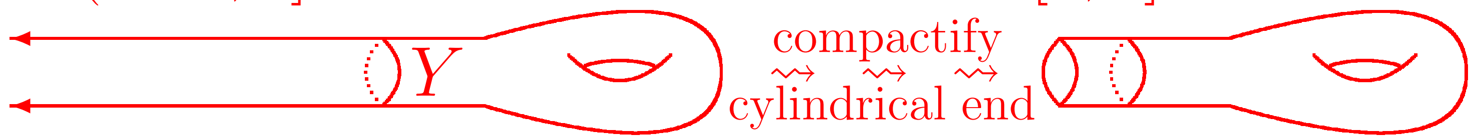
$$dg = ds\, dh \;\rightsquigarrow\; dg = \frac{dx}{x}\, dh \quad (b\text{-measure})$$

$$D = \Gamma(\partial_s + D_Y) \;\rightsquigarrow\; D = \Gamma(x\partial_x + D_Y) \quad (b\text{-diff. operator})$$

$$L^2(M) \;\rightsquigarrow\; L^2_b(X) \quad (L^2_b \text{ space})$$

$$H^k(M) \;\rightsquigarrow\; H^k_b(X) \quad (b\text{-Sobolev space}).$$

$$M \cong (-\infty, 0]_s \times Y \qquad\qquad X \cong [0,1]_x \times Y$$




$M$ $\qquad$ $X$

$$x = e^s \Longleftrightarrow s = \log x$$

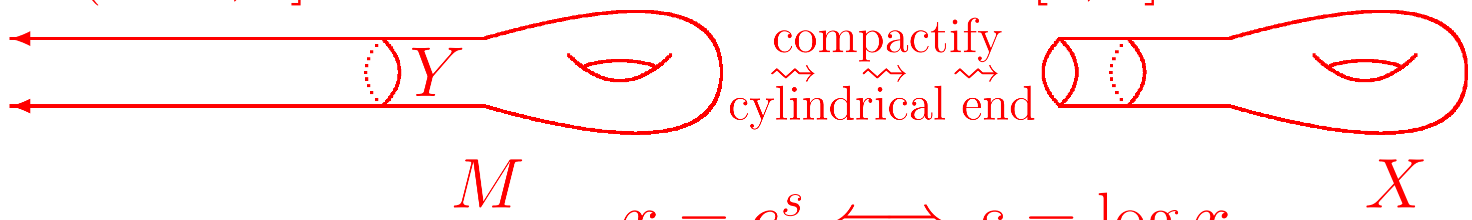
**APS index theorem:** For $\varepsilon > 0$ suff. small,

$D : e^{\varepsilon s} H^1(M, E) \to e^{\varepsilon s} L^2(M, F)$ is Fredholm, and

$$\boxed{\mathrm{ind} D = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

# I. The $b$-geometry

$M \cong (-\infty, 0]_s \times Y$ $\qquad$ $X \cong [0,1]_x \times Y$




$$x = e^s \iff s = \log x$$

**APS index theorem:** For $\varepsilon > 0$ suff. small,

$D : e^{\varepsilon s} H^1(M, E) \to e^{\varepsilon s} L^2(M, F)$ is Fredholm, and

$$\boxed{\operatorname{ind} D = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

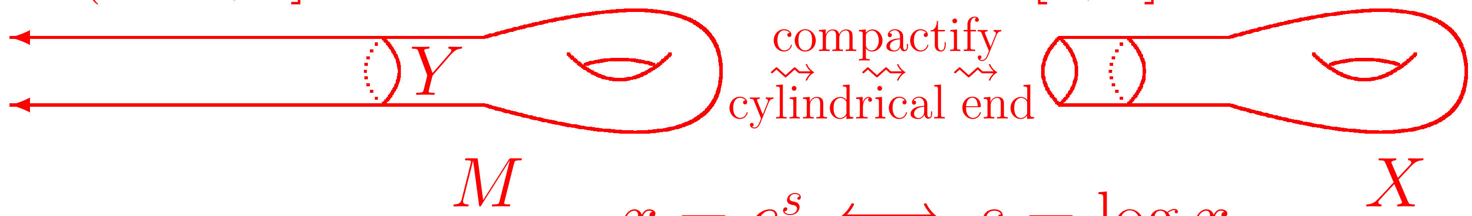
Transforms $\rightsquigarrow$ *weighted* $b$-Sobolev spaces:

$D : x^\varepsilon H_b^1(X, E) \to x^\varepsilon L_b^2(M, F)$ is Fredholm, and

$$\boxed{\operatorname{ind} D = \int_X K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

## Summary of Part I

$M \cong (-\infty, 0]_s \times Y$ $\qquad$ $X \cong [0,1]_x \times Y$




$$x = e^s \Longleftrightarrow s = \log x$$

- We started with an operator on a manifold with cylindrical end:

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

with $D = \Gamma(\partial_s + D_Y)$ over the collar.

- Changing variables, we ended with an operator on a compact manifold with boundary with “$b$-objects”:

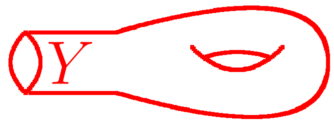
$$D : C^\infty(X, E) \to C^\infty(X, F)$$

with $D = \Gamma(x\partial_x + D_Y)$ over the collar.

## Summary of Part I

- The APS theorem on $M$ can be expressed as a statement about *weighted* Sobolev spaces on $X$. We'll see why we need *weighted* spaces later.

$X \cong [0,1)_x \times Y$




$X$

- Henceforth our focus will mostly be on compact manifold with boundary with "$b$-objects"; e.g. a Dirac operator

$$D : C^\infty(X,E) \to C^\infty(X,F)$$

with $D = \Gamma(x\partial_x + D_Y)$ over the collar.

**Question:** Why study $b$-objects? We'll see in $\ldots$

# II. $b$-pseudodifferential operators

## Preview of Part II

- There is a “global” geometric definition of $\Psi$dos on compact manifolds *without* boundary in terms of their Schwartz kernels.

- $b$-$\Psi$dos are a very close analog on compact manifolds *with* boundary.

# II. $b$-pseudodifferential operators

- $\Psi$dos on $\mathbb{R}^n$

$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a **$\Psi$do of order** $m \in \mathbb{R}$ means:

$$Au = \int_{\mathbb{R}^n} e^{it\cdot\xi} a(t,\xi)\, \widehat{u}(\xi)\, đ\xi,$$

where $đ\xi = \frac{1}{(2\pi)^n} d\xi$,

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-it\cdot\xi} u(t)\, dt = \text{Four. Trans of } u,$$

and $a \in S^m$:

$$\left|\partial_t^\alpha \partial_\xi^\beta a(t,\xi)\right| \le C\,(1+|\xi|)^{m-|\beta|}.$$

**Can you remind us what's the Schwartz kernel?**

## II. $b$-pseudodifferential operators

- Schwartz kernel.

Since $\widehat{u}(\xi) = \displaystyle\int_{\mathbb{R}^n} e^{-it'\cdot\xi} u(t')\,dt'$, we have

$$
\begin{aligned}
Au = \int e^{it\cdot\xi} a(t,\xi)\,\widehat{u}(\xi)\,đ\xi &= \int\int e^{it\cdot\xi - it'\cdot\xi} a(t,\xi)\,u(t')\,dt'\,đ\xi \\
&= \int \Big( \int e^{i(t-t')\cdot\xi} a(t,\xi)\,đ\xi \Big) u(t')\,dt' \\
&= \int K_A(t,t')\,u(t')\,dt'.
\end{aligned}
$$

- Schwartz kernel.

Since $\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-it'\cdot\xi} u(t')\,dt'$, we have

$$Au = \int e^{it\cdot\xi} a(t,\xi)\,\widehat{u}(\xi)\,đ\xi = \int\int e^{it\cdot\xi - it'\cdot\xi} a(t,\xi)\,u(t')\,dt'\,đ\xi$$

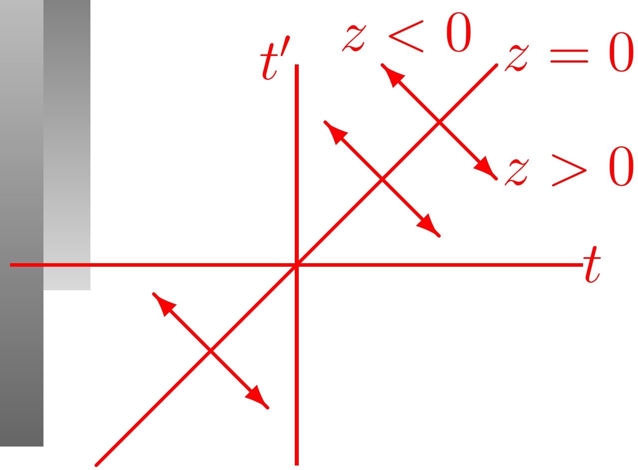
$$= \int \Big( \int e^{i(t-t')\cdot\xi} a(t,\xi)\,đ\xi \Big) u(t')\,dt'$$

$$= \int K_A(t,t')\,u(t')\,dt'.$$

- $K_A(t,t') := \int e^{iz\cdot\xi} a(t,\xi)\,đ\xi \ , \qquad z = t - t'.$

Called the **Schwartz kernel** of $A$.
**What is the geometric description of $\Psi$dos?**

## II. $b$-pseudodifferential operators




(Picture when $n=1$)

$z = t - t'$ is a normal variable to diag.

- Notice that $z = t - t'$ is a normal variable to the diagonal, and

$$K_A(t,t') = \int e^{iz\cdot\xi} a(t,\xi)\, đ\xi$$

$=$ I.F.T. of a symbol in a direction *normal* to diag.

$K_A$ is a distribution on $\mathbb{R}^n \times \mathbb{R}^n$, said to be **conormal** to the diagonal of order $m$. $K_A$ is a **conormal distribution**.

## II. $b$-pseudodifferential operators

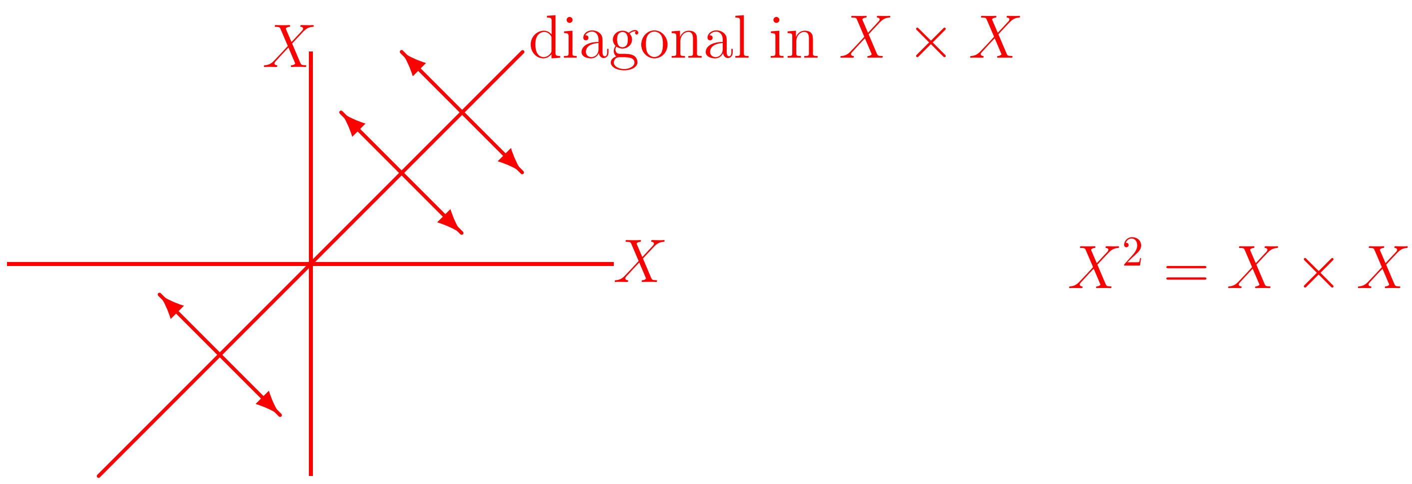
Let $X$ be a compact manifold without boundary. Recall that the Schwartz kernel of an operator

$$A : C^\infty(X) \to C^\infty(X),$$

is a distribution on $X \times X$ that satisfies

$$Au = \int_X K_A(t,t')\, u(t')\, dt'.$$

**Theorem:** $A$ is an $m$-th order $\Psi$do iff $K_A$ is a distribution conormal to the diagonal of order $m$.



## II. $b$-pseudodifferential operators

- $b$-$\Psi$dos

$(-\infty, \infty)$

**Ex:** On $(-\infty, \infty)$, consider $D = \partial_s$. Let's find $K_D$.

## II. $b$-pseudodifferential operators

- $b$-$\Psi$dos

$$(-\infty, \infty)$$

**Ex:** On $(-\infty, \infty)$, consider $D = \partial_s$. Let's find $K_D$.

Writing $u = \int e^{is\tau}\widehat{u}(\tau)\, đ\tau$, then $Du = \int e^{is\tau} i\tau\widehat{u}(\tau)\, đ\tau$,

$$\therefore \quad K_D = \int e^{i(s-s')\tau} i\tau \, đ\tau.$$

- $b$-$\Psi$dos

$$(-\infty, \infty)$$

**Ex:** On $(-\infty, \infty)$, consider $D = \partial_s$. Let's find $K_D$.

Writing $u = \int e^{is\tau} \widehat{u}(\tau)\, đ\tau$, then $Du = \int e^{is\tau} i\tau \widehat{u}(\tau)\, đ\tau$,

$$\therefore \quad K_D = \int e^{i(s-s')\tau} i\tau \, đ\tau.$$

Change variables:

$$(-\infty, \infty) \quad \underset{s = \log x}{\overset{x = e^s}{\rightsquigarrow}} \quad [0, \infty)$$

Then $D = x\partial_x$, and

$$K_D = \int e^{iz\tau} i\tau \, đ\tau$$

where $z = \log x - \log x' = \log\left(\dfrac{x}{x'}\right)$.

## II. $b$-pseudodifferential operators

- A **$b$-$\Psi$do of order** $m \in \mathbb{R}$ on $X = [0, \infty)$ is an operator

$$A : \mathcal{S}(X) \to \mathcal{S}(X)$$

such that

$$\boxed{K_A = \int e^{iz\tau} a(x,\tau)\, đ\tau\ , \qquad z = \log\left(\frac{x}{x'}\right),}$$

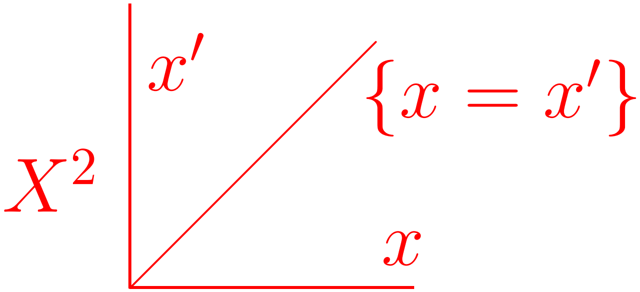
where $a \in S^m$:

$$\left|\partial_x^\alpha \partial_\tau^\beta a(x,\tau)\right| \le C\,(1 + |\tau|)^{m-|\beta|}.$$

(Also require $a$ to be holomorphic in $\tau$ ... a longer story.)

**Is there a geometric description of $b$-$\Psi$dos?**

## II. $b$-pseudodifferential operators




$$X^2 = [0,\infty) \times [0,\infty)$$

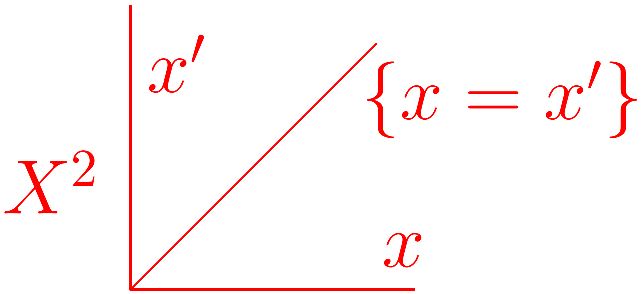
- The Schwartz kernel lives a priori on $X \times X$.

$$K_A = \int e^{iz\tau} a(x,\tau)\, đ\tau \ , \qquad z = \log\left(\frac{x}{x'}\right).$$

**Note:** Want to say $K_A$ is conormal to the diagonal but
**1)** There is no “normal” to the diagonal at the origin!
**2)** $z = \log(\frac{x}{x'})$ is bad at the origin — e.g. $\log(\frac{0}{0}) =$?

What do we do?

# II. $b$-pseudodifferential operators




$$X^2 = [0, \infty) \times [0, \infty)$$

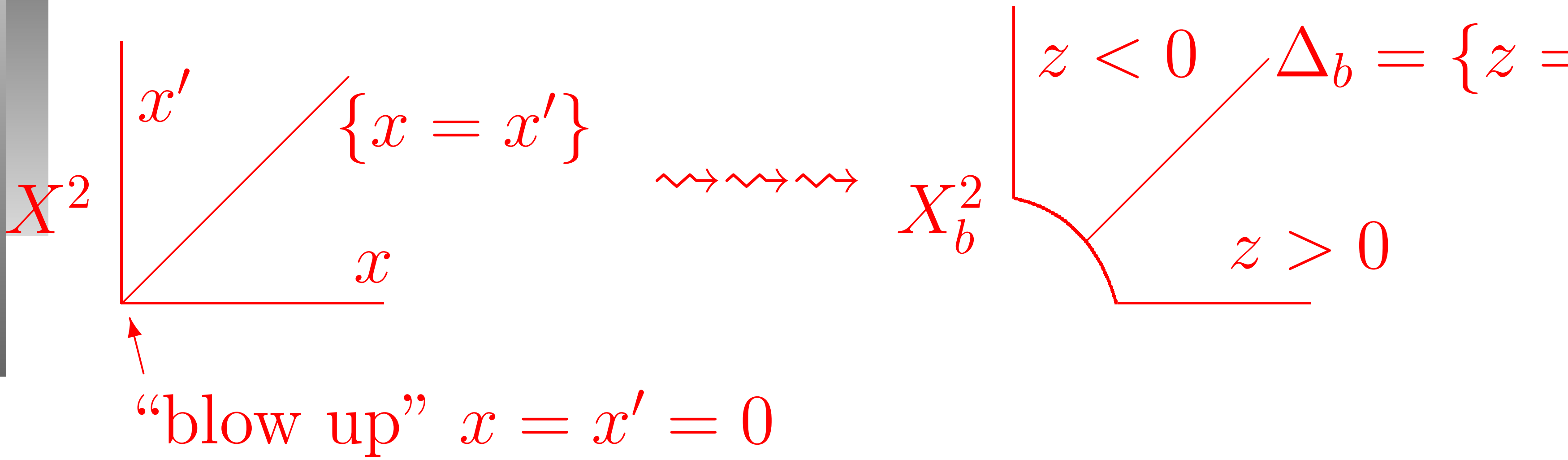
- The Schwartz kernel lives a priori on $X \times X$.

$$K_A = \int e^{iz\tau} a(x,\tau)\, đ\tau\ , \qquad z = \log\left(\frac{x}{x'}\right).$$

**Note:** Want to say $K_A$ is conormal to the diagonal but
**1)** There is no "normal" to the diagonal at the origin!
**2)** $z = \log(\frac{x}{x'})$ is bad at the origin — e.g. $\log(\frac{0}{0}) =$?

What do we do? **Blow-up origin!**

## II. $b$-pseudodifferential operators

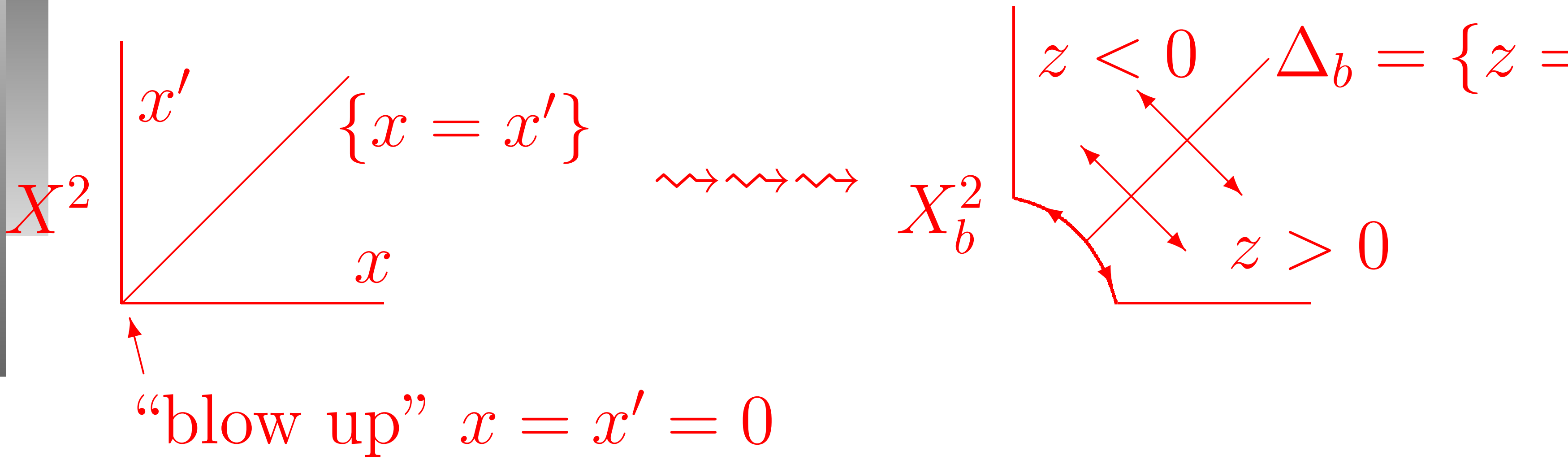



- Blow-up origin to get $X_b^2$: Introduce polar coordinates

$$x = r\cos\theta \quad , \quad x' = r\sin\theta$$

$$\Longrightarrow \qquad z = \log\left(\frac{x}{x'}\right) = \log\left(\frac{\cos\theta}{\sin\theta}\right).$$

**Note:** $z$ is a normal variable to $\Delta_b$.




- Therefore,

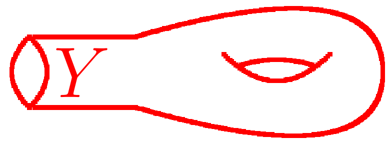
$$K_A = \int e^{iz\tau} a(x,\tau)\, đ\tau\ , \qquad z = \log\left(\frac{x}{x'}\right)$$

$=$ I.F.T. of a symbol in normal direction to $b$-diag.

$\therefore K_A$ is a distribution on $X_b^2$, conormal to the $b$-diagonal of order $m$.

- $b$-$\Psi$dos on manifolds with boundary.

$$X \cong [0,1)_x \times Y$$



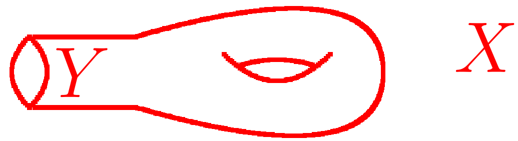

$X$

**What is a $b$-$\Psi$do $A : C^\infty(X) \to C^\infty(X)$?**

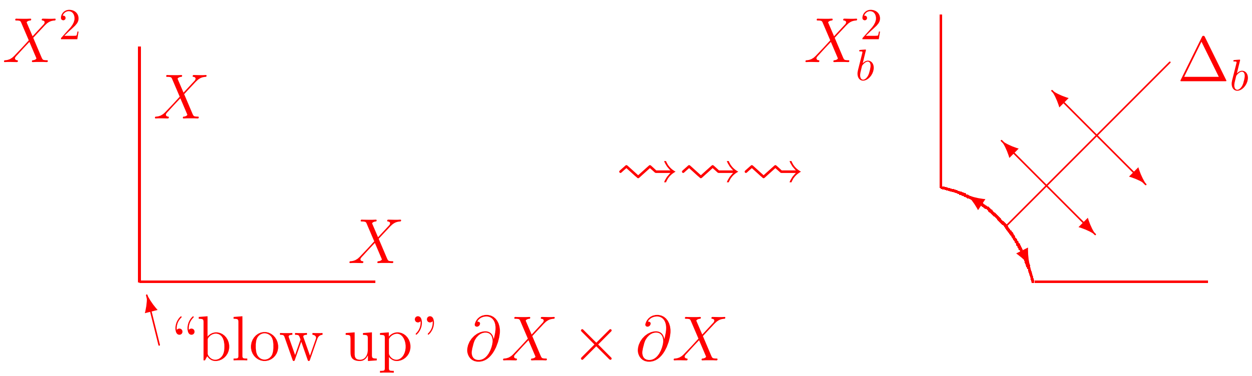
# II. $b$-pseudodifferential operators

- $b$-$\Psi$dos on manifolds with boundary.

$$X \cong [0,1)_x \times Y$$




**What is a $b$-$\Psi$do $A : C^\infty(X) \to C^\infty(X)$?**




**"Global" Geometric Definition:** An operator

$$A : C^\infty(X) \to C^\infty(X)$$

is an element of $\Psi_b^m(X)$, the space of **$b$-$\Psi$dos** of order $m \in \mathbb{R}$ if its Schwartz kernel $K_A$ is a distribution on $X_b^2$, conormal to the $b$-diagonal of order $m$.

### Summary of Part II

- The def. of $b$-$\Psi$dos on compact mwb imitates the global geometric definition of $\Psi$dos on compact manifolds without boundary.

- $b$-$\Psi$do's enjoy (most of) the usual properties you know and love; e.g. there is a symbol map and they behave well under composition, adjoints, etc.

- For $D = x\partial_x$ on $[0,\infty)$ , we have

$$K_D = \int e^{iz\tau} i\tau \, đ\tau \ , \qquad z = \log\left(\frac{x}{x'}\right).$$

## Preview of Part III

**Question:** When does an elliptic operator $A \in \Psi_b^m(X)$ define a Fredholm map

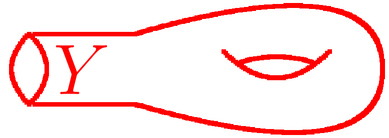
$$A : H_b^m(X) \to L_b^2(X) \quad ?$$

**Answer:** In terms of the normal operator.

**Question:** Are $b$-$\Psi$dos of order $-\infty$ of trace class?

**Answer:** No, but they are "$b$-trace class"

$X \cong [0,1)_x \times Y$



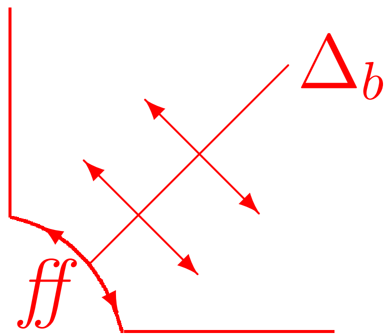

$X$

- Given $A \in \Psi_b^m(X)$ and $\tau \in \mathbb{R}$, the normal operator is a map

$$N(A)(\tau) : C^\infty(Y) \to C^\infty(Y).$$

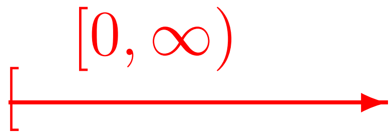
**Geometric Definition:** Recall that $K_A$ is a distribution on $X_b^2$, conormal to $\Delta_b$:

$X_b^2$



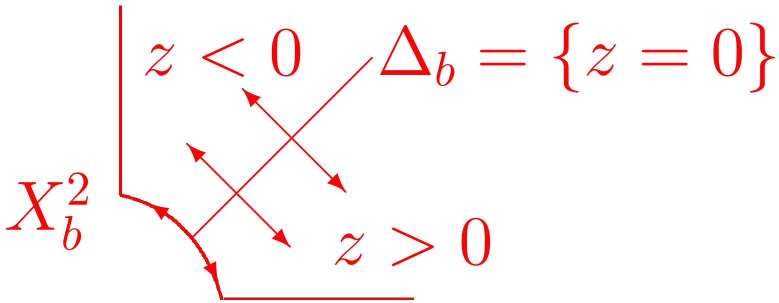

The Schwartz kernel of $N(A)(\tau)$ is obtained by restricting $K_A$ to $ff = \{r = 0\}$ and taking its Fourier transform in $z = \log(x/x')$ evaluated at $\tau$.

## ***III.*** $b$***-trace and normal operator***




**Ex:** On $[0,\infty)$, consider $D = x\partial_x$. Recall that

$$K_D = \int e^{iz\tau} i\tau \, \bar{d}\tau \ , \qquad z = \log\left(\frac{x}{x'}\right).$$

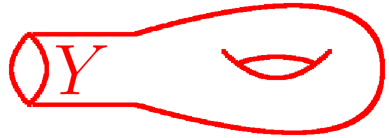



**Exercise:** Show that $\quad N(D)(\tau) = i\tau$

$$\therefore \quad D = x\partial_x \quad \Longrightarrow \quad N(D)(\tau) = i\tau.$$

# III. $b$-trace and normal operator

$$X \cong [0,1)_x \times Y$$




$X$

**Ex:** In the APS situation, recall that over the collar, the Dirac operator $D : C^\infty(X, E) \to C^\infty(X, F)$ takes the form

$$D = \Gamma(x\partial_x + D_Y).$$

Generalizing the previous example, one can show that

$$\boxed{N(D)(\tau) = \Gamma(i\tau + D_Y).}$$

**Now, what are normal oprs good for?**

# III. $b$-trace and normal operator

**Theorem:** For an elliptic $b$-$\Psi$do $A$ of order $m \in \mathbb{R}$,

$$A : H_b^m(X) \to L_b^2(X) \text{ is Fredholm}$$

if and only if for all $\tau \in \mathbb{R}$,

$$N(A)(\tau) : C^\infty(Y) \to C^\infty(Y) \text{ is invertible.}$$

**Theorem:** For an elliptic $b$-$\Psi$do $A$ of order $m \in \mathbb{R}$,

$$A : H_b^m(X) \to L_b^2(X) \text{ is Fredholm}$$

if and only if for all $\tau \in \mathbb{R}$,

$$N(A)(\tau) : C^\infty(Y) \to C^\infty(Y) \text{ is invertible.}$$

**Ex:** $D : H_b^1(X, E) \to L_b^2(X, F)$ is Fredholm iff $\forall \tau \in \mathbb{R}$

$$N(D)(\tau) = \Gamma(i\tau + D_Y)$$

is invertible.

$N(D)(\tau)$ is always invertible for $\tau \neq 0$.
$N(D)(0) = \Gamma D_Y$, which is invertible iff $\ker D_Y = 0$.

**Conclusion:**

$$D : H_b^1(X, E) \to L_b^2(X, F) \text{ is Fredholm} \iff \ker D_Y = 0.$$

# III. $b$-trace and normal operator

**Ex:** For $\varepsilon \in \mathbb{R}$, when is

$$D : x^{\varepsilon} H_b^1(X, E) \to x^{\varepsilon} L_b^2(X, F) \text{ Fredholm?}$$

Equivalently, when is

$$x^{-\varepsilon} D x^{\varepsilon} : H_b^1(X, E) \to L_b^2(X, F) \text{ Fredholm?}$$

## III. $b$-trace and normal operator

**Ex:** For $\varepsilon \in \mathbb{R}$, when is

$$D : x^{\varepsilon} H_b^1(X, E) \to x^{\varepsilon} L_b^2(X, F) \;\text{ Fredholm?}$$

Equivalently, when is

$$x^{-\varepsilon} D x^{\varepsilon} : H_b^1(X, E) \to L_b^2(X, F) \;\text{ Fredholm?}$$

**Exercise:** Show that on the collar,

$$x^{-\varepsilon} D x^{\varepsilon} = \Gamma(x\partial_x + D_Y + \varepsilon).$$

$$\therefore \quad N(x^{-\varepsilon} D x^{\varepsilon})(\tau) = \Gamma(i\tau + D_Y + \varepsilon).$$

**Ex:** For $\varepsilon \in \mathbb{R}$, when is

$$D : x^\varepsilon H_b^1(X,E) \to x^\varepsilon L_b^2(X,F) \text{ Fredholm?}$$

Equivalently, when is

$$x^{-\varepsilon} D x^\varepsilon : H_b^1(X,E) \to L_b^2(X,F) \text{ Fredholm?}$$

**Exercise:** Show that on the collar,

$$x^{-\varepsilon} D x^\varepsilon = \Gamma(x\partial_x + D_Y + \varepsilon).$$

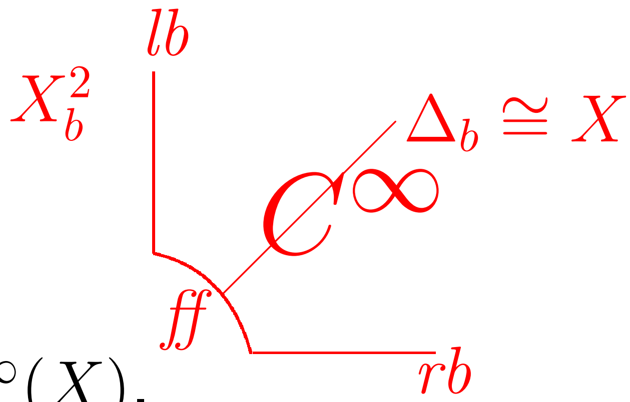
$$\therefore \quad N(x^{-\varepsilon} D x^\varepsilon)(\tau) = \Gamma(i\tau + D_Y + \varepsilon).$$

$$\begin{aligned} \therefore \quad \text{Fredholm} \quad &\iff \quad \ker(D_Y + \varepsilon) \neq 0 \\ &\iff \quad -\varepsilon \text{ is not an e.v. of } D_Y. \end{aligned}$$

In part., Fredholm if we take $\varepsilon \neq 0$ small enough.

# III. $b$-trace and normal operator

- $b$-Trace. Let $X$ be a compact manifold with boundary. An operator $A \in \Psi_b^{-\infty}(X)$ has a Schwartz kernel that is smooth on $X_b^2$:




Thus, $K_A|_{\Delta_b} \in C^\infty(X)$.

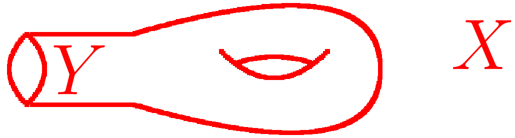
The “obvious” trace is therefore

$$\operatorname{Tr} A := \int_X K_A|_{\Delta_b}.$$

Unfortunately, the RHS is in general not convergent $\ldots$

## III. $b$-trace and normal operator

$X \cong [0, 1)_x \times Y$

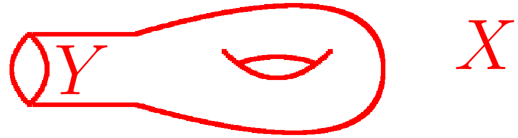



Let $(x, y)$ be coordinates on the collar, and expand $K_A|_{\Delta_b}$ in Taylor series at $x = 0$ to first order:

$$K_A|_{\Delta_b} = a(y) + x\alpha(x, y).$$

## *III. $b$-trace and normal operator*

$X \cong [0,1)_x \times Y$




Let $(x, y)$ be coordinates on the collar, and expand $K_A|_{\Delta_b}$ in Taylor series at $x = 0$ to first order:

$$K_A|_{\Delta_b} = a(y) + x\alpha(x, y).$$

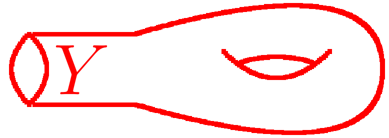
**Recalling:** The measure on $X$ is the $b$-measure $\frac{dx}{x}dh$,

$$\begin{aligned}
\int_{\text{collar}} K_A|_{\Delta_b} &= \int_Y \int_0^1 \Big(a(y) + x\alpha(x, y)\Big)\, \frac{dx}{x}\, dh \\
&= \int_0^1 \frac{dx}{x} \int_Y a(y)\, dh + \int_Y \int_0^1 \alpha(x, y)\, dx\, dh \\
&= \text{a big problem!}
\end{aligned}$$

because $\int_0^1 \frac{dx}{x} = \infty$.

# III. $b$-trace and normal operator

$X \cong [0,1)_x \times Y$




$X$

Remove "big problem" and define

$$\boxed{{}^b\mathrm{Tr}(A) := \int_0^1 \int_Y \alpha(x,y)\, dx\, dh + \int_{X\setminus \text{collar}} K_A|_{\Delta_b}.}$$

Called the $b$-**trace** of $A$.

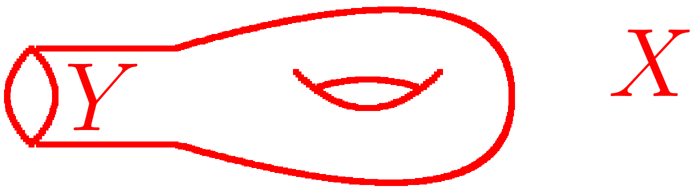
**Warning:** The $b$-trace is *not* a "trace" (e.g. like a trace for matrices or for trace-class operators) because ${}^b\mathrm{Tr}[A,B] \neq 0$ in general. But, we do have a formula $\ldots$

**Theorem:** We have

$$\boxed{{}^b\mathrm{Tr}[A,B] = \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}_Y\left(\partial_\tau N(A)(\tau)\circ N(B)(\tau)\right)\,d\tau,}$$

where $\mathrm{Tr}_Y$ is the trace on $Y$, a compact manifold without boundary.

**Idea:** ${}^b\mathrm{Tr}$ should be a trace on the interior of $X$. Hence, ${}^b\mathrm{Tr}[A,B]$ should only depend only on the boundary $Y$.



## Summary of Part III

- $b$-$\Psi$dos of order $-\infty$ are not trace class in general, but they are always $b$-trace class.

- For $A \in \Psi_b^m(X)$, the normal operator is an operator on the boundary $Y$ depending on a parameter $\tau$:

$$N(A)(\tau) : C^\infty(Y) \to C^\infty(Y).$$

- The normal operator is important for two reasons:

**1)** It determines the Fredholmness of elliptic operators.

**2)** It enters into the formula for ${}^b\mathrm{Tr}[A, B]$.

## Preview of Part IV

- We prove the A-P-S index formula in 3 steps:

Define

$$\boxed{h(t) := {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*}).}$$

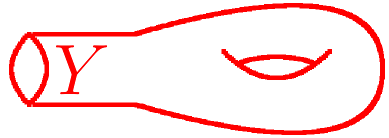
1) Look at $h(\infty) := \lim_{t\to\infty} h(t)$

2) Look at $h(0) := \lim_{t\to 0} h(t)$.

3) Use FTC: $h(\infty) - h(0) = \displaystyle\int_0^\infty h'(t)\,dt$.

# IV. $b$-proof of APS

$X \cong [0,1)_x \times Y$




$X$

**Given:** A Dirac operator

$$D : C^\infty(X, E) \to C^\infty(X, F)$$

such that over the collar,

$$D = \Gamma(x\partial_x + D_Y),$$

where $D_Y : C^\infty(Y, E|_Y) \to C^\infty(Y, E|_Y)$ is a s.a. Dirac operator, and $\Gamma : E|_Y \to F|_Y$ is a bundle map satisfying $\Gamma^*\Gamma = \mathrm{Id}$.

We assume $\ker D_Y = 0$. Then,

$$D : H_b^1(X, E) \to L_b^2(X, F) \text{ is Fredholm.}$$

**Goal:** Compute $\operatorname{ind} D$.

## ***IV. $b$-proof of APS***

**Step 0:** Heat operators.

Consider the Laplacian $L : C^\infty(X) \to C^\infty(X)$. Given $f \in C^\infty(X)$, the **heat equation** is:

$$(\partial_t + L)u(x,t) = 0 \ , \quad u(x,0) = f(x).$$

**Notes:**

- $f$ represents the initial temp. distribution of $X$.
- There always exists a unique solution $u(x,t)$, and this function is the temp. distribution at a future time $t$.

## IV. $b$-proof of APS

**Step 0:** Heat operators.

- For each $t \geq 0$, the solution $u(x,t)$ of

$$(\partial_t + L)u(x,t) = 0 \ , \quad u(x,0) = f(x)$$

can be written as

$$u(x,t) = e^{-tL} f$$

for an operator $e^{-tL} : C^\infty(X) \to C^\infty(X)$. This operator is called the **heat operator**.

- For fixed $t > 0$, $e^{-tL}$ is a $b$-$\Psi$do of order $-\infty$.

- Heat oprs exist in many other cases, not just for Laplacians.

**Step 0:** Heat operators.

- In particular, the heat operators $e^{-tD^*D}$ and $e^{-tDD^*}$ exist and are $b$-$\Psi$dos of order $-\infty$ for all $t > 0$.

- If $X$ were compact *without* boundary, the traces of the heat operators are defined and

$$\mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*})$$

is constant in $t$ and equals $\mathrm{ind}\, D$ — we'll prove this!

**Step 0:** Heat operators.

- In particular, the heat operators $e^{-tD^*D}$ and $e^{-tDD^*}$ exist and are $b$-$\Psi$dos of order $-\infty$ for all $t > 0$.

- If $X$ were compact *without* boundary, the traces of the heat operators are defined and

$$\mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*})$$

is constant in $t$ and equals $\operatorname{ind} D$ — we'll prove this!

- However, in our boundary case, these traces are NOT defined! So, instead we consider

$$\boxed{h(t) := {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*}).}$$

# IV. $b$-proof of APS

**Step 1:** Find $\lim_{t\to\infty} h(t)$.

$$\begin{aligned}\lim_{t\to\infty} h(t) &= \lim_{t\to\infty}\Big({}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})\Big)\\ &= {}^b\mathrm{Tr}(\mathrm{Id}_{\ker D^*D}) - {}^b\mathrm{Tr}(\mathrm{Id}_{\ker DD^*}) \qquad (1)\\ &= \dim\ker(D^*D) - \dim\ker(DD^*) \qquad (2)\\ &= \dim\ker(D) - \dim\ker(D^*) \qquad (3)\\ &= \mathrm{ind}\, D.\end{aligned}$$

**Ideas:**

$(1)$ For $a \geq 0$, $\lim_{t\to\infty} e^{-ta} = 0 = \begin{cases} 0 & \text{if } a > 0 \\ 1 & \text{if } a = 0. \end{cases}$

$(2)$ $\mathrm{Tr}(k \times k$ identity matrix$) = k$.

$(3)$ $\ker(D^*D) = \ker D$ and $\ker(DD^*) = \ker(D)$.

**Step 2:** Find $\lim_{t\to 0} h(t)$.

$$\lim_{t\to 0} h(t) = \lim_{t\to 0} \left( {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*}) \right)$$
$$= \int_X K_{AS}.$$

**Accept this by faith!**

The first "easy" proof of this fact is due to Getzler (1986).

## IV. $b$-proof of APS

**Step 3:** Use FTC:

$$\begin{array}{ccc} h(\infty) & - & h(0) = \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow \\ \operatorname{ind} D & & \int_X K_{AS} \end{array}$$

**Step 3:** Use FTC:

$$\begin{array}{ccc} h(\infty) & - & h(0) = \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow \\ \operatorname{ind} D & & \int_X K_{AS} \end{array}$$

Recalling $h(t) = {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$,

$$h'(t) = -\,{}^b\mathrm{Tr}(D^*De^{-tD^*D}) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*})$$

**Step 3:** Use FTC:

$$\begin{array}{ccc} h(\infty) & - & h(0) = \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow \\ \operatorname{ind} D & & \int_X K_{AS} \end{array}$$

Recalling $h(t) = {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$,

$$\begin{aligned} h'(t) &= -\,{}^b\mathrm{Tr}(D^*De^{-tD^*D}) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*}) \\ &= -\,{}^b\mathrm{Tr}(D^*e^{-tDD^*}D) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*}) \end{aligned}$$

## IV. $b$-proof of APS

**Step 3:** Use FTC:

$$\begin{array}{ccc} h(\infty) & - & h(0) = \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow \\ \operatorname{ind} D & & \int_X K_{AS} \end{array}$$

Recalling $h(t) = {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$,

$$\begin{aligned} h'(t) &= -\,{}^b\mathrm{Tr}(D^*De^{-tD^*D}) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*}) \\ &= -\,{}^b\mathrm{Tr}(\boxed{D^*e^{-tDD^*}}D) + {}^b\mathrm{Tr}(D\boxed{D^*e^{-tDD^*}}) \end{aligned}$$

# IV. $b$-proof of APS

**Step 3:** Use FTC:

$$\begin{array}{ccc} h(\infty) & - & h(0) = \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow \\ \operatorname{ind} D & & \int_X K_{AS} \end{array}$$

Recalling $h(t) = {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$,

$$\begin{aligned} h'(t) &= -\,{}^b\mathrm{Tr}(D^*De^{-tD^*D}) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*}) \\ &= -\,{}^b\mathrm{Tr}(\boxed{D^*e^{-tDD^*}}D) + {}^b\mathrm{Tr}(D\boxed{D^*e^{-tDD^*}}) \\ &= {}^b\mathrm{Tr}\left[D\,,\ D^*e^{-tDD^*}\right]. \end{aligned}$$

**Step 3:** Use FTC:

$$\underset{\substack{\uparrow \\ \operatorname{ind} D}}{h(\infty)} \quad - \quad \underset{\substack{\uparrow \\ \int_X K_{AS}}}{h(0)} = \int_0^\infty h'(t)\,dt$$

Recalling $h(t) = {}^b\mathrm{Tr}(e^{-tD^*D}) - {}^b\mathrm{Tr}(e^{-tDD^*})$,

$$\begin{aligned} h'(t) &= -\,{}^b\mathrm{Tr}(D^*De^{-tD^*D}) + {}^b\mathrm{Tr}(DD^*e^{-tDD^*}) \\ &= -\,{}^b\mathrm{Tr}(\boxed{D^*e^{-tDD^*}}D) + {}^b\mathrm{Tr}(D\boxed{D^*e^{-tDD^*}}) \\ &= {}^b\mathrm{Tr}\left[D\,,\ D^*e^{-tDD^*}\right]. \end{aligned}$$

- If ${}^b\mathrm{Tr}$ *were* a true trace, then $h'(t) = 0 \Longrightarrow$

$$\operatorname{ind} D = \int_X K_{AS} \quad \text{A-S thm.}$$

However, the $b$-trace is NOT a trace! Recall

**Theorem:** We have

$$\boxed{{}^b\mathrm{Tr}[A,B] = \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}_Y\left(\partial_\tau N(A)(\tau)\circ N(B)(\tau)\right)\,d\tau,}$$

where $\mathrm{Tr}_Y$ is the trace of operators on $Y$.

Now,

$$\begin{array}{ccccc} h(\infty) & - & h(0) & = & \int_0^\infty h'(t)\,dt \\ \uparrow & & \uparrow & & \\ \mathrm{ind}\,D & & \int_X K_{AS} & & \end{array}$$

where

$$h'(t) = {}^b\mathrm{Tr}\left[D\,,\ D^* e^{-tDD^*}\right].$$

**Use Theorem to find $h'(t)$!**

## IV. $b$-proof of APS

**Exercise:** Recalling that $N(D)(\tau) = \Gamma(i\tau + D_Y)$, use the theorem to prove that

$$h'(t) = {}^b\mathrm{Tr}\left[D\,,\ D^* e^{-tDD^*}\right] = -\frac{t^{-1/2}}{2\sqrt{\pi}}\,\mathrm{Tr}_Y\left(D_Y e^{-tD_Y^2}\right).$$

## IV. $b$-proof of APS

**Exercise:** Recalling that $N(D)(\tau) = \Gamma(i\tau + D_Y)$, use the theorem to prove that

$$h'(t) = {}^b\mathrm{Tr}\left[D\,,\ D^* e^{-tDD^*}\right] = -\frac{t^{-1/2}}{2\sqrt{\pi}}\,\mathrm{Tr}_Y\left(D_Y e^{-tD_Y^2}\right).$$

**Finish Proof:**

$$\begin{aligned} \operatorname{ind} D - \int_X K_{AS} = h(\infty) - h(0) &= \int_0^\infty h'(t)\,dt \\ &= -\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}_Y\left(D_Y e^{-tD_Y^2}\right) dt \\ &= -\frac{1}{2}\eta(D_Y). \end{aligned}$$

**This is exactly the APS formula when $\dim\ker D_Y = 0$!**

## Summary of Part IV

- The $b$-proof of the APS theorem is the “same” as for the AS theorem in the boundaryless case.

- Only difference: The $b$-trace is used instead of the regular trace.

- The appearance of the eta-invariant is just a simple computation involving the $b$-trace formula for a commutator.

## *Summary of Talk*

**Question:** What are $b$-objects?

**Answer:** The geometric objects obtained from a manifold with cylindrical end by compactifying it.

**Question:** What is the $b$-calculus?

**Answer:** $\Psi$dos on a compact manifolds with boundary obtained by imitating the geometric definition of $\Psi$dos for boundaryless manifolds + tools like the normal operator and $b$-trace $\ldots$.

**Question:** How is the APS theorem proved?

**Answer:** In the same way as the AS theorem $\ldots$ using the FTC!